The Annals of Probability
2003, Vol. 31, No. 2, 592–608

ON ASYMPTOTIC ERRORS IN DISCRETIZATION OF PROCESSES

BY J. JACOD, A. JAKUBOWSKI AND J. MÉMIN

Université Paris 6, Nicholas Copernicus University and Université de Rennes-1

We study the rate at which the difference $X_t^n = X_t - X_{[nt]/n}$ between a process X and its time-discretization converges. When X is a continuous semimartingale it is known that, under appropriate assumptions, the rate is $\sqrt{n}$, so we focus here on the discontinuous case. Then $\alpha_n X^n$ explodes for any sequence α_n going to infinity, so we consider "integrated errors" of the form $Y_t^n = \int_0^t X_s^n\,ds$ or $Z_t^{n,p} = \int_0^t |X_s^n|^p\,ds$ for $p \in (0,\infty)$: we essentially prove that the variables $\sup_{s\le t} |nY_s^n|$ and $\sup_{s\le t} nZ_s^{n,p}$ are tight for any finite t when X is an arbitrary semimartingale, provided either $p \ge 2$ or $p \in (0,2)$ and X has no continuous martingale part and the sum $\sum_{s\le t} |\Delta X_s|^p$ converges a.s. for all $t < \infty$, and in addition X is the sum of its jumps when $p < 1$. Under suitable additional assumptions, we even prove that the discretized processes $nY_{[nt]/n}^n$ and $nZ_{[nt]/n}^{n,p}$ converge in law to nontrivial processes which are explicitly given.

As a by-product, we also obtain a generalization of Itô's formula for functions that are not twice continuously differentiable and which may be of interest by itself.

1. Introduction. Let X be a càdlàg real-valued process on a space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$, and consider the associated discretized process $\widetilde{X}^n$ and the "error process" X^n:

$$\widetilde{X}_t^n = X_{[nt]/n}, \qquad X_t^n = X_t - \widetilde{X}_t^n, \tag{1.1}$$

where $[r]$ denotes the integer part of any positive real r. It is well known that $\widetilde{X}^n$ converges pathwise to X for the Skorokhod J_1 topology. Then a natural question arises, namely at which rate does this convergence take place.

When X is continuous, then $\sup_{s\le t} |X_s^n|$ is in between half the modulus of continuity of X for the size $1/n$ and this modulus over the time interval $[0,t]$, so the problem above is solved in a trivial way (see Remark 7 for discussion of this case). On the other hand, as soon as X has discontinuities, the error process X^n does not even converge to 0 in the Skorokhod sense, and we thus have to use a different sort of measurement for the discrepancy if we wish to obtain convergence rates.

A possibility among others is to consider integrated errors of the following type, where $p \in (0,\infty)$:

$$Y^n(X)_t = \int_0^t X_s^n\,ds, \qquad Z^{n,p}(X)_t = \int_0^t |X_s^n|^p\,ds. \tag{1.2}$$

Received June 2001; revised April 2002.
AMS 2000 subject classifications. 60F17, 60H99.
Key words and phrases. Discretization, semimartingales, Lévy processes.

Let us start with the case when X is a Lévy process, with Lévy exponent φ_X, that is, $E(e^{iuX_t}) = e^{t\varphi_X(u)}$. Then one can prove in a very elementary way (see Section 2) the following.

THEOREM 1.1. *If X is a Lévy process with Lévy exponent φ_X, then $nY^n(X)$ converges finite-dimensionally (but not functionally, unless X is continuous) in law to another Lévy process Y, whose Lévy exponent is $\varphi_Y(u) = \int_0^1 \varphi_X(uy)\,dy$.*

The process Y is continuous iff X itself is continuous, and otherwise we cannot have functional convergence (in the J_1 Skorokhod sense) since the processes $nY^n(X)$ are always continuous themselves.

Note that the laws of all Y_t are s-selfdecomposable, or equivalently of "class $\mathcal{U}$," a class of infinitely divisible distributions introduced by Jurek [5]: see in particular Theorem 2.9 in Jurek [6]. Conversely any Lévy process with s-selfdecomposable distribution may be obtained as the limit of processes $nY^n(X)$ as above.

Of course $Y^n(X)$ is not a genuine measure of the discrepancy, since there might be compensations between positive and negative contributions within the integral. So let us examine $Z^{n,p}(X)$. For this we denote by (b, c, F) the characteristics of the law of X_1 w.r.t. some truncation function h (a bounded function with compact support, equal to the identity in a neighborhood of 0), that is $\varphi_X(u) = iub - \frac{u^2c}{2} + \int(e^{iux} - 1 - iuh(x))F(dx)$. Then we set for $p \in (0, \infty)$:

$$V_t^p = \begin{cases} \sum_{s\le t} |\Delta X_s|^p, & \text{if } p \neq 2, \\ [X, X]_t = ct + \sum_{s\le t} |\Delta X_s|^2, & \text{if } p = 2. \end{cases} \tag{1.3}$$

Observe that V_t^p is either a.s. infinite for all $t > 0$, or a.s. finite for all t. The later holds always when $p \ge 2$, and when $p < 2$ it holds if and only if F integrates $x \mapsto |x|^p$ near the origin. In this case, the process V^p is again a Lévy process, whose Lévy exponent is denoted by φ_{V^p}.

THEOREM 1.2. *If X is a Lévy process, then the sequence $nZ^{n,p}(X)$ converges finite-dimensionally in law to a Lévy process Z^p whose Lévy exponent is $\varphi_{Z^p}(u) = \int_0^1 \varphi_{V^p}(uy)\,dy$, in the following cases*:

(i) $p \ge 2$,

(ii) $1 < p < 2$, *if $c = 0$ and F integrates $x \mapsto |x|^p$ near the origin*,

(iii) $0 < p \le 1$, *if $c = 0$ and $b = \int F(dx)h(x)$ and F integrates $x \mapsto |x|^p$ near the origin*. (Note that in this case we have $X_t = \sum_{s\le t} \Delta X_s$.)

This result is somewhat unexpected: one would rather imagine that there exists a sequence u_n going to infinity and such that $\int_0^t |u_n X_s^n|^p\,ds$ converges in law, or is tight, for all p in a suitable range; here, the sequence u_n is $u_n = n^{1/p}$, depending on p, and thus there is no "convergence rate" in the usual sense.

This behavior is due to the jumps of X, and is already present when X is a Poisson process. In this case indeed, for all n big enough (depending on the path), X^n takes only the values 0 and 1 and thus $Z^{n,p}(X)_t = Y^n(X)_t$ does not depend on p and equals the Lebesgue measure of the set $\{s : 0 \le s \le t,\ X^n_s \ne 0\}$.

The above two theorems can be generalized in three directions. First, we can obtain functional convergence in law for the processes $nY^n(X)$ and $nZ^{n,p}(X)$, provided we discretize them in time; so we will consider in fact the following processes:

$$\widetilde{Y}^n(X)_t = nY^n(X)_{[nt]/n}, \qquad \widetilde{Z}^{n,p}(X)_t = nZ^{n,p}(X)_{[nt]/n}. \tag{1.4}$$

Second, we obtain joint convergence in law for the triples $(\widetilde{X}^n, \widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$ towards a limit of the form (X, Y, Z): this gives more insight, in particular because it makes the dependence of Y or Z^p upon X explicit. Even slightly stronger than this, we obtain stable convergence in law of the pair $(\widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$, a notion introduced by Renyi [8] and for which we refer to [4]. Third, we extend the results for X being a semimartingale, for which we still use the notations (1.2) and (1.4).

We can state two results: the first one is a tightness result, true for *any* semimartingale X; the second one is a limit theorem and needs additional structure for X, and also for the underlying probability space. We always denote by (B, C, ν) the predictable characteristics of X, w.r.t. a fixed truncation function h (see, e.g., [4] for this notion). Two conditions will play a role below. The first one is

$$\int_0^t \int_{\{|x|\le 1\}} |x|^p \nu(ds, dx) < \infty \qquad \text{a.s.}, \forall t \in \mathbb{R}_+. \tag{1.5}$$

This is always satisfied for $p \ge 2$, and if it holds for some p it also holds for all $p' > p$. This condition is equivalent to the following one (see Section 3 below):

$$\sum_{s \le t} |\Delta X_s|^p < \infty \qquad \text{a.s.}, \forall t \in \mathbb{R}_+. \tag{1.6}$$

The second condition makes sense as soon as the previous one holds for some $p \le 1$:

$$B_t = \int_0^t \int h(x)\nu(ds, dx) \qquad \text{a.s.}, \forall t \in \mathbb{R}_+. \tag{1.7}$$

When (1.5) holds for $p = 1$ and $C = 0$, then (1.7) is equivalent to having $X_t = X_0 + \sum_{s \le t} \Delta X_s$. Note that (1.7) does not depend on the chosen truncation h.

THEOREM 1.3. *Let X be a semimartingale.*

(a) *The sequence of two-dimensional processes $(\widetilde{X}^n, \widetilde{Y}^n(X))$ is tight (for the Skorokhod J_1 topology), and further the sequence of real random variables $\sup_{s\in[0,t]} |nY^n(X)_s|$ is tight for all $t < \infty$ and $nY^n(X)_t - \widetilde{Y}^n(X)_t \to 0$ a.s. for each t such that $P(\Delta X_t = 0) = 1$.*

(b) *The sequence of three-dimensional processes $(\widetilde{X}^n, \widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$ is tight in the following three cases*:

(i) $p \geq 2$,
(ii) $1 \leq p < 2$, *when $C = 0$ (equivalently $X^c = 0$) and* (1.5) *holds*,
(iii) $0 < p < 1$, *when $C = 0$ and* (1.5) *and* (1.7) *hold.*

Further the sequence of real random variables $\sup_{s\in[0,t]} nZ^{n,p}(X)_s$ is tight for all $t < \infty$, and $nZ^{n,p}(X)_t - \widetilde{Z}^{n,p}(X)_t \to 0$ in probability for each t such that $P(\Delta X_t = 0) = 1$.

REMARK. We can of course extract convergent subsequences in (a) and (b) above, but the original sequences themselves do not converge in general. Take for example the deterministic process $X_t = \mathbb{1}_{[a,\infty]}(t)$, where a is an irrational number; then $\widetilde{Y}^n(X)_t = 1 + [an] - an$ for all $t \geq a + 1/n$, and the sequence $(1 + [na] - na)_{n\geq 1}$ does not converge.

For describing the limiting processes of the above sequences, when we can prove that they converge, we need additional notation. Recall that we can write our semimartingale as

$$\begin{aligned} X_t = X_0 + B_t + X_t^c + \int_0^t \int h(x)(\mu - \nu)(ds, dx) \\ + \int_0^t \int \big(x - h(x)\big)\mu(ds, dx), \end{aligned} \tag{1.8}$$

where X^c is the continuous martingale part of X and μ is its jump measure. We also denote by (T_n) a sequence of stopping times which exhausts the jumps of X: that is, $T_n \neq T_m$ if $n \neq m$ and $T_n < \infty$, and $\Delta X_s \neq 0$ iff there exists n (necessarily unique) such that $s = T_n$.

We consider an extension of the original space, on which we define a Brownian motion W and a sequence (U_n) of variables uniformly distributed over [0, 1], all mutually independent and independent of $\mathcal{F}$. We consider the random measure on $\mathbb{R}_+ \times \mathbb{R} \times [0, 1]$:

$$\widehat{\mu}(ds, dx, du) = \sum_{n\geq 1: T_n < \infty} \varepsilon_{(T_n, \Delta X_{T_n}, U_n)}(ds, dx, du),$$

whose predictable compensator is

$$\widehat{\nu}(ds, dx, du) = \nu(ds, dx) \otimes du. \tag{1.9}$$

We also need two additional properties. First, we say that the *martingale representation property holds w.r.t. X* if any martingale on our original space can be written as $N_t = N_0 + \int_0^t v_s\, dX_s^c + \int_0^t \int_{\mathbb{R}} U(s,x)(\mu - \nu)(ds, dx)$ for some predictable process v and some predictable function U on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$. Next, we consider a factorization property of the characteristics (B, C, ν), namely that

$$
\begin{aligned}
B_t(\omega) &= \int_0^t b_s(\omega)\, ds, \\
C_t(\omega) &= \int_0^t c_s(\omega)\, ds, \qquad \nu(\omega, ds, dx) = ds \times F_s(\omega, dx).
\end{aligned}
\tag{1.10}
$$

Observe that any Lévy process X satisfies (1.10) with $b_s(\omega) = b$ and $c_s(\omega) = c$ and $F_s(\omega, dx) = F(dx)$, and also the martingale representation property when the filtration is the one generated by X itself.

THEOREM 1.4. *Assume* (1.10) *and the martingale representation property w.r.t. X. Then the sequence $(\widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$ converges stably in law to a limiting process (Y, Z^p)* [and thus $(\widetilde{X}^n, \widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$ converges in law to (X, Y, Z^p)], *in the following three cases*:

(i) $p \geq 2$,
(ii) $1 < p < 2$ *and* $C = 0$ (*equivalently* $X^c = 0$) *and* (1.5) *holds*,
(iii) $0 < p \leq 1$ *and* $C = 0$ *and* (1.5) *and* (1.7) *hold.*

In these cases the limiting process can be defined on the above extension of our space by

$$
\begin{aligned}
Y_t = \frac{1}{2} B_t + \frac{1}{2} X_t^c &+ \frac{1}{\sqrt{12}} \int_0^t \sqrt{c_s}\, dW_s \\
&+ \int_0^t \int_{\mathbb{R}} \int_{[0,1]} h(x) u (\widehat{\mu} - \widehat{\nu})(ds, dx, du) \\
&+ \int_0^t \int_{\mathbb{R}} \int_{[0,1]} \big(x - h(x)\big) u \widehat{\mu}(ds, dx, du),
\end{aligned}
\tag{1.11}
$$

$$
Z_t^p = \begin{cases} \frac{1}{2} C_t + \int_0^t \int_{\mathbb{R}} \int_{[0,1]} u x^2 \widehat{\mu}(ds, dx, du), & \text{if } p = 2, \\ \int_0^t \int_{\mathbb{R}} \int_{[0,1]} u |x|^p \widehat{\mu}(ds, dx, du), & \text{if } p \neq 2. \end{cases}
\tag{1.12}
$$

Moreover, the pairs $(nY^n(X), nZ^{n,p}(X))$ converge finite-dimensionally stably in law to (Y, Z^p).

We can also write the last integral in (1.11) and the integrals in (1.12), respectively, as follows:

$$
\sum_{n \geq 1: T_n \leq t} U_n \big(\Delta X_{T_n} - h(\Delta X_{T_n})\big), \qquad \sum_{n \geq 1: T_n \leq t} U_n |\Delta X_{T_n}|^p,
$$

but such a simple expression is in general not available for the second stochastic integral arising in (1.11).

REMARKS. (1) Of course the expressions for Y and Z^p do not depend on the particular choice of the truncation function h, since changing h changes B accordingly.

(2) When X is a Lévy process, the triple (X, Y, Z^p) is also a Lévy process (on the extended space), and an elementary computation shows that the Lévy exponents of Y and Z^p are $\varphi_Y(u) = \int_0^1 \varphi_X(uy)\,dy$ and $\varphi_{Z^p}(u) = \int_0^1 \varphi_{V^p}(uy)\,dy$: hence Theorems 1.1 and 1.2 are particular cases of Theorem 1.4. When X is not a Lévy process the pair (Y, Z^p) is not a Lévy process either, but it is an $\mathcal{F}$-conditional process with independent increments, in the sense of [2].

(3) If we are interested only in the convergence in law of $(\widetilde{X}^n, \widetilde{Y}^n(X), \widetilde{Z}^{n,p}(X))$ to (X, Y, Z^p), it is enough to have the martingale representation property w.r.t. X holds for the filtration generated by X itself (which may be smaller than the original one). More generally, we could probably drop the martingale representation property, which is a priori unrelated with our result. But this would require results which are not explicitly stated in [2].

(4) There is a gap between Theorems 1.3 and 1.4: when $p = 1$ we have tightness as soon as $C = 0$ and (1.5) holds, while for the convergence of $(\widetilde{Z}^{n,p})$ we need in addition (1.7). When (1.5) does not hold for some $p \in (0, 2)$ we do not know whether the sequence $(\widetilde{Z}^{n,p})$ is tight, but since then the last expression in (1.9) is infinite we conjecture that it is not the case.

(5) We could prove even more, indeed: let p be as in Theorem 1.4. The family $Z^{n,p'}(X)_t$ is defined simultaneously for all values of p', while the $Z_t^{p'}$'s are defined simultaneously for all $p' \geq p$, and further these processes depend continuously on p'. Then we could prove that the pair $(\widetilde{Y}^n(X), (\widetilde{Z}^{n,p'}(X))_{p' \geq p})$ converges stably in law to $(Y, (Z^{p'})_{p' \geq p})$, on the Skorokhod space of functions taking their values in $\mathbb{R} \times C([p, \infty), \mathbb{R})$ equipped with the product of the usual topology on $\mathbb{R}$ and the local uniform topology on the space $C([p, \infty), \mathbb{R})$ of real-valued continuous functions on $[p, \infty)$.

(6) When X is continuous and unless $p = 2$ the limiting process Z^p vanishes. In fact, one could prove that for any $p \geq 0$ the sequence $(n^{p/2} Z^{n,p}(X))_n$ is tight as soon as X is a continuous semimartingale, and it converges in law if in addition (1.10) holds.

(7) The limiting processes obtained in Theorem 1.4 are reminiscent of those in [3], but the context is different: in the quoted paper, and unlike here, we have genuine rates of convergence. However, it is quite likely that *any* type of discretization for discontinuous processes gives rise to the same kind of limiting processes, after a normalization which of course depends on the way the discretization is done.

The paper is organized as follows: in Section 2 we give an elementary proof of Theorem 1.1, which does not use Theorem 1.4. In Section 3 we give an extension of Itô's formula which has interest of its own and which allows us to prove the result when $p < 2$. Then Theorems 1.3 and 1.4 are proved in Sections 4 and 5 respectively.

2. An elementary proof of Theorem 1.1. Set $Y^n = Y^n(X)$ and $\widetilde{Y}^n = \widetilde{Y}^n(X)$ [recall (1.4)]. It is well known that when X is a Lévy process with exponent φ_X, then $\int_0^t X_s\,ds$ has an infinitely divisible law with characteristic exponent $\int_0^t \varphi_X(us)\,ds$. Then the characteristic exponent of $nY_t^n - \widetilde{Y}_t^n$ is

$$\int_0^{t-[nt]/n} \varphi_X(nus)\,ds = \frac{1}{n}\int_0^{nt-[nt]} \varphi_X(ul)\,dl,$$

which goes to 0 as $n \to \infty$. Hence $nY_t^n - \widetilde{Y}_t^n$ goes to 0 in probability, and we are left to prove that the sequence $\widetilde{Y}^n$ converges finite-dimensionally to Y.

Now each process $\widetilde{Y}^n$ has (nonstationary) independent increments, and for all $s < t$ the variable $\widetilde{Y}_t^n - \widetilde{Y}_s^n$ has the same law as $\widetilde{Y}^n_{u_n(s,t)}$ for some $u_n(s,t)$ going to $t - s$. Therefore it is enough to prove that $\widetilde{Y}^n_{t_n}$ converges in law to Y_t as soon as $t_n \to t$. But the $\widetilde{Y}^n_{i/n} - \widetilde{Y}^n_{(i-1)/n}$ are i.i.d. (when $i = 1, 2, \dots$) with characteristic exponents $\int_0^{1/n} \varphi_X(us)\,ds$, hence

$$E\big(\exp\{iu\widetilde{Y}^n_{t_n}\}\big) = \exp\bigg\{[nt_n]\int_0^{1/n} \varphi_X(uns)\,ds\bigg\} = \exp\bigg\{\frac{[nt_n]}{n}\int_0^1 \varphi_X(uy)\,dy\bigg\},$$

and the result readily follows.

3. An extension of Itô's formula. Let X be any semimartingale. The process $H(p)_t = \sum_{s\le t} |\Delta X_s|^p \mathbb{1}_{\{|\Delta X_s|\le 1\}}$ has bounded jumps and admits the left-hand side of (1.5) for predictable compensator. Hence (1.5) holds if and only if $H(p)$ is a.s. finite-valued: since obviously $\sum_{s\le t} |\Delta X_s|^p \mathbb{1}_{\{|\Delta X_s|>1\}} < \infty$ for all t, we have that (1.5) and (1.6) are equivalent.

For proving Theorems 1.2 and 1.3 we need to apply Itô's formula with the function $f(x) = |x|^p$, which is not of class C^2 when $p < 2$. To be more precise, remember that

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,dX_s + \tfrac{1}{2}\int_0^t f''(X_{s-})\,dC_s + W(f)_t \tag{3.1}$$

for any C^2 function f, where

$$W(f)_t = \sum_{s\le t} \eta_f(X_{s-}, \Delta X_s) \quad \text{and} \quad \eta_f(x,y) = f(x+y) - f(x) - f'(x)y. \tag{3.2}$$

Here since (1.6) holds for $p \ge 2$ and $\eta_f(x,y)$ behaves at most like y^2 for small y, the sum defining $W(f)_t$ is a.s. absolutely convergent.

We would like to have (3.1) for more general functions f, *without additional terms* like local times (in contrast with the generalized Itô's formula for convex functions f when X is continuous). Since f'' explicitly shows in (3.1) unless $C = 0$, we will have to assume first that $C = 0$ (or equivalently $X^c = 0$), in which case (3.1) becomes

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,dX_s + W(f)_t. \tag{3.3}$$

If further (1.5) for $p = 1$ and (1.7) hold, then indeed $X_t = X_0 + \sum_{s\le t} \Delta X_s$; then the first derivative in (3.1) also disappears and we have

$$f(X_t) - f(X_0) = \sum_{s\le t} \bigl(f(X_{s-} + \Delta X_s) - f(X_{s-})\bigr). \tag{3.4}$$

The next result shows indeed that (3.3) or (3.4) hold for functions f which are not C^2 but have some weaker regularity, in connection with the reals p for which (1.5) holds. Recall that g is said to be Hölder continuous with index ρ if for all $K > 0$ there is a constant C_K such that $|g(x + y) - g(x)| \le C_K |y|^\rho$ whenever $|x| \le K$ and $|y| \le 1$.

THEOREM 3.1. *Assume that* (1.5) *holds for some* $p \in [0, 2)$ *and that* $C = 0$.

(i) (3.3) *holds when* $p \in (1, 2)$ *and* f *is a differentiable function whose derivative* f' *is Hölder continuous with index* $p - 1$, *and also when* $p = 1$ *and* f *is the difference of two convex functions*, *provided that in this case we take everywhere* f' *to be the right derivative* (*or everywhere the left derivative*).

(ii) (3.4) *holds when* $p \in (0, 1)$ *and* f *is Hölder continuous with index* p, *and also when* $p = 0$ *and* f *is an arbitrary function*, *if in both cases we assume further* (1.7).

Of course (ii) with $p = 0$ is trivial, since then X has finitely many jumps only on finite intervals: it is given here for completeness. In general, the conditions on f are exactly the conditions under which the right-hand sides of (3.3) or (3.4) are meaningful. For the next proof, and also further on, we need the sets

$$\Omega_{t,K} = \Bigl\{\sup_{s\le t} |X_s| \le K\Bigr\} \qquad \text{which satisfy } \lim_{K\uparrow\infty} \Omega_{t,K} = \Omega. \tag{3.5}$$

PROOF. (i) Let $p \in [1, 2)$. By hypothesis there are constants C_K such that $|f'(x + y) - f'(x)| \le C_K |y|^{p-1}$ whenever $|x| \le K$ and $|y| \le 1$ (when $p = 1$ we take for example the right derivative f', which is locally bounded). Then the definition of η_f allows to deduce $|\eta_f(x, y)| \le C_K |y|^p$ for all $|x| \le K$ and $|y| \le 1$: this and (1.6) imply that the series defining $W(f)_t$ is absolutely convergent on each set $\Omega_{t,K}$, hence everywhere.

Denote by f_n the convolution of f with a C^∞ nonnegative function ϕ_n with support in $[0, 1/n]$ and integral 1: we have $f_n \to f$ and $f'_n \to f'$ pointwise [with f' the right derivative in case (ii)]. We have

$$(3.6)\qquad |x| \le K, \qquad |y| \le 1 \quad\Longrightarrow\quad |\eta_{f_n}(x, y)| \le C_{K+1}|y|^{p-1}$$

for all n. Further each f_n is C^∞, so the usual Itô's formula yields

$$(3.7)\qquad f_n(X_t) - f_n(X_0) = \int_0^t f'_n(X_{s-})\,dX_s + W(f_n)_t.$$

The left-hand side of (3.7) converges to the left-hand side of (3.3) because $f_n \to f$ pointwise. Since $f'_n \to f'$ as well and since by (3.6) the sequence (f'_n) is locally bounded, uniformly in n, the stochastic integral in (3.7) converges to the stochastic integral in (3.3), in probability (dominated convergence theorem for stochastic integrals). Finally, $\eta_{f_n}(X_{s-}, \Delta X_s) \to \eta_f(X_{s-}, \Delta X_s)$ pointwise and $|\eta_{f_n}(X_{s-}, \Delta X_s)| \le C_{K+1}|\Delta X_s)|^p$ for all $s \le t$ on the set $\Omega_{t,K}$, so an application of the dominated convergence theorem yields that $W(f)^n_t \to W(f)_t$ pointwise, and we are done.

(ii) When $p < 1$ we write (3.4) as $f(X_t) - f(X_0) = W(f)_t$ with $W(f)$ given by (3.2) and $\eta_f(x, y) = f(x + y) - f(x)$: we again have $|\eta_f(x, y)| \le C_K|y|^p$ for all $|x| \le K$ under our assumptions, so $W(f)$ is well defined. Also, the convergence argument works as for (i), with (3.6) unchanged and (3.7) replaced by $f_n(X_t) - f_n(X_0) = W(f_n)_t$. $\square$

Let us now specialize the above results when $f(x) = |x|^p$. For each $p \in \mathbb{R}$, we define the following function on $\mathbb{R}$:

$$\rho_p(x) = \begin{cases} |x|^p \operatorname{sign}(x), & \text{if } p > 0, \\ \operatorname{sign}(x), & \text{if } p = 0, \\ 0, & \text{if } p < 0, \end{cases}$$

where $\operatorname{sign}(x)$ equals 1 if $x \ge 0$ and equals -1 if $x < 0$. Then for $p > 0$ we define the processes

$$(3.8)\qquad \begin{gathered} W^p_t = \sum_{s \le t} \psi_p(X_{s-}, \Delta X_s) \qquad \text{where} \\ \psi_p(x, y) = |x + y|^p - |x|^p - p\rho_{p-1}(x)y, \end{gathered}$$

with the convention $W^p_t = +\infty$ whenever the sum above is not absolutely convergent.

Then suppose that (1.5) holds for some $p \in (0, \infty)$ (this is always the case when $p \ge 2$). In this case W^p is a.s. finite-valued, and if further $C = 0$ when $p < 2$ and (1.7) holds when $p < 1$, by applying (3.1) when $p \ge 2$ and Theorem 3.1 when $p < 2$ we get

$$(3.9)\qquad |X_t|^p - |X_0|^p = p \int_0^t \rho_{p-1}(X_{s-})\,dX_s + \frac{p(p-1)}{2} \int_0^t |X_{s-}|^{p-2}\,dC_s + W^p_t$$

(if $p < 2$ the second integral above vanishes, and the first integral as well if $p < 1$).

4. Proof of Theorem 1.3. We assume that X is an arbitrary semimartingale with characteristics (B, C, ν), and take a $p > 0$. If $p < 2$ we assume that (1.5) holds and that $C = 0$; if $p < 1$ we assume further that (1.7) holds.

First, by Itô's formula, we have for $t \in (\frac{i}{n}, \frac{i+1}{n}]$ [recall (1.1) for X^n]:

$$\int_{i/n}^{t} X_r^n\,dr = \left(t - \frac{i}{n}\right)(X_t - X_{i/n}) - \int_{i/n}^{t}\left(r - \frac{i}{n}\right)dX_r = \int_{i/n}^{t}(t-r)\,dX_r. \tag{4.1}$$

Similarly, if we set

$$W_t^{n,p} = \sum_{s \le t} \psi_p(X_{s-}^n, \Delta X_s),$$

an application of (3.9) for the process $\widetilde{X}^n$ gives for $t \in (\frac{i}{n}, \frac{i+1}{n}]$:

$$\begin{aligned} &\int_{i/n}^{t} |X_r^n|^p\,dr \\ &\quad = \int_{i/n}^{t}(t-r)\left(p\rho_{p-1}(X_{r-}^n)\,dX_r + \frac{p(p-1)}{2}|X_{r-}^n|^{p-2}\,dC_r + dW_r^{n,p}\right). \end{aligned} \tag{4.2}$$

Then if we set

$$\phi_n(s) = i + 1 - ns \qquad \text{if } \frac{i}{n} < s \le \frac{i+1}{n},\ \ \phi_n(0) = 0$$

and

$$\widetilde{Y}'^n(X)_t = \int_0^t \phi_n(s)\,dX_s, \tag{4.3}$$

we have, by (4.1),

$$\begin{aligned} \widetilde{Y}^n(X)_t &= \widetilde{Y}'^n(X)_{[nt]/n}, \\ nY^n(X)_t &= \widetilde{Y}'^n(X)_t - \phi_n(t)(X_t - X_{[nt]/n}). \end{aligned} \tag{4.4}$$

Similarly, if

$$\widetilde{Z}'^{n,p}(X)_t = \int_0^t \phi_n(s)\left(p\rho_{p-1}(X_{r-}^n)\,dX_r + \frac{p(p-1)}{2}|X_{r-}^n|^{p-2}\,dC_r + dW_r^{n,p}\right),$$

we get, by (4.2),

$$\begin{aligned} \widetilde{Z}^{n,p}(X)_t &= \widetilde{Z}'^{n,p}(X)_{[nt]/n}, \\ nZ^{n,p}(X)_t &= \widetilde{Z}'^{n,p}(X)_t - \phi_n(t)\big(\widetilde{Z}'^{n,p}(X)_t - \widetilde{Z}'^{n,p}(X)_{[nt]/n}\big). \end{aligned} \tag{4.5}$$

Finally, let us introduce the following process [the same as (1.3) in the Lévy case], which is a.s. finite-valued:

$$V_t^p = \begin{cases} \sum_{s \le t} |\Delta X_s|^p, & \text{if } p \ne 2, \\ [X, X]_t = C_t + \sum_{s \le t} |\Delta X_s|^2, & \text{if } p = 2. \end{cases} \tag{4.6}$$

Now, for any $K > 0$ there is a constant C_K (depending also on p) such that when $|x| \leq 2K$, then $|p\rho_{p-1}(x)| \leq C_K$ and $\frac{p(p-1)}{2}|x|^{p-2} \leq C_K$ if $p \geq 2$ and $|\psi_p(x, y)| \leq C_K|y|^p$. Consider the triple $U^n = (X, \widetilde{Y}'^n(X), \widetilde{Z}'^{n,p}(X))$: on the set $\Omega_{T,K}$ of (3.5), and over the time interval $[0, T]$, its components are stochastic integrals of predictable processes, depending on n but smaller than C_K, with respect to X and to C, plus (for the third component) the process $\int_0^t \phi_n(s)\, dW_s^{n,p}$ whose total variation satisfies for $t \leq T$:

$$C_K V_t^p - \int_0^t \phi_n(s)|dW_s^{n,p}| \qquad \text{is nondecreasing.}$$

Then it is an easy consequence of Theorem VI-5.10 of [4], with the Condition (C3), plus the last part of (3.5), that the three-dimensional sequence U^n is tight for the Skorokhod topology, and in particular, the real random variables $\sup_{s\leq t} |U_s^n|$ are tight for all $t < \infty$.

Further, Lemma 2.2 of [3] and its proof yield that if the sequence U^n is tight, then so is the sequence of discretized processes $(U^n_{[nt]/n})_{t\geq 0}$: in view of (4.4) and (4.5), this finishes the proof of the first and second claims in (a) and (b) of Theorem 1.3.

Finally, let t be such that $P(\Delta X_t = 0) = 1$. Then obviously $X_{[nt]/n} \to X_t$ a.s., and $\widetilde{Z}'^{n,p}(X)_{[nt]/n} - \widetilde{Z}'^{n,p}(X)_t \to 0$ in probability, so the last claims in (a) and (b) follow from the last equalities in (4.4) and (4.5) and from $0 \leq \phi_n \leq 1$.

5. Proof of Theorem 1.4. In this section we assume that X satisfies (1.10) and the martingale representation property. Let $p > 0$: if $p < 2$ we assume $C = 0$ and (1.5), and if $p \leq 1$ we assume further (1.7). By virtue of Lemma 2.2 of [3] (and of its proof), in order to obtain Theorem 1.4 it is enough to prove that the pair $(\widetilde{Y}'^n(X), \widetilde{Z}'^{n,p}(X))$ converges stably in law to (Y, Z^p).

If $C' = C$ when $p \neq 2$ and $C' = 0$ when $p = 2$, we observe that

$$\begin{aligned} &\widetilde{Z}'^{n,p}(X)_t - \widetilde{Y}'^n(V^p)_t \\ &\quad = \int_0^t \phi_n(s)\Big(p\rho_{p-1}(X^n_{s-})\, dX_s + \frac{p(p-1)}{2}|X^n_{s-}|^{p-2}\, dC'_s\Big) \\ &\qquad + \sum_{s\leq t} \phi_n(s)\big(\psi_p(X^n_{s-}, \Delta X_s) - |\Delta X_s|^p\big). \end{aligned} \tag{5.1}$$

We have that $X^n_{s-} \to 0$, hence $\psi_p(X^n_{s-}, \Delta X_s) - |\Delta X_s|^p \to 0$ when $p \neq 1$ and $C' = 0$ when $p \leq 2$ and $|\psi_p(X^n_{s-}, \Delta X_s) - |\Delta X_s|^p| \leq (C_K + 1)|\Delta X_s|^p$ for $s \leq T$ on the set $\Omega_{T,K}$ of (3.5): therefore the dominated convergence theorems for stochastic and ordinary integrals and series yield that $\sup_{s\leq t} |\widetilde{Z}'^{n,p}(X)_s - \widetilde{Y}'^n(V^p)_s| \to 0$ in probability as soon as $p \neq 1$. When $p = 1$ the property (1.7) yields $X_t = X_0 + \sum_{s\leq t} \Delta X_s$; therefore (5.1) writes as

$$\widetilde{Z}'^{n,1}(X)_t - \widetilde{Y}'^n(V^1)_t = \sum_{s\leq t} \phi_n(s)\big(|X^n_{s-} + \Delta X_s| - |X^n_{s-}| - |\Delta X_s|\big).$$

Exactly as before, the last sum above goes to 0 in probability, uniformly over each finite time interval, hence again $\sup_{s\leq t}|\widetilde{Z}'^{n,1}(X)_s - \widetilde{Y}'^n(V^1)_s| \to 0$ in probability: so in all cases we are left to proving that the pair $(\widetilde{Y}'^n(X), \widetilde{Y}'^n(V^p))$ converges stably in law to (Y, Z^p) [recall (4.3) for the definition of $\widetilde{Y}'^n(V^p)$].

Let us also state the following trivial consequence of (4.3), of $0 \leq \phi_n \leq 1$ and of the very definition of Emery's topology (see, e.g., the books of Dellacherie and Meyer [1] or Protter [7] for a definition of this topology, also called "topology of semimartingales"):

LEMMA 5.1. *If $U(q)$ is a sequence of semimartingales converging to a limiting semimartingale U in Emery's topology as $q \to \infty$, then we have for all $\varepsilon > 0$ and $t \in \mathbb{R}_+$:*

$$\lim_q \sup_n P\left(\sup_{s\leq t}\left|\widetilde{Y}'^n(U(q))_s - \widetilde{Y}'^n(U)_s\right| \geq \varepsilon\right) = 0.$$

Below we choose a truncation function h which is Lipschitz continuous and with $h(x) = x$ if $|x| \leq 1$ and $h(x) = 0$ if $|x| > 2$. For any $q \geq 2$ set $R_q = \{x : \frac{1}{q} < |x| \leq q\}$ and

$$\begin{aligned} X(q)_t = {} & X_0 + B_t + X_t^c + \int_0^t \int_{\{|x|>1/q\}} h(x)(\mu - \nu)(ds, dx) \\ & + \int_0^t \int_{\{|x|\leq q\}} \big(x - h(x)\big)\mu(ds, dx). \end{aligned} \tag{5.2}$$

We associate with $X(q)$ the process $V(q)^p$ defined as in (4.6), that is,

$$V(q)_t^p = \begin{cases} \sum_{s\leq t} |\Delta X(q)_s|^p, & \text{if } p \neq 2, \\ [X(q), X(q)]_t = C_t + \sum_{s\leq t} |\Delta X(q)_s|^2, & \text{if } p = 2. \end{cases} \tag{5.3}$$

Then $X(q)$ and $V(q)^p$ converge to X and V^p for Emery's topology as $q \to \infty$ [compare (5.2) with (1.8)]. Similarly, on our extended space we define the processes [recall (1.11) and (1.12)]:

$$\begin{aligned} Y(q)_t = {} & \frac{1}{2}B_t + \frac{1}{2}X_t^c + \frac{1}{\sqrt{12}}\int_0^t \sqrt{c_s}\, dW_s \\ & + \int_0^t \int_{\{|x|>1/q\}} \int_{[0,1]} h(x)u(\widehat{\mu} - \widehat{\nu})(ds, dx, du) \\ & + \int_0^t \int_{\{|x|\leq q\}} \int_{[0,1]} \big(x - h(x)\big)u\widehat{\mu}(ds, dx, du), \end{aligned} \tag{5.4}$$

$$Z(q)_t^p = \begin{cases} \frac{1}{2}C_t + \int_0^t \int_{R_q} \int_{[0,1]} ux^2 \widehat{\mu}(ds, dx, du), & \text{if } p = 2, \\ \int_0^t \int_{R_q} \int_{[0,1]} u|x|^p \widehat{\mu}(ds, dx, du), & \text{if } p \neq 2. \end{cases} \tag{5.5}$$

Then $Y(q)$ and $Z(q)^p$ go to Y and Z^p for Emery's topology as well. So by virtue of Lemma 5.1 it is then enough to prove that for any fixed q we have stable convergence in law of $(\widetilde{Y}'^n(X(q)), \widetilde{Y}'^n(V(q)^p))$ to $(Y(q), Z(q)^p)$.

Below, we fix $q \in \mathbb{N}^\star$. We choose two other truncation functions h' and h'' that are Lipschitz continuous and satisfies $h'(x) = x$ if $|x| \leq q$ and $h''(x) = x$ if $|x| \leq q^p$. Since the processes $Y(q)$ and $Z(q)^p$ have jumps smaller than q and q^p respectively, we easily deduce from (5.4) and (5.5) that the characteristics of the triple $(X, Y(q), Z(q)^p)$ (w.r.t. the truncation function $\overline{h}$ on $\mathbb{R}^3$ having the components h, h' and h'') are $(\widehat{B}, \widehat{C}, \eta)$, given by

$$\widehat{B} = \begin{pmatrix} B \\ B'/2 \\ B''/2 \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} C & C/2 & 0 \\ C/2 & C/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

where, with the notation $G_t^p = \int_0^t ds \int_{R_q} F_s(dx)|x|^p$,

$$\begin{aligned} B'_t &= B_t + \int_0^t \int_{R_q} (x - h(x))\nu(ds, dx), \\ B'' &= \begin{cases} C + G^2, & \text{if } p = 2, \\ G^p, & \text{if } p \neq 2, \end{cases} \end{aligned} \tag{5.7}$$

and

$$\eta([0,t] \times A) = \int_0^t ds \int_{\mathbb{R}} F_s(dx) \int_0^1 \mathbb{1}_A\big(x, ux\mathbb{1}_{R_q}(x), u|x|^p \mathbb{1}_{R_q}(x)\big)\, du. \tag{5.8}$$

Note that these characteristics are predictable on the original probability space and not only on the extended space.

Next we set for simplicity $U^n = \widetilde{Y}'^n(X(q))$ and $U'^n = \widetilde{Y}'^n(V(q)^p)$. In view of (4.3), (5.2), (5.3) and of the fact that the jump measure of $X(q)$ is the restriction of the jump measure of X to $\mathbb{R}_+ \times R_q$, it is an easy computation to check that the characteristics of the triple (X, U^n, U'^n) w.r.t. the truncation function $\overline{h}$ are $(\widehat{B}^n, \widehat{C}^n, \eta^n)$, given by

$$\widehat{B}^n = \begin{pmatrix} B \\ B'^n \\ B''^n \end{pmatrix}, \qquad \widehat{C}^n = \begin{pmatrix} C & C'^n & 0 \\ C'^n & C''^n & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.9}$$

where

$$
\begin{aligned}
B_t'^n &= \int_0^t \phi_n(s)\, dB_s - \int_0^t \phi_n(s) \int_{\{|x|\geq q\}} \big(x - h(x)\big)\nu(ds, dx), \\
B_t''^n &= \int_0^t \phi_n(s)\, dB_s'',
\end{aligned} \tag{5.10}
$$

$$
C_t'^n = \int_0^t \phi_n(s)\, dC_s, \qquad C_t''^n = \int_0^t \phi_n(s)^2\, dC_s \tag{5.11}
$$

and

$$
\eta^n([0,t]\times A) = \int_0^t ds \int_{\mathbb{R}} F_s(dx) 1_A\big(x, \phi_n(s) x 1_{R_q}(x), \phi_n(s)|x|^p 1_{R_p}(x)\big). \tag{5.12}
$$

Finally, we also introduce the following processes, taking values in the set of all symmetric nonnegative 3×3 matrices and nondecreasing in this set:

$$
\begin{aligned}
H_t &= \widehat{C}_t + \int_0^t \int_{\mathbb{R}^3} (\overline{h}\,\overline{h}^*)(x,y,z)\eta(ds,dx,dy,dz), \\
H_t^n &= \widehat{C}_t^n + \int_0^t \int_{\mathbb{R}^3} (\overline{h}\,\overline{h}^*)(x,y,z)\eta^n(ds,dx,dy,dz),
\end{aligned}
$$

where $\overline{h}^*$ denotes the transpose of the row vector-valued function $\overline{h}$.

Note that the extension on which our limiting processes are defined is trivially a "very good extension" in the sense of [2]. By assumption we also have the martingale representation property w.r.t. X. Then, by virtue of Theorem 2.1 of [2], in order to prove our convergence result it is enough to prove the following three convergences (pointwise in ω) for all $t\in\mathbb{R}_+$ and every function g which is bounded Lipschitz on $\mathbb{R}^3$ and null on a neighborhood of 0:

$$
\sup_{r\leq t} |\widehat{B}_r^n - \widehat{B}_r| \to 0, \tag{5.13}
$$

$$
H_t^n \to H_t, \tag{5.14}
$$

$$
\int_0^t \int_{\mathbb{R}^3} g(x,y,z)\eta^n(ds,dx,dy,dz) \to \int_0^t \int_{\mathbb{R}^3} g(x,y,z)\eta(ds,dx,dy,dz). \tag{5.15}
$$

Let us prove an auxiliary result: If f is a locally integrable (w.r.t. Lebesgue measure) function on $\mathbb{R}_+$, then for each $t>0$ we have for any $\alpha>0$:

$$
\sup_{r\leq t} \left| \int_0^r \phi_n(s)^\alpha f(s)\, ds - \frac{1}{\alpha+1}\int_0^r f(s)\, ds \right| \to 0. \tag{5.16}
$$

Indeed, $\int_{(i-1)/n}^{i/n} \phi_n(s)^\alpha\, ds = \frac{1}{n(\alpha+1)}$, so for any $u<v$ we have $|\int_u^v \phi_n(s)^\alpha\, ds - \frac{v-u}{\alpha+1}| \leq \frac{2}{n}$. It follows that (5.16) holds when f is piecewise constant. For a general (locally integrable) f there exists a sequence of piecewise constant function f^m

on $\mathbb{R}_+$ which converges in L^1 for the Lebesgue measure on any compact interval $[0,t]$ to f. Since $0 \le \phi_n \le 1$ we clearly have as $m \to \infty$:

$$\sup_n \int_0^t \phi_n(s)^\alpha |f(s) - f_m(s)|\,ds \to 0.$$

Then (5.16) holds for f as well.

In view of (1.10) and of (5.7) and (5.10), we readily obtain (5.13).

Next we turn to (5.15). We prove a stronger form, namely when g is bounded, Lipschitz continuous and $g(0,0,0)=0$, and when the process $L_t = \int_0^t \int_{\mathbb{R}} g(x,0,0)\nu(ds,dx)$ is well defined (i.e., the integral defining L_t is absolutely convergent): when g is null on a neighborhood of 0 and bounded, all these conditions are obviously satisfied.

Set $\gamma_s(\omega) = F_s(\omega, R_q)$, which is finite-valued and Lebesgue-locally integrable. Any finite measure being the image of Lebesgue measure on an appropriate interval, and since $(\omega,s) \mapsto F_s(\omega,A)$ is predictable for any Borel set A, then γ is a predictable process, and we can find a predictable map $\beta = \beta(\omega,t,v)$ from $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ into $R_q \cup \{0\}$ such that

$$\begin{aligned} &\beta(\omega,t,v) = 0 \quad \Longleftrightarrow \quad v \ge \gamma_t(\omega), \\ &\int_{R_q} f(x) F_t(\omega,dx) = \int_0^{\gamma_t(\omega)} f\big(\beta(\omega,t,v)\big)\,dv \end{aligned} \tag{5.17}$$

for any Borel and locally bounded function f. Then using (5.17), (5.8) and (5.12), we see that the left- and right-hand sides of (5.15) are respectively $L_t + \delta_n(t)$ and $L_t + \delta(t)$, where

$$\delta(t) = \int_0^t ds \int_0^{\gamma_s} dv \int_0^1 g\big(\beta(s,v), u\beta(s,v), u|\beta(s,v)|^p\big)\,du, \tag{5.18}$$

$$\delta_n(t) = \int_0^t ds \int_0^{\gamma_s} g\big(\beta(s,v), \phi_n(s)\beta(s,v), \phi_n(s)|\beta(s,v)|^p\big)\,dv, \tag{5.19}$$

so it remains to prove that

$$\delta_n(t) \to \delta(t). \tag{5.20}$$

We fix ω. Let us denote by $\mathcal{H}$ the set of all functions on $\mathbb{R}_+ \times \mathbb{R}_+$ of the form $\sum_{i=1}^m a_i \mathbb{1}_{A_i}$, where $a_i \in \mathbb{R}$ and the A_i's are bounded rectangles and $m \in \mathbb{N}$. We can find a sequence β_m of functions in $\mathcal{H}$ which converges to β in L^1 for the two-dimensional Lebesgue measure, on each compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$. We define $\delta(m,t)$ and $\delta_n(m,t)$ by (5.18) and (5.19) again, with β_m instead of β. Then since g is Lipschitz and bounded, we have

$$|\delta(t) - \delta(m,t)| \le \alpha_{m,K}(t) + \beta_K(t), \qquad |\delta_n(t) - \delta_n(m,t)| \le \alpha_{m,K}(t) + \beta_K(t),$$

for all $K > 0$, where $\beta_K(t) = \|g\| \int_0^t (\gamma_s - K)^+\,ds$ and with $\alpha_{m,K}(t)$ not depending on n and going to 0 as $m \to \infty$ for each K. Since further γ is Lebesgue-integrable over $[0,t]$, we have $\beta_K(t) \to 0$ as $K \to \infty$. Therefore we are left to prove that $\delta_n(m,t) \to \delta(m,t)$ for all m and t.

In other words, we need to prove (5.20) when $\beta \in \mathcal{H}$. But then we have $\beta(t,v) = \sum_{i=1}^k a_i \mathbb{1}_{(x_i,y_i]}(t)\mathbb{1}_{(w_i,z_i]}(v)$ where $a_i \in \mathbb{R}$ and where the rectangles $(x_i,y_i] \times (w_i,z_i]$ are pairwise disjoint. Then since $g(0,0,0)=0$, we get

$$\delta_n(t) = \sum_{i=1}^k \int_{x_i\wedge t}^{y_i\wedge t} (z_i \wedge \gamma_s - w_i \wedge \gamma_s) g\big(a_i, \phi_n(s)a_i, \phi_n(s)|a_i|^p\big)\, ds, \tag{5.21}$$

$$\delta(t) = \sum_{i=1}^k \int_{x_i\wedge t}^{y_i\wedge t} (z_i \wedge \gamma_s - w_i \wedge \gamma_s)\, ds \int_0^1 g(a_i, ua_i, u|a_i|^p)\, du. \tag{5.22}$$

If γ is piecewise constant, the two quantities in (5.21) and (5.22) differ at most by $2r\|g\| \sum_{i=1}^k \frac{z_i - w_i}{n}$, where r is the number of dicontinuities of the function γ over the time interval $[0,t]$: this proves (5.20) when γ is piecewise constant, and one deduces that it holds in general by approximating the locally integrable function γ by a sequence of piecewise constant functions converging to γ in L^1 for the Lebesgue measure.

So we have completed the proof of (5.20), hence of (5.15) when g is bounded, Lipschitz continuous and $g(0,0,0)=0$ and L_t is well defined.

It remains to prove (5.14). First, (5.6), (5.9) and (5.11) together with (5.16) show that $\widehat{C}^n_t \to \widehat{C}_t$ for all t. Therefore it remains to prove that (5.15) holds for the functions $g_{ij} = \overline{h}^i \overline{h}^j$, for $i,j = 1,2,3$. But these functions are bounded Lipschitz null at 0, and the process $L_t = \int_0^t \int_{\mathbb{R}} g_{ij}(x,0,0)\nu(ds,dx)$ is well defined (and indeed vanishes except when $i=j=1$). So we can apply step 6 and we are done.

Acknowledgments. We wish to thank Z. Jurek and an anonymous referee for very valuable and constructive remarks.

REFERENCES

[1] DELLACHERIE, C. and MEYER, P. A. (1978). *Probabilités et Potentiel II*. Hermann, Paris.

[2] JACOD, J. (2003). On processes with conditional independent increments and stable convergence in law. *Séminaire de Probabilités XXXVI. Lecture Notes in Math.* **1801** 383–401. Springer, New York.

[3] JACOD, J. and PROTTER, P. (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann. Probab.* **26** 267–307.

[4] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.

[5] JUREK, Z. J. (1977). Limit distributions for sums of shrunken random variables. In *Second Vilnius Conference on Probability Theory and Mathematical Statists, Abstracts of Communications* **3** 95–96. Vilnius.

[6] JUREK, Z. J. (1985). Relations between the s-selfdecomposable and selfdecomposable measures. *Ann. Probab.* **13** 592–609.

[7] PROTTER, P. (1990). *Stochastic Integration and Differential Equations. A New Approach*. Springer, Berlin.

[8] RENYI, A. (1963). On stable sequences of events. *Sankyā* **25** 293–302.

J. JACOD
LABORATOIRE DE PROBABILITES
ET MODELES ALEATOIRES (UMR 7599)
UNIVERSITE PIERRE ET MARIE CURIE
BOITE 188
4 PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE

A. JAKUBOWSKI
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCES
NICHOLAS COPERNICUS UNIVERSITY
UL. CHOPINA 12/18
87-100 TORUN
POLAND

J. MÉMIN
IRMAR
CAMPUS DE BEAULIEU
UNIVERSITÉ DE RENNES-1
35042 RENNES
FRANCE