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FIRST PASSAGE TIMES FOR THRESHOLD GROWTH DYNAMICS ON $\mathbf{Z}^2$

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In the *threshold growth model* on an integer lattice, the occupied set grows according to a simple local rule: a site becomes occupied iff it sees at least a threshold number of already occupied sites in its prescribed neighborhood. In this paper, we analyze the behavior of two-dimensional threshold growth dynamics started from a sparse Bernoulli density of occupied sites. We explain how nucleation of rare centers, invariant shapes and interaction between growing droplets influence the first passage time in the *supercritical* case. We also briefly address scaling laws for the *critical* case.

1. Introduction. In the *discrete threshold growth model*, an empty site joins the occupied set if it sees enough occupied sites around it. These deterministic dynamics have two parameters:

1. Parameter $\mathcal{N}$, a finite subset of $\mathbf{Z}^2$, is the *neighborhood* of the origin; $x+\mathcal{N}$ is then the neighborhood of a site x. We will always assume that $0 \in \mathcal{N}$.
2. Parameter θ, a positive integer, is the *threshold* value.

For $\mathcal{N}$, most of our examples will use either the range ρ diamond neighborhood, $\mathcal{N} = \{x\colon ||x||_1 \le \rho\}$, or the range ρ box neighborhood, $\mathcal{N} = \{x\colon ||x||_\infty \le \rho\}$.

Given a set $A \subset \mathbf{Z}^2$, define

$$\mathcal{T}(A) = A \cup \{x\colon |(x+\mathcal{N}) \cap A| \ge \theta\}.$$

Start from an initial set $A_0 \subset \mathbf{Z}^2$ and iterate $A_{n+1} = \mathcal{T}(A_n)$ to generate *discrete threshold growth dynamics*. This extremely simple model arose in connection with our previous empirical and theoretical studies of mathematical prototypes for excitable media and crystal growth. In various contexts it captures the essential qualitative and quantitative features of wave propagation. For applications to other basic cellular automata and interacting particle systems, we refer the reader to [2] and [9]–[11] and the many references contained therein.

The subset $\mathcal{N}$ is called *symmetric* if $-\mathcal{N} = \mathcal{N}$. We will often make this assumption to simplify the statements and proofs of our theorems. (See [11] for some discussion of this issue.)

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The following definitions distinguish three fundamentally different ways in which threshold growth dynamics can behave. Let $A_\infty = \mathscr{T}^\infty(A_0) = \bigcup_{n=0}^\infty A_n$; of course A_∞ depends implicitly on A_0. We say that the dynamics are *supercritical* if there exists a finite A_0 such that A_n eventually occupies every site in $\mathbf{Z}^2$. That is, $A_\infty = \mathbf{Z}^2$. If $A_\infty \neq \mathbf{Z}^2$ for every finite A_0, but $A_\infty = \mathbf{Z}^2$ for every A_0 with finite *complement*, then we call the dynamics *critical*. Finally, in the case of *subcritical* dynamics, there exists a nonempty finite set H (a *hole*) so that the dynamics cannot fill H even when started with all of H^c occupied.

Critical threshold growth dynamics have been widely studied in the mathematics and physics literature as a model of nucleation and metastability, usually under the name *bootstrap percolation*. Papers [1]–[4] and [13]–[17] provide a representative sample of the literature and include a large number of additional references.

Probability enters the picture when one considers the dynamics started from the random set $\Pi(p)$ which contains each site in $\mathbf{Z}^2$ independently with probability $p > 0$. Most of our analysis will focus on threshold growth started from $\Pi(p)$. The first question that arises here is whether the dynamics started from such a random seeding will eventually occupy every site of $\mathbf{Z}^2$ for any $p > 0$ (with probability 1). This is obviously true in the supercritical case, since for any $p > 0$, $\Pi(p)$ contains arbitrarily large sets (by the monkey-at-the-typewriter theorem). On the other hand, since $\Pi(p)$ also has arbitrarily large holes, the answer is clearly no in the subcritical case. Indeed, the next proposition shows that by far the best chance for a site to be in A_∞ is for it to belong to A_0. We use the standard notation for balls: $B_r(x, R) = \{y: \|x-y\|_r \leq R\}$.

PROPOSITION 1.1. *Assume the threshold growth dynamics to be subcritical and start from $A_0 = \Pi(p)$. Then, for each site x, $P(x \in A_\infty) = p + \mathscr{O}(p^\theta)$.*

PROOF. Let R be large enough so that $0 \notin \mathscr{T}^\infty(B_\infty(0, R)^c)$ and $\mathscr{N} \subset B_\infty(0, R)$.

Assume now that $x \notin A_0$ and $|A_0 \cap B_\infty(x, 3\theta R)| < \theta$. We claim that x cannot be in $\mathscr{T}^\infty(A_0)$ under these conditions. Indeed, they imply existence of an integer $i \in [1, \theta]$ such that, initially, there is no site at all in $B_\infty(x, 3iR) \setminus B_\infty(x, 3(i-1)R)$. This means that no site in $B_\infty(x, (3i-1)R) \setminus B_\infty(x, (3i-2)R)$ can ever become occupied. Consequently, the dynamics restricted to $B_\infty(x, (3i-2)R)$ would need to eventually occupy x on their own, since they cannot get help from the outside, but there are less than θ sites in $B_\infty(x, (3i-2)R)$, so that not even one site in this box can ever be added. Since x is not initially occupied, it will therefore never be. We conclude that, for $\theta < |\mathscr{N}|$,

$$p + (1-p)p^\theta \leq P(x \in A_\infty) \leq p + (6\theta R + 1)^{2\theta} p^\theta. \qquad \square$$

To illustrate our classification, let us consider range 1 box dynamics. The reader can easily verify that these are supercritical for $\theta = 1, 2, 3$, critical for $\theta = 4$ and subcritical for $\theta \geq 5$. Techniques from [2] and [17] imply that in the $\theta = 4$ case, these critical dynamics are able to fill $\mathbf{Z}^2$ starting from any $\Pi(p)$. Is this convergence to total occupancy always the case for critical threshold growth started from a random seeding with positive density? Not in full generality, as we will now illustrate. (A similar example appears in Section 4 of [17].)

EXAMPLE 1.2. Assume that $\mathscr{N}$ consists of the origin and three points immediately below: $(0, -1)$, $(\pm 1, -1)$ and that $\theta = 3$. This is clearly a critical case. For $M > 0$, let G_M denote the event that $0 \in A_\infty$ for the dynamics started from $\Pi(p) \cap (\mathbf{Z} \times [-M, 0])$. Then starting from $A_0 = \Pi(p)$, it follows that $\{0 \in A_\infty\} = \bigcup_{M=0}^{\infty} G_M$.

Now introduce a two-state (0, 1-valued) cellular automaton η_t on $\mathbf{Z}$ in which a 1 remains 1 if both its nearest neighbors are 1 and otherwise remains 1 with probability p, while a 0 becomes a 1 with probability p in any case. Start this automaton from density p of 1's. If it happens that $\eta_t(x) = \eta_t(x \pm 1) = 1$, then we call (x, t), $(x-1, t)$ and $(x+1, t)$ *predecessors* of $(x, t+1)$. Say that an $(x, 0)$ such that $\eta_0(x) = 1$ *survives until time* t if there is a space–time chain of predecessors linking $(x, 0)$ to (y, t) for some y. Using standard methods from oriented percolation (see [8]), one can obtain an $\alpha(p)$, with $\alpha(p) \to \infty$ as $p \to 0$, such that $P((x, 0)$ survives until time $t) \leq e^{-\alpha t}$.

It is easy to check that $G_M \setminus G_{M-1} \subset \bigcup_{x \in [-M, M]} \{(x, 0)$ survives until time $M\}$, so $P(x \in A_\infty) \to 0$ as $p \to 0$.

In conclusion, we note that the models with the same neighborhood and $\theta = 1$ or $\theta = 2$ are still critical, but fill $\mathbf{Z}^2$ from every $\Pi(p)$ with $p > 0$. This follows from the fact that single occupied sites for $\theta = 1$ and horizontally adjacent occupied pairs for $\theta = 2$ are able to propagate upwards indefinitely.

In contrast to the previous example, our next result shows that critical *symmetric* dynamics fill $\mathbf{Z}^2$ for every $p > 0$. In the process, we determine precisely which thresholds give rise to critical dynamics. Our criterion involves the quantity $\iota(\mathscr{N}) = \max\{|\mathscr{N} \cap \ell| \colon \ell$ a line through $0\}$.

PROPOSITION 1.3. *Threshold growth dynamics with* $-\mathscr{N} = \mathscr{N}$ *are as follows*:

(i) *supercritical, iff* $\theta \leq \frac{1}{2}(|\mathscr{N}| - \iota(\mathscr{N}))$;
(ii) *subcritical iff* $\theta > \frac{1}{2}(|\mathscr{N}| - 1)$.

Furthermore, in the critical case and for every $p > 0$, *the dynamics started from* $A_0 = \Pi(p)$ *fill the lattice* (*i.e.*, $A_\infty = \mathbf{Z}^2$) *a.s.*

For example, for range ρ box neighborhoods, the dynamics are supercritical if $\theta \leq 2\rho^2 + \rho$ (or, in the terminology of [10], below *boot*), critical if $\theta \in [2\rho^2 + \rho + 1, 2\rho^2 + 2\rho]$ and subcritical if $\theta > 2\rho^2 + 2\rho$. This complete trichotomy improves results in Section 5 of [11].

Once we know that every site becomes occupied eventually, a natural question is to ask when. Most of the present paper is devoted to answering this question in the supercritical case. Our main theorem concerns T, the first time that the origin is occupied. The mechanism whereby nucleating droplets cover space is suggested in Figure 1. The rule there is range 1 box, threshold 3. The initial seeding has density $p = 0.01$ and the contours represent the extent of growth after regularly spaced time intervals. In supercritical cases such as this, we will see that T obeys a power law. More precisely, there exists a $\gamma > 0$ such that $T p^{\gamma/2}$ converges weakly to a nondegenerate random variable. Not surprisingly, this is a much more complete result than has been established for even the simplest critical cases, where it has been shown that T is exponentially large in a power of $1/p$ (e.g., [2], [15]). We will briefly address the scaling laws for critical dynamics in Section 7.

There are several combinatorial issues that come into play in our analysis. We will introduce two *nucleation parameters* that measure how easy it is for the dynamics to start growing. However, there are delicate issues connected with the initial stages of growth. At least for sufficiently nice neighbor sets, threshold growth would appear to spread out in an essentially regular manner. However, our next definition will help clarify a thorny combinatorial issue that remains unsettled.

Say that A_0 *generates persistent growth* if $A_{t+1} \neq A_t$ for every t. The dynamics are *omnivorous* if for every A_0 which generates persistent growth, $A_n \uparrow \mathbf{Z}^2$.

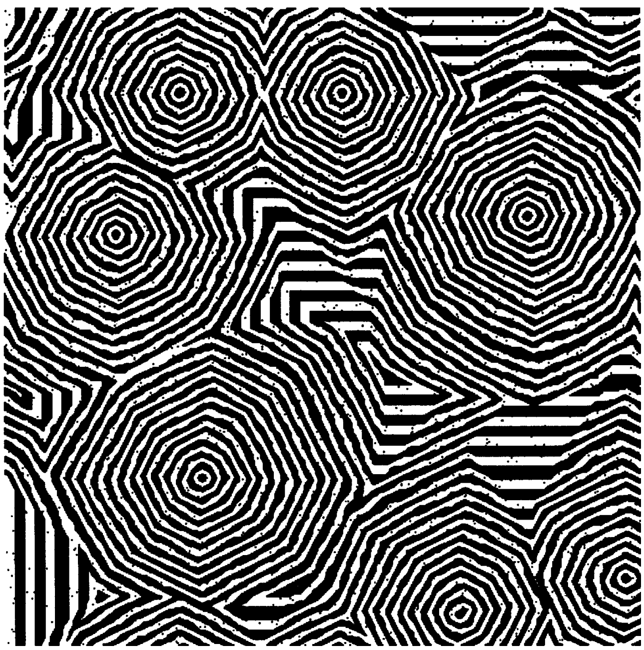

FIG. 1. *Supercritical threshold growth from sparse random seeds.*

A simple (albeit artificial) example that is not omnivorous can be fashioned by choosing $\mathcal{N}$ and A_0 to contain only sites from the even sublattice of $\mathbf{Z}^2$. However, the following natural conjecture remains unproved.

CONJECTURE 1.4. If $\mathcal{N}$ is an obese neighborhood (see [11] for a definition), then the dynamics are omnivorous.

In recent and quite remarkable work, Bohman [7] proved this conjecture for box neighborhoods of arbitrary range. This substantial advance bodes well for future development of a comprehensive regularity theory for threshold growth dynamics.

We now proceed to define the key nucleation parameters. Let $\gamma = \gamma(\mathcal{N}, \theta)$ be the minimal number of sites needed for persistent growth, that is, the smallest i for which there exists an A_0 with $|A_0| = i$ and $A_{n+1} \neq A_n$ for every n. Moreover, let $\nu = \nu(\mathcal{N}, \theta)$ be the number of sets A_0 of size γ that generate persistent growth and have their leftmost lowest sites at the origin. The phrase "leftmost lowest site" (meaning the leftmost among the lowest sites) is included to assure that translations of A_0 are not counted as different, so that ν counts the number of distinct smallest "shapes" that grow.

It turns out that omnivorous dynamics are not required for our theorems. Instead, all we need is the following much more checkable condition. Call a threshold growth model *voracious* if, started from any of the ν initial sets A_0 described above, $A_n \uparrow \mathbf{Z}^2$.

For relatively small θ, $\gamma = \theta$. For example, in the range ρ box case, $\gamma = \theta$ as long as $\theta \leq \rho^2$. The smallest box neighborhood example where $\gamma > \theta$ is range 2 with $\theta = 10$. In that case, $A_0 = \mathcal{N}$ does not generate persistent growth, hence $\gamma > \theta$. On the other hand, there does exist an A_0 with 11 sites that generates persistent growth, proving that $\gamma = 11$. We leave it as a (surprisingly challenging) puzzle for the reader to try to find such an A_0. Gluttons for punishment are invited to compute the corresponding ν. The size of γ for large neighborhoods will be addressed in a forthcoming paper.

Needless to say, a little technology is quite helpful in determining these parameter values for small cases. One can write a computer program, which computes ν for small neighborhoods and thresholds, and checks voracity in the process. (This is of course no longer necessary for box neighborhoods, due to Bohman's theorem.) In all cases listed in Table 1, $\gamma = \theta$, and voracity is established.

As is common in "lattice animal" combinatorial problems, the parameters and computation times grow very quickly. In the range ρ box case, only for $\theta = 2$ do we know an explicit general formula: $\nu = 4\rho(2\rho + 1)$.

The main result of this paper is the following theorem:

THEOREM 1.5. *Assume that a threshold growth model is symmetric, supercritical and voracious, with nucleation parameters γ and ν. Starting from*

TABLE 1
The number ν in some small cases

	ρ	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$
Box	2	20	136	398			
Diamond	1	12	42	$\nexists$	$\nexists$	$\nexists$	$\nexists$
	2	40	578	4,683	24,938	94,050	259,308
	3	84	2,602	46,704	574,718		
	4	144	7,702	241,151			
	5	220	18,038				

$\Pi(p)$, *let T be the first time the origin is occupied. Then, as $p \to 0$,*

$$\sqrt{\nu p^{\gamma}}T$$

converges in distribution to a nontrivial random variable τ.

A detailed description of the random variable τ is a long story. For now, suffice it to say that it is a functional of a Poisson point location on $\mathbf{R}^2$ (which is guaranteed to have unit intensity by the normalization factor ν). To see why, consider the easiest example: $\theta = 1$. This is the instance of *additive* dynamics, which simply means that $\mathscr{T}(A \cup B) = \mathscr{T}(A) \cup \mathscr{T}(B)$. Also, it is not hard to see that started from $A_0 = \{0\}$, A_n/n converges to the set $\mathrm{co}(\mathscr{N})$ (the convex hull of $\mathscr{N}$). Supercriticality is equivalent to $\mathrm{co}(\mathscr{N})$ being a neighborhood of the origin, in which case it defines a norm $||\cdot||$ as its Minkowski functional (given by $||z|| = \inf\{\lambda > 0\colon z \in \lambda \cdot \mathrm{co}(\mathscr{N})\}$). Since the sites of $\mathbf{Z}^2$ are independently occupied with density p, the locations of occupied sites, multiplied by $\sqrt{p}$, converge to the unit–intensity Poisson point location $\wp$. Additivity then almost immediately implies the following result.

COROLLARY 1.6. *Assume that $\mathscr{N}$ is symmetric and* $\mathrm{co}(\mathscr{N})$ *is a neighborhood of the origin. Assume also that $\theta = 1$. Then*

$$\sqrt{p}T \to_d \tau = \inf\{||z||\colon z \in \wp\}.$$

It is not immediately clear why Corollary 1.6 is not true in general, with some appropriate choice of the norm $||\cdot||$. In fact, cases where the dynamics are close enough to additive that this is true are rare, but we do mention one next.

COROLLARY 1.7. *Assume range ρ box neighborhood with $\theta = 2$. Then*

$$\rho\sqrt{4\rho(2\rho+1)}pT$$

converges as $p \to 0$ to a random variable with distribution function $F(r) = 1 - \exp(-4r^2(1-\rho^{-2}/2))$.

It should be noted that the results of this paper are limited to two dimensions. In higher dimensions, the technical difficulties associated with supercritical growth increase dramatically, due to the much more complex interaction between droplets. In critical cases the scaling behavior is actually expected to change significantly [16].

Let us conclude this introduction with a remark about *random* growth. For additive random processes such as Eden's model (where, at each time, every site with an occupied neighbor becomes occupied with some fixed probability q), Corollary 1.6 is true except that our norm needs to be replaced by another given as the Minkowski functional of the asymptotic shape. Unfortunately, that shape seems impossible to compute explicitly (but see [12] for a result on how the shape in a random model can approximate, say, a square). On the one hand, our dynamics are simpler since their determinism precludes stochastic fluctuations. On the other hand, lacking additivity, they are somewhat more complicated than the most commonly studied random growth models. Indeed, our work is largely motivated by the tractability of threshold cellular automata dynamics as a prototype for nonlinear growth.

The rest of the paper is organized as follows. The next section gives a brief exposition of shape theory for bounded initial sets and proves Proposition 1.3. Section 3 then establishes a shape theorem for certain infinite initial sets. These results are used in Section 4 to define the limiting first passage time τ. Section 5 exploits standard Poisson convergence machinery to establish convergence of nucleation centers. Then, to complete the proof of Theorem 1.5, we need to analyze threshold growth in a slightly polluted environment; this is the agenda of Section 6. Some representative examples of *critical* dynamics are presented in Section 7, indicating various possible scalings for the first passage time. Finally, Section 8 gives a recipe for choosing parameters $\mathcal{N}$ and θ in order to obtain a prescribed limiting shape.

2. Classification and the shape theorem. The paper [11] proves the shape theorem for a continuous space version of our dynamics, acting on subsets of $\mathbf{R}^2$. It is not immediately clear how the techniques from that analysis translate to this case, but the following observation makes the transition quite easy.

Let B be any subset of $\mathbf{R}^2$ and define

$$\bar{\mathcal{T}}(B) = B \cup \{x \in \mathbf{R}^2 \colon |(x + \mathcal{N}) \cap B| \geq \theta\}.$$

(Note that $\mathcal{N}$ is still the same finite subset of $\mathbf{Z}^2$ and $|\cdot|$ still means cardinality.) The point is that $\mathcal{T}$ and $\bar{\mathcal{T}}$ are *conjugate*; that is, for any $B \subset \mathbf{R}^2$,

$$\bar{\mathcal{T}}(B) \cap \mathbf{Z}^2 = \mathcal{T}(B \cap \mathbf{Z}^2). \tag{2.1}$$

Moreover, $\bar{\mathcal{T}}$ translates half-planes: if u is a two-dimensional unit vector and $H_u^- = \{x \colon \langle x, u\rangle \leq 0\}$, then $\bar{\mathcal{T}}(H_u^-) = H_u^- + w(u)u$ for some $w(u) \geq 0$ (known as the *speed* of the half-space H_u^-). As in [11], introduce the set $K_{1/w} = \bigcup_u [0, 1/w(u)]u$ and define $L = L(\mathcal{N}, \theta)$ by

$$L = K_{1/w}^* = \{y \colon \langle x, y\rangle \leq 1 \text{ for every } x \in K_{1/w}\}. \tag{2.2}$$

Note that L is a closed convex subset of $\mathbf{R}^2$ that includes the origin. Moreover, as we will see shortly (see the proof of Proposition 2.4), the origin is in the interior of L if and only if the dynamics are supercritical. In this case, the Minkowski functional

$$||z|| = \inf\{\lambda\colon z \notin \lambda L\}$$

is a norm on $\mathbf{R}^2$.

Since our next theorem deals with convergence of subsets of $\mathbf{R}^2$, we recall that closed sets $F_n \subset \mathbf{R}^2$ converge to a closed set $F \subset \mathbf{R}^2$ *in the Hausdorff metric* if for every $\varepsilon > 0$, there exists an $n_0(\varepsilon)$ so that $F_n \subset F + B_2(0, \varepsilon)$ and $F \subset F_n + B_2(0, \varepsilon)$ for $n \geq n_0$. See the Appendix in [11] for some relevant theoretical properties of this kind of convergence.

THEOREM 2.1. *If A_0 is finite and $A_n \uparrow \mathbf{Z}^2$, then $A_n/n \subset \mathbf{R}^2$ converge* (*in the Hausdorff metric*) *to L.*

PROOF. The proof for iterations of $\bar{\mathscr{T}}$ consists of approximating $K^*_{1/w}$ with smooth convex sets and is virtually identical to the proof of Theorem 1 in [11]. The final step is use of conjugacy formula (2.1). $\square$

Let us now mention some consequences of these observations.

PROPOSITION 2.2. *L is always a polygon.*

PROOF. For each fixed vector u, one of the following alternatives occurs:

(i) All translates of the line $\{x\colon \langle x, u\rangle = 0\}$ contain at most one point of $\mathscr{N}$.
(ii) A translate of the above line contains two or more points of $\mathscr{N}$.

There are only finitely many u's that fall in case (ii). However, all directions u that fall in the case (i) point to a location on the boundary of $K_{1/w}$ which is inside a straight-line segment included in the boundary. To check this, observe that $\{\langle x, u\rangle = -w(u)\}$ contains exactly one integer point z. Therefore, $\{\langle x, u'\rangle = -w(u)\}$ contains only one integer point (namely, z) for u' close to u. This implies that $w(u') = w(u)\langle u, u'\rangle$, so that $1/w(u')$ gives an equation of a straight line (in polar coordinates). Therefore, $K_{1/w}$ is a polygon and so is $K^*_{1/w}$. $\square$

EXAMPLE 2.3. We illustrate the shape theorem with a few simple examples. Range 1 box neighborhood has three shapes. The $\theta = 1$ shape is of course $B_\infty(0, 1)$, the $\theta = 2$ shape is $B_1(0, 1)$ and the $\theta = 3$ shape is an octagon with vertices $(\frac{1}{2}, 0)$ and $(\frac{1}{3}, \frac{1}{3})$ in the first quadrant on or below $y = x$. (We need only specify these, by symmetry.) The $\theta = 3$ case (see [11] for a picture) is the only one of these with a nonconvex $K_{1/w}$, which has vertices $(1, 0)$, $(2, 1)$ and $(1, 1)$.

Figure 2 illustrates the 10 shapes for a range 2 box neighborhood. This picture was generated by running the dynamics from the small initial seed

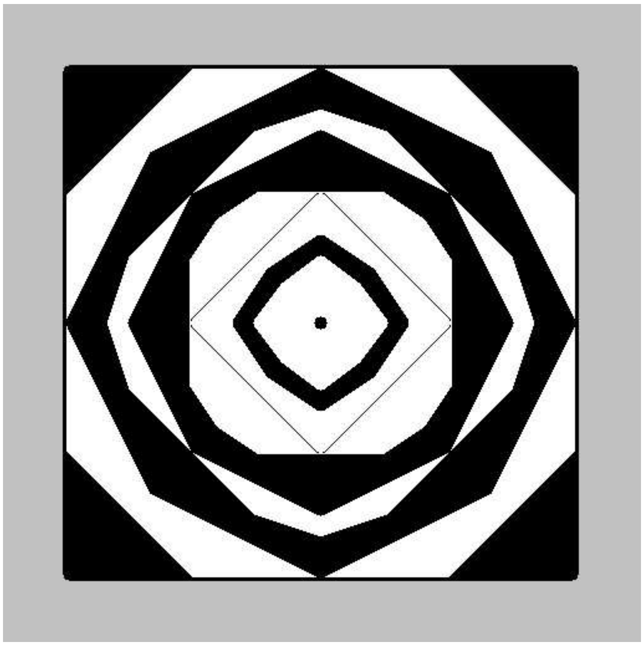

FIG. 2. *Range* 2 *box polygonal shapes.*

shown in the middle of the figure rather than using (2.2). The most interesting observation is that the shapes for $\theta = 7$ and $\theta = 8$ appear to both equal $B_1(0, 1)$. This is confirmed by a computation of $K_{1/w}$, which in the $\theta = 7$ case has vertices $(1, 0)$, $(\frac{3}{4}, \frac{1}{4})$, $(1, \frac{1}{2})$, $(\frac{3}{4}, \frac{1}{2})$ and $(1, 1)$, while $K_{1/w} = B_\infty(0, 1)$ in the $\theta = 8$ case. The only other cases with convex $K_{1/w}$ are $\theta = 1$ and $\theta = 2$ [this last has an octagonal shape L with vertex $(2, 1)$].

PROPOSITION 2.4. *If* $-\mathscr{N} = \mathscr{N}$ *and* $\theta \le \frac{1}{2}(|\mathscr{N}| - \iota(\mathscr{N}))$, *then the dynamics are supercritical*.

PROOF. The first observation to make is that the dynamics are supercritical iff $w(u) > 0$ for every u. For, if this is true, there exists a bounded subset $B \subset \mathbf{R}^2$ so that $\bar{\mathscr{T}}^n(B) \uparrow \mathbf{R}^2$ (see Lemma 1 on page 853 of [11]). The converse is obvious since an edge speed of 0 constrains growth to a corresponding half-space.

Now for any u, there are at most $\iota(\mathscr{N})$ sites in $\mathscr{N} \cap \{x: \langle x, u\rangle = 0\}$. Hence, by symmetry of $\mathscr{N}$, there are at least $\frac{1}{2}(|\mathscr{N}| - \iota(\mathscr{N}))$ sites in $\mathscr{N} \cap \{x: \langle x, u\rangle < 0\}$. For y in a small neighborhood of the origin, then

$$|(y + \mathscr{N}) \cap \{x: \langle x, u\rangle < 0\}| = |\mathscr{N} \cap \{x: \langle x, u\rangle < 0\}| \ge \tfrac{1}{2}(|\mathscr{N}| - \iota(\mathscr{N})) \ge \theta. \qquad \square$$

PROPOSITION 2.5. *If* $-\mathcal{N} = \mathcal{N}$ *and* $\theta > \frac{1}{2}(|\mathcal{N}| - 1)$, *then the dynamics are subcritical.*

PROOF. In this case it immediately follows that $w(u) = 0$ for every u. Even more, we claim there exists an $\varepsilon > 0$ so that $\bar{\mathcal{T}}(H_u \cup H_v) = H_u \cup H_v$ for every u and v such that $\|u - v\| < \varepsilon$. The easiest way to see this is to consider open half-spaces, denoted by H_u^o, instead of closed ones (as any point slightly outside $H_u \cup H_v$ cannot see the points on the boundary anyway). Now it is clear that there exists an $\varepsilon > 0$ so that if u and v are closer than ε, and z is an arbitrary real point with $\langle z, u\rangle = 0$, then $|(H_u^o \cup (z + H_v^o)) \cap \mathcal{N}| \le \frac{1}{2}(|\mathcal{N}| - 1)$. This means that real points outside $H_u \cup H_v$, but very near its boundary, do not see θ sites, but then other points do not either. This proves the claim.

Now we can construct a large open polygon $\mathcal{O}$, with sides longer than the diameter of $\mathcal{N}$ and interior angles close enough to π (so that their cosines are at least $1 - \varepsilon^2/2$), so that no real point in $\mathcal{O}$ sees θ points outside. Thus $\bar{\mathcal{T}}$ can never add a point in $\mathcal{O}$, even if the complement is entirely occupied, but then neither can $\mathcal{T}$. □

PROPOSITION 2.6. *If* $-\mathcal{N} = \mathcal{N}$ *and* $\theta \in [\frac{1}{2}(|\mathcal{N}| - \iota(\mathcal{N})) + 1, \frac{1}{2}(|\mathcal{N}| - 1)]$, *the dynamics are critical. Moreover, if* $A_0 = \Pi(p)$, *then* $A_\infty = \mathbf{Z}^2$ *a.s. for every* $p > 0$.

PROOF. In this regime $w(u) = 0$ for the u such that $|\mathcal{N} \cap \{x: \langle x, u\rangle = 0\}| = \iota(\mathcal{N})$, so supercriticality is out of the question. So is subcriticality: in this regime $w(u) > 0$ as soon as $\mathcal{N} \cap \{x: \langle x, u\rangle = 0\} = \{0\}$—this means $w(u) > 0$ for all except finitely many directions. To prove the last assertion, pick any unit vector u. Although $w(u)$ may be 0, we claim that a judicious addition of a bounded set actually makes H_u^- progress a little. To be more precise, we claim that there exists a bounded $F \subset \mathbf{R}^2$ such that $\bar{\mathcal{T}}^\infty(F \cup H_u^-) \supset w'(u)u + H_u^-$ for some $w'(u) > 0$.

To do this, assume for convenience (and without loss of generality) that u is on the left side of the unit ball ($\langle u, e_1\rangle < 0$). Now define $w'(u)$ to be the speed of H_u^- when the threshold equals $\frac{1}{2}(|\mathcal{N}| - \iota(\mathcal{N}))$. Pick an integer point $x_0 \in w'(u)u + H_u^-$ and put in F all sites in $(x_0 + \mathcal{N}) \cap (w'(u)u + H_u^-)$ *below* x_0. Then also add real points below x_0 and in $w'(u) + H_u^-$ (this yields an infinite set, which can be cut at some point far from x_0). It is easy to see that such an F does the required job. Note, for later use, that if, say, $\mathcal{N}$ is a box neighborhood, then $|F \cap \mathbf{Z}^2| \le \theta - \frac{1}{2}(|\mathcal{N}| - \iota(\mathcal{N}))$.

Let a (a finite integer) be the number of directions with $w(u) = 0$ and let now F be a finite set that works for all of them. Now it is possible to make a finite polygon $B \subset \mathbf{R}^2$ with angles between edges close enough to π so that the following is true. Fix a $t > 0$ and add a translates of F anywhere along the edges of tB perpendicular to u's with 0 speed, making sure they are away from the corners. Then apply the growth dynamics. By the preceding paragraph, the dynamics will eventually cover $(t + \delta)B$ for some $\delta > 0$ independent of t.

At this point, the standard bootstrap methods from [2] can be adapted to finish the proof. □

3. Infinite initial sets. Now that we understand the essential behavior of threshold growth started from finite sets and sets with finite complement, we need to know how the dynamics affect infinite sets that are convex or have convex complement. For fixed unit vectors u_1 and u_2, a wedge and a complement of a wedge are defined, respectively, as

$$W = W_{u_1, u_2} = H_{u_1}^- \cap H_{u_2}^-,$$
$$Q = Q_{u_1, u_2} = H_{u_1}^- \cup H_{u_2}^-.$$

THEOREM 3.1. *The following convergence results hold as $n \to \infty$, in the Hausdorff metric*:

$$\frac{1}{n}\bar{\mathscr{T}}^n(W) \to \bigcap\{w(u)u + H_u^-: W \subset H_u^-\} = (K_{1/w} \cap W^*)^*, \tag{3.1}$$

$$\frac{1}{n}\bar{\mathscr{T}}^n(Q) \to \bigcup\{w(u)u + H_u^-: H_u^- \subset Q\}. \tag{3.2}$$

Result (3.1) is not so surprising. Essentially, it is analogous to the "finite" shape result Theorem 2.1, with some w's set to ∞. Result (3.2) is a little tricky. One would expect the curvature of Q near the origin to "smooth out" and stabilize quickly, so that it should not matter on a linear scale. This is indeed the case, but the smoothed out "corner" can have any inclination (within the bounds determined by Q) in general. In one case, however, (3.2) simplifies significantly.

PROPOSITION 3.2. *If $K_{1/w}$ is convex, then*

$$\frac{1}{n}\bar{\mathscr{T}}^n(Q) \to L + Q = v + Q, \tag{3.3}$$

where v is the vector

$$v = \frac{1}{1 - \langle u_1, u_2\rangle^2}[(w(u_1) - w(u_2)\langle u_1, u_2\rangle)u_1 + (w(u_2) - w(u_1)\langle u_1, u_2\rangle)u_2].$$

In fact, the proof will show that (3.3) is valid for every Q if and only if $K_{1/w}$ is convex.

PROOF OF PROPOSITION 3.2. One should start by checking that the v above is the correct translation vector. This follows from the fact that $\langle v, u_1\rangle = w(u_1)$ and $\langle v, u_2\rangle = w(u_2)$.

Choose two unit vectors u_1 and u_2 and an $\alpha \in (0, 1)$, and denote $\bar{u} = \alpha u_1 + (1 - \alpha)u_2$. Now, $K_{1/w}$ is convex if and only if

$$w(\bar{u}/||\bar{u}||)||\bar{u}|| \le \alpha w(u_1) + (1 - \alpha)w(u_2). \tag{3.4}$$

On the other hand, (3.3) is valid (by Theorem 3.1) iff v does not fall into the interior of $w(\bar{u}/||\bar{u}||)\bar{u}/||\bar{u}|| + H^-_{\bar{u}/||\bar{u}||}$, which is true iff

$$\langle v, \bar{u}\rangle \geq w(\bar{u}/||\bar{u}||)||\bar{u}||. \tag{3.5}$$

Clearly, (3.4) and (3.5) are identical. □

The easiest way to check convexity of $K_{1/w}$ is the following. Pick a point $x \in \mathbf{R}^2$ and a line ℓ through x so that the closed connected component of the complement of the line which does not include the origin contains θ sites. Suppose that a small rotation of the line ℓ around x in *either direction* makes this complement contain fewer than θ sites. This never happens iff $K_{1/w}$ is convex. It suffices to check finitely many lines ℓ that contain two or more points of $\mathscr{N}$. In the range ρ box neighborhood case, it is easy to see that $K_{1/w}$ is convex if $\theta = 1$ or 2. Much more surprisingly, $K_{1/w}$ is also convex [in fact, equal to $B_\infty(0,1)$] if $\theta = 2\rho^2$.

In passing we note that $K_{1/w}$ always becomes convex in the *threshold-range limit*. More precisely, if $\theta_k \to \infty$ and $\mathscr{N}_k/\sqrt{\theta_k} \to \bar{\mathscr{N}} \subset \mathbf{R}^2$, with w_k the speeds of the $(\mathscr{N}_k, \theta_k)$ dynamics, then $\sqrt{\theta_k}K_{1/w_k}$ converges to a convex set. See [11] for a proof.

Proof of Theorem 3.1. We start by proving (3.1). The first step is to verify the right-hand equality. Call the two sets L' and L'_1, respectively. Then $z \in L'$ iff $\langle u, W\rangle \leq 0$ implies $\langle u, z\rangle \leq w(u)$. On the other hand, $z \in L'_1$ iff $\langle u, W\rangle \leq \lambda$ and $\lambda \in [w(u), \infty)$ imply $\langle u, z\rangle \leq w(u)$. These two are equivalent since $\langle u, W\rangle \leq \lambda$ is equivalent to $\langle u, W\rangle \leq 0$ (W is a cone).

Introduce an auxiliary transformation $\bar{\mathscr{T}_a}$ that acts on the subsets of $\mathbf{R}^2$ in the following fashion:

$$\bar{\mathscr{T}_a}(B) = \bigcap\{w(u)u + \lambda u + H^-_u\colon B \subset H^-_u,\ \lambda > 0\}.$$

To prove convergence in (3.1), we begin by claiming that L' is *invariant* for $\bar{\mathscr{T}_a}$. More precisely, for every $r > 0$,

$$\bar{\mathscr{T}_a}(rL') = (r+1)L'. \tag{3.6}$$

To show this, note that $z \in (r+1)L'$ iff

$$\forall u, \qquad \langle u, W\rangle \leq 0 \quad \Rightarrow \quad \langle u, z\rangle \leq (r+1)w(u) \tag{3.7}$$

and $z \in \bar{\mathscr{T}_b}(rL')$ iff

$$\forall u,\ \forall\lambda, \qquad \langle u, rL'\rangle \leq \lambda \quad \Rightarrow \quad \langle u, z\rangle \leq w(u) + \lambda. \tag{3.8}$$

Assume first that (3.8) holds. To show (3.7), we also assume $\langle u, W\rangle \leq 0$. If $y \in L'$, then, since $\langle u, W\rangle \leq 0$, $\langle u, y\rangle \leq w(u)$. This proves that $\langle u, L'\rangle \leq w(u)$, which, if we choose $\lambda = rw(u)$, implies $\langle u, rL'\rangle \leq \lambda$. Now, (3.8) implies $\langle u, z\rangle \leq w(u) + \lambda = (r+1)w(u)$.

Now assume that (3.7) holds. To show (3.8), pick a $z \in (r+1)L'$ and λ and u so that $\langle u, rL'\rangle \leq \lambda$. Since $W \subset rL'$, $\langle u, W\rangle \leq \lambda$, but then, since W is a cone, $\langle u, W\rangle \le 0$. This, by (3.7), implies that $\langle u, z\rangle \le (r+1)w(u)$. Thus (3.8) holds if $\lambda \ge rw(u)$. On the other hand, if $\lambda \le rw(u)$, then just use the fact that $r/(r+1)z \in L'$, so that $\langle u, z\rangle \le ((r+1)/r)\lambda \le \lambda + w(u)$, so again (3.8) holds.

The final step in proving (3.1) is to quantify the fact that if r is large, $\bar{\mathscr{T}}(rL') \approx \bar{\mathscr{T}}_a(rL')$. Fix an $\varepsilon > 0$. First, we find L'_ε and Q such that L_ε is convex, has a $\mathscr{C}^1$ boundary, $(1-\varepsilon)L' \subset L'_\varepsilon \subset L'$ and $L'_\varepsilon \cap B_2(0, R)^c = L' \cap B_2(0, R)^c$ for some large R. (It is intuitively clear that such a set should exist, but see [11] for more details.) In particular, by enlarging R if necessary, we can assume that L'_ε is flat outside $B_2(0, R)$.

Now we claim that there exists an $r_0 = r_0(\varepsilon)$ so that for $r \ge r_0$,

$$\bar{\mathscr{T}}_a((r-\varepsilon)L'_\varepsilon) \subset \bar{\mathscr{T}}(rL'_\varepsilon). \tag{3.9}$$

Let $M = \max\{w(u)\} + 2\,\mathrm{diam}(\mathscr{N})$. Pick $r_0 = r_0(\varepsilon)$ large enough so that for $r \ge r_0$ and every point x on the boundary of $(r-\varepsilon)L'_\varepsilon$, $(x + H^-_{n(x)}) \cap B_2(0, M) \subset rL'_\varepsilon$, where $n(x)$ is the outward normal to $(r-\varepsilon)L$ at x. (See Figure 5 and Lemma 1 in [11].) Then pick a $z \in \bar{\mathscr{T}}_a((r-\varepsilon)L'_\varepsilon)$. Let y be the closest point to z in $(r-\varepsilon)L'_\varepsilon$ and set $u = n(y)$. Then $(r-\varepsilon)L_\varepsilon \subset y + H^-_u = \langle u, y\rangle u + H^-_u$. Therefore, $z \in y + w(u)u + H^-_u$ by definition of $\bar{\mathscr{T}}_a$. Now it follows that $z \in \bar{\mathscr{T}}(rL'_\varepsilon)$, since z sees at least θ sites in $y + H^-_u$ and is at distance $w(u)$ from y. This shows (3.9).

Iterating (3.9) and using (3.6) and the properties of L_ε, we find that

$$\begin{aligned}(n + (r_0-\varepsilon)(1-\varepsilon))L' &= \bar{\mathscr{T}}_a^n((r_0-\varepsilon)(1-\varepsilon)L') \subset \bar{\mathscr{T}}_a^n((r_0-\varepsilon)L'_\varepsilon)\\ &\subset \bar{\mathscr{T}}^n(r_0L'_\varepsilon) \subset \bar{\mathscr{T}}^n(r_0L) \subset \bar{\mathscr{T}}^n(x_0 + W)\\ &= x_0 + \bar{\mathscr{T}}^n(W)\end{aligned}$$

for some vector x_0 (which depends on r_0, and hence on ε). Such a vector exists because w is bounded both above and away from 0. Therefore,

$$-x_0 + (n + (r_0-\varepsilon)(1-\varepsilon))L' \subset \bar{\mathscr{T}}^n(W) \subset nL',$$

which, after dividing by n and sending $n \to \infty$, implies (3.1).

The proof of (3.2) is very similar in spirit, but the technical details are sufficiently different that we provide a sketch. The starting point is another auxiliary transformation:

$$\bar{\mathscr{T}}_b(B) = \bigcup\{w(u)u + \lambda u + H^-_u\colon \lambda u + H^-_u \subset B,\ \lambda \ge 0\}.$$

These dynamics leave L'', the right-hand side of (3.2), invariant for all $r > 0$:

$$\bar{\mathscr{T}}_b(rL'') = (r+1)L''. \tag{3.10}$$

To prove (3.10) we need to show that, for $z \in \mathbf{R}^2$, the statement

$$\exists u, \qquad H^-_u \subset Q \quad \text{and} \quad \langle u, z\rangle \le (r+1)w(u) \tag{3.11}$$

is equivalent to

$$\exists u,\ \exists\lambda \ge 0, \qquad \lambda u + H^-_u \subset rL'' \quad \text{and} \quad \langle u, z\rangle \le w(u) + \lambda. \tag{3.12}$$

To see that (3.11) implies (3.12), we choose the same u and $\lambda = rw(u)$ to get (3.12). Conversely, if (3.12) is true, we again choose the same u and note that there exists an x_0 so that

$$(H_u^-)^c \supset (rL'' - \lambda u)^c \supset x_0 + Q^c.$$

This implies that $(H_u^-)^c \supset Q^c$ and $H_u^- \subset Q$. Now finish the argument by contradiction: if $z \notin (r+1)L''$, then $r/(r+1)z \notin rL''$, so by (3.12), $r/(r+1)z \notin \lambda u + H_u^-$. This last is the same as $\langle u, r/(r+1)z\rangle > \lambda$, which in turn implies that $\langle u, z\rangle \leq w(u) + r/(r+1)\langle u, z\rangle$, so (3.11) holds.

Equipped with invariance (3.10), virtually the same argument as for (3.1) goes through. (Now, however, the tricky part is the upper bound.) $\square$

4. A description of $\boldsymbol{\tau}$. Assume that $z_1, \dots, z_n \in \mathbf{R}^2$ are distinct points. The aim of this section is to define a "continuous movie": from every point z_i a separate expanding copy of L is started. Limiting dynamics from (3.1) and (3.2) are then used to govern the interaction when the growing sets collide. Our asymptotic passage time $\tau = \tau(z_1, \dots, z_n)$ is represented as the first time, starting from a Poisson field, that the movie reaches the origin.

To be more precise, we will define dynamics $\mathbb{M}_t = \mathbb{M}_t(z_1, \dots, z_n)$. Let $\mathbb{M}_t^1$ be $\bigcup_i (z_i + tL)$ and let T_1 be the first time $(z_i + tL) \cap (z_j + tL) \neq \varnothing$ for some $i \neq j$. Define $\mathbb{M}_t = \mathbb{M}_t^1$ for $0 \leq t \leq T_1$. At the time $t \geq T_1$, think of the boundary of $\mathbb{M}_t$ as the finite sequence of corners $k_0(t), k_1(t), \dots, k_{m(t)}(t)$ connected by straight lines (it is useful to be reminded of the fact that there are only finitely many possible slopes). Note that it may happen that $k_i(t) = k_j(t)$ for some $0 \leq i < j \leq m(t)$ since the boundary may consist of some disconnected pieces and since it is possible that even connected components of $\mathbb{M}_t$ might be "pinched" at some point.

Every corner $k_{i-1}(t), k_i(t), k_{i+1}(t)$ is either convex or concave. This merely means that $\mathbb{M}_t$ is a locally convex or concave set at $k_i(t)$. Now make every convex corner evolve according to (3.1) and every concave corner according to (3.2). At time $t = T_1$ this may (and probably will, unless $K_{1/w}$ is convex) cause branching of corners, as illustrated in Figure 3. Note that these dynamics are well defined at least for some small time after T_1, as both limiting sets in (3.1) and (3.2) have straight-line boundaries outside a neighborhood of the origin. Indeed, consider the subsequent time T_2, when either two corners coalesce or new ones are created or the nature of an existing one changes by corners bumping into straight-line parts of the boundary or other corners. These interactions define the set $\mathbb{M}_t$ for times $t \in [T_1, T_2]$. Now redefine the set of corners and continue with the same dynamics as above until time T_3, defined analogously to T_2. Continue this procedure indefinitely.

Our movie is well defined forever since infinitely many edges cannot arise in finite time: there are only finitely many possibilities for convex corners, and new edges are created only when such convex corners collide with straight lines. Thus, define $\tau = \tau(z_1, \dots, z_n) = \inf\{t\colon 0 \in \mathbb{M}_t\}$. We will now state two propositions that establish needed continuity properties of τ.

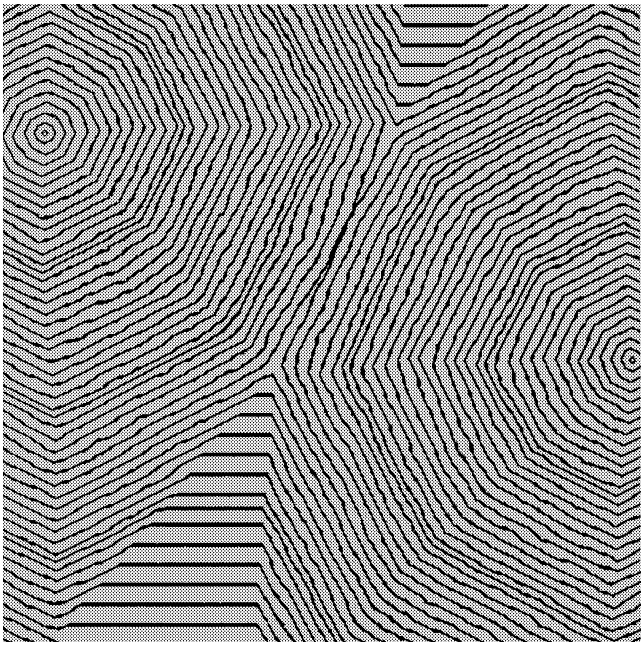

FIG. 3. *Interaction of two range* 1 *box threshold* 3 *droplets.*

PROPOSITION 4.1. *Fix $z_1, \dots, z_n$. For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if $z'_1, \dots, z'_m$ is a finite collection of points with $z'_i \in \bigcup_{i=1}^n B(z_i, \delta)$, then $\tau(z'_1, \dots, z'_m) \le \tau(z_1, \dots, z_n) + \varepsilon$.*

This implies that τ is continuous with respect to small perturbation of existing centers, or even adding new centers close to existing ones.

PROOF OF PROPOSITION 4.1. This follows simply by noting that the sets $\mathbb{M}_t$ are continuous in t (in the Hausdorff metric) and that after a short initial time, $\mathbb{M}_t(z_1, \dots, z_n)$ covers all the z'_i. The final step is to use monotonicity of the sets defined in (3.1) and (3.2). $\square$

The sets in (3.1) and (3.2) depend only on the speed function w. Assume now that the speed is perturbed to w' and let τ' be the new first passage time. The next proposition states that if the perturbation is small enough, then the first passage time does not change much.

PROPOSITION 4.2. *Fix $z_1, \dots, z_n$. For every $\varepsilon > 0$ there exists a $\delta > 0$ so that $||w - w'||_\infty < \delta$ implies $|\tau - \tau'| < \varepsilon$.*

PROOF. If $w' = w(1 + \delta)$, then $\tau' = \tau/(1 + \delta)$, by rescaling of time. Now again use the monotonicity in (3.1) and (3.2) to conclude that if $w' < w(1+\delta)$, then $\tau' > \tau/(1 + \delta)$. $\square$

In the easiest case, if $K_{1/w}$ is convex, then there is no branching of corners in the dynamics $\mathbb{M}_t$. In fact, Proposition 3.2 implies the following result.

PROPOSITION 4.3. *If $K_{1/w}$ is convex, then $\mathbb{M}_t = \bigcup_{i=1}^n (z_i + tL)$ and so $\tau = \inf\{\|z_i\|: i = 1, \dots, n\}$.*

The final task of this section is to replace the finitely many centers in our construction by infinitely many. If $\mathscr{Z}$ is any infinite subset of $\mathbf{R}^2$ without limit points, let $z_1 \in \mathscr{Z}$ be such that $\|z_1\| = \min\{\|z\|: z \in \mathscr{Z}\}$ and let $z_1, \dots, z_n$ be all points in $\mathscr{Z} \cap B_2(0, \|z_1\|(1 + 2\max\{w\}))$. Then $\tau(z_1, \dots, z_1)$ is the same as τ obtained by adding any finite number of points from $\mathscr{Z}$ to $z_1, \dots, z_n$. Therefore, it makes sense to define $\tau(\mathscr{Z}) = \tau(z_1, \dots, z_n)$. We say that $\tau(\mathscr{Z})$ is *decided* by $z_1, \dots, z_n$.

5. Convergence of nucleation centers. Call site $x \in \mathbf{Z}^2$ a *nucleus* if it is the leftmost lowest site of γ sites in $\Pi(p)$ which generate persistent growth. Let $Q(p)$ be the set of all such nuclei. Since $P(x \text{ is a nucleus}) \sim \nu p^\gamma$ as $p \to 0$ and different sites are nuclei almost independently, $\sqrt{\nu p^\gamma}Q(p)$ should converge to the unit-intensity Poisson point location. The right machinery to prove this is the Chen–Stein method ([5], [6]), which we make use of in the next proposition.

PROPOSITION 5.1. *Fix a threshold growth model, any $R > 0$, and let $B = B_\infty(0, R)$. Then there exists a constant $C > 0$ (independent of p) so that the total variation distance between $Q(p) \cap (p^{-\gamma/2}B)$ and $\Pi(\nu p^\gamma) \cap (p^{-\gamma/2}B)$ is bounded above by Cp.*

PROOF. Let G_x be the event that there is a nucleus at x, $p_x = P(G_x)$ and $p_{xy} = P(G_x \cap G_y)$. Let M be large enough that $\mathscr{N} \subset B_\infty(0, M)$ and also, if $|A_0| < \gamma$, then $A_\infty \subset A_0 + B_\infty(0, M)$. It follows that if x is the leftmost lowest site of a set of γ points which generate persistent growth, then that set has to be included in $B_\infty(x, 2\gamma M)$. The crucial point is that if $B_x = B_\infty(x, (4\gamma+1)M)$, then $y \notin B_x$ implies that G_x and G_y are independent. Also, let $I = p^{-\gamma/2}B$.

Then

$$p_x = \nu p^\gamma + \mathscr{O}(p^{\gamma+1}). \tag{5.1}$$

Moreover,

$$b_1 = \sum_{x\in I} \sum_{y \in B_x} p_x p_y \le (8\gamma M + 1)^2 (2Rp^{-\gamma/2} + 1)^2 p_x^2 = \mathscr{O}(p^\gamma) \tag{5.2}$$

and

$$b_2 = \sum_{x\in I} \sum_{y \in B_x \setminus \{x\}} p_{xy} \le |I||B_x| P(\gamma + 1 \text{ sites in } \Pi(p) \cap B_x) = \mathscr{O}(p). \tag{5.3}$$

Theorem 2 from [5] implies that the variation distance between $Q(p) \cap (p^{-\gamma/2} \times B)$ and $\Pi(\nu p^\gamma) \cap (p^{-\gamma/2}B)$ is bounded above by $4b_1 + 4b_2$. Hence (5.2) and (5.3) complete the proof. □

Our next proposition provides one of the main steps in the proof of Theorem 1.5. Recall that $\wp$ denotes a unit-intensity Poisson process and τ denotes the first passage functional of the previous section.

PROPOSITION 5.2. *Given a threshold voter model, let* $\tau_p = \tau(\sqrt{\nu p^\gamma}Q(p))$ *and* $\tau = \tau(\wp)$. *Then* $\tau_p \to_d \tau$ *as* $p \to 0$.

PROOF. Let $q = \nu p^\gamma$ and $q' = -\ln(1-q)$. Then $\Pi(q)$ and $\wp$ can be coupled by declaring that $x \in \Pi(q)$ iff $B_\infty(\sqrt{q'}x, \sqrt{q'}/2) \cap \wp \neq \varnothing$. Let $\tau'_p = \tau(\sqrt{q}\Pi(q))$ and $\tau''_p = \tau(\sqrt{q'}\Pi(q))$. Since every point in $\wp$ has a point in $\sqrt{q'}\Pi(q)$ at $\|\cdot\|_\infty$ distance at most $\sqrt{q'}/2$, and vice versa, results from Section 4 immediately imply that $|\tau''_p - \tau|$ converges to 0 a.s. as $p \to 0$.

To see that $|\tau''_p - \tau'_p|$ goes to 0 in probability as $p \to 0$, note that $|\sqrt{q'} - \sqrt{q}| \le q^{3/2}$ for small q, so that $\|\sqrt{q'/q}z - z\|_\infty \le \sqrt{q}$ as soon as $\|z\|_\infty < q^{-1/2}$. However, the probability that the finite set which decides the value of τ' is not found in $B_\infty(0, q^{-1/2})$ is exponentially small in $1/q$. It follows that with high probability the values of τ' and τ'' are decided by an event in which the respective sets of centers are close to each other. Therefore, for every $\varepsilon > 0$, $P(|\tau''_p - \tau'_p| > \varepsilon) \to 0$ as $p \to 0$.

What remains to be shown, therefore, is that for every $\varepsilon > 0$, $P(|\tau'_p - \tau_p| > \varepsilon)$ converges to 0 as $p \to 0$. Start by fixing a large $R > 0$. Now the probability that both τ'_p and τ_p are decided on $B_\infty(0, R)$ *and* differ by more than ε is bounded above by Cp, using Proposition 5.1. On the other hand, either passage time is decided on $B_\infty(0, R)$ as soon as it has a center in $R/(1+\max w)$, which implies that the probability that one of them is *not* decided in this ball converges to 0 as $R \to \infty$. □

6. Growth of half–spaces in polluted environments. In this section we finish the proof of Theorem 1.5. If we take the set of nuclei $Q(p)$, together with the sites that make them nuclei, and erase all other sites, then results from Sections 2 and 4, combined with Proposition 5.2, would make the proof immediate. Thus, the missing step is to show that a small density of helpful additional occupied sites cannot achieve much. This last step will be made more precise below, but first we need a simple large deviation result. The argument is completely standard, but quick enough that we give it here.

LEMMA 6.1. *Assume that* $X_1, X_2, \dots$ *are i.i.d. nonnegative integer-valued random variables, whose common distribution depends on a parameter* $p > 0$. *Assume, moreover, that* $P(X_1 \ge k) \le e^{-Kk}$ *for all* $k \ge 0$ *and* $K = K(p) \to \infty$ *as* $p \to 0$. *Then for every* $\varepsilon > 0$ *the large deviation rate*

$$\Gamma_p(\varepsilon) = -\lim_{n\to\infty} \frac{1}{n} \log P(X_1 + \cdots + X_n \ge \varepsilon n)$$

converges to ∞ *as* $p \to 0$.

PROOF. Write $\phi_p(\lambda) = \ln E(e^{\lambda X_1})$. Then $\Gamma_p(\varepsilon) = \sup\{\varepsilon\lambda - \phi_p(\lambda): \lambda > 0\}$. It is enough to prove that for every $M > 0$ there exists a small enough p so that $\phi_p(\lambda) < 1$ for some $\lambda \geq M$. This follows from a simple computation:

$$E(e^{\lambda X_1}) \leq \sum_{k=0}^{\infty} e^{(\lambda-K)k} = \frac{1}{1-e^{\lambda-K}} < e$$

as soon as $\lambda < K - 1$. □

Let us now define what we mean by a slightly helpful environment. Fix a positive integer $r > 0$, and choose a small $p > 0$. Call sites in $\Pi(p)$ *open*. Think of $\Pi(p)$ as a random environment, chosen before the growth dynamics start. For $A \subset \mathbf{Z}^2$, define

$$\begin{aligned}\mathscr{T}_p(A) = \{x \in \mathbf{Z}^2\colon & [x - re_1, x + re_1] \cap A \neq \varnothing \text{ or } x - e_2 \in A \text{ or} \\ & \text{there is a } y \in A \text{ so that } B_\infty(y, r) \cap \Pi(p) \cap A^c \neq \varnothing \\ & \qquad \text{and } \|x - y\|_\infty \leq r\}.\end{aligned}$$

To see what this dynamics does, note that $\mathscr{T}_0^n(H_{e_2}^-) = ne_2 + H_{e_2}^-$. Our goal is to establish that a small $p > 0$ perturbation is negligible on a linear scale. It turns out that this situation is general enough to handle any threshold growth dynamics and any half–space.

LEMMA 6.2. *Fix an arbitrary $\varepsilon > 0$. There exists a constant $\Gamma = \Gamma(p)$ which goes to ∞ as $p \to 0$ such that*

$$P(\mathscr{T}_p^n(H_{e_2}^-) \text{ includes a site } ke_2 \text{ with } k \geq (1+\varepsilon)n) \leq e^{-\Gamma n}.$$

PROOF. We start with an observation that simplifies our arguments. Namely, for any n, $\mathscr{T}_p^n(H_{e_2}^-)$ is *vertically regular*: if $x \in \mathscr{T}_p^n(H_{e_2}^-)$, so is $x - e_2$. A case by case check easily demonstrates this fact.

The next step is to prove the following claim by induction on n. It is a somewhat technical statement about how many boosting sites are needed to advance substantially beyond the $p = 0$ rate.

CLAIM. Assume that $\mathscr{T}_p^n(H_{e_2}^-)$ includes a site $(h, n + 4Nr)$, where $N \geq 1$ is an integer. Then there exist N open sites $(x_1, y_1), \dots, (x_n, y_N)$ such that

$$0 < y_0 < y_1 < \cdots < y_N \leq n + 4Nr + 2r \tag{6.1}$$

and

$$\sum_{i=2}^{N} |x_i - x_{i-1}| + |h - x_N| \leq r(n + 4N). \tag{6.2}$$

For $n = 0$ the claim is vacuous, so we assume its validity for $n - 1$ and prove (6.1) and (6.2). First, we can assume $h = 0$ and also that $(0, n + 4Nr)$ gets occupied *at* time n. (Otherwise it becomes occupied at some previous time and the induction hypothesis applies.) As there are three ways for this site to become occupied, the proof naturally divides into three cases.

Case 1. There is an $(a,b) \in \mathscr{T}_p^{n-1}(H_{e_2}^-)$ and an $(x_N, y_N) \in \Pi(p)$ such that $\|(a,b) - (x_N, y_N)\|_\infty \le r$ and $\|(0, n+4Nr) - (a,b)\|_\infty \le r$.

Clearly $y_N \le n + 4Nr + 2r$. Now $b \ge n + 4Nr - r > n - 1 + 4(N-1)r$, so by the vertical regularity and the induction hypothesis there are open sites $(x_1, y_1), \dots, (x_{N-1}, y_{N-1})$, such that

$$0 < y_0 < \cdots < y_{N-1} \le n - 1 + 4(N-1)r + 2r = n - 1 + (4N-2)r,$$

$$\sum_{i=2}^{N-2} |x_i - x_{i-1}| + |x_{N-1} - a| \le r(n-1) + 4(N-1)r.$$

It follows that $y_{N-1} < n + 4Nr - 2r \le y_N$, proving (6.1). To show (6.2), use the triangle inequality:

$$\begin{aligned}
&\sum_{i=2}^{N-1} |x_i - x_{i-1}| + |x_N - x_{N-1}| + |x_N| \\
&\qquad \le \sum_{i=2}^{N-1} |x_i - x_{i-1}| + |x_{N-1} - a| + |x_N - a| + |x_N| \\
&\qquad \le r(n-1) + 4(N-1)r + r + 2r < rn + 4Nr.
\end{aligned}$$

Case 2. $(0, n+4Nr-1) \in \mathscr{T}_p^{n-1}(H_{e_2}^-)$.

The claim is trivially valid for the same (x_i, y_i) guaranteed by the induction hypothesis for $(0, n-1+4Nr)$.

Case 3. For some $r' \in [-r, r]$, $(r', n+4Nr) \in \mathscr{T}_p^{n-1}(H_{e_2}^-)$.

Here the claim is valid for the same (x_i, y_i) guaranteed by the induction hypothesis for $(r', n+4Nr)$, as is easy to check.

With the claim in place, the rest of the proof is a more or less standard ("Peierls type") counting argument. To get an upper bound on the probability that such sites (with $h = 0$) exist, we need to first choose N lines which contain (x_i, y_i) among $n + (4N+2)r$: a crude upper bound for the number of such choices is $2^{n+(4N+2)r}$. Imagine these N lines stacked on top of each other. Then the (x_i, y_i) have to lie on a path of at most $r(n+4N)$ sites that starts on the line at the top of the stack [at a point with absolute value of its x–coordinate bounded by $r(n+4N)$] and goes either left, right or down at every turn, ending at the bottom of the stack. The number of such paths is bounded above by $2r(n+4N)3^{2r(n+4N)} < 4^{2r(n+4N)}$. Therefore, the probability that the sites (x_i, y_i) which satisfy (6.1) and (6.2) exist is at most

$$2^{n+r(4n+12N+2)} P(\text{a fixed set of } r(n+4N) \text{ sites includes } N \text{ open sites}).$$

Plug in $N = \varepsilon n/4$ and use Lemma 6.1 for Bernoulli random variables with small p. $\square$

Next we use Lemma 6.2 to prove a result about another comparison process, characterized by a transformation $\bar{\mathcal{T}}_p$, which is much closer to the actual threshold growth dynamics. Again, fix an $r > 0$ and define, for $p \in [0, 1]$ and $B \subset \mathbf{R}^2$,

$$\begin{aligned}\bar{\mathcal{T}}_p(B) = B \cup \{x \in \mathbf{R}^2\colon |(x + \mathcal{N}) \cap B| \geq \theta\} \\ \cup \{x \in \mathbf{R}^2\colon \operatorname{dist}_\infty(x, B) \leq r \text{ and } \operatorname{dist}_\infty(x, \Pi(p)) \leq r\}.\end{aligned}$$

In words, under these dynamics, sites in $\Pi(p)$ help out only once the growth dynamics reach them. The idea is that if we pretend there are no nuclei in $\Pi(p)$, then for large enough r this provides an upper bound on threshold growth in a polluted environment.

LEMMA 6.3. *Fix any unit vector u and $\varepsilon > 0$. The probability that there exist an (integer or noninteger) point in $(n(w(u) + \varepsilon)u + H_u^-)^c \cap \bar{\mathcal{T}}_p^n(H_u^-)$ is bounded above by $e^{-\Gamma n}$. Here, $\Gamma = \Gamma(p)$ is some constant which goes to ∞ as $p \to 0$.*

Note that this is exactly what we need: the complement of the event in question is that there are no occupied sites outside $n(w(u) + \varepsilon)u + H_u^-$.

PROOF OF LEMMA 6.3. To reproduce the situation of Lemma 6.2, rotate the space so that u becomes e_2. Then rescale time so that $w(u) = 1$. Finally, think of H_u^- and its iterates under $\bar{\mathcal{T}}_p$ as being divided into 1×1 blocks. Then every block which does not see an open site advances vertically by at most 1 and horizontally by some amount for which there exists a strict upper bound. If a block does see an open site, however, then it cannot proceed past a certain square of blocks (whose radius again is uniformly bounded by a bound that depends on r). This means that the dynamics defined by $\mathcal{T}_p$ (for a suitably large r, perhaps much larger than that in the definition of $\bar{\mathcal{T}}_p$) is an upper bound for this dynamics, so we can apply Lemma 6.2 to complete the proof. □

LEMMA 6.4. *Fix a finite number of unit vectors $u_1, \dots, u_n$ and a small $p > 0$. Consider the following event. There exists a site $x \in B_\infty(0, p^{-2\gamma})$ so that $\bar{\mathcal{T}}_p^n(x + H_u^-)$ includes a point in $(n(w(u) + \varepsilon)u + H_u)^c \cap B_\infty(0, p^{-2\gamma})$ for some $n \geq p^{-1/2}$. The probability of this event is bounded above by $e^{-C/\sqrt{p}}$ for some constant $C > 0$.*

The proof of Lemma 6.4 follows immediately from Lemma 6.3.

PROOF OF THEOREM 1.5. Fix an $\varepsilon > 0$. By Proposition 5.2, we need to show that

$$P(|\tau_p - \sqrt{\nu p^\gamma} \cdot T| > \varepsilon) \to 0 \quad \text{as } p \to 0.$$

This, however, follows easily from Theorem 3.1, Proposition 4.2 and Lemma 6.4. □

7. Notes on the critical case. In this section we will briefly consider some examples of critical dynamics. As usual, we start from a small density of occupied sites $\Pi(p)$. The proof of Proposition 2.6 and techniques from the vintage paper [2] then easily imply that there exists a constant $c_2 > 0$ so that

$$P(\ln T \le c_2/p^{\alpha}) \to 1 \quad \text{as } p \to 0.$$

Here $\alpha = \theta - \frac{1}{2}(|\mathscr{N}| - \iota(\mathscr{N}))$. A first guess might be that this upper bound always gives the accurate exponential scale. This is indeed the right rate in many cases, but we will see that the answer may also be considerably different (and more interesting). Below we present three examples which we suspect exhaust all possible types of scaling for first passage times in critical cases. These examples certainly do not close the book: we plan to investigate critical dynamics more closely in a forthcoming paper.

EXAMPLE 7.1. Assume that $\mathscr{N}$ consists of the seven points

$$\mathscr{N} = \begin{array}{ccccc} & & \bullet & & \\ \bullet & \bullet & 0 & \bullet & \bullet \\ & & \bullet & & \end{array}$$

and take $\theta = 2$. Note that $w(e_2) = w(-e_2) = 0$, but these are the only directions with speed 0. This causes T to obey a power law; in fact, T clearly is between the orders of $1/p$ and $1/p^2$. On the basis of the next proposition, we conjecture that $T \cdot p^{3/2}$ should converge weakly to a nontrivial random variable.

PROPOSITION 7.2. *For every $\varepsilon > 0$ there exists constants $c_1, c_2 > 0$ so that*

$$\liminf_{p\to 0} P\left(T \in \left[\frac{c_1}{p^{3/2}}, \frac{c_2}{p^{3/2}}\right]\right) \ge 1 - \varepsilon.$$

PROOF. In all the proofs of this section C will be a generic large constant and δ will be a generic small constant; they may both change from one appearance to the next. Also, our statements are crude enough so that we will fail to distinguish between p and a constant multiple of p.

To prove the upper bound, pick a $Cp^{-3/2} \times \delta p^{-1/2}$ box which has the origin in the middle of its top. The probability that this box includes two horizontally adjacent sites is very close to 1 if $C\delta$ is large. Then, after time $Cp^{-3/2}$ all sites on a horizontal line in the box will be occupied. By choosing a bottommost such pair, we can also assume that the occupied sites above this line are specified by a product measure with density p. On our time scale, then, we can assume that the bottom of our box is completely occupied initially. Let ξ_i be the smallest $j \ge 0$ such that $(i, j) \in \Pi(p)$. Then $\xi_1, \xi_2, \dots$ are i.i.d. geometric random variables with expectation $1/p$. Moreover, if the bottom line is full and $\xi_i \le Cp^{-3/2}$ for all $i \le \delta/\sqrt{p}$, then progress toward the top is made by a series of lateral interpolations between occupied sites at positions ξ_i. Thus, the time to reach the origin is at most $2(\xi_1 + \cdots + \xi_{\delta/\sqrt{p}})$. The upper bound then follows from the computation

$$P(T \geq Cp^{-3/2}) \leq P(\xi_1 + \cdots + \xi_{\delta/\sqrt{p}} \geq Cp^{-3/2}) + \varepsilon$$
$$\leq \frac{p^{3/2}}{C} \frac{\delta}{\sqrt{p}} E(\xi_1) + \varepsilon = \frac{\delta}{C} + \varepsilon.$$

To prove the lower bound, start with the same size box as before, except that now the origin is in its *center*. If δC is small, then there is no pair of sites in $\Pi(p)$ within a neighborhood of any point inside the box. We can also assume that there are no sites at all to the left or to the right of the box since those sites cannot influence the origin before time $Cp^{-3/2}$ anyway. In this case, the origin can be reached either from the top or from the bottom, but it suffices to show that the probability the origin is reached from the bottom is small. To sum up, we have all lines below the box fully occupied, no other occupied sites outside the box and density p product measure conditioned on no pairs inside the box. However, {no pairs} is a negative event and {the origin reached by time $\delta p^{-3/2}$} is a positive event, so the two are negatively correlated by the FKG inequality. Lemma 6.2 now finishes the proof of the lower bound. □

EXAMPLE 7.3. Take the same $\mathscr{N}$ as in the Example 7.1 and $\theta = 3$. Now $w(e_1) = 0$ as well, but $|\mathscr{N} \cap \{x: \langle x, u\rangle = 0\}| = \iota(N)$ only if $u = \pm e_2$. This seems to be a typical situation in which T has exponential scaling in a power of $1/p$ with logarithmic corrections.

In this context we should mention a result of Mountford [14], which deals with the asymmetric critical case $\mathscr{N} = \{(0,0), (0,-1), (\pm 1, 0)\}$, $\theta = 2$, for which it proves that the correct scaling for $\ln T$ is $1/p(\ln(1/p))^2$. However the logarithmic corrections in [14] originate from the size of critical droplets, whereas in the present example they are a consequence of the way these droplets grow.

PROPOSITION 7.4. *There exist constants* $c_1, c_2 > 0$ *so that*

$$P\left(\ln T \in \left[c_1 \frac{1}{p} \ln \frac{1}{p}, c_2 \frac{1}{p} \ln \frac{1}{p}\right]\right) \to 1 \quad \textit{as } p \to 0.$$

PROOF. Some of the ingredients in this proof are straightforward adaptations of the original ideas from [2]. We will largely skip over those, emphasizing the differences which arise in our case. Following [2], we call a set $A \subset \mathbf{Z}^2$ *internally spanned* (IS) if the dynamics restricted to A (with free boundary conditions) eventually fill in every site of A.

Again, we start with the upper bound. Fix a sequence of positive even integers $a_1, a_2, \dots$ and let $s_k = a_1 + \cdots + a_k$. Let R_k be the $s_{k-1} \times k$ rectangle with leftmost lowest point $(1,1)$. Denote by G_k the event that the horizontal line segment $\{(j, k+1)\colon 1 \leq j \leq s_k\}$ includes two horizontally adjacent occupied sites and that each of the a_k vertical line segments $\{(s_{k-1} + i, j)\colon 1 \leq i \leq k\}$,

$j = 1, \ldots, a_k$, contains at least one site. Then R_{k+1} is internally spanned, given that $G_1 \cap G_2 \cap \cdots \cap G_k$ happens. If we can choose a_k so that

$$P\left(\bigcap_{k=1}^{\infty} G_k\right) \geq \exp\left(-C\frac{1}{p}\ln\frac{1}{p}\right), \tag{7.1}$$

then a percolation argument (see [2]) establishes the upper bound. However, the G_k are independent, so that the probability in (7.1) is at least

$$\prod_{k=1}^{\infty}(1-(1-p^2)^{s_k/2})(1-(1-p)^k)^{a_k}. \tag{7.2}$$

Setting $a_k = 2\lceil e^{kp/2}\rceil$ and taking $-\log$ of the expression (7.2), one finds, after a straightforward computation (replacing all multiples of p by p), that what needs to be shown is

$$\sum_{k=1}^{\infty}\exp(-pe^{kp}) + \sum_{k=1}^{\infty}\exp(-kp) \leq C\frac{1}{p}\ln\frac{1}{p}. \tag{7.3}$$

Verifying (7.3) is an elementary exercise in estimating sums by integrals.

To prove the lower bound, call a rectangle R in $\mathbf{Z}^2$ *potentially internally spanned* (PIS) if it is either a single site in $\Pi(p)$ or (i) for every (integer) vertical line ℓ through R there exist two sites $x, y \in \Pi(p) \cap R$ such that $||x-y||_\infty \leq 4$ and they are both at ℓ^∞–distance at most 4 from ℓ and (ii) every horizontal line ℓ through it has a site in $\Pi(p) \cap R$ at ℓ^∞–distance at most 2 from ℓ.

To see the reason for this slightly convoluted definition, note that R can never be IS unless it is PIS [because a line that fails either (i) or (ii) could never be crossed]. If fact, much more is true.

CLAIM. Let $L < M$ be positive integers. Assume that the origin is not eventually occupied if the dynamics are restricted to $[-L, L]^2$, but is eventually occupied if the dynamics are restricted to $[-M, M]^2$. For every integer $a \in [4, L/4]$, there exists a PIS rectangle R included in $[-M, M]^2$ whose longest side is between a and $4a$.

To prove the claim, successively define collections $\mathscr{C}_1, \mathscr{C}_2, \ldots$ of PIS rectangles which are subsets of $[-M, M]^2$. Start with

$$\mathscr{C}_1 = \{\text{all PIS rectangles with length of both sides} \leq a\}.$$

Now, given $\mathscr{C}_i$, there are three possibilities.

Case 1. Two rectangles R_1 and R_2 from $\mathscr{C}_i$ intersect. Then let R_3 be the smallest rectangle containing R_1 and R_2. It is clear that R_3 must be PIS as well. Let $\mathscr{C}_{i+1} = (\mathscr{C}_i \setminus \{R_1, R_2\}) \cup \{R_3\}$.

Case 2. No two rectangles from $\mathscr{C}_i$ intersect, but there exists a site $x \in [-M, M]^2$ and a minimal collection $R_1, \ldots, R_k \in \mathscr{C}_i$ with $k \geq 2$ so that

$$|(x + \mathscr{N}) \cap (R_1 \cup \cdots \cup R_k)| \geq 3. \tag{7.4}$$

By a minimal collection we mean that if any one of $R_1, \dots, R_k$ is discarded, (7.4) no longer holds. (This of course makes $k \le 3$.) Note that $k \ne 1$ is necessary to avoid the trivial case of one rectangle interacting with itself. Then let R_{k+1} be the smallest rectangle containing $R_1 \cup \cdots \cup R_k$ and $x + \mathscr{N}$. To see that R_{k+1} is PIS, note that every line which intersects R_{k+1} intersects one of $R_1, \dots, R_k$ or $x + \mathscr{N}$. (Otherwise, $x + \mathscr{N}$ and, say, R_k would be on different sides of a line, but then $|(x + \mathscr{N}) \cap (R_1 \cup \cdots \cup R_{k-1})| \ge 3$, contradicting minimality.) Let $\mathscr{C}_{i+1} = (\mathscr{C}_i \setminus \{R_1, \dots, R_k\}) \cup \{R_{k+1}\}$.

Case 3. Neither of the cases above happens. Assume that the longest side of any rectangle in $\mathscr{C}_i$ is strictly below L. This means that all the occupied sites in $[-M, M]^2$ are covered by disjoint rectangles in $\mathscr{C}_i$ so that even if all these rectangles are completely filled, no site outside them gets added by the dynamics. Even more, any site that gets added inside any one of these rectangles does so without any help from the outside. Moreover, none of these rectangles which has at least one site outside $[-L, L]^2$ can contain the origin. The only sites which can contribute to the origin being occupied are therefore those in rectangles inside $[-L, L]^2$. This contradiction shows that there must be a rectangle in $\mathscr{C}_i$ with its longest side at least L.

Let r_i^* be the longest side of a rectangle in $\mathscr{C}_i$. Then $r_{i+1}^* \le 3r_i^* + 4$, which completes the proof of the claim.

Notice that

$$\begin{aligned} P(k, l) &= P(\text{a fixed } k \times l \text{ rectangle is PIS}) \\ &\le \min\{(1 - e^{-kp^2})^l, (1 - e^{-lp})^k\}. \end{aligned}$$

From this one easily obtains that

$$\max_l P(p^{-2}, l) \le \exp\left(-\delta \frac{1}{p} \ln \frac{1}{p}\right) \tag{7.5}$$

and

$$\max_k P\left(k, \frac{1}{2}\frac{1}{p} \ln \frac{1}{p}\right) \le \exp\left(-\delta \frac{1}{p} \ln \frac{1}{p}\right). \tag{7.6}$$

Let $M = \exp(\delta(1/p)\ln(1/p))$. To prove the lower bound, it is enough to show that the event that the dynamics restricted to $[-M, M]^2$ (with free boundary) ever occupies the origin goes to 0 as $p \to 0$. For the said dynamics to occupy the origin, one of the following three events has to happen:

$$\begin{aligned} G_1 &= \{\text{there is an occupied site in } [-200, 200]^2\}, \\ G_2 &= \{\text{the dynamics restricted to } [-1/p^3, 1/p^3]^2 \\ &\qquad\qquad \text{eventually occupies the origin}\} \setminus G_1, \\ G_3 &= \{\text{the dynamics in } [-M, M]^2 \\ &\qquad \text{eventually occupies the origin}\} \setminus (G_1 \cup G_2). \end{aligned}$$

Obviously, $P(G_1) = \mathscr{O}(p)$. Moreover, by the claim,

$$P(G_2) \le P\big(\text{there exists a PIS rectangle in } [-1/p^3, 1/p^3]^2 \text{ with longest side in } [50, 200]\big)$$
$$\le C p^{-6} P(\text{at least 10 of 40,000 fixed sites are occupied}) = \mathscr{O}(p^4).$$

Finally, the claim, (7.5) and (7.6) imply that $P(G_3) \le \exp(-\delta(1/p)\ln(1/p))$, which proves the lower bound and hence completes the proof of Proposition 7.4. □

EXAMPLE 7.5. Now let $\mathscr{N}$ be a range ρ box neighborhood and let $\theta \in [2\rho^2 + \rho + 1, 2\rho^2 + 2\rho]$. Then $|\mathscr{N} \cap \{x\colon \langle x, u\rangle = 0\}| = \iota(N)$ for two linearly independent vectors, namely, e_1 and e_2. This is, in a sense, the simplest situation.

PROPOSITION 7.6. *Let $\alpha = \theta - 2\rho^2 - \rho$. Then there exist constants $c_1, c_2 > 0$ so that*

$$P(\ln T \in [c_1/p^\alpha, c_2/p^\alpha]) \to 1 \quad \textit{as } p \to 0.$$

PROOF. This proof is very similar to the proof of Proposition 7.4. We point out the main difference and omit further details.

In this case, a rectangle $R \subset \mathbf{Z}^2$ is PIS if every vertical or horizontal line ℓ through R has α sites which are all both included in some translate of $\mathscr{N}$ and within ℓ^∞–distance at most $2\rho + 1$ of ℓ. Then the claim is valid in exactly the same form, with essentially the same proof. □

8. The inverse shape problem. Professor V. Drobot has posed the following "inverse shape problem" (private communication). Fix a set $\Lambda \subset \mathbf{R}^2$. Is there a discrete threshold growth dynamics given by $\mathscr{N}$ and θ such that $L(\mathscr{N}, \theta) = \Lambda$? To make the question meaningful, assume that Λ is a symmetric closed convex polygon which contains a neighborhood of the origin and has all vertices at rational points. This last requirement follows from the fact that all vertices of $K_{1/w}$ are rational, so the same must be true for $K^*_{1/w}$.

However, these assumptions are still not sufficient. For instance, $\Lambda = B_\infty(0, 1/2)$ cannot be the limiting set for any discrete threshold growth dynamics. That would imply that $w(0) = \frac{1}{2}$, which is impossible since $w(0)$ must be an integer. We must therefore allow ourselves the freedom to replace Λ by a large magnification. With this proviso, we obtain the following answer to Drobot's question.

THEOREM 8.1. *Assume that $\Lambda \subset \mathbf{R}^2$ is a symmetric closed convex polygonal neighborhood of* 0 *with its vertices at rational points. Fix an integer $\theta \ge 1$. Then there exists a symmetric finite $\mathscr{N} \subset \mathbf{Z}^2$ and an integer $a > 0$ such that* $L(\mathscr{N}, \theta) = a\Lambda$.

PROOF. First note that the proof is trivial if $\theta = 1$: take an a so that $a\Lambda$ has integer vertices and take the neighborhood to be $a\Lambda$. For various reasons, however, this solution is not very satisfactory; $\theta = 1$ is a rather trivial setting in many models related to threshold growth. To handle the case of general θ, first strip the above neighborhood to bare essentials: let $\mathcal{N}'$ consist of the extreme points of $a\Lambda$. Denote the speed function with neighborhood $\mathcal{N}'$ and $\theta = 1$ by w'.

Now build $\mathcal{N}$ by adding $\theta - 1$ points at every vertex of $\mathcal{N}'$ in the following way. Enumerate sites of $\mathcal{N}'$ counterclockwise: $v_0, \dots, v_n = v_0$. For every $i = 0, \dots, n-1$, take the ray starting at v_i that goes through v_{i+1} and add the first $\theta - 1$ integer points past v_{i+1} on this ray. In order to prevent interference, enlarge a if necessary to make sure that the open line segment from v_{i+1} to the last of these $\theta - 1$ points does not intersect the line determined by v_j and v_{j+1} for any $j \neq i$. In this way, $\mathcal{N}$ ends up consisting of $n\theta$ sites.

Let w be the speed function of the $(\mathcal{N}, \theta)$ growth model. Since exactly θ points are added at each vertex and there is no interference, for every u which points at an extreme point of $K_{1/w'}$ we have $w(u) = w'(u)$. (Recall that such u's are normals to lines defined by v_i and v_{i+1}.) Moreover, by a similar reasoning, $w'(u) \leq w(u)$ for every u. This immediately implies that $\mathrm{co}(K_{1/w}) = \mathrm{co}(K_{1/w'})$ and therefore $K^*_{1/w} = K^*_{1/w'} = a\Lambda$. $\square$

Software availability. Figures in this paper were generated by *WinCA*, a Windows-based program for cellular automata experimentation written by R. Fisch and D. Griffeath. Readers interested in observing these dynamics in action are invited to download *WinCA* from `http://math.wisc.edu/~griffeat/sink.html`.

Note added in proof. The authors have recently learned that Theorem 2.1 and Proposition 2.2 of the present paper have been proved in a 1978 paper by Stephen J. Willson, On convergence of configurations, *Discrete Mathematics* **23** 279–300. We thank Prof. Willson for bringing this work to our attention.

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