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LIMIT LAWS OF MODULUS TRIMMED SUMS

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Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables. Let ${}^{(1)}X_n, \dots, {}^{(n)}X_n$ be an arrangement of $X_1, X_2, \dots, X_n$ in decreasing order of magnitude, and set ${}^{(r_n)}S_n = {}^{(r_n+1)}X_n + \cdots + {}^{(n)}X_n$. This is known as the modulus trimmed sum. We obtain a complete characterization of the class of limit laws of the normalized modulus trimmed sum when the underlying distribution is symmetric and $r_n \to \infty$, $r_n n^{-1} \to 0$.

1. Introduction. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables. Arrange $X_1, X_2, \dots, X_n$ in decreasing order of magnitude as ${}^{(1)}X_n, \dots, {}^{(n)}X_n$, that is,

$$|{}^{(1)}X_n| \geq \cdots \geq |{}^{(n)}X_n|,$$

and set

$$ {}^{(r_n)}S_n = {}^{(r_n+1)}X_n + \cdots + {}^{(n)}X_n. \tag{1.1}$$

Ties are broken according to the order in which the random variables occur. That is, for $1 \leq j \leq n$, let $m_n(j)$ be the number of i for which either $|X_i| > |X_j|$, $i \leq n$ or $|X_i| = |X_j|$ and $i \leq j$. Then define ${}^{(r)}X_n = X_j$ if $m_n(j) = r$.

${}^{(r_n)}S_n$ is known as the modulus trimmed sum. Its behavior depends on the sequence r_n. Considerable work has been done on the problem of describing the asymptotic distribution of appropriately normalized modulus trimmed sums. The case when r_n is bounded is referred to as light trimming. Generally, light trimming does not improve weak convergence results but does improve the almost sure behavior of S_n; see, for example, Mori (1976, 1984) and Maller (1982, 1984). For heavy trimming, that is, when r_n is proportional to n, the trimmed sum can be made to converge to a mixture of normal distributions provided only a smoothness condition, but no moment condition, is satisfied by the distribution of X; see Maller (1988).

We will consider r_n for which

$$r_n \to \infty, \qquad r_n n^{-1} \to 0. \tag{1.2}$$

This is known as intermediate trimming and offers an interesting and rich collection of results. Some of the results that have been obtained for intermediate

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trimming, under the assumption that X is symmetric, include necessary and sufficient conditions for asymptotic normality as well as a solution to the more general problem of describing the class of subsequential limit laws and their domains of partial attraction; see Griffin and Pruitt (1987) or, for the quantile-transfrom approach, see Csörgő, Haeusler and Mason (1991). Griffin and Pruitt (1991) also solved a special case of characterizing the class of limit laws along the entire sequence when r_n is nondecreasing. In this paper we will characterize the class of limit laws in the general case, where r_n is not necessarily monotone but still assuming that X is symmetric. This is considerably more complex than in the monotone case in that several new classes of limit laws arise and the dependence on the sequence $\{r_n\}$ is much more involved. The problem when X is not symmetric remains open, as is the case for many problems relating to modulus trimmed sums.

Other methods of trimming have also been considered in the literature. Among those most closely related to the modulus trimmed sum include trimming upper and lower order statistics [see Csörgő, Haeusler and Mason (1988) or Griffin and Pruitt (1989)], and the variations on modulus trimming surveyed in Hahn, Kuelbs and Weiner (1991). The methods of this paper can probably be adapted to solve the analogous limit law problem for some of these other forms of trimming. An overview of many aspects of trimming can be found in the volume edited by Hahn, Mason and Weiner (1991).

2. Preliminaries. Throughout this paper we will assume that X is symmetric and (1.2) holds. For $x \geq 0$ let

$$G(x) = P(|X| > x) \tag{2.1}$$

and define the right continuous inverse $\tilde{G}$ by

$$\tilde{G}(u) = \sup\{x : G(x) > u\} \qquad \text{if } 0 \leq u < 1,$$

with $\tilde{G}(1) = \tilde{G}(1-)$, where $\tilde{G}(u) = 0$ if $\{x : G(x) > u\} = \varnothing$. Note that $\tilde{G}$ is nonincreasing in u, and if U is uniform on $(0, 1)$, then $|X| \stackrel{d}{=} \tilde{G}(U)$.

For $\alpha \in \mathbb{R}$ let

$$u_n(\alpha) = 0 \vee (r_n - \alpha {r_n}^{1/2})n^{-1} \wedge 1 \tag{2.2}$$

and

$$\tau_n(\alpha) = \big(E\big(\tilde{G}(U)^2; U \geq u_n(\alpha)\big)\big)^{1/2}. \tag{2.3}$$

If G is continuous, then (2.3) can be rewritten as

$$\tau_n(\alpha) = \big(E\big(X^2; |X| \leq c_n(\alpha)\big)\big)^{1/2}$$

where $c_n(\alpha) = \tilde{G}(u_n(\alpha))$.

From (2.3) it follows that

$$\tau_n \text{ is nonnegative, nondecreasing and continuous} \tag{2.4}$$

while by (2.5) of Griffin and Pruitt (1991), for any $\lambda > 0$, if n is sufficiently large

$$\tau_n^2 \text{ is convex on } [-\lambda, \lambda]. \tag{2.5}$$

Thus if $\tau_n \to \tau$ pointwise, or more generally $\tau_n \gamma_n^{-1} \to \tau$ pointwise for some scalar sequence γ_n, then $\tau \in L$ where

$$L = \{\tau : \tau \text{ is nonnegative, nondecreasing and } \tau^2 \text{ is convex}\}. \tag{2.6}$$

Furthermore, the convergence will be uniform on compact sets (u.c.), since any convergent sequence of convex functions converges u.c.

Let

$$\mathcal{I}(\{r_n\}) = \{Z : \text{ there exist X symmetric, } \gamma_n \text{ and } n_k \text{ such that } {}^{(r_{n_k})}S_{n_k}\gamma_{n_k}{}^{-1} \xrightarrow{d} Z\}$$

and

$$\mathcal{L}(\{r_n\}) = \{Y : \text{ there exist X symmetric and } \gamma_n \text{ such that } {}^{(r_n)}S_n\gamma_n{}^{-1} \xrightarrow{d} Y\}.$$

Thus $\mathcal{I}(\{r_n\})$ is the class of all possible subsequential limit laws and $\mathcal{L}(\{r_n\})$ is the class of all possible limit laws along the entire sequence. The families $\mathcal{I}(\{r_n\})$ and $\mathcal{L}(\{r_n\})$, which a priori depend on r_n, may be considered analogues of the infinitely divisible laws and the stable laws respectively for modulus trimmed sums.

We will now state the aforementioned results of Griffin and Pruitt (1987) concerning the class of subsequential limit laws and their domains of partial attraction. Let N_1, N_2 be independent standard normal random variables, and write $Z \sim \tau$ if $Z \stackrel{d}{=} N_1\tau(N_2)$. Set

$$\tilde{L} = \{Z : Z \sim \tau \text{ for some } \tau \in L\}.$$

Recall that we are assuming throughout that X is symmetric and r_n satisfies (1.2).

THEOREM 2.1. *Fix a sequence r_n. Then*

$$\mathcal{I}(\{r_n\}) = \tilde{L}.$$

It is easily seen that if $Z \sim \tau_1$ and $Z \sim \tau_2$ where $\tau_1, \tau_2 \in L$, then $\tau_1 = \tau_2$ [Theorem 3.7, Griffin and Pruitt (1987)]. Thus the subsequential limit laws can be identified with a unique $\tau \in L$. In particular Z has a normal distribution (possibly degenerate at 0) if and only if $\tau \equiv a$ for some $a \geq 0$.

One rather surprising consequence of Theorem 2.1 is that the subsequential limit laws $\mathcal{I}(\{r_n\})$, do not depend on the sequence r_n. Criteria for convergence to a given law in $\mathcal{I}(\{r_n\})$ is the subject of the next result.

THEOREM 2.2. *Assume that* $Z \sim \tau$ *where* $\tau \in L$. *Fix a subsequence* n_k. *Then*

$$\text{there exists } \gamma_n \text{ such that } {}^{(r_{n_k})}S_{n_k}\gamma_{n_k}{}^{-1} \xrightarrow{d} Z$$

if and only if

$$\text{for some (all) } \alpha_0 \text{ with } \tau(\alpha_0) > 0,\ \frac{\tau_{n_k}(\alpha)}{\tau_{n_k}(\alpha_0)} \to \frac{\tau(\alpha)}{\tau(\alpha_0)} \qquad \text{for all } \alpha \in \mathbb{R}. \tag{2.7}$$

In the special case where $\tau \equiv a$ for some $a > 0$, we obtain the following necessary and sufficient conditions for asymptotic normality:

$$\frac{\tau_{n_k}(\alpha)}{\tau_{n_k}(0)} \to 1 \qquad \text{for all } \alpha.$$

The characterization of $\mathcal{L}(\{r_n\})$, analogous to Theorem 2.1, is much harder as the class of limit laws may depend on the sequence r_n. For some r_n, $\mathcal{L}(\{r_n\}) = \mathcal{I}(\{r_n\})$ and for others the inclusion $\mathcal{L}(\{r_n\}) \subseteq \mathcal{I}(\{r_n\})$ can be strict. The characterization of $\mathcal{L}(\{r_n\})$ for r_n nondecreasing, given by Griffin and Pruitt (1991), depends on the following condition:

(S): for any sequences $m_k, n_k \to \infty$, if

$$\frac{r_{m_k}}{m_k} = \frac{r_{n_k}}{n_k} + O\left(\frac{r_{n_k}^{1/2}}{n_k}\right)$$

then

$$\frac{m_k}{n_k} \to 1.$$

The delicate nature of this condition can be seen from the examples given by Griffin and Pruitt [(1991), page 65]. Before stating their result we introduce two classes of distributions;

$$\mathcal{N} = \{Y : Y \stackrel{d}{=} aN_1, a \geq 0\} \tag{2.8}$$

and

$$\mathcal{E} = \{Y : Y \stackrel{d}{=} aN_1 \exp(\lambda N_2), a \geq 0, \lambda \geq 0\}. \tag{2.9}$$

THEOREM 2.3. *Fix a nondecreasing sequence* r_n.

$$\text{If (S) fails, then } \mathcal{L}(\{r_n\}) = \mathcal{N}. \tag{2.10}$$

$$\text{If (S) holds, then } \mathcal{L}(\{r_n\}) = \mathcal{E}. \tag{2.11}$$

By taking X to have a symmetric distribution in the Feller class it follows that $\mathcal{N} \subset \mathcal{L}(\{r_n\})$ for any sequence r_n [Griffin and Pruitt (1987), Corollary 3.12]. Thus the failure of Condition (S) restricts the class of limit laws to the smallest possible class.

3. Main results and proofs. In this section we will characterize the class of limit laws $\mathcal{L}(\{r_n\})$ in the general case, that is, when r_n is not necessarily monotone. The nature of $\mathcal{L}(\{r_n\})$ depends on the structure of the set S where

$$S = S(\{r_n\}) = \left\{(\beta,\rho) \in \mathbb{R} \times [0,\infty) : \text{ there exist } \{n_k\}, \{m_k\} \to \infty \right.$$
$$\left. \text{such that } (\beta,\rho) = \lim\left(\frac{u_{n_k}(0) - u_{m_k}(0)}{v_{n_k}}, \left(\frac{n_k}{m_k}\right)^{1/2}\right)\right\}$$

and

$$v_n = \frac{r_n^{1/2}}{n}.$$

If r_n is assumed to be nondecreasing then one can show that S must contain the line $\rho = 1$ [Griffin and Pruitt (1991), Lemma 4.2]. It then follows from Theorems 3.7 and 3.8 below that this limits the possibilities for $\mathcal{L}(\{r_n\})$ to just two, namely those given in Theorem 2.3. If, however, the only assumptions on r_n are those of (1.2), then it turns out that S can be almost anything, subject to a few obvious restrictions, as we now show.

First note that S is closed, being a set of limit points, and $(0, 1) \in S$ since we can take $n_k = m_k$ in the definition of S. Furthermore, S satisfies the following condition; for any $\beta \in \mathbb{R}$ and $\rho > 0$,

$$(\beta,\rho) \in S \quad \Longrightarrow \quad \left(-\frac{\beta}{\rho}, \frac{1}{\rho}\right) \in S. \tag{3.1}$$

To see this suppose $(\beta,\rho) \in S$ and let $\{n_k\}, \{m_k\}$ satisfy

$$\frac{u_{n_k}(0) - u_{m_k}(0)}{v_{n_k}} \to \beta, \qquad \left(\frac{n_k}{m_k}\right)^{1/2} \to \rho. \tag{3.2}$$

Then

$$1 - \frac{u_{m_k}(0)}{u_{n_k}(0)} = \frac{u_{n_k}(0) - u_{m_k}(0)}{v_{n_k}} \frac{1}{r_{n_k}^{1/2}} \sim \frac{\beta}{r_{n_k}^{1/2}} \to 0,$$

giving

$$\frac{r_{n_k}}{n_k} = u_{n_k}(0) \sim u_{m_k}(0) = \frac{r_{m_k}}{m_k}.$$

Since

$$n_k{}^{1/2} \sim m_k{}^{1/2}\rho,$$

this leads to

$$\frac{v_{m_k}}{v_{n_k}} = \left(\frac{r_{m_k}/m_k}{r_{n_k}/n_k}\right)^{1/2} \left(\frac{n_k}{m_k}\right)^{1/2} \sim \left(\frac{n_k}{m_k}\right)^{1/2} \sim \rho. \tag{3.3}$$

Thus by interchanging the roles of $\{n_k\}$ and $\{m_k\}$ in (3.2) we get (3.1).

The three conditions, that S is closed, $(0, 1) \in S$ and (3.1), are in fact the only restrictions on S.

THEOREM 3.1. *Given any closed set $C \subseteq \mathbb{R} \times [0, \infty)$, containing $(0, 1)$ and having the property that for every $\beta \in \mathbb{R}$ and $\rho > 0$,*

$$(\beta, \rho) \in C \quad \textit{implies} \quad \left(-\frac{\beta}{\rho}, \frac{1}{\rho}\right) \in C, \tag{3.4}$$

there exists a sequence $\{r_n\}$ with $r_n \to \infty,\ r_n n^{-1} \to 0$ such that $S(\{r_n\}) = C$.

Since this result is not used in the remainder of the paper, its proof is omitted. It can be found in Qazi (2001). It does show however that no simplifications are likely to be obtained from any further study of the structure of the set S.

In order to describe the class of all possible limit laws $\mathcal{L}(\{r_n\})$, we need to introduce some further notation. Let

$$L_c = \{\tau \in L : \tau \equiv a \text{ for some constant } a \geq 0\}, \tag{3.5}$$

and for $(\beta, \rho) \in \mathbb{R} \times [0, \infty)$ let

$$L_{(\beta,\rho)} = \{\tau \in L : \tau(\alpha\rho + \beta)\tau(\alpha_0) = \tau(\alpha)\tau(\alpha_0\rho + \beta) \text{ for all } \alpha, \alpha_0 \in \mathbb{R}\} \tag{3.6}$$

and

$$\tilde{L}_{(\beta,\rho)} = \{Y : Y \sim \tau \text{ for some } \tau \in L_{(\beta,\rho)}\}. \tag{3.7}$$

From (3.5) and (3.6) it follows trivially that $L_c \subseteq L_{(\beta,\rho)}$, and so $\mathcal{N} \subseteq \tilde{L}_{(\beta,\rho)}$ for every (β, ρ). Also, for later reference, note that $L_{(0,1)} = L$ and

$$L_{(\beta,\rho)} = L_{(-\beta/\rho, 1/\rho)} \qquad \text{for all } \beta \in \mathbb{R},\ \rho > 0. \tag{3.8}$$

THEOREM 3.2. *Fix a sequence r_n. Then*

$$\mathcal{L}(\{r_n\}) = \bigcap_{(\beta,\rho)\in S} \tilde{L}_{(\beta,\rho)}. \tag{3.9}$$

In this paper we will prove the inclusion from left to right. In addition we will give a simple description of $\bigcap_{(\beta,\rho)\in S} \tilde{L}_{(\beta,\rho)}$ depending on the form of the set S. The inclusion from right to left, which requires the detailed construction of distributions satisfying (2.7), can be found in Qazi (2001).

As the statement of Theorem 3.2 makes clear, each point $(\beta, \rho) \in S$, places a different restriction on the function τ representing a possible limit law. If S is sufficiently large, in a certain sense, then it can happen that

$$\bigcap_{(\beta,\rho)\in S} L_{(\beta,\rho)} = L_c. \tag{3.10}$$

In that case the class of limit laws contains only the normal distributions. This is the situation, for example, in (2.10). On the other hand if S is sufficiently small then other limit laws are possible. Table 1, following Theorem 3.10, describes the various possibilities together with the appropriate reference in the paper.

Before giving the proof of Theorem 3.2, we need to make some preliminary observations about the sets $L_{(\beta,\rho)}$.

LEMMA 3.3. *Assume that $\tau \in L_{(\beta,\rho)}$ and $\tau \notin L_c$.*

$$\text{If } \rho = 0, \text{ then } \tau(\alpha) = 0 \text{ for all } \alpha \le \beta. \tag{3.11}$$

$$\text{If } \rho = 1 \text{ and } \beta \ne 0, \text{ then } \tau(\alpha) > 0 \text{ for all } \alpha. \tag{3.12}$$

$$\text{If } \rho \ne 0, 1, \text{ then } \tau(\alpha) = 0 \text{ for all } \alpha \le \beta^* \text{ and } \tau(\alpha) > 0 \text{ for all } \alpha > \beta^*, \quad \text{where } \beta^* = \beta(1-\rho)^{-1}. \tag{3.13}$$

PROOF. Setting $\rho = 0$ in (3.6) gives $\tau(\beta)\tau(\alpha_0) = \tau(\alpha)\tau(\beta)$ for all $\alpha, \alpha_0 \in \mathbb{R}$. Since $\tau \notin L_c$, this forces $\tau(\beta) = 0$, which together with the monotonicity of τ proves (3.11).

For (3.12), let $\bar{\alpha} = \sup\{\alpha : \tau(\alpha) = 0\}$. Note that $\bar{\alpha} < \infty$ since $\tau \not\equiv 0$. If $\bar{\alpha} > -\infty$ then by continuity of τ we have $\tau(\bar{\alpha}) = 0$. But if we now let $\alpha = \bar{\alpha}$ and $\bar{\alpha} - \beta$ respectively in (3.6), we obtain (since $\rho = 1$)

$$\tau(\bar{\alpha} + \beta)\tau(\alpha_0) = \tau(\bar{\alpha})\tau(\alpha_0 + \beta),$$
$$\tau(\bar{\alpha})\tau(\alpha_0) = \tau(\bar{\alpha} - \beta)\tau(\alpha_0 + \beta)$$

for all α_0. Since $\tau(\bar{\alpha}) = 0$, this means that $\tau(\bar{\alpha} + |\beta|) = 0$, contradicting the definition of $\bar{\alpha}$. Hence $\bar{\alpha} = -\infty$ and (3.12) holds.

For (3.13) we first let $\alpha = \beta^*$ in (3.6). This gives

$$\tau(\beta^*)\tau(\alpha_0) = \tau(\beta^*)\tau(\alpha_0\rho + \beta) \tag{3.14}$$

for all α_0. Since $\tau \notin L_c$, this forces $\tau(\beta^*) = 0$. Now again let $\bar{\alpha} = \sup\{\alpha : \tau(\alpha) = 0\}$. Then $\bar{\alpha} \ge \beta^*$ and by continuity $\tau(\bar{\alpha}) = 0$. If $\bar{\alpha} > \beta^*$ then either $\bar{\alpha}\rho + \beta > \bar{\alpha}$ or $\bar{\alpha}\rho + \beta < \bar{\alpha}$, depending on whether $\rho > 1$ or $\rho < 1$ respectively. If $\bar{\alpha}\rho + \beta > \bar{\alpha}$, then setting $\alpha = \bar{\alpha}$ in (3.6) we obtain

$$\tau(\bar{\alpha}\rho + \beta)\tau(\alpha_0) = \tau(\bar{\alpha})\tau(\alpha_0\rho + \beta) \tag{3.15}$$

for all α_0. Since $\tau(\bar{\alpha}\rho + \beta) > 0$ this implies $\tau(\alpha_0) = 0$ for all α_0, contradicting $\tau \notin L_c$. If $\bar{\alpha}\rho + \beta < \bar{\alpha}$, then $(\bar{\alpha} - \beta)\rho^{-1} > \bar{\alpha}$. Thus setting $\alpha = (\bar{\alpha} - \beta)\rho^{-1}$ in (3.6) we obtain

$$\tau(\bar{\alpha})\tau(\alpha_0) = \tau\big((\bar{\alpha} - \beta)\rho^{-1}\big)\tau(\alpha_0\rho + \beta) \tag{3.16}$$

for all α_0. Since $\tau((\bar{\alpha} - \beta)\rho^{-1}) > 0$ this implies $\tau(\alpha_0\rho + \beta) = 0$ for all α_0, again contradicting $\tau \notin L_c$. $\square$

COROLLARY 3.4. *Fix $\tau \in L$. The following are equivalent*:

$$\tau \in L_{(\beta,\rho)}; \tag{3.17}$$

$$\begin{aligned} &\tau(\alpha\rho+\beta)\tau(\alpha_0) = \tau(\alpha)\tau(\alpha_0\rho+\beta) \textit{ for all } \alpha \in \mathbb{R}, \\ &\qquad \textit{and for all } \alpha_0 \textit{ satisfying } \tau(\alpha_0) > 0 \textit{ and } \tau(\alpha_0\rho+\beta) > 0. \end{aligned} \tag{3.18}$$

PROOF. We need only consider the implication that (3.18) implies (3.17) as the other direction is obvious. Thus fix $\tau \in L$ satisfying (3.18). If $\tau \in L_c$, then the result is trivial since $L_c \subseteq L_{(\beta,\rho)}$ for all (β, ρ). Thus we may assume $\tau \notin L_c$. If $(\beta, \rho) = (0, 1)$ the result is again trivial since $L_{(0,1)} = L$. We now consider separately the three cases of Lemma 3.3. If $\rho = 0$, then $\tau(\beta) = 0$ by (3.11). But this clearly forces the functional equation in (3.6) to hold. Hence (3.17) holds. If $\rho = 1$ and $\beta \neq 0$, then by (3.12), τ is never 0. Hence the result holds in this case also. Finally if $\rho \neq 0, 1$ then the result follows from (3.13), since $\alpha_0 > \beta^*$ implies $\alpha_0\rho + \beta > \beta^*$ and $\alpha_0 \leq \beta^*$ implies $\alpha_0\rho + \beta \leq \beta^*$. □

PROOF OF THEOREM 3.2 (Inclusion from left to right). Fix $Y \in \mathcal{L}(\{r_n\})$ and $(\beta, \rho) \in S$. By Theorem 2.1, $Y \sim \tau$ for some $\tau \in L$. We must show $\tau \in L_{(\beta,\rho)}$.

By Theorem 2.2 there exists a symmetric distribution X such that for all α_0 satisfying $\tau(\alpha_0) > 0$,

$$\frac{\tau_n(\alpha)}{\tau_n(\alpha_0)} \to \frac{\tau(\alpha)}{\tau(\alpha_0)} \tag{3.19}$$

for all $\alpha \in \mathbb{R}$.

Since $(\beta, \rho) \in S$ we can find $\{n_k\}$ and $\{m_k\}$ satisfying (3.2). Then by (3.3), for all α, if k is sufficiently large,

$$\begin{aligned} u_{m_k}(\alpha) &= u_{m_k}(0) - \alpha\nu_{m_k} \\ &= u_{n_k}(0) - \big(\beta + o(1)\big)\nu_{n_k} - \alpha\big(\rho + o(1)\big)\nu_{n_k} \\ &= u_{n_k}\big(\alpha\rho + \beta + o(1)\big). \end{aligned}$$

Thus

$$\tau_{m_k}(\alpha) = \tau_{n_k}\big(\alpha\rho + \beta + o(1)\big) \tag{3.20}$$

for all $\alpha \in \mathbb{R}$. Since the convergence in (3.19) is u.c., it follows from (3.20) that for all α,

$$\frac{\tau(\alpha)}{\tau(\alpha_0)} = \frac{\tau(\alpha\rho+\beta)}{\tau(\alpha_0\rho+\beta)}, \tag{3.21}$$

provided $\tau(\alpha_0) > 0$ and $\tau(\alpha_0\rho + \beta) > 0$. The result now follows from Corollary 3.4. □

Theorem 3.2 gives a characterization of $\mathcal{L}(\{r_n\})$ in terms of the multiple conditions, one for each $(\beta,\rho)\in S$, that the function τ representing the limit law, must satisfy. This characterization is quite complicated, and it is not immediately clear how to even recover the earlier result of Griffin and Pruitt described in Theorem 2.3. The aim of the remainder of the paper is to provide a simple description of $\bigcap_{(\beta,\rho)\in S}\tilde{L}_{(\beta,\rho)}$ depending on the form of the set S. We begin with a more informative description of the set $L_{(\beta,\rho)}$ depending on the particular value of (β,ρ).

LEMMA 3.5.

$$L_{(\beta,0)} = L_c \cup \{\tau \in L : \tau(\beta) = 0\}, \tag{3.22}$$

$$\begin{aligned} \underset{\beta\neq 0}{L_{(\beta,1)}} = L_c \cup \big\{\tau \in L : \tau > 0 \textit{ on } (-\infty,\infty) \textit{ and for all } \alpha\in\mathbb{R} \\ \tau(\alpha+\beta)\tau(0) = \tau(\alpha)\tau(\beta)\big\}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \underset{\rho\neq 0,1}{L_{(\beta,\rho)}} = L_c \cup \big\{\tau \in L : \tau(\beta^*) = 0, \tau > 0 \textit{ on } (\beta^*,\infty) \textit{ and for all } \alpha > 0 \\ \tau(\beta^*+\alpha)\tau(\beta^*+\rho) = \tau(\beta^*+1)\tau(\beta^*+\alpha\rho)\big\}. \end{aligned} \tag{3.24}$$

PROOF. To prove (3.22), observe the left inclusion follows immediately from (3.11). For the converse, let $\tau \in L_c \cup \{\tau \in L : \tau(\beta) = 0\}$. We may assume $\tau \notin L_c$ else there is nothing to prove since $L_c \subseteq L_{(\beta,\rho)}$ for all (β,ρ). Hence $\tau(\beta) = 0$. But then $\tau \in L_{(\beta,0)}$ by (3.6).

For (3.23), the left inclusion follows immediately by taking $\alpha_0 = 0$ and $\rho = 1$ in (3.6) and using (3.12). For the converse assume $\tau \notin L_c$ (otherwise there is nothing to prove), $\tau > 0$ on $(-\infty,\infty)$ and for all $\alpha\in\mathbb{R}$,

$$\tau(\alpha+\beta) = \frac{\tau(\alpha)\tau(\beta)}{\tau(0)}.$$

Then for any $\alpha, \alpha_0 \in \mathbb{R}$,

$$\tau(\alpha+\beta)\tau(\alpha_0) = \frac{\tau(\alpha)\tau(\beta)}{\tau(0)}\tau(\alpha_0) = \tau(\alpha)\frac{\tau(\alpha_0)\tau(\beta)}{\tau(0)} = \tau(\alpha)\tau(\alpha_0+\beta).$$

Hence $\tau \in L_{(\beta,1)}$ by (3.6).

For (3.24) first suppose $\tau \in L_{(\beta,\rho)}$. If $\tau \in L_c$ then the left inclusion is trivially true. So assume that $\tau \notin L_c$. From (3.13) we have $\tau(\beta^*) = 0$ and $\tau(\alpha) > 0$ for all $\alpha > \beta^*$. Thus the only condition remaining to show is that for any $\alpha' > 0$,

$$\tau(\beta^*+\alpha')\tau(\beta^*+\rho) = \tau(\beta^*+1)\tau(\beta^*+\alpha'\rho).$$

But this follows easily by setting $\alpha_0 = \beta^*+1$ and $\alpha = \beta^*+\alpha'$ in (3.6) and noting that $\alpha_0\rho+\beta = \beta^*+\rho$ and $\alpha\rho+\beta = \beta^*+\alpha'\rho$.

Conversely, suppose that $\tau \notin L_c$, $\tau(\beta^*) = 0$, $\tau > 0$ on (β^*, ∞) and for all $\alpha > 0$,

$$\tau(\beta^* + \alpha)\tau(\beta^* + \rho) = \tau(\beta^* + 1)\tau(\beta^* + \alpha\rho).$$

Since $\alpha > \beta^*$ if and only if $\alpha\rho + \beta > \beta^*$, to prove $\tau \in L_{(\beta,\rho)}$, it is enough in view of (3.13), to prove (3.18) for all $\alpha, \alpha_0 > \beta^*$. In this case we can write $\alpha = \beta^* + \alpha'$ and $\alpha_0 = \beta^* + \alpha'_0$ for some $\alpha', \alpha'_0 > 0$. Hence $\alpha\rho + \beta = \beta^* + \alpha'\rho$ and $\alpha_0\rho + \beta = \beta^* + \alpha'_0\rho$. Thus

$$\begin{aligned}
\tau(\alpha\rho + \beta)\tau(\alpha_0) &= \tau(\beta^* + \alpha'\rho)\tau(\beta^* + \alpha'_0) \\
&= \frac{\tau(\beta^* + \alpha')\tau(\beta^* + \rho)}{\tau(\beta^* + 1)}\tau(\beta^* + \alpha'_0) \\
&= \tau(\beta^* + \alpha')\tau(\beta^* + \alpha'_0\rho) \\
&= \tau(\alpha)\tau(\alpha_0\rho + \beta).
\end{aligned}$$

Hence $\tau \in L_{(\beta,\rho)}$ by (3.18). $\square$

We are now ready to simplify the characterization of $\mathcal{L}(\{r_n\})$ given in Theorem 3.2. First observe that by (3.1) and (3.8),

$$\bigcap_{(\beta,\rho)\in S} \tilde{L}_{(\beta,\rho)} = \bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)}$$

where

$$S' = S \cap (\mathbb{R} \times [0, 1]).$$

We begin with the simplest case $|S'| = 1$, in which case $S' = \{(0, 1)\}$. Since $L_{(0,1)} = L$ it then follows that

$$\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \tilde{L}. \tag{3.25}$$

The nature of $\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)}$ when $|S'| > 1$, is described in Theorems 3.7–3.10. We first deal with the case where only normal limits can arise.

PROPOSITION 3.6. *Assume that* $S' \supseteq \{(0, 1), (\beta_1, \rho_1), (\beta_2, \rho_2)\}$ *where* $(\beta_i, \rho_i) \neq (0, 1)$ *are distinct for* $i = 1, 2$. *If any of the following conditions hold*:

$$\rho_i \neq 0, 1, \ \textit{for } i = 1, 2 \quad \textit{and} \quad \frac{\beta_1}{1 - \rho_1} \neq \frac{\beta_2}{1 - \rho_2}; \tag{3.26}$$

$$\rho_1 \neq 1, \qquad \rho_2 = 1; \tag{3.27}$$

$$\rho_1 \neq 0, 1, \quad \rho_2 = 0, \quad \textit{and} \quad \frac{\beta_1}{1 - \rho_1} < \beta_2; \tag{3.28}$$

then $\bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_c$ *and consequently* $\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \mathcal{N}$.

PROOF. Let $\beta_i^* = \beta_i(1-\rho_i)^{-1}$ if $\rho_i \neq 1$ for $i = 1, 2$. Assume that $\tau \in \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)}$. We must show $\tau \in L_c$.

If (3.26) holds then (3.24) implies that either $\tau \in L_c$ or $\tau(\alpha) = 0$ for $\alpha \leq \beta_i^*$ with $\tau(\alpha) > 0$ for $\alpha > \beta_i^*$. Since $\beta_1^* \neq \beta_2^*$, we conclude that $\tau \in L_c$.

Now assume that (3.27) holds and $\tau \notin L_c$. Since $\rho_2 = 1$, (3.23) implies that τ is always positive, but at the same time because $\rho_1 \neq 1$, (3.22) and (3.24) imply that $\tau(\alpha) = 0$ for some α. Hence $\tau \in L_c$.

Finally assume that (3.28) holds and $\tau \notin L_c$. Then $\tau(\beta_2) = 0$ by (3.22), while $\tau(\beta_2) > 0$ by (3.24). Hence $\tau \in L_c$. □

For $-\infty < m \leq \infty$, let

$$S_m = \{(\beta,\rho) : \rho = m\beta + 1,\ \rho \geq 0\} \cup \{(\beta, 0) : \beta \leq -m^{-1}\} \tag{3.29}$$

where we interpret $m = \infty$ to mean $\{(0,\rho) : \rho \geq 0\} \cup \{(\beta, 0) : \beta \leq 0\}$ and $m = 0$ to mean $\{(\beta,\rho) : \rho = 1\}$. It will be convenient to introduce

$$m^* = \begin{cases} 0, & \text{if } m = \infty, \\ -m^{-1}, & \text{if } -\infty < m < \infty, m \neq 0. \end{cases}$$

Observe that if $\rho \neq 1$ and $(\beta,\rho) \in S_m$ then $m \neq 0$, and either $\beta^* = m^*$ or else, $\rho = 0$ and $\beta \leq m^*$.

THEOREM 3.7. *If $S' \not\subseteq S_m$ for any $-\infty < m \leq \infty$, then $\bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_c$ and consequently $\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \mathcal{N}$.*

PROOF. If $\sup\{\beta : (\beta, 0) \in S'\} = \infty$, then by (3.22) it follows that $\bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_c$ and so $\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \mathcal{N}$. Hence we may assume $\sup\{\beta : (\beta, 0) \in S'\} \neq \infty$ where $\sup \varnothing = -\infty$. In that case we claim there exist $(\beta_i, \rho_i) \in S'$ distinct and not equal to (0,1) for $i = 1, 2$, such that one of (3.26)–(3.28) hold. First suppose there exists $(\beta_2, \rho_2) \in S'$ with $\beta_2 \neq 0$ and $\rho_2 = 1$. Since $S' \not\subseteq S_0$ there must exist $(\beta_1, \rho_1) \in S'$ with $\rho_1 \neq 1$. Hence (3.27) holds. Thus we may assume $S' \cap S_0 = \{(0, 1)\}$. If $S' \cap (\mathbb{R} \times (0, 1]) = \{(0, 1)\}$ then clearly $S' \subseteq S_m$ for any $m \in (\infty, \infty]$ with $m^* \geq \sup\{\beta : (\beta, 0) \in S'\}$. Thus we may assume $S' \cap (\mathbb{R} \times (0, 1]) \neq \{(0, 1)\}$. If there exist two points in S' satisfying (3.26) then we are done. If not, $S' \cap (\mathbb{R} \times (0, 1]) \subseteq S_m$ for some m. Let $(\beta_1, \rho_1) \in S' \cap (\mathbb{R} \times (0, 1))$. Then in order that $S' \not\subseteq S_m$, there must exist $(\beta_2, 0) \in S'$ with $\beta_1(1-\rho_1)^{-1} < \beta_2$ which means that (3.28) holds. Thus one of (3.26)–(3.28) must hold, and the result then follows from Proposition 3.6. □

We are left then to consider the case where $S' \subseteq S_m$ for some m. We first consider the case $m = 0$. Set

$$L_e = \{\tau \in L : \tau(\alpha) = a \exp(\lambda\alpha),\ a \geq 0,\ \lambda \geq 0\}.$$

Let $\Gamma = \{\beta : (\beta, 1) \in S'\} = \{\beta : (\beta, 1) \in S\}$ and let $\mathcal{A}(\Gamma)$ be the additive subgroup of $\mathbb{R}$ generated by Γ. Since every additive subgroup of $\mathbb{R}$ is either cyclic or dense in $\mathbb{R}$, we have that if $|S'| > 1$ then either $\mathcal{A}(\Gamma) = \gamma\mathbb{Z}$ for some $\gamma \neq 0$, or $\mathcal{A}(\Gamma)$ is dense.

THEOREM 3.8. *Assume that $|S'| > 1$ and $S' \subseteq S_0$.*

$$\text{If } \mathcal{A}(\Gamma) = \gamma\mathbb{Z}, \text{ then } \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_{(\gamma,1)} \quad \text{and so } \bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \tilde{L}_{(\gamma,1)}. \tag{3.30}$$

$$\text{If } \mathcal{A}(\Gamma) \text{ is dense is } \mathbb{R}, \text{ then } \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_e \quad \text{and so } \bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \mathcal{E}. \tag{3.31}$$

PROOF. We begin with the observation that for any $\beta_1, \beta_2 \in \mathbb{R}$,

$$L_{(\beta_1,1)} \cap L_{(\beta_2,1)} \subseteq L_{(\beta_1+\beta_2,1)}. \tag{3.32}$$

To see this let $\tau \in L_{(\beta_1,1)} \cap L_{(\beta_2,1)}$. Since $L_c \subseteq L_{(\beta,\rho)}$ for any (β, ρ), we may assume $\tau \notin L_c$. Since $L_{(0,1)} = L$ we may also assume $\beta_i \neq 0$ for $i = 1, 2$ and $\beta_1 + \beta_2 \neq 0$, else the result is immediate. From (3.23) we then obtain, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\tau(\alpha + \beta_1 + \beta_2)\tau(0) &= \tau(\alpha + \beta_1)\tau(\beta_2)\\ &= \frac{\tau(\alpha)\tau(\beta_1)\tau(\beta_2)}{\tau(0)}\\ &= \tau(\alpha)\tau(\beta_1 + \beta_2),\end{aligned}$$

and so (3.32) holds. It then follows by induction that for any $\beta_1, \beta_2, \dots, \beta_k \in \Gamma$,

$$L_{(\beta_1,1)} \cap \cdots \cap L_{(\beta_k,1)} \subseteq L_{(\beta_1+\cdots+\beta_k,1)}. \tag{3.33}$$

Now assume $\mathcal{A}(\Gamma) = \gamma\mathbb{Z}$. If $\tau \in L_{(\gamma,1)}$, then by (3.33), $\tau \in L_{(k\gamma,1)}$ for any $k \in \mathbb{Z}$ [since $L_{(\gamma,1)} = L_{(-\gamma,1)}$ by (3.8)]. In particular $\tau \in L_{(\beta,1)}$ for all $\beta \in \Gamma$, and so $\tau \in \bigcap_{(\beta,1)\in S'} L_{(\beta,1)}$. Conversely assume $\tau \in L_{(\beta,1)}$ for all $\beta \in \Gamma$. Using (3.1) we may write $\gamma = \beta_1 + \beta_2 + \cdots + \beta_k$ for some $\beta_1, \beta_2, \dots, \beta_k \in \Gamma$, where the β_i need not be distinct. It then follows from (3.33) that $\tau \in L_{(\gamma,1)}$, thus proving (3.30).

Next assume $\mathcal{A}(\Gamma)$ is dense in $\mathbb{R}$. If $\tau(\alpha) = a \exp(\lambda\alpha)$ for some $a \geq 0$ and $\lambda \geq 0$, then trivially $\tau \in L_{(\beta,1)}$ for all $\beta \in \mathbb{R}$ by (3.23). Hence $\tau \in \bigcap_{(\beta,1)\in S'} L_{(\beta,1)}$. Conversely let $\tau \in \bigcap_{(\beta,1)\in S'} L_{(\beta,1)}$ and $\tau \notin L_c$ (else there is nothing to prove). For every $\eta \in \mathbb{R}$ there exists a sequence $\nu_n \in \mathcal{A}(\Gamma)$ such that $\nu_n \to \eta$. Furthermore we can find $\beta_1, \beta_2, \dots, \beta_k \in \Gamma$ such that $\nu_n = \beta_1 + \cdots + \beta_k$. Hence $\tau \in L_{(\nu_n,1)}$ by (3.33). But then by (3.23), for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\tau(\alpha + \eta)\tau(0) &= \tau\Big(\alpha + \lim_{n\to\infty} \nu_n\Big)\tau(0) = \lim_{n\to\infty} \tau(\alpha + \nu_n)\tau(0)\\ &= \lim_{n\to\infty} \tau(\alpha)\tau(\nu_n) = \tau(\alpha)\tau(\eta).\end{aligned}$$

Since this holds for every $\alpha, \eta \in \mathbb{R}$, we conclude that $\tau(\alpha) = a \exp(\lambda\alpha)$. Since $\tau \in L$ it then follows that $a \geq 0$ and $\lambda \geq 0$. Hence (3.31) holds. This completes the proof. $\square$

Next, we consider the case where $m \neq 0$ in (3.29), and S' contains at least one point (β, ρ) with $\rho \neq 0, 1$. Let $\Theta = \{\rho : (\beta, \rho) \in S, \ \ \rho \neq 0, 1\}$ and let $\mathcal{G}(\Theta)$ be the multiplicative subgroup of $(0, \infty)$ generated by Θ. Observe that by (3.1), the same subgroup is generated if S is replaced by S' in the definition of Θ. Since the multiplicative subgroups of $(0, \infty)$ are either cyclic or dense in $(0, \infty)$, we have that either $\mathcal{G}(\Theta) = \langle\theta\rangle$ for some $\theta \in (0, 1)$ or $\mathcal{G}(\Theta)$ is dense in $(0, \infty)$. Let

$$L_m = L_c \cup \{\tau \in L : \tau(\alpha) = a(\alpha - m^*)_+^p \text{ for some } a \geq 0, \ \ p \geq 1/2\}$$

and

$$\mathcal{P}_m = \{Y : Y \sim \tau \text{ for some } \tau \in L_m\}. \tag{3.34}$$

THEOREM 3.9. *Assume that* $|S'| > 1$, $S' \subseteq S_m$ *where* $m \neq 0$, *and* S' *contains at least one point* (β, ρ) *with* $0 < \rho < 1$.

$$\begin{aligned} &\textit{If } \mathcal{G}(\Theta) = \langle\theta\rangle \textit{ then } \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_{((1-\theta)m^*,\theta)} \\ &\qquad\qquad \textit{and so } \bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \tilde{L}_{((1-\theta)m^*,\theta)}. \end{aligned} \tag{3.35}$$

$$\begin{aligned} &\textit{If } \mathcal{G}(\Theta) \textit{ is dense in } (0,\infty) \textit{ then } \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_m \\ &\qquad\qquad \textit{and so } \bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \mathcal{P}_m. \end{aligned} \tag{3.36}$$

PROOF. Recall that if $(\beta, \rho) \in S_m$ and $\rho \neq 0, 1$ then $\beta^* = m^*$. Thus points in S_m with $\rho \neq 0, 1$ are precisely those of the form $((1 - \rho)m^*, \rho)$. This is also the case if $\rho = 1$ because then $\beta = 0$ since $m \neq 0$.

Now assume that $(\beta_i, \rho_i) \in S_m$ and $\rho_i \neq 0$ for $i = 1, 2$. We claim that

$$L_{(\beta_1,\rho_1)} \cap L_{(\beta_2,\rho_2)} \subseteq L_{((1-\rho_1\rho_2)m^*,\rho_1\rho_2)}. \tag{3.37}$$

To see this let $\tau \in L_{(\beta_1,\rho_1)} \cap L_{(\beta_2,\rho_2)}$. If $\tau \in L_c$ then trivially $\tau \in L_{((1-\rho_1\rho_2)m^*,\rho_1\rho_2)}$, thus we may assume $\tau \notin L_c$. Also we may assume $\rho_1 \neq 1$, $\rho_2 \neq 1$ and $\rho_1\rho_2 \neq 1$ else the result is trivial since $L_{(0,1)} = L$. Then by (3.24) we have $\tau(m^*) = 0$, $\tau > 0$ on (m^*, ∞), and for any $\alpha > 0$,

$$\begin{aligned} \tau(m^* + \alpha) &= \frac{\tau(m^* + 1)}{\tau(m^* + \rho_1)} \tau(m^* + \rho_1\alpha) \\ &= \frac{\tau(m^* + 1)^2}{\tau(m^* + \rho_1)\tau(m^* + \rho_2)} \tau(m^* + \rho_1\rho_2\alpha) \\ &= \frac{\tau(m^* + 1)}{\tau(m^* + \rho_1\rho_2)} \tau(m^* + \rho_1\rho_2\alpha). \end{aligned}$$

Thus (3.24) is satisfied with $\beta = (1-\rho_1\rho_2)m^*$ and $\rho = \rho_1\rho_2$, which proves (3.37). Since $((1-\rho_1\rho_2)m^*, \rho_1\rho_2) \in S_m$ it then follows by induction that for any $(\beta_1, \rho_1), \dots, (\beta_k, \rho_k) \in S_m$ with $\rho_i \neq 0$ for $1 \le i \le k$, we have

$$L_{(\beta_1,\rho_1)} \cap \cdots \cap L_{(\beta_k,\rho_k)} \subseteq L_{((1-\rho_1\cdots\rho_k)m^*, \rho_1\cdots\rho_k)}. \tag{3.38}$$

Now assume $\mathcal{G}(\Theta) = \langle\theta\rangle$. If $\tau \in L_{((1-\theta)m^*,\theta)}$, then by (3.38) and (3.8), $\tau \in L_{((1-\theta^k)m^*,\theta^k)}$ for all $k \in \mathbb{Z}$. Thus $\tau \in L_{(\beta,\rho)}$ for all $(\beta,\rho) \in S'$, $\rho \neq 0$. Since $L_{((1-\theta)m^*,\theta)} \subseteq L_{(\beta,0)}$ for all $\beta \le m^*$ by (3.22) and (3.24), it then follows that $\tau \in \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)}$. Conversely assume $\tau \in \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)}$. Using (3.1) we can write $\theta = \rho_1 \cdots \rho_k$ for some $\rho_1, \dots, \rho_k \in \Theta$. Thus by (3.38), $\tau \in L_{((1-\theta)m^*,\theta)}$.

Now assume $\mathcal{G}(\Theta)$ is dense in $(0,\infty)$. If $\tau \in L_m$, then it follows immediately from (3.24) that $\tau \in \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)}$. Conversely assume $\tau \in \bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)}$ and $\tau \notin L_c$. For every $\xi \in (0,\infty)$ there exists a sequence $\zeta_n \in \mathcal{G}(\Theta)$ such that $\zeta_n \to \xi$. Furthermore we can find $\rho_1, \dots, \rho_k \in \Theta$ such that $\zeta_n = \rho_1 \cdots \rho_k$. Hence $\tau \in L_{((1-\zeta_n)m^*,\zeta_n)}$ by (3.38). In particular $\tau(m^*) = 0$, $\tau > 0$ on (m^*, ∞) and by taking limits in (3.24), we obtain, for any $\alpha > 0$,

$$\tau(m^* + \alpha)\tau(m^* + \xi) = \tau(m^* + 1)\tau(m^* + \alpha\xi).$$

Setting $h(\alpha) = \tau(m^* + \alpha)\tau(m^* + 1)^{-1}$ this becomes

$$h(\alpha)h(\xi) = h(\alpha\xi) \tag{3.39}$$

for all $\alpha, \xi \in (0,\infty)$. Now $h(\alpha) > 0$ for all $\alpha \in (0,\infty)$, therefore taking natural log of both sides and writing g for $\ln \circ h$ we obtain

$$g(\alpha\xi) = g(\alpha) + g(\xi).$$

As g is strictly increasing, g^{-1} exists. Thus

$$\alpha\xi = g^{-1}(g(\alpha) + g(\xi)).$$

Since the range of g is $\mathbb{R}$, this implies

$$g^{-1}(x)g^{-1}(y) = g^{-1}(x + y)$$

for all $x, y \in \mathbb{R}$. Hence g^{-1} is exponential, say $g^{-1}(x) = e^{qx}$ for some constant q. This gives $h(\alpha) = \alpha^{1/q}$, for $\alpha \in (0,\infty)$ where $0 < q \le 1/2$ since τ is increasing and τ^2 is convex. Thus

$$\tau(m^* + \alpha) = \begin{cases} 0, & \text{for } \alpha \le 0, \\ a\alpha^p, & \text{otherwise,} \end{cases}$$

for some $a > 0$ and $p \ge 1/2$. This completes the proof. $\square$

The only remaining case is the following.

THEOREM 3.10. *Assume that* $|S'| > 1$, $S' \subset S_m$ *where* $m \neq 0$, *and* S' *does not contain any point* (β, ρ) *with* $0 < \rho < 1$. *Then* $\bigcap_{(\beta,\rho)\in S'} L_{(\beta,\rho)} = L_{(\delta,0)}$ *and so* $\bigcap_{(\beta,\rho)\in S'} \tilde{L}_{(\beta,\rho)} = \tilde{L}_{(\delta,0)}$ *where* $\delta = \sup\{\beta : (\beta, 0) \in S'\}$.

TABLE 1

	S		$\mathcal{L}(\{r_n\})$	Reference
$\lvert S\rvert = 1$	$S = \{(0, 1)\}$		$\tilde{L}$	(3.25)
$\lvert S\rvert > 1$	$S \subseteq S_m, m = 0$	$\mathcal{A}(\Gamma) = \gamma\mathbb{Z}$	$\tilde{L}_{(\gamma,1)}$	Theorem 3.8
		$\mathcal{A}(\Gamma)$ dense	$\mathcal{E}$	Theorem 3.8
	$S \subseteq S_m, m \neq 0$	$\Theta = \varnothing$	$\tilde{L}_{(\delta,0)}$	Theorem 3.10
		$\mathcal{G}(\Theta) = \langle\theta\rangle$	$\tilde{L}_{((\theta-1)m^{-1},\theta)}$	Theorem 3.9
		$\mathcal{G}(\Theta)$ dense	$\mathcal{P}_m$	Theorem 3.9
	$S \not\subseteq S_m$		$\mathcal{N}$	Theorem 3.7

PROOF. This follows immediately from (3.22). □

Table 1 summarizes our results.

The table is expressed in terms of S rather than S', but the results are equivalent since $|S| > 1$ if and only if $|S'| > 1$ and $S \subseteq S_m$ if and only if $S' \subseteq S_m$. Thus we have a complete description of the limiting behavior of ${}^{(r_n)}S_n$ when $r_n \to \infty$, $r_n n^{-1} \to 0$ and X is symmetric. In particular the only possible classes of limit laws are $\mathcal{N}$, $\mathcal{E}$, $\mathcal{P}_m$, $\tilde{L}$ and $\tilde{L}_{(\beta,\rho)}$. The description of the first four classes, given in (2.8), (2.9), (3.34) and (2.6) respectively, is explicit. To make the description of the fifth class equally explicit, we conclude by constructing all functions satisfying the functional equation in (3.6). We need to consider separately the cases $\rho = 1$ and $0 < \rho < 1$.

PROPOSITION 3.11. *For $\beta \neq 0$ the following are equivalent*:

(i) $\tau \in L_{(\beta,1)}$;

(ii) *there exists a nondecreasing, positive convex function $g : [0, |\beta|] \to \mathbb{R}$ such that*

$$g'_R(0) \geq \frac{g(0)}{g(|\beta|)} g'_L(|\beta|) \tag{3.40}$$

and

$$\tau^2(\alpha) = \left(\frac{g(|\beta|)}{g(0)}\right)^k g(\alpha - k|\beta|) \tag{3.41}$$

for all $\alpha \in [k|\beta|, (k+1)|\beta|]$ and any $k \in \mathbb{Z}$.

PROOF. Since $L_{(\beta,1)} = L_{(-\beta,1)}$, we may assume that $\beta > 0$. If $\tau \equiv c$, for some constant c, then we may take $g \equiv c^{1/2}$ on $[0, \beta]$. Conversely, if $g \equiv$ constant on $[0, \beta]$, then we must have $\tau \in L_c$. Hence we may also assume that $\tau \notin L_c$ and $g \not\equiv$ constant.

We will first prove that (i) $\Rightarrow$ (ii). Assume $\tau \in L_{(\beta,1)}$ and let $f = \tau^2$. Then by (3.23), $f > 0$ on $(-\infty, \infty)$ and for all $\alpha \in \mathbb{R}$,

$$f(\alpha + \beta) f(0) = f(\alpha) f(\beta). \tag{3.42}$$

Let $\lambda = \frac{f(\beta)}{f(0)}$. Thus (3.42) becomes, for every $\alpha \in \mathbb{R}$,

$$f(\alpha + \beta) = \lambda f(\alpha). \tag{3.43}$$

It then follows by induction that

$$f(\alpha) = \lambda^k f(\alpha - k\beta) \qquad \text{for all } \alpha \in \mathbb{R},\ k \in \mathbb{Z}. \tag{3.44}$$

Now f is convex and thus its right-hand and left-hand derivatives exist everywhere. Taking the right-hand derivative of (3.43) with respect to α and then setting $\alpha = 0$ we get

$$f_R'(0) = \lambda^{-1} f_R'(\beta). \tag{3.45}$$

Now set

$$g(\alpha) = f(\alpha) \qquad \text{for all } \alpha \in [0, \beta].$$

Clearly g is nondecreasing, convex on $[0, \beta]$ and $g > 0$. Then (3.40) follows easily from (3.45) by using $f_R'(0) = g_R'(0)$, $f_L'(\beta) = g_L'(\beta)$, $f_R'(\beta) \ge f_L'(\beta)$ and $\lambda = \frac{g(\beta)}{g(0)}$. Also, (3.41) follows from (3.44) by observing that if $\alpha \in [k\beta, (k+1)\beta]$, then $\alpha - k\beta \in [0, \beta]$ and so $f(\alpha - k\beta) = g(\alpha - k\beta)$. Hence (i) $\Rightarrow$ (ii).

To prove the reverse direction let g be a function defined on $[0, \beta]$ with the given properties. We must show the function $\tau : \mathbb{R} \to \mathbb{R}$ defined by (3.41), satisfies $\tau \in L_{(\beta,1)}$. Let $f = \tau^2$. Then by taking $k = 0$ in (3.41) we get $f = g$ on $[0, \beta]$. Letting $\lambda = \frac{g(\beta)}{g(0)}$, (3.41) then becomes

$$f(\alpha) = \lambda^k g(\alpha - k\beta) \tag{3.46}$$

for any $\alpha \in [k\beta, (k+1)\beta]$, $k \in \mathbb{Z}$. Observe that $f(k\beta) = \lambda^k g(0)$ and also $f(k\beta) = \lambda^{k-1} g(\beta) = \lambda^{k-1} \lambda g(0) = \lambda^k g(0)$. Thus f is a well defined function and clearly continuous and positive on $\mathbb{R}$. Furthermore, f is nondecreasing on every interval of the form $(k\beta, (k+1)\beta)$ since g is nondecreasing on $(0, \beta)$. Then continuity of f ensures that it is nondecreasing for all $\alpha \in \mathbb{R}$.

By (3.23) it remains to show that f is convex and (3.42) holds. We will first show that (3.42) holds. Let $\alpha \in [k\beta, (k+1)\beta]$ for some $k \in \mathbb{Z}$. Then $\alpha = r + k\beta$ for some $0 \le r \le \beta$. Hence, by (3.46),

$$\begin{aligned}
f(\alpha + \beta) f(0) &= f\big(r + (k+1)\beta\big) f(0) \\
&= \lambda^{k+1} g(r) g(0) \\
&= \lambda^k g(r) \lambda g(0) \\
&= f(r + k\beta) f(\beta) \\
&= f(\alpha) f(\beta),
\end{aligned}$$

and so (3.42) holds.

Finally, we need to show that f is convex. Since $\lambda^k > 0$ and g is convex on $(0, \beta)$, we must have f convex on $(k\beta, (k+1)\beta)$ for every $k \in \mathbb{Z}$. So the only points to check are those of the form $k\beta$. To ensure convexity at these points it is enough to show that $f'_L(k\beta) \le f'_R(k\beta)$. Taking the right-hand and left-hand derivatives of (3.46) with respect to α we obtain

$$f'_R(\alpha) = \lambda^k g'_R(\alpha - k\beta) \qquad \text{for all } \alpha \in \bigl[k\beta, (k+1)\beta\bigr) \text{ and all } k \in \mathbb{Z} \tag{3.47}$$

and

$$\begin{aligned} f'_L(\alpha) &= \lambda^{k-1} g'_L\bigl(\alpha - (k-1)\beta\bigr) \\ &\qquad \text{for all } \alpha \in \bigl((k-1)\beta, k\beta\bigr] \text{ and all } k \in \mathbb{Z}. \end{aligned} \tag{3.48}$$

Setting $\alpha = k\beta$ in (3.47) and (3.48) and using (3.40) then gives

$$\begin{aligned} f'_R(k\beta) &= \lambda^k g'_R(0) \\ &\ge \lambda^{k-1} g'_L(\beta) \\ &= f'_L(k\beta) \end{aligned}$$

as required. $\square$

PROPOSITION 3.12. *For* $0 < \rho < 1$ *the following are equivalent*:

(i) $\tau \in L_{(\beta,\rho)}$;

(ii) *there exists a nondecreasing*, *positive convex function* $g : [\beta^* + \rho, \beta^* + 1] \to \mathbb{R}$ *such that*

$$g'_R(\beta^* + \rho) \ge \frac{g(\beta^* + \rho)}{\rho g(\beta^* + 1)} g'_L(\beta^* + 1) \tag{3.49}$$

and

$$\tau^2(\beta^* + \alpha) = \left(\frac{g(\beta^* + 1)}{g(\beta^* + \rho)}\right)^j g(\beta^* + \alpha\rho^j) \tag{3.50}$$

for all $\alpha \in [\rho^{-j+1}, \rho^{-j}]$, $j \in \mathbb{Z}$, *and* $\tau(\beta^* + \alpha) = 0$ *for all* $\alpha \le 0$.

PROOF. The general idea of the proof is similar to Proposition 3.11. As in Proposition 3.11, we may assume that $\tau \notin L_c$ and $g \not\equiv$ constant.

To prove (i) $\Rightarrow$ (ii) we assume $\tau \in L_{(\beta,\rho)}$. Then by (3.24), $\tau(\beta^* + \alpha) = 0$ for $\alpha \le 0$. Let $f = \tau^2$. Again by (3.24), $f > 0$ on (β^*, ∞) and for all $\alpha > 0$,

$$f(\beta^* + \alpha) f(\beta^* + \rho) = f(\beta^* + 1) f(\beta^* + \alpha\rho). \tag{3.51}$$

Let $\lambda = \frac{f(\beta^*+1)}{f(\beta^*+\rho)}$. Then (3.51) becomes, for every $\alpha > 0$,

$$f(\beta^* + \alpha) = \lambda f(\beta^* + \alpha\rho). \tag{3.52}$$

It then follows by induction that

$$f(\beta^* + \alpha) = \lambda^j f(\beta^* + \alpha\rho^j) \qquad \text{for all } \alpha > 0,\ j \in \mathbb{Z}. \tag{3.53}$$

Now f is convex and thus its right-hand and left-hand derivatives exist everywhere. Taking the right-hand derivative of (3.52) with respect to α and then setting $\alpha = 1$ we get

$$f_R'(\beta^* + 1) = \lambda\rho f_R'(\beta + \rho). \tag{3.54}$$

Now set

$$g(\alpha) = f(\alpha) \qquad \text{for all } \alpha \in [\beta^* + \rho, \beta^* + 1].$$

It is clear that g is nondecreasing, convex and $g > 0$ on $[\beta^* + \rho, \beta^* + 1]$. Then (3.49) follows easily from (3.54) by using $f_R'(\beta^* + \rho) = g_R'(\beta^* + \rho)$, $f_L'(\beta^* + 1) = g_L'(\beta^* + 1)$, $f_R'(\beta^* + 1) \geq f_L'(\beta^* + 1)$ and $\lambda = \frac{g(\beta^*+1)}{g(\beta^*+\rho)}$. Also, (3.50) follows from (3.53) by observing that if $\alpha \in [\rho^{-j+1}, \rho^{-j}]$, then $\alpha\rho^j \in [\rho, 1]$ and so $f(\beta^* + \alpha\rho^j) = g(\beta^* + \alpha\rho^j)$. Hence (i) $\Rightarrow$ (ii).

To prove the reverse direction let g be any function defined on $[\beta^* + \rho, \beta^* + 1]$ with the given properties. We must show the function $\tau : \mathbb{R} \to \mathbb{R}$ defined by (3.50) with $\tau(\beta^* + \alpha) = 0$ for $\alpha \leq 0$, satisfies $\tau \in L_{(\beta,\rho)}$. Let $f = \tau^2$. Then by taking $j = 0$ in (3.50) we get $f = g$ on $[\beta^* + \rho, \beta^* + 1]$. Let $\lambda = \frac{g(\beta^*+1)}{g(\beta^*+\rho)}$, then (3.50) becomes

$$f(\beta^* + \alpha) = \lambda^j g(\beta^* + \alpha\rho^j) \tag{3.55}$$

for any $\alpha \in [\rho^{-j+1}, \rho^{-j}]$, $j \in \mathbb{Z}$. Observe that $f(\beta^* + \rho^{-j}) = \lambda^j g(\beta^* + 1)$ and also $f(\beta^* + \rho^{-j}) = \lambda^{j+1} g(\beta^* + \rho) = \lambda^j \lambda g(\beta^* + \rho) = \lambda^j g(\beta^* + 1)$ and so f is a well defined function and clearly continuous and positive on (β^*, ∞). Furthermore, f is nondecreasing on every interval of the form $(\beta^* + \rho^{-j+1}, \beta^* + \rho^{-j})$ since g is nondecreasing on $(\beta^* + \rho, \beta^* + 1)$. Then continuity of f ensures that it is nondecreasing on $(-\infty, \infty)$.

By (3.24) it remains to show that f is convex and (3.51) holds. We will first show that (3.51) holds. Let $\alpha \in [\rho^{-j+1}, \rho^{-j}]$ for some $j \in \mathbb{Z}$. Then $\alpha\rho^j \in [\rho, 1]$ and so by (3.55),

$$\begin{aligned}
f(\beta^* + \alpha) f(\beta^* + \rho) &= \lambda^j g(\beta^* + \alpha\rho^j) g(\beta^* + \rho) \\
&= \lambda^{j-1} g(\beta^* + \alpha\rho^j) \lambda g(\beta^* + \rho) \\
&= \lambda^{j-1} g(\beta^* + \alpha\rho^j) g(\beta^* + 1) \\
&= \lambda^{j-1} g(\beta^* + (\alpha\rho)\rho^{j-1}) f(\beta^* + 1) \\
&= f(\beta^* + \alpha\rho) f(\beta^* + 1)
\end{aligned}$$

for all $\alpha > 0$.

Finally, we need to show that f is convex. Again for any $\alpha \in (\rho^{-j+1}, \rho^{-j})$, since $\lambda^j > 0$ and g is convex on $(\beta^* + \rho, \beta^* + 1)$, we must have f convex on $(\beta^* + \rho^{-j+1}, \beta^* + \rho^{-j})$ for every $j \in \mathbb{Z}$. So the only points to check are β^* and those of the form $\beta^* + \rho^{-j}$, $j \in \mathbb{Z}$. To ensure convexity at these points it is enough to show that $f'_L(\beta^*) \le f'_R(\beta^*)$ and $f'_L(\beta^* + \rho^{-j}) \le f'_R(\beta^* + \rho^{-j})$, $j \in \mathbb{Z}$. The first inequality is trivial since $f'_L(\beta^*) = 0$ while $f'_R(\beta^*) \ge 0$ since f is nondecreasing and convex on $[\beta^*, \infty)$. For the remaining inequalities we take the right-hand and left-hand derivatives of (3.55) with respect to α to obtain

$$f'_R(\beta^* + \alpha) = (\lambda\rho)^{j+1} g'_R(\beta^* + \alpha\rho^{j+1}) \quad \text{for all } \alpha \in [\rho^{-j}, \rho^{-j-1}) \text{ and all } j \in \mathbb{Z} \tag{3.56}$$

and

$$f'_L(\beta^* + \alpha) = (\lambda\rho)^{j} g'_L(\beta^* + \alpha\rho^{j}) \quad \text{for all } \alpha \in (\rho^{-j+1}, \rho^{-j}] \text{ and all } j \in \mathbb{Z}. \tag{3.57}$$

Setting $\alpha = \rho^{-j}$ in (3.56) and (3.57) and using (3.49) then gives

$$\begin{aligned} f'_R(\beta^* + \rho^{-j}) &= (\lambda\rho)^{j+1} g'_R(\beta^* + \rho) \\ &\ge (\lambda\rho)^{j+1} (\lambda\rho)^{-1} g'_L(\beta^* + 1) \\ &= (\lambda\rho)^{j} g'_L(\beta^* + 1) \\ &= f'_L(\beta^* + \rho^{-j}) \end{aligned}$$

as required. □

When $\rho > 1$, the same characterization holds provided ρ is replaced by ρ^{-1} in (ii) and g is now defined on the interval $[\beta^* + 1, \beta^* + \rho]$. This follows easily from the previous proposition and (3.8).

Finally when $\rho = 0$, the characterization of $L_{(\beta,0)}$ in (3.22) is already explicit, since there is no functional equation satisfied by the functions in $L_{(\beta,0)}$.

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