The Annals of Probability
1997, Vol. 25, No. 3, 1180–1209

CAPACITY AND PRINCIPAL EIGENVALUES: THE METHOD OF ENLARGEMENT OF OBSTACLES REVISITED

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We describe a coarse graining method, which provides lower bounds on the principal Dirichlet eigenvalue of the Laplacian in regions receiving small obstacles, and sharpens the previous method of enlargement of obstacles. Based on a quantitative Wiener criterion, one replaces the actual obstacles by obstacles of a much larger size. Controls on the shift of principal eigenvalues and capacity estimates on the locus where the Wiener criterion breaks down are derived. The results are written in a self-contained fashion.

Introduction. Capacity and principal eigenvalues have a well known interplay which has been studied by many authors (e.g., Ancona [1], Kac [7], Ozawa [8], Rauch and Taylor [10] and Swanson [11]). The present article describes another instance of this connection in the context of the "method of enlargement of obstacles."

The method of enlargement of obstacles (see [14] for a review) aims at deriving lower estimates on the principal Dirichlet eigenvalue of the Laplacian in regions of $\mathbb{R}^d$, $d \geq 1$, which receive many small obstacles. These obstacles are often produced by some random mechanism. A typical instance to keep in mind is the case of closed balls of small radius ε centered at the points of a Poisson cloud with possibly high intensity, which are "deleted" from some given region of $\mathbb{R}^d$.

The rough idea of the method is to replace the original configuration of small obstacles with size ε by a configuration of obstacles with a much bigger size ε'. This replacement is performed in such a fashion that one does not increase too much the eigenvalue under study, at least when its level in the original configuration is not too high. The point is that the configurations of enlarged obstacles now have a reduced combinatorial complexity and are easier to analyze.

It turns out that in general one cannot enlarge every obstacle, for it may cause a substantial upward shift of the eigenvalue (see [14]). One needs to distinguish between good obstacles, which are "well surrounded," and "bad obstacles," which are "poorly surrounded." One then enlarges only the

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Received December 1995; revised September 1996.

AMS 1991 *subject classifications.* 60J45, 35P15, 82D30.

Key words and phrases. Principal eigenvalues, quantitative Wiener criterion, enlargement of obstacles.

obstacles corresponding to good points and discards the obstacles corresponding to bad points.

When using this coarse graining method, there are essentially three quantities which measure the efficiency of the procedure:

(I.1a) The uniform upper bounds one has on the possible upward shift caused by the replacement of some of the true obstacles by enlarged obstacles.

(I.1b) The ratio of the typical size ε of the true obstacles to the size ε' of the enlarged obstacles. This ratio in essence controls the reduction of combinatorial complexity achieved by considering enlarged obstacles instead of true obstacles.

(I.1c) The uniform controls on the volume of enlarged bad obstacles. These are crucial when deriving probabilistic estimates on configurations of enlarged obstacles.

The point of the present article is the description of a new way of constructing such a coarse grained picture, which leads to estimates on quantities like (I.1a)–(I.1c) quite better than those coming from the method described in [12–14]. For instance, in the two-dimensional case, the method previously used provides controls of order $\exp\{-\text{const}\sqrt{\log(1/\varepsilon)}\}$, as $\varepsilon \to 0$, on the quantities showing up in (I.1a)–(I.1c). These estimates become worse in higher dimension. The construction we present here, instead, produces controls of order $\varepsilon^{\text{const}}$, as $\varepsilon \to 0$, regardless of the dimension.

At the heart of this improvement lies a new way of enlarging obstacles. Roughly speaking, we decompose $\mathbb{R}^d$ into L-adic boxes of size less than or equal to 1. We use this decomposition to partition $\mathbb{R}^d$ into a collection of density boxes (size approximately ε^{γ}), of bad boxes (size approximately ε^{β}) and of boxes receiving no point of the cloud (size approximately ε^{β}), with $1 > \beta > \gamma > 0$. The scales ε^{γ} and ε^{β} which show up in this coarse grained picture are large compared to the scale ε of the true obstacles. This addresses (I.1b).

The density boxes are those boxes of size approximately ε^{γ} which fulfill a certain quantitative Wiener criterion [see (1.15)]; the closure of their union is precisely the set where we "enlarge obstacles." This leads to eigenvalue estimates developed in Section 1, which take care of (I.1a).

The bad boxes on the other hand are the boxes of size approximately ε^{β}, contained in the complement of density boxes and receiving some point of the cloud [see (2.32)]. We derive sharp controls on the volume of bad boxes; see (I.1c). This is essentially performed by means of capacity estimates developed in Section 2. In particular, we show in Theorem 2.1 some rather general exponential controls on the average capacity of obstacles attached to rarefaction boxes (i.e., boxes of size approximately ε^{γ} which are not density boxes). The estimates are worked out through the successive generations of boxes. This has some flavor of renormalization theory or of the methods developed in the study of harmonic measures (see [3, 6]), and is also reminiscent of the arguments involving trees and capacity present in the series of work by Benjamini, Pemantle and Peres (see, e.g., [2], [9] and references therein).

The motivation for the method we develop in the present article stemmed from a problem, which we now briefly describe. Consider a Poisson cloud of points (x_i) in $\mathbb{R}^d$, $d \geq 2$, with constant intensity $\nu > 0$. Let $W(\cdot)$ be a nonnegative, compactly supported, bounded measurable function, which is not a.e. equal to 0. It was shown in [11] that the principal Dirichlet eigenvalue λ_u of $-\Delta/2 + \sum_i W(\cdot - x_i)$ in the box $(-u/2, u/2)^d$ has asymptotic behavior described by

$$\text{(I.2)} \qquad \mathbb{P}\text{-a.s.} \quad \lambda_u \sim \frac{c(d,\nu)}{(\log u)^{2/d}} \quad \text{as } u \to \infty,$$

where $c(d,\nu) \in (0,\infty)$ is a constant solely depending on d and ν. What can now be said about the fluctuations of the random variable λ_u? In particular, what bounds can be derived on the spread of the distribution of λ_u around a median? These questions, for instance, turn out to be of importance for the fine study of Brownian motion in a Poissonian potential. Some preliminary applications to the control of fluctuations of λ_u are developed in Section 3, but the main body of applications will be developed in [15].

As a result of this motivation, we wrote our results in the context of "soft obstacles," where the obstacle attached to a point x_i of the "cloud" is $\varepsilon^{-2} W(\varepsilon^{-1}(\cdot - x_i))$. However our results can routinely be adapted to the case of "hard obstacles." In this case we would instead delete the set $\bigcup_i x_i + \varepsilon K$ from $\mathbb{R}^d$, with K a fixed nonpolar compact subset of $\mathbb{R}^d$ (e.g., the closed ball of radius $a > 0$ centered at 0).

Let us explain how the article is organized. Section 1 introduces the precise notation, describes the notion of density and rarefaction boxes and derives the eigenvalue estimates. The main results are Theorems 1.2 and 1.4. In Theorem 1.2 a control as in (I.1a) is discussed. In Theorem 1.4 it is shown that one can discard regions of space which are distant enough from places in which the coarse grained configuration presents a noticeable hole (the so-called clearing boxes) without causing a substantial shift of the eigenvalue.

In Section 2 we define the notion of bad boxes. The main objective is to control their volume; see (I.1c). This is performed with the help of the capacity estimates on rarefaction boxes shown in Theorem 2.1 together with certain "solidification estimates" presented in Proposition 2.3.

Section 3 develops some first applications of our results to the study of fluctuations of the above-mentioned principal Dirichlet eigenvalue λ_u [see (I.2)]. It also tries to address the reader's legitimate question "what is all this good for?"

The Appendix collects some results which are used in Section 1 in the derivation of eigenvalue estimates. These are streamlined and reinforced versions of the arguments developed in [12] or [13].

Let us finally mention that the present article has been written in an essentially self-contained way, which in particular does not require knowledge of [12–14]. We thank A. Ancona for mentioning [6] and I. Benjamini for [3].

1. Eigenvalue estimates. We begin with a description of our notations and setting. We let Ω stand for the set of locally finite simple pure point measures $\omega = \sum_i \delta_{x_i}$ on $\mathbb{R}^d$, $d \geq 1$. To each point x_i of a "cloud configuration $\omega = \sum_i \delta_{x_i}$," we attach a soft obstacle. This soft obstacle is defined by means of a fixed shape function $W(\cdot)$ which is suitably scaled and translated to the point x_i under consideration. We assume that $W(\cdot)$ is nonnegative, bounded, measurable, compactly supported and not a.e. equal to 0. The scaling is governed by a parameter $\varepsilon \in (0,1)$, and our soft obstacles are defined by the potential function

$$V_\varepsilon(x,\omega) = \varepsilon^{-2} \sum_i W\big(\varepsilon^{-1}(x - x_i)\big), \qquad x \in \mathbb{R}^d,\ \omega \in \Omega. \tag{1.1}$$

It is convenient to denote by $a = a(W) > 0$ the radius of the smallest closed ball centered at the origin outside which $W(\cdot)$ vanishes:

$$a(W) = \inf\big\{r > 0,\ W(\cdot) = 0 \text{ on } \overline{B}(0,a)^c\big\}. \tag{1.2}$$

We let P_x, $x \in \mathbb{R}^d$, stand for the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting from x and let $(Z_t)_{t \geq 0}$ stand for the canonical Brownian motion. Given U, a nonempty open subset of $\mathbb{R}^d$, $\varepsilon > 0$ and $\omega \in \Omega$, a central object of interest in what follows is $\lambda^\varepsilon(U)$, the bottom of the spectrum of the generator of the strongly continuous self-adjoint semigroup on $L^2(U, dx)$ induced by

$$R_t^{U,\varepsilon} f(x) = E_x\left[T_U > t,\ \exp\left\{-\int_0^t V_\varepsilon(Z_s,\omega)\, ds\right\} f(Z_t)\right], \tag{1.3}$$

with $t > 0$, $x \in \mathbb{R}^d$, $f \in L^2(U, dx)$ and $T_U = \inf\{s \geq 0,\ Z_s \notin U\}$, the exit time from U. It is known (see [5], Proposition 3.1) that

$$\lambda^\varepsilon(U) = \inf\left\{\int \left(\tfrac{1}{2}|\nabla f|^2 + V_\varepsilon f^2\right) dx,\ f \in C_c^\infty(U),\ \int f^2\, dx = 1\right\}. \tag{1.4}$$

[In fact $\lambda^\varepsilon(U)$ is the bottom of the spectrum of the Friedrichs extension of $-\Delta/2 + V_\varepsilon$ on $C_c^\infty(U)$.] When B is a closed subset of $\mathbb{R}^d$, we denote by H_B the entrance time of $Z_\cdot$ in B:

$$H_B = \inf\{s \geq 0,\ Z_s \in B\}. \tag{1.5}$$

We are now ready to describe how we "enlarge obstacles." We consider an integer $L \geq 2$ [which later will be chosen large enough; see (2.9)] and introduce an L-adic decomposition of $\mathbb{R}^d$: For $m = (i_0, i_1, \dots, i_k)$ with $k \geq 0$, $i_0 \in \mathbb{Z}^d$ and $i_1, \dots, i_k \in \{0, \dots, L-1\}^d$, we consider the box of generation k with size L^{-k} associated with m:

$$C_m = i_0 + \frac{i_1}{L} + \cdots + \frac{i_k}{L^k} + \frac{1}{L^k}[0,1)^d. \tag{1.6}$$

We denote by I (resp., I_k) the collection of indices m of the above form (resp. of indices m of generation k). We shall sometimes write $C_{m,k}$ to recall the generation k of m. There is a natural tree-like structure on I and we shall write

$$m \preccurlyeq m', \tag{1.7}$$

when m' extends m. That is,

$$m = (i_0, \dots, i_k), \qquad m' = (i'_0, \dots, i'_{k'}) \quad \text{with } k \le k' \text{ and } i_0 = i'_0, \dots, i_k = i'_k.$$

When $m = (i_0, \dots, i_k) \in I$, $j \in \{0, \dots, L-1\}^d$. We shall also write

$$m \cdot j \text{ instead of } (i_0, \dots, i_k, j). \tag{1.8}$$

Finally when $m = (i_0, \dots, i_k) \in I_k$ and $0 \le k' \le k$, $[m]_{k'}$ will stand for the truncation of m to the k'th generation:

$$[m]_{k'} = (i_0, \dots, i_{k'}) \in I_{k'}. \tag{1.9}$$

We shall now introduce two supplementary length scales which are intermediate between 1 and ε. Namely, we pick

$$0 < \alpha < \gamma < 1, \tag{1.10}$$

and the above-mentioned scales will roughly be $1 \gg \varepsilon^\alpha \gg \varepsilon^\gamma \gg \varepsilon$. The scales ε^γ will correspond to the size of enlarged obstacles, whereas the scale ε^α, which should be thought of as being close to 1, will come in the definition of the quantitative Wiener criterion in (1.15) below. Since we are working with the L-adic decomposition of $\mathbb{R}^d$, it is convenient to introduce

$$\begin{gathered} n_\alpha(\varepsilon) = \left[\alpha \frac{\log(1/\varepsilon)}{\log L}\right], \qquad n_\gamma(\varepsilon) = \left[\gamma \frac{\log(1/\varepsilon)}{\log L}\right] \quad \text{and} \\ l(\varepsilon) = n_\gamma(\varepsilon) - n_\alpha(\varepsilon). \end{gathered} \tag{1.11}$$

We then have

$$\begin{gathered} L^{-n_\alpha - 1} < \varepsilon^\alpha \le L^{-n_\alpha}, \qquad L^{-n_\gamma - 1} < \varepsilon^\gamma \le L^{-n_\gamma}, \\ l(\varepsilon) \sim \frac{(\gamma - a)}{\log L} \log \frac{1}{\varepsilon} \quad \text{as } \varepsilon \to 0. \end{gathered} \tag{1.12}$$

We are now going to define density and rarefaction boxes of the configuration $\omega \in \Omega$. To this end we still need some notation. For $m \in I_k$, an index of generation k and $\omega \in \Omega$ we define

$$\begin{gathered} K_m = L^k \Big(\bigcup_{x_i \in C_m} \overline{B}(x_i, a\varepsilon) \Big), \\ \mathrm{cap}_m = \mathrm{cap}(K_m), \end{gathered} \tag{1.13}$$

provided that for a compact subset K of $\mathbb{R}^d$, $\mathrm{cap}(K)$ denotes the usual Brownian capacity when $d \geq 3$ and the 1-capacity when $d = 1, 2$. That is,

$$\mathrm{cap}(K) = \left(\inf\left\{\int\int g(x, y)\nu(dx)\nu(dy);\ \nu \text{ probability on } K\right\}\right)^{-1}, \tag{1.14}$$

where for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} g(x, y) &= \int_0^\infty p(s, x, y)\, ds = c(d)|y - x|^{2-d} \quad (\text{when } d \geq 3) \\ &= \int_0^\infty e^{-s} p(s, x, y)\, ds \quad (\text{when } d = 1, 2), \end{aligned}$$

provided $p(s, x, y)$ stands for the Brownian motion transition density and

$$c(d) = \Gamma\left(\frac{d}{2} - 1\right) \Big/ (2\pi^{d/2}) \quad \text{when } d \geq 3.$$

In other words, K_m corresponds to blowing up by a factor L^k the "skeleton of obstacles" $\bigcup_{x_i \in C_m} \overline{B}(x_i, a\varepsilon)$ attached to the points of the cloud falling in the box C_m of size L^{-k}, and cap_m is the capacity of this set.

The quantitative Wiener criterion is now as follows. We pick $\delta > 0$ (it will later turn out that a useful δ should not be too large; see Theorem 2.1). Given d, ε, ω, $W(\cdot)$, L, α, γ and δ, we shall say that a box C_m of size L^{-n_γ} is a density box when

$$\sum_{n_\alpha(\varepsilon) < k \leq n_\gamma(\varepsilon)} \mathrm{cap}_{[m]_k} \geq \delta l(\varepsilon) = \delta\big(n_\gamma(\varepsilon) - n_\alpha(\varepsilon)\big). \tag{1.15}$$

When (1.15) fails, we shall say that C_m is a rarefaction box. It is convenient to write

$$\begin{aligned} D &= \bigcup_{\substack{m \in I_{n_\gamma} \\ m \text{ density index}}} C_m, \\ R &= \bigcup_{\substack{m \in I_{n_\gamma} \\ m \text{ rarefaction index}}} C_m. \end{aligned} \tag{1.16}$$

The interest in the notion of density box stems from a lemma which we shall now state. It plays a crucial role in the eigenvalue estimates we shall derive. For $k \geq 0$, we denote by H_k the stopping time

$$H_k = \inf\left\{s \geq 0, \|Z_s - Z_0\| \geq L^{-k}\right\}, \tag{1.17}$$

where

$$\|x\| = \sup_{i = 1, \dots, d} |x_i| \quad \text{for } x \in \mathbb{R}^d. \tag{1.18}$$

LEMMA 1.1. *There exists $c_1(d, W) > 0$, such that when*

$$4a\varepsilon < L^{-n_\gamma(\varepsilon)}, \tag{1.19}$$

$$E_x\left[\exp\left\{-\int_0^{H_{n_\alpha(\varepsilon)}} V_\varepsilon(Z_s, \omega)\, ds\right\}\right] \le \exp\left\{-c_1 \sum_{n_\alpha < k \le n_\gamma} \mathrm{cap}_{[m]_k}\right\} \tag{1.20}$$

for all $\omega \in \Omega$, $m \in I_{n_\gamma(\varepsilon)}$ and $x \in \overline{C}_m$.

PROOF. Assume (1.19) and consider $m \in I_{n_\gamma(\varepsilon)}$, $x \in \overline{C}_{\underline{m}}$, k with $n_\alpha < k \le n_\gamma$, z with $\|z - x\| \le L^{-k}$ and $K = L^{-k} K_{[m]_k} = \bigcup_{x_i \in C_{[m]_k}} \overline{B}(x_i, \alpha\varepsilon)$. It follows from (1.19) that the $3a\varepsilon$ closed $\|\cdot\|$-neighborhood of K is contained in the open $\|\cdot\|$-ball of radius L^{-k+1} centered at x. Define $H = \inf\{s \ge 0, \|Z_s - x\| > L^{-k+1}\}$. Then

$$\begin{aligned} &E_z\left[\exp\left\{-\int_0^H V_\varepsilon(Z_s, \omega)\, ds\right\}\right] \\ &\quad \le P_z[H < H_K] + E_z\left[H_K < H,\ E_{Z_{H_K}}\left[\exp\left\{-\int_0^H V_\varepsilon(Z_s, \omega)\, ds\right\}\right]\right] \\ &\quad \le 1 - P_z[H_K < H]\left(1 - \sup_{z' \in K} E_{z'}\left[\exp\left\{-\int_0^H V_\varepsilon(Z_s, \omega)\, ds\right\}\right]\right) \\ &\quad \le 1 - P_z[H_K < H](1 - K(d, W)), \end{aligned} \tag{1.21}$$

where in the last step we used the observation preceding (1.21), scaling and the notation

$$K(d, W) = \sup_{z' \in \overline{B}(0,a)} E_{z'}\left[\exp\left\{-\int_0^{T_{B(0,3a)}} W(Z_s)\, ds\right\}\right] \in (0, 1).$$

Now observe that $L^k K = K_{[m]_k}$, thanks to (1.19), is included in the closed $1/4\|\cdot\|$-neighborhood of some box C_q of size 1 ($q \in \mathbb{Z}^d$) such that $L^k x \in \overline{C}_q$. If we now use scaling, the correspondence between equilibrium charge and last visit of killed Brownian motion (see [4], Chapter 5), together with standard comparisons of Green functions, we find that for z with $\|z - x\| \le L^{-k}$,

$$P_z[H_K < H] \ge K'(d)\mathrm{cap}_{[m]_k}. \tag{1.22}$$

As a result, the left member of (1.21) is smaller than $1 - c_1(d, W)\mathrm{cap}_{[m]_k}$. If we now use the strong Markov property at the times H_k defined in (1.17), we find

$$\begin{aligned} E_x\left[\exp\left\{-\int_0^{H_{n_\alpha}} V_\varepsilon(Z_s, \omega)\, ds\right\}\right] &\le \prod_{n_\alpha < k \le n_\gamma} \left(1 - c_1 \mathrm{cap}_{[m]_k}\right) \\ &\le \exp\left\{-c_1 \sum_{n_\alpha < k \le n_\gamma} \mathrm{cap}_{[m]_k}\right\}, \end{aligned}$$

which proves our claim (1.20). □

We can now define our enlarged obstacles. We shall "solidify" each density box. More precisely, with each nonempty open subset U of $\mathbb{R}^d$, we associate the open set

$$U' = U \setminus \overline{D}. \tag{1.23}$$

When U' is empty $\lambda^{\varepsilon}(U')$ will be $+\infty$ by convention. Our first main result is the following theorem:

THEOREM 1.2. *If*

$$\rho \in \left(0, \delta c_1 \frac{(\gamma - \alpha)}{(d+2)\log L}\right),$$

then for $M > 0$,

$$\lim_{\varepsilon \to 0} \sup_{\omega, U} \varepsilon^{-\rho} \big(\lambda^{\varepsilon}(U') \wedge M - \lambda^{\varepsilon}(U) \wedge M \big) = 0. \tag{1.24}$$

PROOF. Pick $M > 0$, $0 < \rho < \rho' < \delta c_1(\gamma - \alpha)\{\log L(d+2)\}^{-1}$ and for given $U, \varepsilon > 0, \omega$ define

$$\lambda = \big(\lambda^{\varepsilon}(U') \wedge M - \varepsilon^{\rho'} \big)_+ . \tag{1.25}$$

If $\lambda > 0$, then $M > \varepsilon^{\rho'}$, so that

$$\lambda \le \lambda^{\varepsilon}(U') \left(1 - \frac{\varepsilon^{\rho'}}{M}\right), \tag{1.26}$$

and from Proposition A.1 in the Appendix, we find that

$$A \stackrel{\text{def}}{=} 1 + \sup_x \int_0^{\infty} \lambda e^{\lambda s} R_s^{U', \varepsilon} 1(x)\, ds \le K(d) \left(\frac{\varepsilon^{\rho'}}{M} \right)^{-1-d/2}, \tag{1.27}$$

since A increases with λ [see (A.16)].

We now want to apply Theorem A.3 in the Appendix. To this end we choose $\tau = H_{n_\alpha(\varepsilon)}$ [with the notations of (1.17)], $U_1 = U'$ and $U_2 = U$. If $\varepsilon \in (0,1)$ is small enough so that (1.19) holds and

$$E_0[\exp\{2M\tau\}] \le K'(d) < \infty, \tag{1.28}$$

then clearly (A.11) holds and the quantities A and B from (A.10) are finite, that is, (A.12) holds. We now consider

$$C \stackrel{\text{def}}{=} \sup_{x \in (U')^c} E_x \left[\tau < T_U, \exp\left\{ \lambda\tau - \int_0^{\tau} V_{\varepsilon}(Z_s, \omega)\, ds \right\} \right] \tag{1.29}$$

and wish to prove that $A \cdot C < 1$ for a suitable choice of parameters. Observe that the expectation in (1.29) vanishes when $x \notin U$ and we thus only need consider $x \in U \setminus U' \subset \overline{D}$.

From the Cauchy–Schwarz inequality and (1.28), it follows that

$$\begin{aligned} C^2 &\le K'(d) \sup_{x \in \overline{D}} E_x \left[\exp\left\{ -\int_0^{\tau} 2V_{\varepsilon}(Z_s, \omega)\, ds \right\} \right] \\ &\le K'(d) \exp\{-c_1 \delta l(\varepsilon)\} \end{aligned}$$

by Lemma 1.1 and (1.15). In view of (1.27), we find

$$AC \le K(d, M)\exp\left\{\rho'\left(\frac{d}{2}+1\right)\log\frac{1}{\varepsilon} - \frac{c_1}{2}\,\delta l(\varepsilon)\right\}. \tag{1.30}$$

Observe that by (1.12), $l(\varepsilon) \sim ((\gamma-\alpha)/\log L)\log(1/\varepsilon)$, as $\varepsilon \to 0$. Our choice of ρ' now ensures that when ε is small enough, $A\cdot C < 1$ and therefore by (A.14), $\lambda \le \lambda^{\varepsilon}(U)$.

We have thus shown that when $\varepsilon < \varepsilon_0(d, \alpha, \gamma, \delta, \rho', L, W(\cdot), M)$ for arbitrary U, ω, either $\lambda = 0$, so that $\lambda^{\varepsilon}(U') \wedge M - \lambda^{\varepsilon}(U) \wedge M \le \varepsilon^{\rho'}$, or $\lambda > 0$ and $\lambda \le \lambda^{\varepsilon}(U)$, which together with the fact that $\lambda \le M$ implies

$$\lambda^{\varepsilon}(U') \wedge M - \varepsilon^{\rho'} \le \lambda^{\varepsilon}(U) \wedge M.$$

Our claim (1.24) easily follows. □

We shall now discuss a second result. We shall show that $\lambda^{\varepsilon}(T)$, provided it has a "reasonable value," does not increase too much when we replace the open set T by its intersection with a suitable neighborhood of the boxes of unit size, where the complement of the enlarged obstacles is not too small (the so-called clearing boxes). For each box C_q, $q \in \mathbb{Z}^d$, of size 1, we define

$$v_q = |C_q \setminus D|, \tag{1.31}$$

the volume of the complement of density boxes within C_q.

We now introduce a parameter $r \in (0, 1/3)$ and call a box C_q of size 1 a clearing box when

$$v_q > r^d. \tag{1.32}$$

If (1.32) fails, we say that C_q is a forest box. We denote by A the closed set

$$A = \bigcup_{q:\, C_q \text{ clearing box}} \overline{C_q} \tag{1.33}$$

and by O the open neighborhood of size $R \in (0, \infty)$ of A:

$$O = \left\{x \in \mathbb{R}^d, \exists\, y \in A, \|y - x\| < R\right\}. \tag{1.34}$$

As we shall now see, we have natural lower bounds on $\lambda^{\varepsilon}(A^c)$.

Proposition 1.3. *There exists a constant $c_2(d) \in (0, \infty)$ such that when* (1.19) *holds together with*

$$L^{-n_{\alpha}(\varepsilon)} < r \tag{1.35}$$

and

$$\delta c_1 l(\varepsilon) > \log 2, \tag{1.36}$$

then

$$\lambda^{\varepsilon}(A^c) > c_2(d)/r^2. \tag{1.37}$$

Proof. We now apply Lemma A.2 from the Appendix, choosing $U = A^c$ and

$$S_1 = \inf\left\{s \ge 0, \|Z_s - Z_0\| \ge 5r\right\}.$$

First observe that when $x \in A^c$ and $x \in C_q$, $q \in \mathbb{Z}^d$, then from the definition of A,

$$|B_\infty(x, 3r) \cap D \cap C_q| \geq \left\{\left(\tfrac{3}{2}\right)^d - 1\right\} r^d, \tag{1.38}$$

where $B_\infty(x, 3r)$ is the $\|\ \|$-ball of radius $3r$ and center x, and $0 < r < 1/3$ was used. Now when (1.19), (1.35) and (1.36) hold, it follows from Lemma 1.1 and (1.38) that

$$\begin{aligned} E_x\left[\exp\left\{-\int_0^{S_1} V_\varepsilon(Z_s, \omega)\, ds\right\}\right] &\leq P_x\left[T_{B_\infty(x,4r)} < H_{\overline{D}}\right] \\ &\quad + P_x\left[H_{\overline{D}} \leq T_{B_\infty(x,4r)},\ E_{Z_{H_{\overline{D}}}}\left[\exp\left\{-\int_0^{H_{n_\alpha}} V_\varepsilon(Z_s, \omega)\, ds\right\}\right]\right] \\ &\leq 1 - \tfrac{1}{2} P_x\left[H_{\overline{D}} \leq T_{B_\infty(x,4r)}\right], \end{aligned}$$

so that

$$E_x\left[\exp\left\{-\int_0^{S_1} V_\varepsilon(Z_s, \omega)\, ds\right\}\right] \leq c(d) < 1. \tag{1.39}$$

Picking $\lambda = c_2(d)/r^2$, we can make sure by picking $c_2(d)$ small enough that for any $z \in \mathbb{R}^d$,

$$E_z[\exp\{2\lambda S_1\}] = c'(d) < \frac{1}{c(d)}.$$

It now follows, in the notation of Lemma A.2, that $\beta < \infty$ and

$$\begin{aligned} \alpha^2 &= \left(\sup_x E_x\left[S_1 < T_{A^c}, \exp\left\{\lambda S_1 - \int_0^{S_1} V_\varepsilon(Z_s, \omega)\, ds\right\}\right]\right)^2 \\ &\leq c'(d) \sup_{x \in A^c} E_x\left[\exp\left\{-\int_0^{S_1} 2V_\varepsilon(Z_s, \omega)\, ds\right\}\right] < 1. \end{aligned}$$

Since (A.5) is clearly fulfilled, it follows from (A.8) that

$$\lambda = c_2(d)/r^2 \leq \lambda^\varepsilon(A^c).$$

This proves our claim. $\square$

We are now ready to state our second main result of this section.

THEOREM 1.4. *There exists* $c_3(d) \in (0, \infty)$ *and* $r_0(d, M) < r_1(d, M) \in (0, 1/5)$ *such that for* $M > 0$ *and* d, α, γ, δ, L *and* $W(\cdot)$ *as above*,

$$\begin{aligned} \limsup_{\varepsilon \to 0} \sup_{\substack{R > 5r_1 \\ L^{-n_\alpha(\varepsilon)} < r < r_0}} \sup_{\omega, T} &\exp\left\{c_3\left[\frac{R}{5r}\right]\right\} \\ &\times(\lambda^\varepsilon(T \cap O) \wedge M - \lambda^\varepsilon(T) \wedge M) \leq 1. \end{aligned} \tag{1.40}$$

PROOF. As in the proof of Theorem 1.2, we shall apply Theorem A.3 from the Appendix. We consider $M > 0$ and assume $\varepsilon \in (0,1)$ is small enough so that (1.19) and (1.36) hold. We choose $r_1(d, M) < 1/5$, such that

$$c_2(d)/r_1^2 > 4M \tag{1.41}$$

and assume from now on that $r \in (L^{-n_\alpha}, r_1(d, M))$.

The nonempty open set T will play the role of U_2 in Theorem A.3, whereas $T \cap O$ will play that of U_1. We now choose

$$\tau = H_A (= T_{A^c}),$$

and observe that (A.11) holds. We pick some $c_3 > 0$ and define

$$\lambda = \left(\lambda^\varepsilon(T \cap O) \wedge M - \exp\left\{-c_3\left[\frac{R}{5r}\right]\right\}\right)_+. \tag{1.42}$$

In the case $\lambda > 0$, we have

$$0 < \lambda \le \lambda^\varepsilon(T \cap O)\left(1 - \exp\left\{-c_3\left[\frac{R}{5r}\right]\right\}\frac{1}{M}\right),$$

and applying Proposition A.1, we find that

$$\begin{aligned} A &\stackrel{\text{def}}{=} 1 + \sup_x \int_0^\infty \lambda e^{\lambda u} R_u^{T \cap O, \varepsilon} 1(x)\, du \\ &\le K(d)\left(M \exp\left\{c_3\left[\frac{R}{5r}\right]\right\}\right)^{(d/2+1)}. \end{aligned} \tag{1.43}$$

A second application of Proposition A.1, together with (1.41), shows that the quantity B from (A.10) is finite, so that (A.12) holds. Now consider

$$C \stackrel{\text{def}}{=} \sup_{x \notin T \cap O} E_x\left[H_A < T_T, \exp\left\{\lambda H_A - \int_0^{H_A} V_\varepsilon(Z_s, \omega)\, ds\right\}\right].$$

The above expectation vanishes when $x \notin T$, and we thus only need consider $x \in T \setminus O$. Consequently,

$$\begin{aligned} C^2 \le &\sup_{x \in T \setminus O} E_x\left[H_A < T_T, \exp\left\{2\lambda H_A - \int_0^{H_A} V_\varepsilon(Z_s, \omega)\, ds\right\}\right] \\ &\times \sup_{x \notin T \cap O} E_x\left[\exp\left\{-\int_0^{H_A} V_\varepsilon(Z_s, \omega)\, ds\right\}\right], \end{aligned}$$

and by (A.16) and Fubini's theorem,

$$\begin{aligned} C^2 \le &\sup_x \left(1 + \int_0^\infty 2Me^{2Ms} R_s^{A^c, \varepsilon} 1(x)\, ds\right) \\ &\times \sup_{x \in T \cap O} E_x\left[\exp\left\{-\int_0^{H_A} V_\varepsilon(Z_s, \omega)\, ds\right\}\right]. \end{aligned} \tag{1.44}$$

Observe that when $x \in T \setminus O$, it requires at least $[R/5r]$ successive displacements of $Z_.$ at $\|\ \|$-distance $5r$ to reach A. In view of (A.2) and (1.39) we obtain

$$C^2 \le K(d) 2^{(d/2+1)} c(d)^{[R/5r]}. \tag{1.45}$$

If we now choose $c_3(d)$ such that

$$\left(\frac{d}{2} + 1\right) c_3 = \frac{1}{4} \log \frac{1}{c(d)}, \tag{1.46}$$

we find

$$AC \le K(d, M) \exp\left\{ -c_3 \left(\frac{d}{2} + 1\right) \left[\frac{R}{5r}\right] \right\}.$$

Since we only consider $R > 5r_1$, we can make sure by picking $r < r_0(d, M) \le r_1(d, M)$ that $AC < 1$. By Theorem A.3, it follows that $\lambda \le \lambda^{\varepsilon}(T)$ and in fact $\lambda \le \lambda^{\varepsilon}(T) \wedge M$ since $\lambda \le M$. Our claim (1.40) now follows by the same reasoning as at the end of the proof of Theorem 1.2. □

REMARK 1.5. (i) The uniformity of the controls in Theorem 1.4 enable us to consider situations where the parameters r and R may depend on ε. This will for instance be of use in Section 3. In fact variations of the above argument can be given to handle other examples of T, O and A. We omit them for the sake of simplicity.

(ii) The eigenvalue estimates of Theorems 1.2 and 1.4 could in fact be combined, essentially by picking a different τ in the application of Theorem A.3 (see also [12]). However we refrained from doing so in order to give more transparency to the arguments.

When T is a nonempty open set of $\mathbb{R}^d$, we define

$$\tilde{T} = T \cap O \setminus \overline{D}. \tag{1.47}$$

That is, we restrict T to the R neighborhood (for $\|\cdot\|$) of the clearing boxes and delete the enlarged obstacles. The combination of Theorems 1.2 and 1.4 now shows the following corollary.

COROLLARY 1.6. *Assume*

$$\rho \in \left(0, \delta c_1 \frac{(\gamma - \alpha)}{\log L(d+2)}\right).$$

Then for $M > 0$,

$$\limsup_{\varepsilon \to 0} \sup_{\substack{R > 5r_1 \\ L^{-n_\alpha(\varepsilon)} < r < r_0}} \sup_{\omega, T} \left(\exp\left\{ c_3 \left[\frac{R}{5r}\right] \right\} \wedge \varepsilon^{-\rho} \right) \times \left(\lambda^{\varepsilon}(\tilde{T}) \wedge M - \lambda^{\varepsilon}(T) \wedge M \right) \le 1. \tag{1.48}$$

2. Capacity and volume estimates. We shall define in this section the notion of bad boxes; see (2.32) below. Roughly speaking, this involves a third length scale ε^{β} which is intermediate between ε and ε^{γ}: the "bad boxes" will be the boxes of size approximately ε^{β} which are included in the rarefaction set R of the cloud [see (1.16)] and receive a point of the cloud. Our main objective is to derive estimates on the volume of bad boxes within each box of size 1 (see Theorem 2.4). These estimates address (I.1c) of the Introduction. They are important in the case where the cloud ω is random, when one wishes for instance to derive controls on the probability that a given box of size 1 is a "clearing box" [see (3.6) of the next section] or on the distribution of the coarse grained picture $\tilde{T}$ of some given bounded open set T.

The volume estimates of Theorem 2.4 will come as consequences of two capacity estimates given in Theorem 2.1 and Proposition 2.4. The first estimate, in Theorem 2.1, is quite general and gives upper bounds on the capacity cap_m [see (1.13)] attached to "the average rarefaction box C_m" (of size L^{-n_γ}). The second estimate, in Proposition 2.4, provides upper bounds on the possible increase of capacity, which might occur when replacing cap_m, when C_m is some rarefaction box, by the capacity of the suitably scaled union of bad boxes contained in C_m.

We are now ready to begin with the first estimate. We assume from now on that $\varepsilon \in (0,1)$ is small enough so that (1.19) holds. This has the effect that for any index m of I_k, with $n_\alpha < k \le n_\gamma$,

(2.1) the closed $1/4$ neighborhood of $L^k C_m$ relative to $\|\ \|$ contains K_m.

(The box $L^k C_m$ has size 1.) Moreover, we have the consistency relation

$$K_m = \bigcup_{j \in \{0,\dots,L-1\}^d} L^{-1} K_{m \cdot j} \quad \text{for any } m \in I_k, \text{ with } n_\alpha < k < n_\gamma. \tag{2.2}$$

The results we shall now prove are quite general. They only rely on the fact that the (possibly empty) compact sets K_m, $m \in \bigcup_{n_\alpha < k \le n_\gamma} I_k$, satisfy (2.1), (2.2) and cap_m is $\mathrm{cap}(K_m)$ [see (1.14)]. The specific definition (1.13) plays no role. Theorem 2.1 applies as well if instead of (1.13) we define K_m via $K_m = L^k(K \cap \overline{C}_m)$, when $m \in I_k$, K being some fixed compact set [(2.1) and (2.2) are automatically fulfilled].

Let us mention that in the one-dimensional case, we shall see that when $\delta < \mathrm{cap}\{0\}$ [see (2.35) below] there are no bad boxes. This is why we only consider the $d \ge 2$ situation in the following theorem.

THEOREM 2.1. *Assume $d \ge 2$ and $L \ge 2$ is large enough so that* (2.9) *holds. Then there exists $\delta_0(d, L) > 0$, $c_4(d, L) > 0$ and $c_5(d, L) > 0$* [*see* (2.20) *and* (2.29) *below*] *such that for K_m, $m \in \bigcup_{n_\alpha < k \le n_\gamma} I_k$, satisfying* (2.1), (2.2) *and $\delta < \delta_0(d, L)$,*

$$\frac{1}{L^{d n_\gamma}} \sum_{\substack{m \succ q \\ m \in I_{n_\gamma} \\ m \text{ rarefaction index}}} \mathrm{cap}_m \le c_4 \exp\left\{-c_5\left(1 - \frac{\delta}{\delta_0}\right) l\right\} \quad \textit{for } q \in \mathbb{Z}^d. \tag{2.3}$$

(*here $l = n_\gamma - n_\alpha$ and $q \in \mathbb{Z}^d \simeq I_0$.*)

PROOF. With no loss of generality we assume that $q = 0 \in \mathbb{Z}^d$, $n_\alpha = 0$ and $n_\gamma = l > 0$. We are first going to derive three recurrence relations on the numbers cap_m, which "govern the interaction of one generation with the next"; this has some flavor of renormalization methods. The first relation is immediate. In view of (2.1), there is a $c_6(d) > 0$ such that

$$\mathrm{cap}_m \le c_6(d) \quad \text{for } m \text{ of generation } k,\ 0 < k \le l. \tag{2.4}$$

Then note that when $d \ge 3$, the Green function $g(\cdot,\cdot)$ in (1.14) has an exact scaling property, whereas when $d = 2$, for $\|x - y\| \le 4$,

$$g\left(\frac{x}{L}, \frac{y}{L}\right) \le c'(d=2)(\log L + g(x, y)) \le c(d=2)\log L g(x, y). \tag{2.5}$$

Now for $m \in I_k$, $0 < k < l$, and $j \in \{0, \dots, L-1\}^d$, (2.2) implies $L^{-1}K_{mj} \subseteq K_m$. It thus follows from the above-mentioned Green function estimates that for $m \in I_k$, $0 < k < l$, and $j \in \{0, \dots, L-1\}^d$,

$$\mathrm{cap}_{mj} \le c_7(d, L)\mathrm{cap}\left(\frac{1}{L}K_{m\cdot j}\right) \le c_7\, \mathrm{cap}_m, \tag{2.6}$$

where

$$c_7(d, L) = \begin{cases} L^{(d-2)}, & \text{when } d \ge 3, \\ c(d=2)\log L, & \text{when } d = 2\ (c_7 > 1 \text{ by construction}). \end{cases} \tag{2.7}$$

This is our second recurrence relation. We are now ready to derive the third and main recurrence relation. We let $G(\cdot)$ stand for the continuous decreasing function from $\mathbb{R}_+$ to $(0, \infty]$, such that $g(x, y) = G(|y - x|)$. We then define $\tilde{\delta}_1(d, L)$ and $\delta_1(d, L)$ via

$$\tilde{\delta}_1 L^d G\left(\frac{1}{2L}\right) = 1, \qquad \delta_1 = \tilde{\delta}_1 c_7, \tag{2.8}$$

as well as

$$c_8(d, L) = \left\{(3^d + 1)c_7\right\}^{-1} L^d = \begin{cases} (3^d + 1)^{-1} L^2, & \text{when } d \ge 3, \\ \dfrac{1}{10c(d=2)} \dfrac{L^2}{\log L}, & \text{when } d = 2. \end{cases}$$

We assume from now on that L is such that

$$c_8(d, L) > 1. \tag{2.9}$$

LEMMA 2.2. *For* $m \in I_k$, $0 < k < l$,

$$\mathrm{cap}_m \ge c_8 \frac{1}{L^d} \sum_{j \in \{0, \dots, L-1\}^d} (\mathrm{cap}_{mj} \wedge \delta_1). \tag{2.10}$$

PROOF. Consider m as above. If for each j, $\mathrm{cap}_{mj} = 0$, there is nothing to prove. Otherwise, we choose for each $j \in \{0, \dots, L-1\}^d$ a compact subset $\tilde{K}_j$

of $(1/L)K_{m_j}$ such that

$$(2.11)\qquad \begin{aligned} \tilde{K}_j &= \frac{1}{L} K_{mj} \quad \text{if } \operatorname{cap}\left(\frac{1}{L} K_{mj}\right) \le \tilde{\delta}_1, \\ \operatorname{cap}\left(\tilde{K}_j\right) &= \tilde{\delta}_1 \quad \text{if } \operatorname{cap}\left(\frac{1}{L} K_{m \cdot j}\right) > \tilde{\delta}_1. \end{aligned}$$

This can for instance be done with the help of the L-adic partitioning, approximating $\tilde{K}_j$ from above and below by an increasing and decreasing sequence of compact sets. We denote by ν_j the equilibrium measure of $\tilde{K}_j$ [i.e., $\nu_j = 0$ if cap $\tilde{K}_j = 0$, and otherwise $(\operatorname{cap} \tilde{K}_j)^{-1}\nu_j$ is the unique minimum of the variational problem (1.14) associated to $\tilde{K}_j$; see [4], Chapter 5, Section 2]. In view of (2.6), not all $\nu_j = 0$. We define

$$(2.12)\qquad \nu = \frac{1}{\sum_j \operatorname{cap}\left(\tilde{K}_j\right)} \sum_j \nu_j,$$

where the sum runs over all j with $\operatorname{cap}(\tilde{K}_j) > 0$. Observe that ν is a probability supported on K_m and

$$(2.13)\qquad \begin{aligned} \operatorname{cap}_m^{-1} &\le \int\int g(x,y)\nu(dx)\nu(dy) \\ &= \frac{1}{\left(\sum_j \operatorname{cap}\left(\tilde{K}_j\right)\right)^2} \left(\sum_j \operatorname{cap}\left(\tilde{K}_j\right) + \sum_j \int \nu_j(dx) \sum_{j' \neq j} \int g(x,y) \nu_{j'}(dy) \right) \\ &= \frac{1}{\sum_j \operatorname{cap}\left(\tilde{K}_j\right)} \left(1 + \sum_j \int \frac{\nu_j(dx)}{\sum_{j''} \operatorname{cap}\left(\tilde{K}_{j''}\right)} \sum_{j' \neq j} \int g(x,y)\nu_{j'}(dy) \right). \end{aligned}$$

Observe now that for fixed j,

$$(2.14)\qquad \sum_{j' \neq j} \int g(x,y)\nu_{j'}(dy) \le 3^d - 1 + \sum_{j'}{}' g(x,y)\nu_{j'}(dy),$$

where $\sum'_{j'}$ stands for the sum over indices $j' \neq j$ such that $C_{mj'}$ is not a neighbor C_{mj}. This and (2.1) imply that when $x \in \operatorname{Supp} \nu_j$ and $y \in \operatorname{Supp} \nu_{j'}$, then $|x-y| \ge 1/2L$. Therefore the left member of (2.14) is smaller than

$$3^d - 1 + \sum_{j'}{}' G\left(\frac{1}{2l}\right)\tilde{\delta}_1 \le 3^d - 1 + L^d G\left(\frac{1}{2L}\right)\tilde{\delta}_1 = 3^d.$$

Inserting this inequality in (2.13) we see that

$$(2.15)\qquad \operatorname{cap}_m^{-1} \le \left(\sum_j \operatorname{cap}\left(\tilde{K}_j\right) \right)^{-1} (1 + 3^d).$$

On the other hand we know from (2.6) that

$$\operatorname{cap}\left(\tilde{K}_j\right) = \operatorname{cap}\left(\frac{1}{L} K_{m \cdot j}\right) \wedge \tilde{\tilde{\delta}}_1 \geq \frac{1}{c_7}\left(\operatorname{cap}_{mj} \wedge \delta_1\right).$$

Combining with (2.15) we find

$$\operatorname{cap}_m \geq \frac{1}{(3^d + 1)c_7} \sum_j \left(\operatorname{cap}_{mj} \wedge \delta_1\right),$$

from which our claim (2.10) immediately follows. □

We are now going to exploit (2.4), (2.6) and (2.10). To this end we consider the auxiliary space $\Sigma = (\{0, \dots, L-1\}^d)^l$ endowed with the uniform probability Q. We denote by $X_1, \dots, X_l$ the canonical $\{0, \dots, L-1\}^d$-valued coordinates on this space and denote by (G_k), $k \geq 0$, the filtration on Σ defined via

$$G_0 = \{\phi, \Sigma\}, \qquad G_k = \sigma(X_1, \dots, X_{k \wedge l}) \quad \text{for } k \geq 1.$$

We now view $(0, X_1, \dots, X_k)$ for $1 \leq k \leq l$ as a random index in I_k and consider the stochastic process

$$Y_k = \operatorname{cap}_{(0, X_1, \dots, X_k)} \quad \text{for } 1 \leq k \leq l. \tag{2.16}$$

Observe that Y_k is G_k-adapted, and our basic relations (2.4), (2.6) and (2.10) now imply that

$$Y_k \leq c_6, \qquad 1 \leq k \leq l, \tag{2.17}$$

$$Y_{k+1} \leq c_7 Y_k, \qquad 1 \leq k \leq l, \tag{2.18}$$

$$Y_k \geq c_8 E\left[Y_{k+1} \wedge \delta_1 | G_k\right], \qquad 1 \leq k < l. \tag{2.19}$$

We now introduce $\delta_0(d, L)$ via

$$\delta_0 = \tfrac{1}{2}\delta_1 c_7^{-1}\left(< \delta_1 c_7^{-1} < \delta_1\right). \tag{2.20}$$

We shall now construct a certain supermartingale based on Y_k, $1 \leq k \leq l$, which will produce our exponential estimates (2.3). We need to consider the excursions of $Y_.$ below δ_0 and above $\delta_1 c_7^{-1}$. Accordingly we introduce two sequences of G_k-stopping times τ_i and σ_i, $i \geq 1$, as follows:

$$\begin{aligned}
\tau_1 &= \inf\{k \geq 1, Y_k \leq \delta_0\} \wedge l, \\
\sigma_1 &= \inf\left\{k \geq \tau_1, Y_k \geq \delta_1 c_7^{-1}\right\} \wedge l, \quad \text{and for } i \geq 2, \\
\tau_i &= \inf\{k \geq \sigma_{i-1}, Y_k \leq \delta_0\} \wedge l, \\
\sigma_i &= \inf\left\{k \geq \tau_i, Y_k \geq \delta_1 c_7^{-1}\right\} \wedge l.
\end{aligned} \tag{2.21}$$

Of course we have

$$1 \leq \tau_1 \leq \sigma_1 \leq \cdots \leq \tau_i \leq \sigma_i \leq \cdots \leq l$$

and these inequalities, except maybe for the first one, are strict when the left member is less than l. The key (elementary) observation is that in fact (2.24) holds. The proof follows a classical supermartingale argument, suggested by (2.19):

$$U_n^{(i)} = c_8^{n \wedge (\sigma_i - \tau_i)} Y_{(\tau_i + n) \wedge \sigma_i}, \; n \geq 0, \quad \text{is a } \big(G_{\tau_i + n}\big)_{n \geq 0}\text{-supermartingale.} \tag{2.22}$$

Indeed, $U_n^{(i)}$ is clearly G_{τ_i+n}-adapted and

$$\begin{aligned} &E\Big[c_8^{(n+1) \wedge (\sigma_i - \tau_i)} Y_{(\tau_i + n + 1) \wedge \sigma_i} \mid G_{\tau_i + n}\Big] \\ &\qquad = E\Big[c_8^{(n+1)} Y_{\tau_i + n + 1}, \tau_i + n < \sigma_i \mid G_{\tau_i+n}\Big] + 1\{\tau_i + n \geq \sigma_i\} U_n^{(i)}. \end{aligned} \tag{2.23}$$

As a consequence of (2.18) and (2.21), on $\{\tau_i + n < \sigma_i\}$,

$$Y_{\tau_i + n + 1} \leq Y_{\tau_i + n} c_7 \leq \frac{\delta_1}{c_7} c_7 = \delta_1.$$

Therefore using (2.19) and the fact that on $\{\tau_i = k\}$, with $k + n < l$,

$$E\Big[Y_{\tau_i + n + 1} \wedge \delta_1, \tau_i + n < \sigma_i \mid G_{\tau_i+n}\Big] = 1\{ k + n < \sigma_i\} E\big[Y_{k+n+1} \wedge \delta_1 \mid G_{k+n}\big],$$

we have

$$\begin{aligned} &E\Big[c_8^{n+1} Y_{\tau_i+n+1}, \tau_i + n < \sigma_i \mid G_{\tau_i+n}\Big] \\ &\quad = c_8^n E\Big[c_8\big(Y_{\tau_i+n+1} \wedge \delta_1\big), \tau_i + n < \sigma_i \mid G_{\tau_i+n}\Big] \\ &\quad \leq c_8^n Y_{\tau_i+n} 1\{\tau_i + n < \sigma_i\}. \end{aligned}$$

Inserting this inequality in (2.23), we now see that

$$E\Big[U_{n+1}^{(i)} \mid G_{\tau_i+n}\Big] \leq U_n^{(i)}, \qquad n \geq 0,$$

which shows (2.22).

If we now choose $n = l$, it follows from (2.22) that

$$E\Big[c_8^{\sigma_i - \tau_i} Y_{\sigma_i} \mid G_{\tau_i}\Big] \leq Y_{\tau_i} \quad \text{for any } i \geq 1. \tag{2.24}$$

Using the convention $Y_{\sigma_i}/Y_{\tau_i} = 0$ on $\{Y_{\tau_i} = 0\}$, it follows by repeated use of (2.24) that

$$E\left[\prod_{i \geq 1} c_8^{\sigma_i - \tau_i} \frac{Y_{\sigma_i}}{Y_{\tau_i}} \,\middle|\, G_{\tau_1}\right] \leq 1 \tag{2.25}$$

(the "infinite product" can of course be reduced to $i \leq l$). Now on the set $\bigcap_{i \geq 1} \{Y_{\tau_i} > 0\} = \{Y_l > 0\}$, we consider

$$H \overset{\text{def}}{=} \prod_{i \geq 1} \frac{Y_{\sigma_i}}{Y_{\tau_i}} = \frac{1}{Y_{\tau_1}} \frac{Y_{\sigma_1}}{Y_{\tau_2}} \cdots \frac{Y_{\sigma_{M-1}}}{Y_{\tau_M}} Y_{\sigma_M},$$

provided $M = \inf\{i \geq 1, \tau_i \geq l\}$. If $M = 1$, then clearly $H = 1 = Y_l / Y_{\tau_1}$. Otherwise, $M \geq 2$ and either $\sigma_{M-1} < l \leq \tau_M$, in which case

$$Y_{\sigma_{M-1}} \geq \delta_1 c_7^{-1}, \qquad Y_l = Y_{\tau_M} \leq c_6, \qquad Y_{\sigma_i} \geq Y_{\tau_{i+1}} \quad \text{for } 1 \leq i < M - 1,$$

[see (2.21)] and therefore

$$H \geq \frac{Y_l}{Y_{\tau_1}} \frac{\delta_1}{c_6 \cdot c_7},$$

or $\tau_{M-1} < l \leq \sigma_{M-1}$, in which case

$$H \geq \frac{Y_l}{Y_{\tau_1}}.$$

It thus follows from (2.25) that

$$\text{(2.26)} \quad c_9 E\left[c_8^{\sum_i (\sigma_i - \tau_i)} \frac{Y_l}{Y_{\tau_1}}, Y_l > 0 \mid G_{\tau_1} \right] \leq 1 \quad \text{with } c_9 = \left(\frac{\delta_1}{c_6 \cdot c_7} \right) \wedge 1.$$

As a result (recall that $Y_l = 0$ on $\{Y_{\tau_1} = 0\}$),

$$\text{(2.27)} \qquad E\left[c_8^{\sum_i (\sigma_i - \tau_i)} Y_l \right] \leq c_9^{-1} E\left[Y_{\tau_1} \right] \leq c_6 c_9^{-1}.$$

We now assume that $\delta < \delta_0(d, L)$ [see (1.15)]. Observe that for

$$k \in [1, \tau_1 - 1] \cup \bigcup_{i \geq 2} [\sigma_{i-1}, \tau_i - 1],$$

$Y_k > \delta_0$ and

$$\tau_1 - 1 + \sum_{i \geq 2} (\tau_i - \sigma_{i-1}) + 1 + \sum_{i \geq 1} (\sigma_i - \tau_i) = l.$$

On the other hand, if we define the event $A = \{(0, X_1, \dots, X_l)$ is a rarefaction index$\}$, then by definition, on A,

$$\delta l \geq \sum_1^l Y_k > \delta_0 \left(\tau_1 - 1 + \sum_{i \geq 2} (\tau_i - \sigma_{i-1}) \right),$$

so that

$$\text{(2.28)} \qquad 1 + \sum_{i > 1} \sigma_i - \tau_i \geq \left(1 - \frac{\delta}{\delta_0} \right) l \quad \text{on } A.$$

This and (2.27) now imply that

$$E[Y_l, A] \leq c_6 c_9^{-1} c_8^{1 - (1 - \delta/\delta_0) l},$$

which is exactly (2.3), provided we define

$$\text{(2.29)} \qquad c_4 = c_6 c_8 c_9^{-1} \quad \text{and} \quad c_5 = \log c_8. \qquad \square$$

REMARK 2.3. (i) Observe that the fact that m is a rarefaction index is not very sensitive to the individual value of cap_m [see (1.15)]. However (2.3) provides a global constraint on the cap_m, for rarefaction subboxes of a given box of size 1.

(ii) It is possible at this point to directly derive estimates on the volume of rarefaction boxes which receive some point of the cloud. Indeed (2.3) is easily seen to imply controls on

$$\frac{1}{L^{d \cdot n_\gamma}} \sum_{\substack{m \succ q,\ m \in I_{n_\gamma} \\ m \text{ rarefaction index}}} |K_m|.$$

On the other hand, it is easy to argue that when ε is small and C_m is a rarefaction box receiving a point of ω,

$$|L^{-n_\gamma} K_m| / |C_m| \geq \text{const}(d, a, L)\, \varepsilon^{d(1-\gamma)}.$$

The combination of the two estimates now shows that the volume of rarefaction boxes within a given box of size 1 which receive some point of Ω is smaller than const ε^{μ}, with $\mu > 0$, at least when γ is chosen close enough to 1. However we can sharpen these estimates with the notion of bad boxes which we now define.

We consider a new parameter β such that

$$1 > \beta > \gamma > \alpha > 0. \tag{2.30}$$

As in (1.12) we introduce for $\varepsilon \in (0,1)$,

$$n_\beta(\varepsilon) = \left[\beta \frac{\log(1/\varepsilon)}{\log L}\right] \quad \text{so that} \quad L^{-n_\beta(\varepsilon)-1} < \varepsilon^{\beta} \leq L^{-n_\beta(\varepsilon)}. \tag{2.31}$$

We now define, for given d, ε, ω, $W(\cdot)$, L, α, γ, δ and β, the bad boxes as the boxes C_m of size L^{-n_β} (i.e., $m \in I_{n_\beta}$) such that

$$C_m \subseteq R \text{ and } C_m \text{ receives a point of } \omega. \tag{2.32}$$

We also write

$$B = \bigcup_{\substack{m \in I_{n_\beta} \\ C_m \text{ bad box}}} C_m. \tag{2.33}$$

Observe that our definitions are such that

$$\begin{aligned} &D \cap B = \phi \quad \text{and} \quad \omega\big(\mathbb{R}^d \setminus (D \cup B)\big) = 0 \\ &\qquad (\omega \text{ has no point in the complement of } D \cup B). \end{aligned} \tag{2.34}$$

Our main objective in this section is to derive uniform estimates showing that the restriction of B to boxes of size 1 has small volume. Observe that in the special case of dimension 1, it follows from (1.16) that

$$B = \phi \text{ as soon as } \delta < \text{cap}\{0\}. \tag{2.35}$$

We are now ready to state our second main capacity estimate. We restrict ourselves to the case $d \geq 2$, thanks to (2.35).

PROPOSITION 2.4. *Assume $d \geq 2$. Then*

$$\limsup_{\varepsilon \to 0} \varepsilon^{(d-2)(1-\beta)} \sup_{\omega,\, m \in I_{n_\gamma}} \frac{\text{cap}\big(L^{n_\gamma}(\overline{B \cap C_m})\big)}{\text{cap}_m} < \infty. \tag{2.36}$$

(*If* $\text{cap}_m = 0$ *and therefore* $B \cap C_m = \phi$, *the above fraction is understood as equal to* 0.)

PROOF. We assume that ε is small enough so that

$$4\alpha\varepsilon < L^{-n_\beta(\varepsilon)}. \tag{2.37}$$

We consider m_0, an index of I_{n_γ}, such that $B \cap C_m \neq \phi$. Consider P_{m_0}, some maximal collection of indices $m \in I_{n_\beta}$, $m \succ m_0$, of bad subboxes of C_{m_0}, such that any two distinct subboxes with indices in P_{m_0} are not neighbors. We consider

$$B_{m_0} = \overline{B \cap C_{m_0}} \quad \text{and} \quad B'_{m_0} = \bigcup_{m \in P_{m_0}} \overline{C}_m. \tag{2.38}$$

The maximality of P_{m_0} implies that B_{m_0} is included in the union of 3^d translates of B'_{m_0} so that

$$\operatorname{cap}\left(L^{n_\gamma} B_{m_0}\right) \leq 3^d \operatorname{cap}\left(L^{n_\gamma} B'_{m_0}\right). \tag{2.39}$$

We let μ denote the equilibrium measure of $L^{n_\gamma} B'_{m_0}$ and define for $m \in P_{m_0}$,

$$\mu_m = 1_{L^{n_\gamma}\overline{C}_m} \cdot \mu, \tag{2.40}$$

so that $\mu = \sum_{m \in P_{m_0}} \mu_m$. Clearly, we have

$$\int g(x, y)\mu_m(dy) \leq 1 \quad \text{on } L^{n_\gamma}\overline{C}_m, \text{ when } m \in P_{m_0},$$

so that for $m \in P_{m_0}$,

$$\begin{aligned} \mu_m(1) &= \mu_m\left(L^{n_\gamma}\overline{C}_m\right) \leq \operatorname{cap}\left(L^{n_\gamma}\overline{C}_m\right) = \operatorname{cap}\left(L^{n_\gamma - n_\beta}[0,1]^d\right) \\ &= L^{-(d-2)(n_\gamma - n_\beta)}\operatorname{cap}\left([0,1]^d\right) \quad (\text{when } d \geq 3) \\ &\leq \frac{C(L)}{n_\beta - n_\gamma} \quad (\text{when } d = 2 \text{ and } \varepsilon \text{ small enough}). \end{aligned} \tag{2.41}$$

For each $m \in P_{m_0}$, we let x_m be a point of the cloud ω falling in C_m. We define

$$\nu = \sum_{m \in P_{m_0}} \mu_m(1)\bar{e}_m, \tag{2.42}$$

where $\bar{e}_m$ stands for the normalized equilibrium measure of $L^{n_\gamma}\overline{B}(x_m, a\varepsilon)$ [i.e., the normalized surface measure on $L^{n_\gamma}\partial\overline{B}(x_m, \alpha\varepsilon)$, when $d \geq 3$]. The measure ν is concentrated on $L^{n_\gamma}(\bigcup_{m \in P_{m_0}}\overline{B}(x_m, \alpha\varepsilon)) \subseteq K_{m_0}$.

On the other hand, when $x \in L^{n_\gamma}\overline{B}(x_m, a\varepsilon)$, with $m \in P_{m_0}$, we have

$$\begin{aligned} &\int g(x, y)\nu(dy) \\ &\quad = \frac{\mu_m(1)}{\operatorname{cap}\left(L^{n_\gamma}\overline{B}(0, a\varepsilon)\right)} + \sum_{\substack{m' \neq m \\ m' \in P_{m_0}}} \mu_{m'}(1)\int g(x, y)\bar{e}_m(dy). \end{aligned} \tag{2.43}$$

Observe that when $x \in L^{n_\gamma}\overline{B}(x_m, a\varepsilon)$, $y \in L^{n_\gamma}\overline{B}(x_{m'}, a\varepsilon)$, where $m \neq m'$ are in P_{m_0}, then $x_m \in C_m$, $x_{m'} \in C_{m'}$, two nonneighboring boxes of size L^{-n_β}. If

y' is some point of $L^{n_\gamma}\overline{C}_{m'}$, we have, thanks to (2.37),

$$|x-y'| \le |x-y| + |y-y'| \le |x-y| + 2\left(\sqrt{d} + \tfrac{1}{4}\right)|x-y| = \left(\tfrac{3}{2} + 2\sqrt{d}\right)|x-y|.$$

It follows that we have $c(d) > 0$, such that for x, y, y' as above,

$$g(x,y) \le c(d)\, g(x,y'). \tag{2.44}$$

Inserting in (2.43) we find that for $x \in L^{n_\gamma}\overline{B}(x_m, a\varepsilon)$, $m \in \boldsymbol{P}_{m_0}$,

$$\begin{aligned} \int g(x,y)\nu(dy) &\le \frac{\mu_m(1)}{\operatorname{cap}\left(L^{n_\gamma}\overline{B}(0,a\varepsilon)\right)} \\ &\quad + c(d)\sum_{m'\ne m}\int g(x,y')\mu_{m'}(dy') \\ &\le \frac{\mu_m(1)}{\operatorname{cap}\left(L^{n_\gamma}\overline{B}(0,a\varepsilon)\right)} + c(d), \end{aligned} \tag{2.45}$$

since $\mu = \sum_{m\in \boldsymbol{P}_{m_0}} \mu_m$ is the equilibrium measure of $L^{n_\gamma} B'_{m_0}$. We can now combine (2.45) with the upper bound on $\mu_m(1)$ given in (2.41) and thus obtain an upper bound on $\int g(x,y)\nu(dy)$ for $x \in \operatorname{Supp}\nu \subseteq K_{m_0}$. It implies that

$$\begin{aligned} \operatorname{cap}_{m_0}\left(\frac{\operatorname{cap}\left(L^{n_\gamma - n_\beta}[0,1]^d\right)}{\operatorname{cap}\left(L^{n_\gamma}\overline{B}(0,a\varepsilon)\right)} + c(d)\right) \ge \nu(1) &= \mu(1) \\ &= \operatorname{cap}\left(L^{n_\gamma} B'_{m_0}\right) \\ &\ge 3^{-d}\operatorname{cap}\left(L^{n_\gamma} B_{m_0}\right) \\ &\qquad [\text{in view of } (2.39)]. \end{aligned} \tag{2.46}$$

Now for small ε, the multiplicative factor to the left of (2.46) is smaller than $c(d,a,L)\varepsilon^{-(d-2)(1-\beta)}$, when $d \ge 3$, and $2((1-\gamma)/(\beta-\gamma)) + c(d)$, when $d = 2$. Our claim (2.36) follows. □

We shall now state our main estimate on the volume of bad boxes. We restrict ourselves to the case of dimension $d \ge 2$, thanks to (2.35), and define

$$\kappa_0 = \begin{cases} \left(1-\dfrac{\delta}{\delta_0}\right)(\gamma-\alpha)\left(2-\dfrac{\log(3^d+1)}{\log L}\right) - (d-2)(1-\beta), \\ \hfill \text{when } d \ge 3, \\ \left(1-\dfrac{\delta}{\delta_0}\right)(\gamma-\alpha)\left(2-\dfrac{\log[10c(d=2)\log L]}{\log L}\right), \\ \hfill \text{when } d = 2 \end{cases} \tag{2.47}$$

[see (2.5) for the notation].

THEOREM 2.5. *Assume that $d \geq 2$ and that L is large enough so that* (2.9) *holds and $\delta < \delta_0(d, L)$. Then*

$$\limsup_{\varepsilon \to 0} \sup_{q \in \mathbb{Z}^d,\, \omega} \varepsilon^{-\kappa_0} |B \cap C_q| < \infty. \tag{2.48}$$

PROOF. Observe that

$$K(d) \overset{\text{def}}{=} \sup \left\{ \frac{|K|}{\operatorname{cap}(K)},\, K \text{ compact},\, K \subset \left[-\frac{1}{4}\frac{5}{4} \right]^d \right\} < \infty \tag{2.49}$$

(see [16], page 58, where the argument is given for the $d = 2$ case, but can easily be extended to the $d \geq 3$ situation). As a result when $q \in \mathbb{Z}^d$ and $\omega \in \Omega$, we find

$$\begin{aligned} |B \cap C_q| &\leq \frac{1}{L^{d n_\gamma}} \sum_{\substack{m \succ q,\, m \in I_{n_\gamma} \\ m \text{ rarefaction index}}} |L^{n_\gamma}(B \cap C_m)| \\ &\leq K(d) \sup_{\omega,\, m \in I_{m_\gamma}} \frac{\operatorname{cap}\big(L^{n_\gamma}(\overline{B \cap C_m})\big)}{\operatorname{cap}_m} \frac{1}{L^{d n_\gamma}} \sum_{\substack{m \succ q \\ m \text{ rarefaction index}}} \operatorname{cap}_m. \end{aligned} \tag{2.50}$$

Our claim now follows from (2.3), (2.36) and the explicit value of the constants involved in (2.3). □

We shall conclude this section with the following remarks.

REMARK 2.6. (i) If β is chosen close enough to 1 and (2.9) holds, then $\kappa_0 > 0$. As a consequence $|B \cap C_q|$ is small uniformly in q and ω for small ε, and Theorem 2.5 takes care of the estimates mentioned in (I.1c) of the Introduction.

(ii) We have introduced the scale ε^β in order to define bad boxes. The main reason why we did not simply define bad boxes as rarefaction boxes receiving a point of ω is that the constant κ_0 of (2.47) is worse if one uses this latter definition. If one follows the method explained in Remark 2.3, one obtains a term $-d(1-\gamma)$ instead of $-(d-2)(1-\beta)$ in (2.47).

(iii) By our very construction, when $q \in \mathbb{Z}^d$, the set $D \cap C_q$ can have at most $2^{L^{d n_\gamma}} \leq 2^{\varepsilon^{-d\gamma}}$ possible shapes. A similar estimate holds of course for $B \cap C_q$, with γ being replaced by β. In fact when $\kappa_0 > 0$ [in (2.47)], one can do better.

The classical Cramér estimates on the binomial distribution show that there are at most $2^{NH(p)}$ subsets of a given set of size N with less than pN elements, when $p \leq 1/2$, where $H(p) = -p \log_2 p - (1-p)\log_2(1-p)$ (logarithm to the base 2). It follows from Theorem 2.5 that when $0 < \kappa < \kappa' < \kappa_0$ for small ε, $B \cap C_q$ for any $q \in \mathbb{Z}^d$ can take no more than $2^{L^{d n_\beta} H(\varepsilon^{\kappa'})} \leq 2^{\varepsilon^{-d\beta + \kappa}}$ distinct shapes. In other words the "combinatorial complexity" of the coarse grained picture made of $D \cap C_q$, $B \cap C_q$ is no bigger than $2^{\varepsilon^{-d\gamma} + \varepsilon^{-d\beta+\kappa}}$

for small ε. The natural complexity of the original cloud of obstacles in a box of size 1 can be viewed as $2^{\varepsilon^{-d}}$. We thus have a reduction of combinatorial complexity due to the coarse graining [see (I.1b)].

3. One example. We shall now discuss some first applications of the results of the previous two sections to the control of fluctuations of the principal Dirichlet eigenvalue of $-\Delta/2 + \sum_i W(\cdot - x_i)$ in a large box $(-t, t)^d$, $d \geq 2$, when the points x_i form a Poisson cloud of constant intensity $\nu > 0$. Intuitively, this principal eigenvalue is very much influenced by the presence in the box $(-t, t)^d$ of certain "big holes" within the cloud. The coarse graining method we have developed in Sections 1 and 2 enable us to study these big holes. Our main goal will be Proposition 3.2.

It is convenient to pick $(\log t)^{1/d}$ as unit scale. After rescaling, we are led to study, with the notation of (1.4),

$$\lambda_t(\omega) \stackrel{\text{def}}{=} \lambda^\varepsilon(T), \tag{3.1}$$
$$\text{with } T = \left(-t(\log t)^{-1/d}, t(\log t)^{-1/d}\right)^d \text{ and } \varepsilon = (\log t)^{-1/d},$$

and ω is now a Poisson point measure on $\mathbb{R}^d$, $d \geq 2$, with intensity $\nu\varepsilon^{-d} = \nu(\log t)$. We shall denote by $\mathbb{P}_\varepsilon$ its law on Ω. We now choose the parameters $0 < \alpha < \gamma < \beta < 1$, L such that (2.9) holds, $\delta < \delta_0(d, L)$ and $\kappa_0 > 0$ in (2.47). As truncation value M (in Theorems 1.2 and 1.4), it is convenient to choose

$$M = 2c(d, \nu) \stackrel{\text{def}}{=} 2\lambda_d \left(\frac{d}{\nu\omega_d}\right)^{-2/d}, \tag{3.2}$$

where $c(d, \nu)$ refers to (I.2), λ_d is the principal Dirichlet eigenvalue of $-\Delta/2$ in $B(0, 1)$ and ω_d is the volume of $B(0, 1)$.

We shall also define clearing boxes, (see discussion before Proposition 1.3) by picking a fixed value $r \in (0, r_0(d, M))$. From the construction of $r_0(d, M)$ [see (1.41) and the end of the proof of Theorem 1.4] this implies

$$c_2(d)/r^2 > 4M. \tag{3.3}$$

In view of Proposition 1.3, this ensures that when t is large, uniformly in ω, $\lambda(A^c) > 4M$. The neighborhood O of the set A [see (1.33) and (1.34)] is now determined by picking $R = R(t, d, \nu) \geq 1$ as the smallest integer for which

$$c_3(d)\left[\frac{R}{5r}\right] \geq 3 \log\log t. \tag{3.4}$$

We shall first derive some controls on the size of connected components of O which meet T. To this end we consider the event:

$$\begin{aligned}&\{\omega \in \Omega, \text{ all connected components of } O \text{ intersecting}\\ &C = T \text{ are contained in some } q + (0, [\gamma_1 \log\log t])^d,\\ &q \in \mathbb{Z}^d\},\end{aligned} \tag{3.5}$$

where $\gamma_1(d, \nu)$ is defined in (3.14) below.

PROPOSITION 3.1. *For large* t,

$$\mathbb{P}_\varepsilon\big[C_q \subseteq A\big] \le \exp\Big\{-\frac{\nu}{2}(\log t)\, r^d\Big\} \quad \text{for any } q \in \mathbb{Z}^d, \tag{3.6}$$

$$\mathbb{P}_\varepsilon[C] \ge 1 - t^{-d}. \tag{3.7}$$

PROOF. We begin with (3.6). We know that $D \cap B = \phi$ and $\omega(\mathbb{R}^d \setminus (D \cup B)) = 0$. Moreover for each box C_q of size 1, the random set $C_q \setminus (D \cup B)$ has no more than $2^{2\varepsilon^{-d\beta}}$ possible shapes (a very rough upper bound); see our discussion of Remark 2.6(iii). Using Theorem 2.5, we find for large t and $\kappa < \kappa_0$,

$$\begin{aligned} \mathbb{P}_\varepsilon\big[C_q \subseteq A\big] &= \mathbb{P}_\varepsilon\big[|C_q \setminus D| > r^d\big] \le \mathbb{P}_\varepsilon\big[\big|C_q/(D \cup B)\big| > r^d - \varepsilon^\kappa\big] \\ &\le \exp\big\{2\varepsilon^{-d\beta}\log 2 - \nu\varepsilon^{-d}(r^d - \varepsilon^\kappa)\big\} \end{aligned}. \tag{3.8}$$

Our claim (3.6) follows. We now shall prove (3.7). To this end we define

$$M_0 = 2\left(\left[\frac{4d}{\nu r^d}\right] + 1\right), \qquad \mu = \left(\frac{\nu r^d}{4d}\right) \wedge \frac{1}{2}, \tag{3.9}$$

as well as

(3.10) K_t, the collection of blocks of the form $q + [0,[t^\mu]]^d$, $q \in \mathbb{Z}^d$, which intersect T

and the event

$$\tilde{C} = \{\omega \in \Omega, \text{ all blocks of } K_t \text{ contain at most } M_0 \text{ clearing boxes}\}. \tag{3.11}$$

Then for large t,

$$\begin{aligned} \mathbb{P}_\varepsilon[\tilde{C}^c] &\le t^d\, t^{d\mu M_0}\,\mathbb{P}_\varepsilon[C_0 \subseteq A]^{M_0} \\ &\le \exp\Big\{\varepsilon^{-d}\Big(d - M_0\Big(\frac{\nu}{2} r^d - d\mu\Big)\Big)\Big\} \\ &\le \exp\Big\{\varepsilon^{-d}\Big(d - M_0\frac{\nu r^d}{4}\Big)\Big\} \le t^{-d}. \end{aligned} \tag{3.12}$$

Our claim (3.7) will now follow from the observation

$$\tilde{C} \subseteq C \quad \text{when } t \text{ is large.} \tag{3.13}$$

Indeed consider some connected component V of O intersecting T. Component V contains a clearing box C_q which is at most within $\|\ \|$-distance R from T. Denote by B_q a block of K_t with center in C_q and by W_q the union of open R-neighborhoods in $\|\ \|$-distance of clearing boxes included in B_q. When t is large, then

$$O \cap \left(q + \left(-\frac{t^\mu}{4}\frac{t^\mu}{4}\right)^d\right) \subseteq W_q.$$

On the other hand on the event $\tilde{C}$ the projection of W_q on each coordinate axis has measure less than $(2R+1)M_0$. Observe that

$$V \cap \left(q + \left(-(t^\mu/4)(t^\mu/4)\right)^d\right)$$

contains q. Thus its connected component containing q has diameter smaller than

$$(2R+1)M_0 < \frac{t^\mu}{4} \quad (t \text{ is large}).$$

It follows that V has diameter smaller than $(2R+1)M_0$, and our claim (3.13) follows once we choose $\gamma_1(d,\nu)$ to be the smallest integer so that

$$[\gamma_1 \log\log t] > (2R(t,d,\nu)+1)M_0 \quad \text{for large } t. \tag{3.14}$$ □

Define

$$\begin{aligned}&\mathbf{B}_t, \text{ the collection of blocks of the form } q + (0, [\gamma_1 \log\log t])^d,\\ &q \in \mathbb{Z}^d, \text{ which intersect } T.\end{aligned} \tag{3.15}$$

The next proposition shows that λ_t can roughly be viewed as a minimum value of the not too dependent random variables $\lambda^\varepsilon(B \cap T)$, $B \in \mathbf{B}_t$ [see (3.15) for the notation]. This will conclude our first applications of the results of Sections 1 and 2 to the study of fluctuations of $\lambda^\varepsilon(T)$. Let us mention that we could adjust the size of R [i.e., replace 3 by a larger number in (3.4)] to produce an arbitrary correction $(\log t)^{-k}$, instead of $(\log t)^{-2}$, in (3.16) below. However, (3.16) will be sufficient for our later use in [15].

PROPOSITION 3.2. *When t is large,*

$$\mathbb{P}_\varepsilon\left[\inf_{B\in \mathbf{B}_t} \lambda^\varepsilon(B\cap T) \geq \lambda_t \geq \inf_{B\in \mathbf{B}_t} \lambda^\varepsilon(B\cap T) - (\log t)^{-2}\right] \geq 1 - 2t^{-d}, \tag{3.16}$$

$$\mathbb{P}_\varepsilon\left[c(d,\nu) + \gamma_2(d,\nu,W)(\log t)^{-1/d} \geq \inf_{B\in \mathbf{B}_t} \lambda^\varepsilon(B\cap T)\right] \geq 1 - t^{-d}. \tag{3.17}$$

PROOF. The inequality $\lambda_t \leq \inf_{B\in K_t} \lambda^\varepsilon(B\cap T)$ is of course automatic. On the other hand, Theorem 1.4 implies that for large t and any ω,

$$\begin{aligned}\lambda_t \wedge M &\geq \lambda^\varepsilon(T\cap O) \wedge M - 2\exp\left\{-c_3\left[\frac{R}{5r}\right]\right\}\\ &\geq \lambda^\varepsilon(T\cap O) \wedge M - 2(\log t)^{-3},\end{aligned} \tag{3.18}$$

where we used (3.4) in the last step, and any number greater than 1 could be used in the place of 2. However, on the event C of (3.5),

$$\lambda^\varepsilon(T\cap O) \wedge M \geq \inf_{B\in \mathbf{B}_t} \lambda^\varepsilon(B\cap T) \wedge M. \tag{3.19}$$

Finally observe that as soon as T contains a ball of radius $(d/\nu\omega_d)^{1/d} - a\varepsilon \overset{\text{def}}{=} R_1$ (t large), receiving no point of the cloud, then

$$\inf_{B\in \mathbf{B}_t} \lambda^\varepsilon(B\cap T) \leq c(d,\nu) + \gamma_2(d,\nu,a(W))\varepsilon \leq 2c(d,\nu) = M.$$

Slicing T in boxes of sizes $2R_1$ and using standard estimates, we see that the probability of occurrence of such a spherical hole of the cloud within T is greater than

$$\begin{aligned}
&1 - \left(1 - \exp\left\{-\nu\varepsilon^{-d}\omega_d R_1^d\right\}\right)^{\mathrm{const}(d,\nu)t^d(\log t)^{-1}} \\
(3.20)\qquad &\geq 1 - \exp\left\{-\mathrm{const}'(d,\nu)t^d(\log t)^{-1}\exp\left\{-\nu\varepsilon^{-d}\omega_d R_1^d\right\}\right\} \quad (t \text{ large}) \\
&\geq 1 - \exp\left\{-\mathrm{const}'(d,\nu)(\log t)^{-1}\exp\left(\gamma_3\varepsilon^{-(d-1)}\right)\right\}.
\end{aligned}$$

Combining this, (3.7) and (3.19), we see our claims (3.16) and (3.17) follow. □

With the help of Theorem 1.2, which we did not use so far, and the Faber–Krahn inequality we shall derive lower bounds for $\inf_{B_t}\lambda^{\varepsilon}(B \cup T)$ in terms of $\sup_{B_t}|B \cup T \setminus D|$, in [15], Section 2. Thanks to Theorem 2.5, this latter quantity can be controlled with $\sup_{B_t}|B \setminus (D \cup B)|$, which is now amenable to probabilistic estimates. This is the rough outline of the strategy we use in Section 2 of [15] to obtain confidence intervals on λ_t.

APPENDIX

The object of this Appendix is to collect in a rather self-contained way the results which were used in Section 1 to derive eigenvalue estimates. We present here streamlined and improved versions of arguments developed in [12, 13].

Throughout the sequel, $V(\cdot)$ is a nonnegative locally bounded measurable function on $\mathbb{R}^d$; this generality will be sufficient for us, and U will be a nonempty open subset of $\mathbb{R}^d$, $d \geq 1$. We denote by $R_t^{U,V}$ the semigroup defined as in (1.3), with V_ε replaced by V. When no confusion arises, we shall simply write R_t. We begin with the following proposition.

PROPOSITION A.1. *There exists a constant $c(d) \in (1,\infty)$ such that for any U, V as above,*

$$\begin{aligned}
(\mathrm{A.1})\qquad &\sup_x E_x\left[\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}, T_U > t\right]\left(= \|R_t^{U,V}\|_{L^\infty \to L^\infty}\right) \\
&\leq c(d)\left(1 + (\lambda_V(U)t)^{d/2}\right)\exp\{-\lambda_V(U)t\}.
\end{aligned}$$

There exist a constant $K(d) \in (1,\infty)$ such that for any $\rho \in (0,1)$ and U, V as above,

$$\begin{aligned}
(\mathrm{A.2})\qquad &\sup_x\left(1 + \int_0^\infty (1-\rho)\lambda_V(U)\exp((1-\rho)\lambda_V(U)s)R_s^{U,V}1(x)\,ds\right) \\
&\leq \frac{K(d)}{\rho^{d/2+1}}.
\end{aligned}$$

PROOF. It can be seen that (A.2) follows from (A.1) by integration. As for the proof of (A.1), using scaling [see (1.4)], we have

$$\lambda_{V_\alpha}(\alpha U) = \alpha^{-2}\lambda_V(U), \qquad \alpha > 0, \quad \text{provided} \quad V_\alpha(\cdot) = \alpha^{-2} V\left(\frac{\cdot}{\alpha}\right). \tag{A.3}$$

Observe that, picking $c(d) > 1$, (A.1) holds automatically when $\lambda_V(U) = 0$. If we then can show (A.1) when $\lambda_V(U) = 1$, (A.1) will hold for a general U, V with $\lambda_V(U) > 0$, as well. Indeed one applies (A.1) to αU, $V_\alpha(\cdot)$, $\alpha^2 t$, with $\alpha = \lambda_V(U)^{1/2}$, and recovers (A.1) for U, V, t.

We are thus reduced to the case $\lambda_V(U) = 1$. Now for $t \geq 1$, $L > 0$, $x \in \mathbb{R}^d$,

$$\begin{aligned} R_t 1(x) &= R_t\big(1_{B(x,L)}\big)(x) + R_t\big(1_{B(x,L)^c}\big)(x) \\ &= \big(r(1,x,\cdot), R_{t-1}\big(1_{B(x,L)}\big)\big)_{L^2} + R_t\big(1_{B(x,L)^c}\big)(x), \end{aligned}$$

provided $r(s,x,y)$ stands for the kernel of R_s [which can be expressed with the help of the Brownian bridge and satisfies $r(s,x,y) \leq (2\pi s)^{-d/2}$]. We now have

$$\begin{aligned} \sup_x R_t 1(x) \leq{}& \| r(1,x,\cdot)\|_{L^2} \|1_{B(x,L)\cap U}\|_{L^2} \exp\big(-\lambda_V(U)(t-1)\big) \\ &+ P_0\big[Z_t \notin B(0,L)\big], \quad \text{with } \lambda_V(U) = 1. \end{aligned}$$

Choosing $L = 2t$, we find for $t \geq 1$,

$$\begin{aligned} \sup_x R_t 1(x) &\leq c_1(d) t^{d/2} e^{-t} + c_2(d) e^{-2t} \\ &= c_3(d)(1 + t^{d/2}) e^{-t}. \end{aligned}$$

Possibly after increasing c_3, this inequality holds for all $t \geq 0$, and this proves our claim. □

We shall now prove a lemma which is preparatory for the main result of this Appendix. We consider a stopping time S_1 for the canonical right continuous filtration on $C(\mathbb{R}_+, \mathbb{R}^d)$. Letting ϑ_t, $t \geq 0$, stand for the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$, we introduce the sequence of iterates of S_1:

$$S_0 = 0, \qquad S_1 \quad \text{and} \quad S_{k+1} = S_k + S_1 \circ \vartheta_{S_k} \leq \infty \quad \text{for } k \geq 1. \tag{A.4}$$

We of course have

$$0 = S_0 \leq S_1 \leq \cdots \leq S_k \leq \cdots \leq \infty.$$

LEMMA A.2. *Let U, V be as above. Assume $\lambda > 0$ and S_1 is such that*

$$\text{for } x \in \mathbb{R}^d, \qquad \lim_k \uparrow S_k \geq T_U, \qquad \mathbb{P}_x\text{-}a.s., \tag{A.5}$$

$$\alpha \overset{\text{def}}{=} \sup_x E_x\left[S_1 < T_U, \exp\left\{\lambda S_1 - \int_0^{S_1} V(Z_s)\, ds\right\}\right] < 1, \tag{A.6}$$

$$\beta \overset{\text{def}}{=} \sup_x \left(\int_0^\infty du\, \lambda e^{\lambda u} E_x\left[S_1 \wedge T_U > u, \exp\left\{-\int_0^u V(Z_s)\, ds\right\}\right]\right) < \infty. \tag{A.7}$$

Then

$$\lambda \leq \lambda_V(U) \quad \textit{and} \quad \sup_x \int_0^\infty \lambda e^{\lambda u} R_u^{U,V} 1(x)\, du \leq \frac{\beta}{1-\alpha}. \tag{A.8}$$

PROOF. Using routine arguments $\lambda_V(U) = \inf_{x \in U} -\lim_t (1/t) \log R_t^{U,V} 1(x)$. It suffices to prove the last statement of (A.8). Define $\tilde{S}_k = S_k \wedge T_U$, $k \geq 0$. We have

$$\tilde{S}_{k+1} = \tilde{S}_k + \tilde{S}_1 \circ \vartheta_{\tilde{S}_k}, \qquad k \geq 1 \quad \text{and} \quad \tilde{S}_0 = 0$$

($\tilde{S}_k$ is the sequence of iterates of $\tilde{S}_1$). Using (A.5), for $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_0^\infty \lambda e^{\lambda u} R_u^{U,V} 1(x)\, du &= E_x\left[\int_0^{T_U} \lambda \exp\left\{\lambda u - \int_0^u V(Z_s)\, ds\right\} du\right]\\ &= \sum_{k \geq 0} E_x\left[\tilde{S}_k < T_U, \int_{\tilde{S}_k}^{\tilde{S}_{k+1}} \lambda \exp\left\{\lambda u - \int_0^u V(Z_s)\, ds\right\} du\right],\end{aligned}$$

and using the strong Markov property, this sum is no greater than

$$\begin{aligned}&\sum_{k \geq 0} E_x\left[\tilde{S}_k < T_U, \exp\left\{\lambda \tilde{S}_k - \int_0^{\tilde{S}_k} V(Z_s)\, ds\right\}\right]\\ &\quad \times \sup_z E_z\left[\int_0^{\tilde{S}_1} \lambda \exp\left\{\lambda u - \int_0^u V(Z_s)\, ds\right\} du\right]\\ &\quad \leq \sum_{k \geq 0} a_k \beta,\end{aligned}$$

provided we define, for $k \geq 0$,

$$a_k = E_x\left[\tilde{S}_k < T_U, \exp\left\{\lambda \tilde{S}_k - \int_0^{\tilde{S}_k} V(Z_s)\, ds\right\}\right].$$

It follows from the strong Markov property that

$$\begin{aligned}a_{k+1} &\leq E_x\left[\tilde{S}_k < T_U, \exp\left\{\lambda \tilde{S}_k - \int_0^{\tilde{S}_k} V(Z_s)\, ds\right\}\right]\\ &\quad \times \sup_z E_z\left[\tilde{S}_1 < T_U, \exp\left\{\lambda \tilde{S}_1 - \int_0^{\tilde{S}_1} V(Z_s)\, ds\right\}\right]\\ &= a_k \alpha,\end{aligned}$$

where we used the fact that $\tilde{S}_1$ and S_1 coincide on $\{\tilde{S}_1 < T_U\} = \{S_1 < T_U\}$. By induction, we thus find

$$\alpha_k \leq \alpha^k, \qquad k \geq 0.$$

Summing over k, we have

$$\int_0^\infty \lambda e^{\lambda u} R_u^{U,V} 1(x)\, du \leq \frac{\beta}{1 - \alpha} < \infty.$$

This proves our claim. $\square$

We shall now apply the above lemma to the case where we consider two open subsets U_1, U_2 of $\mathbb{R}^d$, $d \geq 1$, $U_2 \neq \phi$, a stopping time $\tau \geq 0$ and a number $\lambda > 0$. The relevant stopping time S_1 is defined via

$$S_1 = \tau \circ \vartheta_{T_{U_1}} + T_{U_1}. \tag{A.9}$$

As above S_k, $k \geq 0$ stand for the sequence of iterates of S_1. We also define

$$
\begin{aligned}
&A = \sup_x \left(1 + \int_0^\infty \lambda e^{\lambda u} R_u^{U_1, V} 1(x)\, du\right),\\
&B = \sup_{x \notin U_1} \int_0^\infty \lambda e^{\lambda u} E_x\left[\tau \wedge T_{U_2} > u, \exp\left\{-\int_0^u V(Z_s)\, ds\right\}\right], \qquad \text{(A.10)}\\
&C = \sup_{x \notin U_1} E_x\left[\tau < T_{U_2}, \exp\left\{\lambda\tau - \int_0^\tau V(Z_s)\, ds\right\}\right]
\end{aligned}
$$

(when $U_1 = \mathbb{R}^d$, then $B = C = 0$; when $U_1 = \phi$, then $A = 1$).

THEOREM A.3. *Assume that*

$$\textit{for } x \in \mathbb{R}^d, \qquad \lim_k \uparrow S_k \geq T_{U_2}, \qquad P_x\text{-a.s.} \tag{A.11}$$

$$A < \infty, \qquad B < \infty \tag{A.12}$$

and

$$AC < 1. \tag{A.13}$$

Then

$$\lambda \leq \lambda_V(U_2). \tag{A.14}$$

PROOF. If U_2 has the role of U in Lemma A.2, it suffices to show that

$$\alpha < 1 \quad \text{and} \quad \beta < \infty. \tag{A.15}$$

First observe the following identity for $t \geq 0$:

$$
\begin{aligned}
1 + \int_0^t &\lambda \exp(\lambda u) \exp\left\{-\int_0^u V(Z_s)\, ds\right\} du\\
&= \int_0^t V(Z_u) \exp\left\{\lambda u - \int_0^u V(Z_s)\, ds\right\} du \qquad \text{(A.16)}\\
&\quad + \exp\left\{\lambda t - \int_0^t V(Z_s)\, ds\right\}.
\end{aligned}
$$

Now using the strong Markov property and the notation $T = T_{U_1} \wedge T_{U_2} (= T_{U_1 \cap U_2})$, we have

$$
\begin{aligned}
\beta &= \sup_x E_x\left[\int_0^{S_1 \wedge T_{U_2}} \lambda \exp\left\{\int_0^u (\lambda - V)(Z_s)\, ds\right\} du\right]\\
&\leq \sup_x E_x\left[\int_0^T \lambda \exp\left\{\int_0^u (\lambda - V)(Z_s)\, ds\right\} du\right]\\
&\quad + \sup_x E_x\Bigg[T < \infty, \exp\left\{\int_0^T (\lambda - V)(Z_s)\, ds\right\}\\
&\qquad\qquad \times E_{Z_T}\left[\int_0^{\tau \wedge T_{U_2}} \lambda \exp\left\{\int_0^u (\lambda - V)(Z_s)\, ds\right\} du\right]\Bigg].
\end{aligned}
$$

Observe that the inner expectation in the last term vanishes when $T = T_{U_2} < \infty$. We thus only need consider the case $T = T_{U_1} < \infty$, for which $Z_T \notin U_1$.

Observe (A.16) implies that $\sup_x E_x[T < \infty, \exp\{\int_0^T (\lambda - V)(Z_s)\,ds\}] \le A$ and thus

$$\beta \le A + AB < \infty$$

[using (A.12)].

Using once more the strong Markov property, we also have

$$\begin{aligned}\alpha = {} & \sup_x E_x\left[S_1 < T_{U_2}, \exp\left\{-\int_0^{S_1} (\lambda - V)(Z_s)\,ds\right\}\right] \\ \le {} & \sup_x E_x\left[T_{U_1} < T_{U_2}, \exp\left\{-\int_0^{T_{U_1}} (\lambda - V)(Z_s)\,ds\right\}\right] \\ & \times \sup_{x \notin U_1} E_x\left[\tau < T_{U_2}, \exp\left\{-\int_0^{\tau} (\lambda - V)(Z_s)\,ds\right\}\right] \le AC < 1,\end{aligned}$$

in view of (A.13). This proves (A.15), and our claim follows. □

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