

ON THE ASYMPTOTIC DISTRIBUTIONS OF PARTIAL SUMS OF FUNCTIONALS OF INFINITE-VARIANCE MOVING AVERAGES

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Dedicated to the memory of Stamatis Cambanis

This paper investigates the asymptotic distribution of the partial sum $S_N = \sum_{n=1}^N [K(X_n) - EK(X_n)]$, as $N \rightarrow \infty$, where $\{X_n\}$ is a moving average stable process and K is a bounded and measurable function. The results show that S_N follows a central or non-central limit theorem depending on the rate at which the moving average coefficients tend to 0.

1. Introduction. Suppose that $\{\varepsilon_i\}$ is an iid sequence of random variables and $\{a_j\}$ is a sequence of constants for which the linear process

$$(1) \quad X_n = \sum_{j=1}^{\infty} a_j \varepsilon_{n-j}, \quad n \geq 1,$$

is well defined. A large class of time series models can be represented as linear processes [cf. Brockwell and Davis (1991)]. Let K be a measurable function such that $E|K(X_n)| < \infty$. The investigation of the asymptotic behavior of

$$S_N := \sum_{n=1}^N (K(X_n) - EK(X_n)),$$

as $N \rightarrow \infty$, is the topic of a number of papers in the literature.

Generally speaking, if $\{X_n\}$ is known to satisfy a certain mixing condition, say a strong mixing condition [cf. Bradley (1986)], with the mixing coefficients decreasing to 0 fast enough, then $\{K(X_n)\}$ inherits the same properties, in which case standard results based on mixing conditions could be used to derive the limiting distribution of S_N [cf. Peligrad (1986)]. In view of that, our focus in this paper has to be processes which do not satisfy (or at least do not obviously satisfy) the standard mixing conditions under which limit theorems are already available. In the context of linear processes, it is well known that, for the mixing conditions to hold, stringent restrictions typically have to be imposed on the rate of decay of a_i . See Gorodetskii (1977), Pham and Tran (1985) and Withers (1981). Accordingly, we will make as few assumptions as possible about how quickly the a_i tend to 0.

In the case where $\{X_n\}$ is Gaussian, one has to distinguish between the short-memory case where $\sum_{j=1}^{\infty} |a_j| < \infty$ and the long-memory case where $\sum_{j=1}^{\infty} |a_j| = \infty$ but $\sum_{j=1}^{\infty} a_j^2 < \infty$. In fact, under the basic assumption that

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$EK^2(\varepsilon_i) < \infty$ the appropriate normalizing constants for S_N in the two cases have different rates. In the short-memory case, S_N is normalized by $N^{1/2}$ to converge weakly to the normal distribution, whereas, in the long-memory case, the proper normalization for S_N has a faster rate than $N^{1/2}$ and the weak limit may or may not be normal depending on the Hermite rank of K , that is, the smallest positive integer j for which

$$\int_{-\infty}^{\infty} K(x) \exp\left(\frac{x^2}{2}\right) \frac{d^j \exp(-x^2/2)}{dx^j} d\Phi(x) \neq 0,$$

where $\Phi(x)$ is the standard normal distribution function. These are sometimes referred to as central and noncentral limit theorems. See Breuer and Major (1983), Dobrushin and Major (1979) and Taqqu (1979) for these results and also Ho and Hsing (1997) for extensions to other finite-variance cases.

In this paper we consider the asymptotic distribution of S_N for the infinite-variance counterpart of the Gaussian linear process. That is, we are interested in the case where the innovations ε_i follow a stable distribution with stable index in $(0, 2)$. Further, we will focus on the case where $EK^2(\varepsilon_i) < \infty$. If $EK^2(\varepsilon_i) = \infty$, then the asymptotic theory of S_N assumes a completely different character and has to be approached from another direction [cf. Davis and Hsing (1995)]. For simplicity of presentation, we will henceforth confine ourselves to the special case where K is bounded and the distribution of ε_1 is standard symmetric α -stable ($S\alpha S$), namely,

$$(2) \quad E \exp(it\varepsilon_1) = \exp(-|t|^\alpha), \quad t \in \Re,$$

where $\alpha \in (0, 2)$. A close inspection of the steps in the proofs will reveal that, with additional technical details which involve essentially no new ideas, the proofs can be readily extended to cover more general K and distributions that are merely attracted to nonnormal stable distributions. See the remark at the end of Section 3 for details. The reader is referred to Feller (1971) and Samorodnitsky and Taqqu (1994) for the background of stable processes.

Although there is not yet a complete agreement on the definitions of short- and long-range dependence for infinite-variance processes, here we say that $\{X_n\}$ has short memory if

$$(3) \quad \sum_{i=1}^{\infty} |a_i|^{\alpha/2} < \infty$$

and long memory if

$$(4) \quad \sum_{i=1}^{\infty} |a_i|^{\alpha/2} = \infty \quad \text{but} \quad \sum_{i=1}^{\infty} |a_i|^\alpha < \infty,$$

where the finiteness of the second sum in (4) is required for the infinite sum in (1) to converge almost surely under (2) [cf. Samorodnitsky and Taqqu (1994)]. Theorem 1 below shows that, in the short-memory case, $N^{-1/2}S_N$ converges weakly to the normal distribution. For the long-memory case, in view of (4), we will assume that $a_j = j^{-\beta}$ for some $\beta \in (\alpha^{-1}, 2\alpha^{-1})$. Note that the stable linear

process then covers the so-called stable fractional ARIMA process which has important applications in finance [cf. Kokoszka and Taqqu (1996)]. Theorem 2 shows that $N^{-(3-\alpha\beta)/2}S_N$ converges weakly to the normal distribution. For the two cases we also address the weak convergence of the partial sum processes in $\mathcal{C}[0, 1]$. The main results are stated in Section 2 and the proofs are gathered in Section 3.

It is interesting to note the striking contrast between the properties of S_N for the finite- and infinite-variance cases. In the long-memory, finite-variance case, S_N generally admits an asymptotic expansion [cf. Ho and Hsing (1997)] and the asymptotic distribution of S_N may be Gaussian or non-Gaussian. However, in the long-memory, infinite-variance case, the asymptotic distribution of S_N can only be Gaussian and asymptotic expansions are not feasible.

2. Main results. Throughout the rest of this paper we will assume the following. Let K be a bounded, measurable function from \mathfrak{R} to \mathfrak{R} . Suppose that ε_i satisfies (2) for some $\alpha \in (0, 2)$. Let $\{X_n\}$ be defined by (1), where either the a_j satisfy (3) or $a_j = j^{-\beta}$ for some $\beta \in (\alpha^{-1}, 2\alpha^{-1})$. Further, define

$$X_{n, j_1, j_2} = \begin{cases} \sum_{j_1 \leq i \leq j_2} a_i \varepsilon_{n-i}, & 1 \leq j_1 \leq j_2 < \infty, \\ \sum_{i \geq j_1} a_i \varepsilon_{n-i}, & 1 \leq j_1 < \infty, j_2 = \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Let F_{j_1, j_2} and F_j be the cdf's of X_{n, j_1, j_2} and $a_j \varepsilon_{n-j}$, respectively. Also define

(5)
$$K_0(x) = K(x),$$

(6)
$$K_j(x) = EK(x + X_{n, 1, j}) = \int K(x + y)dF_{1, j}(y), \quad 1 \leq j \leq \infty$$

and

(7)
$$K_j(x) = EK(x + X_n - a_j \varepsilon_{n-j}) \\ = \int K(x + y + z)dF_{1, j-1}(y)dF_{j+1, \infty}(z), \quad 1 \leq j < \infty.$$

As mentioned in the Introduction, our main concerns in this paper are the asymptotic properties of the centered partial sum

$$S_N = \sum_{n=1}^N (K(X_n) - EK(X_n)).$$

For $\gamma \in [1/2, 1)$, denote by \mathcal{B}_γ the fractional Brownian motion on $[0, 1]$ with the self-similar index γ ; that is, \mathcal{B}_γ is a zero-mean Gaussian process on $[0, 1]$ with

$$\text{cov}(\mathcal{B}_\gamma(s), \mathcal{B}_\gamma(t)) = \frac{1}{2}(s^{2\gamma} + t^{2\gamma} - |s - t|^{2\gamma}).$$

Note that $\mathcal{B}_{1/2}$ is the standard Brownian motion. See Samorodnitsky and Taqqu (1994).

We first consider the short-memory case. Define the truncated partial sum

$$S_{N,l} = \sum_{n=1}^N (K(X_{n,1,l}) - EK(X_{n,1,l})), \quad 1 \leq l \leq N,$$

and the partial sum process

$$\mathcal{U}_N(t) = N^{-1/2} \left(S_{[Nt]} + (Nt - [Nt])(K(X_{[Nt]+1}) - EK(X_1)) \right), \quad t \in [0, 1].$$

Regard \mathcal{U}_N as a member of $\mathcal{C}[0, 1]$, the space of continuous functions on $[0, 1]$ equipped with the sup-topology and Borel σ -field [cf. Billingsley (1968)].

The essence of the following result is that the asymptotic behavior of S_N can be approximated by that of $S_{N,l}$ for large N, l . Intuitively, this is the case because both $\{X_n\}$ and $\{X_{n,1,l}\}$ have short memory. Since $\{X_{n,1,l}, n \geq 1\}$ is l -dependent (with l fixed), that $S_{N,l}$ follows a \sqrt{N} -central limit theorem is immediate. Consequently, S_N can be expected also to follow a \sqrt{N} -central limit theorem and the limiting distribution can be obtained as an iterated limit.

THEOREM 1. *Suppose that (3) holds and*

$$(8) \quad \lim_{l \rightarrow \infty} E(K(X_1) - K(X_{1,1,l}))^2 = 0.$$

Then

$$(9) \quad \lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \text{var} (S_N - S_{N,l}) = 0$$

and

$$(10) \quad N^{-1/2} S_N \longrightarrow_d \text{Normal} (0, \sigma^2),$$

where

$$(11) \quad \sigma^2 = \lim_{N \rightarrow \infty} N^{-1} \text{var} (S_N) = \lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \text{var} (S_{N,l}),$$

which exists and is finite. Further, if

$$(12) \quad \sum_{i=1}^{\infty} |a_i|^{\alpha/3} < \infty,$$

then \mathcal{U}_N converges in distribution in the space $\mathcal{C}[0, 1]$ to $(\sigma^2)^{1/2} \cdot \mathcal{B}_{1/2}$.

Next we consider the long-memory case, which is the harder of the two. As explained in the Introduction, we assume in this context that $a_j = j^{-\beta}$, where $\beta \in (\alpha^{-1}, 2\alpha^{-1})$. First, the strategy of the previous result completely fails here since $\{X_n, n \geq 1\}$ now has long memory but $\{X_{n,1,l}, n \geq 1\}$ has

short memory, and hence, for any fixed l , $S_{N,l}$ is a poor approximation of S_N . This is where the following quantity comes in:

$$T_N = \sum_{n=1}^N \sum_{j=1}^{\infty} (K_{\infty}(a_j \varepsilon_{n-j}) - EK_{\infty}(a_j \varepsilon_{n-j})), \quad N \geq 1.$$

To see a superficial connection between S_N and T_N , one can think of T_N as being obtained from S_N by bringing the summation on j out of $K(\sum_{j=1}^{\infty} a_j \varepsilon_{n-j})$ and then replacing K by K_{∞} . Theorem 2 below shows that T_N and S_N have the same asymptotic distribution. Unfortunately, we discovered this approach through trial and error by carrying out detailed computations. As far as we can see, there is no obvious logic or intuition which gives a hint that T_N is a good approximation of S_N . It is clear, though, that the asymptotic properties of T_N are much more straightforward to investigate than those of S_N .

Define the partial sum process in $\mathcal{C}[0, 1]$:

$$\mathcal{V}_N(t) = N^{-(3-\alpha\beta)/2} (S_{[Nt]} + (Nt - [Nt])(K(X_{[Nt]+1}) - EK(X_1))).$$

Also define

$$\begin{aligned} \omega^2 = & C_{\alpha} \int_{x=0}^1 \int_{u=-\infty}^{\infty} \left(\int_{y=0}^x [K_{\infty}(y^{-\beta}u) - K_{\infty}(0)] dy \right)^2 |u|^{-\alpha-1} du dx \\ & + C_{\alpha} \int_{x=0}^{\infty} \int_{u=-\infty}^{\infty} \left(\int_{y=x}^{1+x} [K_{\infty}(y^{-\beta}u) - K_{\infty}(0)] dy \right)^2 |u|^{-\alpha-1} du dx, \end{aligned}$$

where

$$C_{\alpha} = \frac{\alpha}{2} \left(\int_0^{\infty} x^{-\alpha} \sin x dx \right)^{-1}.$$

Lemma 4, in Section 3, establishes that ω^2 is finite and it is clear that ω^2 is nonzero if K is not a constant function. The result for the long-memory case is the following.

THEOREM 2. *Suppose that $a_j = j^{-\beta}$, where $\beta \in (\alpha^{-1}, 2\alpha^{-1})$. Then, as $N \rightarrow \infty$,*

$$(13) \quad \text{var}(S_N - T_N) = o(N^{3-\alpha\beta})$$

and

$$(14) \quad \text{var}(T_N) \sim N^{3-\alpha\beta} \omega^2.$$

Further, \mathcal{V}_N converges in distribution in the space $\mathcal{C}[0, 1]$ to $(\omega^2)^{1/2} \cdot \mathcal{B}_{(3-\alpha\beta)/2}$.

The proofs of Theorems 1 and 2 are collected in Section 3.

3. Proofs. We need a few lemmas first. In the following, for convenience, we denote by C a generic constant whose value varies from line to line. The first lemma is instrumental.

LEMMA 3. For each $k \geq 1$, the k th derivatives of K_j and $K_{\hat{j}}$, $j \geq 1$, exist and are uniformly bounded.

PROOF. Let f be the density that corresponds to the stable characteristic function $\exp(-c|t|^\alpha)$ for some $c \in (0, \infty)$. We first establish that f has bounded and integrable derivatives of all orders. By (2), Fourier inversion gives

$$(15) \quad f^{(k)}(x) = \frac{(-i)^k}{2\pi} \int e^{-itx} \varphi_k(t) dt,$$

where

$$\varphi_k(t) = t^k \exp(-c|t|^\alpha).$$

Now, for $t \neq 0$,

$$\varphi_k^{(1)}(t) = kt^{k-1} \exp(-c|t|^\alpha) - c\alpha t^k |t|^{\alpha-1} \text{sign}(t) \exp(-c|t|^\alpha)$$

and

$$\begin{aligned} \varphi_k^{(2)}(t) &= k(k-1)t^{k-2} \exp(-c|t|^\alpha) - 2c\alpha kt^{k-1} |t|^{\alpha-1} \text{sign}(t) \exp(-c|t|^\alpha) \\ &\quad - c\alpha(\alpha-1)t^k |t|^{\alpha-2} \exp(-c|t|^\alpha) + c^2 \alpha^2 t^k |t|^{2(\alpha-1)} \exp(-c|t|^\alpha). \end{aligned}$$

Observe that both $\varphi_k^{(1)}$ and $\varphi_k^{(2)}$ are integrable for $k \geq 1$. Performing two integrations by parts on the integral in (15), we obtain

$$f^{(k)}(x) = -\frac{(-i)^k}{2\pi x^2} \int e^{-itx} \varphi_k^{(2)}(t) dt.$$

By the Riemann–Lebesgue lemma,

$$f^{(k)}(x) = o(1/x^2) \text{ as } x \rightarrow \infty.$$

See Feller [(1971), pages 513–514]. Since $f^{(k)}$ is also bounded by (15), we conclude that $f^{(k)}$ is integrable. Now consider the function

$$G(x) = \int K(x+u)f(u) du = \int K(v)f(v-x) dv.$$

Since K is bounded and $f^{(1)}$ is integrable, Fubini’s theorem gives

$$\begin{aligned} &\int_{x=0}^y \int_{v=-\infty}^{\infty} K(v)f^{(1)}(v-x) dv dx \\ (16) \quad &= \int_{v=-\infty}^{\infty} \int_{x=0}^y K(v)f^{(1)}(v-x) dx dv \\ &= \int_{v=-\infty}^{\infty} K(v)[f(v-y) - f(v)] dv \\ &= G(y) - G(0). \end{aligned}$$

It is clear that $\int_{v=-\infty}^{\infty} K(v)f^{(1)}(v-x)dv$ is continuous in x . Hence, it follows from (16) that

$$G^{(1)}(y) = \int_{v=-\infty}^{\infty} K(v)f^{(1)}(v-y)dv = \int_{u=-\infty}^{\infty} K(u+y)f^{(1)}(u)du,$$

which is bounded. In view of the manner in which this bound depends on the scale parameter c of the stable distribution, the uniformity statement in the lemma can be easily verified. That $G^{(k)}$ is bounded for a general k can be obtained by an induction based on this argument. \square

LEMMA 4. Assume that $\alpha\beta \in (1, 2)$. Then:

- (i) $\int_{x=0}^1 \int_{u=-\infty}^{\infty} \left(\int_{y=0}^x [K_{\infty}(y^{-\beta}u)dy - K_{\infty}(0)] \right)^2 |u|^{-\alpha-1} du dx < \infty,$
- (ii) $\int_{x=0}^{\infty} \int_{u=-\infty}^{\infty} \left(\int_{y=x}^{1+x} [K_{\infty}(y^{-\beta}u) - K_{\infty}(0)]dy \right)^2 |u|^{-\alpha-1} du dx < \infty.$

PROOF. Assume without loss of generality that $K_{\infty}(0) = 0$ for notational convenience.

(i) First write

$$\int_{x=0}^1 \int_{u=-\infty}^{\infty} \left(\int_{y=0}^x K_{\infty}(y^{-\beta}u)dy \right)^2 |u|^{-\alpha-1} dudx = I_1 + I_2,$$

where

$$I_1 = \int_{x=0}^1 \int_{|u|>1} \left(\int_{y=0}^x K_{\infty}(y^{-\beta}u)dy \right)^2 |u|^{-\alpha-1} du dx$$

and

$$I_2 = \int_{x=0}^1 \int_{|u|\leq 1} \left(\int_{y=0}^x K_{\infty}(y^{-\beta}u)dy \right)^2 |u|^{-\alpha-1} du dx.$$

Since K_{∞} is bounded, it is clear that $I_1 < \infty$. Now, if $\beta < 1$, then, by the fact that $K_{\infty}^{(1)}$ is bounded,

$$I_2 \leq C \int_{x=0}^1 \int_{|u|\leq 1} \left(\int_{y=0}^x y^{-\beta} \right)^2 |u|^{1-\alpha} dudx < \infty.$$

If $\beta \geq 1$, then, by change of variables and the facts that K_{∞} and $K_{\infty}^{(1)}$ are bounded and $2/\beta - \alpha - 1 > -1$,

$$\begin{aligned} I_2 &= \int_{x=0}^1 \int_{|u|\leq 1} \left(\int_{v=x^{-\beta}u}^{\infty} K_{\infty}(v)v^{-1/\beta-1} dv \right)^2 |u|^{2/\beta-\alpha-1} du dx \\ &\leq C \int_{x=0}^1 \int_{|u|\leq 1} \left(\int_{v=x^{-\beta}u \wedge 1}^1 v^{-1/\beta} dv + \int_{v=1}^{\infty} v^{-1/\beta-1} dv \right)^2 |u|^{2/\beta-\alpha-1} du dx < \infty. \end{aligned}$$

(ii) Write

$$\int_{x=0}^{\infty} \int_{u=-\infty}^{\infty} \left(\int_{y=x}^{1+x} K_{\infty}(y^{-\beta}u) dy \right)^2 |u|^{-\alpha-1} du dx = \sum_{j=1}^4 I_j,$$

where

$$I_1 = \int_{x=1}^{\infty} \int_{|u| \leq x^{\beta}} \left(\int_{y=x}^{1+x} K_{\infty}(y^{-\beta}u) dy \right)^2 |u|^{-\alpha-1} du dx,$$

$$I_2 = \int_{x=1}^{\infty} \int_{|u| > x^{\beta}} \left(\int_{y=x}^{1+x} K_{\infty}(y^{-\beta}u) dy \right)^2 |u|^{-\alpha-1} du dx,$$

$$I_3 = \int_{x=0}^1 \int_{|u| > 1} \left(\int_{y=x}^{1+x} K_{\infty}(y^{-\beta}u) dy \right)^2 |u|^{-\alpha-1} du dx,$$

$$I_4 = \int_{x=0}^1 \int_{|u| \leq 1} \left(\int_{y=x}^{1+x} K_{\infty}(y^{-\beta}u) dy \right)^2 |u|^{-\alpha-1} du dx.$$

Now,

$$I_1 \leq C \int_{x=1}^{\infty} \int_{|u| \leq x^{\beta}} (x^{-\beta}u)^2 |u|^{-\alpha-1} du dx = [C/(2-\alpha)] \int_{x=1}^{\infty} x^{-\alpha\beta} dx < \infty$$

since $\alpha\beta > 1$. Similarly,

$$I_2 \leq C \int_{x=1}^{\infty} \int_{|u| > x^{\beta}} |u|^{-\alpha-1} du dx = (C/\alpha) \int_{x=1}^{\infty} x^{-\alpha\beta} dx < \infty.$$

That $I_3 < \infty$ is trivial. It remains to consider I_4 , for which the treatment is similar to that of the term I_2 in the proof of (i) and is omitted. \square

LEMMA 5. Under the conditions of Theorem 2, as $N \rightarrow \infty$,

$$(17) \quad \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{j'=j}^{N-n+j} \left(a_j^{\alpha} \sum_{i=j+1}^{\infty} a_i^{\alpha} \right)^{1/2} \left(a_{j'}^{\alpha} \sum_{i=j'+1}^{\infty} a_i^{\alpha} \right)^{1/2} = o(N^{3-\alpha\beta}).$$

PROOF. Replacing a_j by $j^{-\beta}$ and approximating sums by integrals, the left-hand side of (17) is asymptotically equivalent to

$$(18) \quad \begin{aligned} \sum_{n=0}^{N-1} \sum_{j=1}^{\infty} \sum_{j'=j}^{n+j} j^{1/2-\alpha\beta} j'^{1/2-\alpha\beta} &\sim \sum_{n=0}^{N-1} \int_{x=1}^{\infty} \int_{y=x}^{n+x} x^{1/2-\alpha\beta} y^{1/2-\alpha\beta} dx dy \\ &= \sum_{n=0}^{N-1} \int_{x=1}^{\infty} \int_{z=1}^{1+n/x} x^{2-2\alpha\beta} z^{1/2-\alpha\beta} dx dz, \end{aligned}$$

where in the last step we changed variables from y to zx . The derivations for the three cases $\alpha\beta > 3/2$, $\alpha\beta = 3/2$ and $1 < \alpha\beta < 3/2$ differ slightly. For

$\alpha\beta > 3/2$, $x^{2-2\alpha\beta} z^{1/2-\alpha\beta}$ is integrable and hence the quantity in (18) is $O(N)$. For the other two cases, write

$$\int_{x=1}^{\infty} \int_{z=1}^{1+n/x} x^{2-2\alpha\beta} z^{1/2-\alpha\beta} dx dz$$

$$= \int_{x=1}^n \int_{z=1}^{1+n/x} x^{2-2\alpha\beta} z^{1/2-\alpha\beta} dx dz + \int_{x=n}^{\infty} \int_{z=1}^{1+n/x} x^{2-2\alpha\beta} z^{1/2-\alpha\beta} dx dz$$

which is bounded by

$$C \left(\int_{x=1}^n x^{-1} \log \left(1 + \frac{n}{x} \right) dx + \int_{x=n}^{\infty} x^{-1} \left(\frac{n}{x} \right) dx \right)$$

if $\alpha\beta = 3/2$ and by

$$C \left(\int_{x=1}^n x^{2-2\alpha\beta} \left(1 + \frac{n}{x} \right)^{3/2-\alpha\beta} dx + \int_{x=n}^{\infty} x^{2-2\alpha\beta} \left(\frac{n}{x} \right) dx \right)$$

if $1 < \alpha\beta < 3/2$. The rest of the proof is purely algebra and is omitted. \square

PROOF OF THEOREM 1. We first prove (9), (10) and (11). Let

$$(19) \quad \sigma_l^2 = \lim_{N \rightarrow \infty} N^{-1} \text{var} (S_{N,l}),$$

which exists and is finite by l -dependence. By the Cauchy–Schwarz inequality,

$$(20) \quad \text{var} (S_{N,l_1} - S_{N,l_2}) \leq 2 \text{var} (S_N - S_{N,l_1}) + 2 \text{var} (S_N - S_{N,l_2}).$$

Supposing (9) holds, it follows from (20) and the triangle inequality that $\{\sigma_l^2, l = 1, 2, \dots\}$ is a Cauchy sequence and therefore σ_l^2 tends to some finite value σ^2 as $l \rightarrow \infty$. Consequently, (11) follows readily from (9). Further, since, for each fixed l , $\{X_{n,1,l}\}$ is l -dependent, we conclude readily that

$$N^{-1/2} S_{N,l} \rightarrow_d \text{Normal} (0, \sigma_l^2).$$

Hence (10) follows by taking limits iteratively [cf. Billingsley (1968), Theorem 4.2]. Thus, we focus on the proof of (9). Let

$$\mathcal{F}_{-\infty,k} = \sigma\text{-field generated by } \varepsilon_i, \quad i \leq k.$$

First write S_N and $S_{N,l}$ as telescoping sums:

$$S_N = \sum_{n=1}^N \sum_{j=1}^{\infty} [E(K(X_n) | \mathcal{F}_{-\infty, n-j}) - E(K(X_n) | \mathcal{F}_{-\infty, n-(j+1)})],$$

$$S_{N,l} = \sum_{n=1}^N \sum_{j=1}^l [E(K(X_{n,1,l}) | \mathcal{F}_{-\infty, n-j}) - E(K(X_{n,1,l}) | \mathcal{F}_{-\infty, n-(j+1)})],$$

and then accordingly

$$S_N - S_{N,l} = \sum_{n=1}^N \sum_{j=1}^{\infty} U_{n,j,l},$$

where

$$\begin{aligned}
 U_{n,j,l} &= [E(K(X_n)|\mathcal{F}_{-\infty,n-j}) - E(K(X_n)|\mathcal{F}_{-\infty,n-(j+1)})] \\
 &\quad - [E(K(X_{n,1,l})|\mathcal{F}_{-\infty,n-j}) - E(K(X_{n,1,l})|\mathcal{F}_{-\infty,n-(j+1)})]I(j \leq l) \\
 &= [K_{j-1}(X_{n,j,\infty}) - K_j(X_{n,j+1,\infty})] \\
 &\quad - [K_{j-1}(X_{n,j,l}) - K_j(X_{n,j+1,l})]I(j \leq l).
 \end{aligned}$$

It is straightforward to verify that

$$E(U_{n,j,l}) = 0 \quad \text{for all } n, j, l$$

and

$$(21) \quad \text{cov}(U_{n,j,l}, U_{n',j',l}) = 0 \quad \text{unless } n - j = n' - j'.$$

By an elementary inequality,

$$E(S_N - S_{N,l})^2 \leq R_{N,1,l} + R_{N,2,l} + R_{N,3,l},$$

where

$$\begin{aligned}
 R_{N,1,l} &= 3 \text{ var} \left(\sum_{n=1}^N U_{n,1,l} \right), \\
 R_{N,2,l} &= 3 \text{ var} \left(\sum_{n=1}^N \sum_{j=2}^l U_{n,j,l} \right), \\
 R_{N,3,l} &= 3 \text{ var} \left(\sum_{n=1}^N \sum_{j=l+1}^{\infty} U_{n,j,l} \right).
 \end{aligned}$$

Thus our goal is to show that

$$(22) \quad \lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} R_{N,i,l} = 0$$

for $i = 1, 2, 3$. It follows readily from (21) and (8) that

$$\lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} R_{N,1,l} \leq 3 \lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N E U_{n,1,l}^2 = 0.$$

Next by (21) and the Cauchy–Schwarz inequality, with $n' = n - j + j'$,

$$R_{N,2,l} \leq 6 \sum_{n=1}^N \sum_{j=2}^l \sum_{j'=j}^l E^{1/2}(U_{n,j,l}^2) E^{1/2}(U_{n',j',l}^2).$$

Similarly,

$$R_{N,3,l} \leq 6 \sum_{n=1}^N \sum_{j=l+1}^{\infty} \sum_{j'=j}^{\infty} E^{1/2}(U_{n,j,l}^2) E^{1/2}(U_{n',j',l}^2).$$

Clearly, that (22) holds for $i = 2, 3$ will follow if we show that there exists a finite C such that

$$(23) \quad EU_{n,j,l}^2 \leq C|a_j|^\alpha \left(\sum_{i=l+1}^\infty |a_i|^\alpha \right) \quad \text{for } 2 \leq j \leq l$$

and

$$(24) \quad EU_{n,j,l}^2 \leq C|a_j|^\alpha \quad \text{for } j \geq l + 1.$$

We first consider (23). Define

$$D(u, v, w, x) = \left([K_{j-1}(u + v + w) - K_{j-1}(v + w + x)] - [K_{j-1}(u + v) - K_{j-1}(v + x)] \right)^2.$$

For $2 \leq j \leq l$,

$$\begin{aligned} EU_{n,j,l}^2 &= E \left([K_{j-1}(X_{n,j,\infty}) - K_j(X_{n,j+1,\infty})] - [K_{j-1}(X_{n,j,l}) - K_j(X_{n,j+1,l})] \right)^2 \\ &= \iiint ([K_{j-1}(u + v + w) - K_j(v + w)] - [K_{j-1}(u + v) - K_j(v)])^2 \\ &\quad \times dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w), \end{aligned}$$

which, by the Cauchy–Schwarz inequality, is bounded by

$$\begin{aligned} &\iiint D(u, v, w, x) dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w) dF_j(x) \\ &= E_1 + E_2 + E_3 + E_4, \end{aligned}$$

where

$$\begin{aligned} E_1 &= \iiint I(|u - x| \leq 1, |w| \leq 1) \\ &\quad \times D(u, v, w, x) dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w) dF_j(x), \\ E_2 &= \iiint I(|u - x| > 1, |w| \leq 1) \\ &\quad \times D(u, v, w, x) dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w) dF_j(x), \\ E_3 &= \iiint I(|u - x| \leq 1, |w| > 1) \\ &\quad \times D(u, v, w, x) dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w) dF_j(x), \\ E_4 &= \iiint I(|u - x| > 1, |w| > 1) \\ &\quad \times D(u, v, w, x) dF_j(u) dF_{j+1,l}(v) dF_{l+1,\infty}(w) dF_j(x). \end{aligned}$$

Now write

$$D(u, v, w, x) = g_1(u) - g_1(x) = g_2(w) - g_2(0),$$

where

$$g_1(z) = K_{j-1}(z + v + w) - K_{j-1}(z + v)$$

and

$$g_2(z) = K_{j-1}(z + u + v) - K_{j-1}(z + v + x).$$

By Taylor expansions and Lemma 3,

$$\begin{aligned} D(u, v, w, x) &= (u - x)^2 (g_1^{(1)}(y^*))^2 = w^2 (g_2^{(1)}(z^*))^2 \\ &= (u - x)^2 w^2 (g_2^{(2)}(z^{**}))^2 \end{aligned}$$

for appropriate choices of y^* , z^* , z^{**} . Since K_{j-1} , $K_{j-1}^{(1)}$ and $K_{j-1}^{(2)}$ are bounded by Lemma 3, there exists some C such that

$$\begin{aligned} E_1 &\leq C \left(\iint I(|u - x| \leq 1) (u - x)^2 dF_j(u) dF_j(x) \right) \\ &\quad \times \left(\int I(|w| \leq 1) w^2 dF_{l+1, \infty}(w) \right), \\ E_2 &\leq C \left(\iint I(|u - x| > 1) dF_j(u) dF_j(x) \right) \\ &\quad \times \left(\int I(|w| \leq 1) w^2 dF_{l+1, \infty}(w) \right), \\ E_3 &\leq C \left(\iint I(|u - x| \leq 1) (u - x)^2 dF_j(u) dF_j(x) \right) \\ &\quad \times \left(\int I(|w| > 1) dF_{l+1, \infty}(w) \right), \\ E_4 &\leq C \left(\iint I(|u - x| > 1) dF_j(u) dF_j(x) \right) \left(\int I(|w| > 1) dF_{l+1, \infty}(w) \right). \end{aligned}$$

By Feller [(1971), XVII.5], if Z has the characteristic function given by (2), then, as $\lambda \downarrow 0$,

$$(25) \quad P(\lambda|Z| > 1) \sim C\lambda^\alpha$$

and

$$(26) \quad E[(\lambda Z)^2 I(\lambda|Z| \leq 1)] \sim C\lambda^\alpha.$$

Hence, by the stable assumption (2), as $j, l \rightarrow \infty$,

$$\begin{aligned} \iint I(|u - x| \leq 1) (u - x)^2 dF_j(u) dF_j(x) &\sim C|a_j|^\alpha, \\ \iint I(|u - x| > 1) dF_j(u) dF_j(x) &\sim C|a_j|^\alpha, \end{aligned}$$

$$\int I(|w| \leq 1)w^2 dF_{l+1, \infty}(w) \sim C \sum_{i=l+1}^{\infty} |a_i|^\alpha,$$

$$\int I(|w| > 1) dF_{l+1, \infty}(w) \sim C \sum_{i=l+1}^{\infty} |a_i|^\alpha.$$

Hence (23) is proved. The proof of (24) is similar (in fact, simpler) and is omitted. This completes the proof of (9) and hence those of (10) and (11).

Next we consider the functional convergence of \mathcal{U}_N in $\mathcal{C}[0, 1]$. Let $\mathcal{U}_{N,l}$ be the partial sum process defined for the truncated sequence $\{X_{n,1,l}\}$ in the same way as \mathcal{U}_N for the nontruncated sequence $\{X_n\}$. Since $\{X_{n,1,l}\}$ is l -dependent, $\mathcal{U}_{N,l}$ clearly converges in distribution to $(\sigma_l^2)^{1/2} \cdot \mathcal{B}_{1/2}$ in $\mathcal{C}[0, 1]$, where σ_l^2 is defined in (19). By (9) it suffices to show that \mathcal{U}_N is also tight in $\mathcal{C}[0, 1]$. By the telescoping decomposition used previously, it is easy to show that

$$E(S_N - S_{N,l})^4 \leq Q_{N,1,l} + Q_{N,2,l} + Q_{N,3,l},$$

where

$$Q_{N,1,l} = CE \left(\sum_{n_1=1}^N \sum_{j=1}^l U_{n, j, l} \right)^4,$$

$$Q_{N,2,l} = C \left(\sum_{n=1}^N \sum_{j=l+1}^{\infty} \sum_{j'=j}^{N-n+j} EU_{n, j, l} U_{n', j', l} \right)^2,$$

$$Q_{N,3,l} = C \sum_{n_1=1}^N \sum_{j_1=l+1}^{\infty} \sum_{j_2=j_1}^{N-n_1+j_1} \sum_{j_3=j_2}^{N-n_1+j_1} \sum_{j_4=j_3}^{N-n_1+j_1} E(U_{n_1, j_1, l} U_{n_2, j_2, l} U_{n_3, j_3, l} U_{n_4, j_4, l}),$$

where in $Q_{N,2,l}$, $n' = n - j + j'$, and in $Q_{N,3,l}$, $n_i = n_1 - j_1 + j_i$. It is readily shown that

$$\lim_{N \rightarrow \infty} N^{-2} Q_{N,1,l} = 0 \quad \text{for each } l$$

and, by the second-moment computations already done above,

$$\lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-2} Q_{N,2,l} = 0.$$

Hence it suffices to consider $Q_{N,3,l}$. Since the $U_{n, j, l}$ are bounded,

$$Q_{N,3,l} \leq C \sum_{n_1=1}^N (N - n_1) \sum_{j_1=l+1}^{\infty} \sum_{j_2=j_1}^{\infty} \sum_{j_3=j_2}^{\infty} E^{1/3}(|U_{n_1, j_1, l}|^3) E^{1/3}(|U_{n_2, j_2, l}|^3) \\ \times E^{1/3}(|U_{n_3, j_3, l}|^3).$$

The same approach as that used in the second-moment computation gives

$$E|U_{n, j, l}|^3 \leq C|a_j|^\alpha, \quad j \geq l + 1.$$

Hence assumption (12) implies that

$$\lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-2} Q_{N,3,l} = 0.$$

Summarizing, we have shown that

$$(27) \quad \lim_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-2} E(S_N - S_{N,l})^4 = 0.$$

By l -dependence,

$$\lim_{N \rightarrow \infty} N^{-2} ES_{N,l}^4 = c_l$$

for some constant $c_l \in [0, \infty)$. By (27) and the Cauchy argument previously used, we obtain

$$\lim_{N \rightarrow \infty} N^{-2} ES_N^4 = \lim_{l \rightarrow \infty} c_l \in [0, \infty),$$

which implies that

$$\sup_{N \geq 1} N^{-2} ES_N^4 \leq C \in [0, \infty).$$

Hence,

$$\frac{1}{N^2} ES_{[Nt]}^4 = \frac{[Nt]^2}{N^2} \frac{1}{[Nt]^2} ES_{[Nt]}^4 \leq Ct^2 \quad \text{for all } N \geq 1, t \in [0, 1].$$

By Billingsley [(1968), Theorem 12.3], $\{\mathcal{Z}_N\}$ is tight in $\mathcal{C}[0, 1]$. The proof is complete. \square

PROOF OF THEOREM 2. We first prove (13). Define

$$A_{N,1} = \sum_{n=1}^N \sum_{j=1}^{\infty} (K_{j-1}(a_j \varepsilon_{n-j}) - K_j(0)),$$

$$A_{N,2} = \sum_{n=1}^N \sum_{j=1}^{\infty} (K_{\hat{j}}(a_j \varepsilon_{n-j}) - EK_{\hat{j}}(a_j \varepsilon_{n-j})),$$

where the K_j and $K_{\hat{j}}$ are defined by (5)–(7). Write

$$S_N - A_{N,1} = \sum_{n=1}^N \sum_{j=1}^{\infty} U_{n,j},$$

$$A_{N,1} - A_{N,2} = \sum_{n=1}^N \sum_{j=1}^{\infty} V_{n,j},$$

$$A_{N,2} - T_N = \sum_{n=1}^N \sum_{j=1}^{\infty} W_{n,j},$$

where

$$\begin{aligned}
 U_{n,j} &= (K_{j-1}(X_{n,j,\infty}) - K_j(X_{n,j+1,\infty})) - (K_{j-1}(a_j\varepsilon_{n-j}) - K_j(0)), \\
 V_{n,j} &= (K_{j-1}(a_j\varepsilon_{n-j}) - K_j(0)) - (K_j(a_j\varepsilon_{n-j}) - EK_j(a_j\varepsilon_{n-j})), \\
 W_{n,j} &= (K_j(a_j\varepsilon_{n-j}) - EK_j(a_j\varepsilon_{n-j})) - (K_\infty(a_j\varepsilon_{n-j}) - EK_\infty(a_j\varepsilon_{n-j})).
 \end{aligned}$$

Note that in the expression for $S_N - A_{N,1}$ above we incorporated the telescoping sum of S_N introduced in the proof of Theorem 1. We first estimate the variance of $U_{n,j}$. Clearly, $U_{n,1}$ has finite variance since $U_{n,1}$ is bounded. For $j \geq 2$, write

$$U_{n,j} = U_{n,j,1} + U_{n,j,2} + U_{n,j,3} + U_{n,j,4},$$

where

$$\begin{aligned}
 U_{n,j,1} &= I(|a_j\varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| \leq 1) \times U_{n,j}, \\
 U_{n,j,2} &= I(|a_j\varepsilon_{n-j}| > 1, |X_{n,j+1,\infty}| \leq 1) \times U_{n,j}, \\
 U_{n,j,3} &= I(|a_j\varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| > 1) \times U_{n,j}, \\
 U_{n,j,4} &= I(|a_j\varepsilon_{n-j}| > 1, |X_{n,j+1,\infty}| > 1) \times U_{n,j}.
 \end{aligned}$$

Note that

$$U_{n,j} = \int_{u \in \mathfrak{R}} [g_u(X_{n,j+1,\infty}) - g_u(0)] dF_j(u),$$

where

$$g_u(x) = K_{j-1}(a_j\varepsilon_{n-j} + x) - K_{j-1}(u + x).$$

Hence, by two Taylor expansions,

$$\begin{aligned}
 U_{n,j,1} &= I(|a_j\varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| \leq 1) X_{n,j+1,\infty} \\
 &\quad \times \int_{u \in \mathfrak{R}} [K_{j-1}^{(1)}(a_j\varepsilon_{n-j} + u^*) - K_{j-1}^{(1)}(u + u^*)] dF_j(u) \\
 &= I(|a_j\varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| \leq 1) X_{n,j+1,\infty} \\
 &\quad \times \left[\int_{|u| \leq 1} (a_j\varepsilon_{n-j} - u) K_{j-1}^{(2)}(u^* + u^{**}) dF_j(u) \right. \\
 &\quad \left. + \int_{|u| > 1} (K_{j-1}^{(1)}(a_j\varepsilon_{n-j} + u^*) - K_{j-1}^{(1)}(u + u^*)) dF_j(u) \right]
 \end{aligned}$$

for appropriate u^*, u^{**} . Consequently,

$$|U_{n,j,1}| \leq CI(|a_j\varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| \leq 1) |X_{n,j+1,\infty}| (|a_j\varepsilon_{n-j}| + b_j),$$

where

$$(28) \quad b_j = a_j^\alpha I(0 < \alpha < 1) + a_j(-\log a_j) I(\alpha = 1) + a_j I(1 < \alpha < 2).$$

The same arguments lead to

$$\begin{aligned} |U_{n,j,2}| &\leq CI(|a_j \varepsilon_{n-j}| > 1, |X_{n,j+1,\infty}| \leq 1) \times |X_{n,j+1,\infty}|, \\ |U_{n,j,3}| &\leq CI(|a_j \varepsilon_{n-j}| \leq 1, |X_{n,j+1,\infty}| > 1)(|a_j \varepsilon_{n-j}| + b_j), \\ |U_{n,j,4}| &\leq CI(|a_j \varepsilon_{n-j}| > 1, |X_{n,j+1,\infty}| > 1). \end{aligned}$$

Thus, it follows again from (25) and (26) that, for $j \geq 2$,

$$EU_{n,j,k}^2 \leq Ca_j^\alpha \sum_{i=j+1}^\infty a_i^\alpha, \quad k = 1, 2, 3, 4,$$

and consequently, for all j ,

$$(29) \quad EU_{n,j}^2 \leq Ca_j^\alpha \sum_{i=j+1}^\infty a_i^\alpha.$$

Next observe that

$$\begin{aligned} V_{n,j} = \int & \left[(K_{j-1}(a_j \varepsilon_{n-j}) - K_j(0)) \right. \\ & \left. - (K_{j-1}(a_j \varepsilon_{n-j} + u) - K_j(u)) \right] dF_{j+1,\infty}(u). \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$(30) \quad EV_{n,j}^2 \leq EU_{n,j}^2 \leq Ca_j^\alpha \sum_{i=j+1}^\infty a_i^\alpha.$$

Similarly one deduces

$$\begin{aligned} EW_{n,j}^2 \leq \int & \left[(K_j(u) - K_j(v)) \right. \\ & \left. - (K_j(u+w) - K_j(v+w)) \right]^2 dF_j(u)dF_j(v)dF_j(w). \end{aligned}$$

Using this and arguments similar to those in dealing with $U_{n,j}$, it is readily shown that

$$(31) \quad EW_{n,j}^2 \leq Ca_j^{2\alpha}.$$

As in (21),

$$\text{cov}(U_{n,j}, U_{n',j'}) = \text{cov}(V_{n,j}, V_{n',j'}) = \text{cov}(W_{n,j}, W_{n',j'}) = 0 \quad \text{if } n-j \neq n'-j'.$$

Hence,

$$\begin{aligned} \text{var}(S_N - A_{N,1}) + \text{var}(A_{N,1} - A_{N,2}) + \text{var}(A_{N,2} - T_N) \\ \leq 2 \sum_{n=1}^N \sum_{j=1}^\infty \sum_{j'=j}^{N-n+j} (EU_{n,j}^2)^{1/2} (EU_{n',j'}^2)^{1/2} \end{aligned}$$

$$\begin{aligned}
 &+ 2 \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{j'=j}^{N-n+j} (EV_{n,j}^2)^{1/2} (EV_{n',j'}^2)^{1/2} \\
 &+ 2 \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{j'=j}^{N-n+j} (EW_{n,j}^2)^{1/2} (EW_{n',j'}^2)^{1/2},
 \end{aligned}$$

where we write $n' = n - j + j'$ and it follows readily from (29), (30), (31) and Lemma 5 that

$$\text{var}(S_N - A_{N,1}) + \text{var}(A_{N,1} - A_{N,2}) + \text{var}(A_{N,2} - T_N) = o(N^{3-\alpha\beta}).$$

Thus, (13) is proved.

Next we show (14). Write

$$\begin{aligned}
 \text{var}(T_N) &= \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} \left\{ E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)] \right. \\
 (32) \quad &\quad \left. - E(K_{\infty}(a_{n_1-i}\varepsilon_1))E(K_{\infty}(a_{n_2-i}\varepsilon_1)) \right\}.
 \end{aligned}$$

Clearly we can assume without loss of generality that $K_{\infty}(0) = 0$, which we do from this point on to simplify notation. Fix a δ and consider

$$\sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)I(|\varepsilon_1| \leq \delta)].$$

Since K_{∞} has a bounded derivative,

$$E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)I(|\varepsilon_1| \leq \delta)] \leq Ca_{n_1-i}a_{n_2-i}.$$

Now

$$\begin{aligned}
 \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} a_{n_1-i}a_{n_2-i} &= \sum_{i=-\infty}^{N-1} \left(\sum_{n=(i+1) \vee 1}^N a_{n-i} \right)^2 \\
 &= \sum_{j=1}^N \left(\sum_{i=1}^j a_i \right)^2 + \sum_{j=1}^{\infty} \left(\sum_{i=j}^{N+j} a_i \right)^2,
 \end{aligned}$$

which is clearly $O(N)$ if $\beta > 1$. Consider the case $\beta < 1$, for which β must be in $(1/2, 1)$ by the assumption $\alpha\beta \in (1, 2)$. It is straightforward to conclude that

$$\sum_{j=1}^N \left(\sum_{i=1}^j a_i \right)^2 = O(N^{3-2\beta}) = o(N^{3-\alpha\beta})$$

and that

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\sum_{i=j}^{N+j} a_i \right)^2 &\leq C \int_0^{\infty} ((N+x)^{1-\beta} - x^{1-\beta})^2 dx \\ &= CN^{3-2\beta} \int_0^{\infty} ((1+x)^{1-\beta} - x^{1-\beta})^2 dx \\ &= O(N^{3-2\beta}) = o(N^{3-\alpha\beta}) \end{aligned}$$

since, by the Taylor expansion, $((1+x)^{1-\beta} - x^{1-\beta})^2 \leq Cx^{-2\beta}$ and $2\beta > 1$. As a result,

$$(33) \quad \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)I(|\varepsilon_1| \leq \delta)] = o(N^{3-\alpha\beta}).$$

Next, it is straightforward to show that, for any constant $a \in (0, 1)$,

$$EK_{\infty}(a\varepsilon_1) = a^{\alpha}I(0 < \alpha < 1) + a(-\log a)I(\alpha = 1) + aI(1 < \alpha < 2).$$

Computations similar to those leading to (33) show that

$$(34) \quad \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} E(K_{\infty}(a_{n_1-i}\varepsilon_1))E(K_{\infty}(a_{n_2-i}\varepsilon_1)) = o(N^{3-\alpha\beta}).$$

To show (14), by (32)–(34), it suffices to show that

$$(35) \quad \begin{aligned} &\lim_{\delta \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-(3-\alpha\beta)} \\ &\times \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)I(|\varepsilon_1| > \delta)] = \omega^2. \end{aligned}$$

Take a large δ and approximate the density of ε_1 by $C_{\alpha}|u|^{-\alpha-1}$ for $|u| > \delta$ [cf. Samorodnitsky and Taqqu (1994), (1.2.9)]. Thus,

$$\begin{aligned} &\sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} E[K_{\infty}(a_{n_1-i}\varepsilon_1)K_{\infty}(a_{n_2-i}\varepsilon_1)I(|\varepsilon_1| > \delta)] \\ &\approx C_{\alpha} \sum_{n_1=1}^N \sum_{n_2=1}^N \sum_{i=-\infty}^{n_1 \wedge n_2 - 1} \int_{|u|>\delta} K_{\infty}(a_{n_1-i}u)K_{\infty}(a_{n_2-i}u)|u|^{-\alpha-1} du \\ &= C_{\alpha} \sum_{j=-\infty}^{N-1} \int_{|u|>\delta} \left(\sum_{n=(j+1)\vee 1}^N K_{\infty}(a_{n-j}u) \right)^2 |u|^{-\alpha-1} du \\ &= C_{\alpha} \sum_{j=1}^N \int_{|u|>\delta} \left(\sum_{i=1}^j K_{\infty}(a_i u) \right)^2 |u|^{-\alpha-1} du \end{aligned}$$

$$\begin{aligned}
 &+ C_\alpha \sum_{j=1}^\infty \int_{|u|>\delta} \left(\sum_{i=1+j}^{N+j} K_\infty(a_i u) \right)^2 |u|^{-\alpha-1} du \\
 &=: A_N + B_N.
 \end{aligned}$$

Now approximating the sums in A_n by integrals and performing changes of variables,

$$\begin{aligned}
 A_N &\sim C_\alpha \int_{x=1}^N \int_{|u|>\delta} \left(\int_{y=1}^x K_\infty(y^{-\beta} u) dy \right)^2 |u|^{-\alpha-1} du \\
 &= C_\alpha N^{3-\alpha\beta} \int_{x=1/N}^1 \int_{|u|>\delta/N^\beta} \left(\int_{y=1/N}^x K_\infty(y^{-\beta} u) dy \right)^2 |u|^{-\alpha-1} du dx.
 \end{aligned}$$

Thus by (i) of Lemma 4 and Lebesgue’s dominated convergence theorem,

$$(36) \quad A_N \sim C_\alpha N^{3-\alpha\beta} \int_{x=0}^1 \int_{u=-\infty}^\infty \left(\int_{y=0}^x K_\infty(y^{-\beta} u) dy \right)^2 |u|^{-\alpha-1} du dx.$$

Similarly by (ii) of Lemma 4 and Lebesgue’s dominated convergence theorem,

$$(37) \quad B_N \sim C_\alpha N^{3-\alpha\beta} \int_{x=0}^\infty \int_{u=-\infty}^\infty \left(\int_{y=x}^{1+x} K_\infty(y^{-\beta} u) dy \right)^2 |u|^{-\alpha-1} du dx.$$

Hence (35) follows from (36) and (37). This completes the proof of (14).

Finally we show the weak convergence of \mathcal{Y}_N in $\mathcal{C}[0, 1]$. For that it suffices to show tightness in $\mathcal{C}[0, 1]$ and convergence of finite-dimensional distributions. By (13) and (14),

$$\begin{aligned}
 E(N^{-(3-\alpha\beta)/2} S_{[Nt]})^2 &= \frac{1}{N^{3-\alpha\beta}} E S_{[Nt]}^2 = \frac{[Nt]^{3-\alpha\beta}}{N^{3-\alpha\beta}} \frac{1}{[Nt]^{3-\alpha\beta}} E(S_{[Nt]}^2) \\
 &\leq C t^{3-\alpha\beta} \quad \text{for all } t \in [0, 1].
 \end{aligned}$$

Since $\alpha\beta < 2$, tightness of \mathcal{Y}_N in $\mathcal{C}[0, 1]$ follows readily from Theorem 12.3 of Billingsley (1968).

Next we show that the finite-dimensional distributions of \mathcal{Y}_N converge to multivariate Gaussian. In view of (13) and (14), it suffices to consider the convergence of the finite-dimensional distributions of $T_{[Nt]}$, $0 \leq t \leq 1$. Let M_N be a sequence of positive integers such that $M_N \sim N^\gamma$ for some $1 < \gamma < 1/(2(1 - \beta))$. Note that $\beta > 1/2$ since $\alpha\beta > 1$ and $\alpha \in (0, 2)$. Write

$$T_N = T_{N,1} + T_{N,2},$$

where

$$\begin{aligned}
 T_{N,1} &= \sum_{n=1}^N \sum_{j=1}^{M_N} (K_\infty(a_j \varepsilon_{n-j}) - EK_\infty(a_j \varepsilon_{n-j})), \\
 T_{N,2} &= \sum_{n=1}^N \sum_{j=M_N+1}^\infty (K_\infty(a_j \varepsilon_{n-j}) - EK_\infty(a_j \varepsilon_{n-j})).
 \end{aligned}$$

The same argument used in deriving (14) shows that

$$\text{var}(T_{N,2}) = o(N^{3-\alpha\beta})$$

and by (14) it suffices to focus on $T_{N,1}$. To do that we first write

$$T_{N,1} = \sum_{n=1}^N \sum_{k=n-M_N}^{n-1} (K_\infty(a_{n-k}\varepsilon_k) - EK_\infty(a_{n-k}\varepsilon_k)) = \sum_{k=1-M_N}^{N-1} \eta_{N,k},$$

where

$$\eta_{N,k} = \sum_{n=(k+1)\vee 1}^{(k+M_N)\wedge N} (K_\infty(a_{n-k}\varepsilon_k) - EK_\infty(a_{n-k}\varepsilon_k)).$$

Clearly, $\eta_{N,1-M_N}, \dots, \eta_{N,N-1}$ are independent random variables with zero means. Hence the convergence of finite-dimensional distributions of $T_{[Nt],1}$ to multivariate Gaussian will follow if we show that Lindeberg's condition holds for the $\eta_{N,k}/\sqrt{\text{var}(T_N)}$, namely that for each $\delta > 0$,

$$(38) \quad \lim_{N \rightarrow \infty} \frac{1}{\text{var}(T_N)} \sum_{k=1-M_N}^{N-1} E \left[\eta_{N,k}^2 I(|\eta_{N,k}| > \delta \sqrt{\text{var}(T_N)}) \right] = 0.$$

It is easy to show that

$$\sum_{n=(k+1)\vee 1}^{(k+M_N)\wedge N} |EK_\infty(a_{n-k}\varepsilon_k) - K_\infty(0)| \leq C \sum_{j=1}^{M_N} b_j,$$

where b_j is as defined in (28) and similarly if $|\varepsilon_k| \leq \text{some } \zeta$, then

$$\sum_{n=(k+1)\vee 1}^{(k+M_N)\wedge N} |K_\infty(a_{n-k}\varepsilon_k) - K_\infty(0)| \leq C\zeta \sum_{j=1}^{M_N} a_j.$$

Hence, by the triangle inequality, if $|\varepsilon_k| \leq \zeta$, then

$$|\eta_{N,k}| \leq C \sum_{j=1}^{M_N} (b_j + a_j),$$

which, in all cases of α , is $o(\sqrt{N})$ if ζ is fixed. As a result, if $|\eta_{N,k}| > \delta \sqrt{\text{var}(T_N)} > \sqrt{N}$, then it must be that $|\varepsilon_k| > \zeta_N$ for some $\zeta_N \rightarrow \infty$. Then (38) follows again from arguments similar to those used in deriving (14). This completes the proof of Lindeberg's condition and hence the proof of Theorem 2. \square

Although we assumed that ε_1 is symmetric α -stable and that K is bounded, these conditions can be considerably relaxed since they are applied essentially through the following two properties:

1. The asymptotic relations (25) and (26) hold approximately for $Z = \varepsilon_1$.

2. For each $j \geq$ some finite j_0 , the convolutions K_j must have two derivatives which are bounded or at least have some high enough moments with respect to the distribution F_{j+1} . In particular, this should allow us to properly bound quantities such as E_1, \dots, E_4 in the proof of Theorem 1.

Needless to say, the details pertaining to how to achieve these in different contexts depend on the intrinsic nature of the problems.

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