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ON THE STOCHASTIC BURGERS' EQUATION IN THE REAL LINE

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In this paper we establish the existence and uniqueness of an $L^2(\mathbb{R})$-valued solution for a one-dimensional Burgers' equation perturbed by a space–time white noise on the real line. We show that the solution is continuous in space and time, provided the initial condition is continuous. The main ingredients of the proof are maximal inequalities for the stochastic convolution, and some a priori estimates for a class of deterministic parabolic equations.

1. Introduction. In order to study the turbulent fluid flow, the model equation (often called Burgers' equation),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x}$$

has been investigated thoroughly in one space dimension by Burgers, Hopf and others. (See, e.g., [2] and the references therein.) One can solve this equation and investigate the behavior of its solution by the Hopf–Cole transformation $u = 2v_x/v$, which reduces it to the heat equation for v (see [12]). In order to model the turbulent flow in the presence of random forces, Burgers' equation perturbed by some noise has become a popular subject of recent investigations. (See, e.g., [1, 5, 6].) In [1], Burgers' equation with additive space–time white noise $\partial^2 W/\partial t \partial x$ is solved by an adaptation of the Hopf–Cole transformation. A different approach is used in [5] to prove the existence and uniqueness of the mild solution of the Dirichlet problem in the interval $[0, 1]$ for Burgers' equation with additive space–time white noise. The method of this paper is based on a suitable property of the semigroup corresponding to the Dirichlet problem for the heat equation in bounded space interval. This property is derived from a known regularization property of the semigroup via Sobolev's embedding valid on bounded domains. The results of [5] are generalized in [6] to the case of the multiplicative noise $\sigma(u)(\partial^2 W/\partial t \partial x)$, in place of the space–time white noise, where σ is a bounded function. In order to unify the theory developed separately for the stochastic reaction-diffusion equations (see, e.g., [7, 8, 10, 14, 15, 17] and the references therein) and for the stochastic Burgers' equations, the class of

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equations,

$$
(1.1)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(t,x,u(t,x)) + \frac{\partial g}{\partial x}(t,x,u(t,x)) \\ &\quad + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}, \end{aligned}
$$

is investigated in [9], where f has linear growth and g has quadratic growth in the third variable. Existence, uniqueness and comparison theorems for the solutions are proved for the Dirichlet problem in bounded space interval. Moreover, a large deviation principle is proved in [4]. These equations, with colored noise in place of the space–time white noise are studied in [11], when g has polynomial growth.

In the present paper we consider the above class of equations but in the whole line, instead of a bounded interval for the space variable. We prove an existence and uniqueness theorem for the Cauchy problem. Our method is influenced by that of [5]. We derive the property we need for the semigroup directly from well-known estimates on the heat kernel, which are valid not only in bounded interval (with Dirichlet, or periodic boundary conditions), but also in the whole real line. Using the corresponding estimates, we establish the equivalence between the mild solution and the generalized solution defined via test functions in the integral form of the equations. Moreover, we get that the solution is continuous in the time and space variable (t, x), if the initial value u_0 is continuous.

For the standard definitions and tools of the theory of stochastic partial differential equations, which we use in the paper, we refer to [7, 16] and [17].

2. Formulation of the problem and preliminaries. Let $W = \{W(t,x), t \in [0,T], x \in \mathbb{R}\}$ be a Brownian sheet defined on a complete probability space $(\Omega, \mathscr{F}, P)$. That is, W is a zero-mean Gaussian random field with covariance

$$
E(W(s,x)W(t,y)) = \tfrac{1}{2}(s \wedge t)(|x| + |y| - |x-y|),
$$

$x, y \in \mathbb{R}$, $s, t \in [0,T]$. For each $t \in [0,T]$, we denote by $\mathscr{F}_t$ the σ-field generated by the family of random variables $W = \{W(s,x), s \in [0,t], x \in \mathbb{R}\}$ and the P-null sets. The family of σ-fields $\{\mathscr{F}_t, 0 \le t \le T\}$ constitutes a stochastic basis on the probability space $(\Omega, \mathscr{F}, P)$. We will denote by $\mathscr{P}$ the corresponding predictable σ-field on $[0,T] \times \Omega$. For the definition of the stochastic Itô integral with respect to the Brownian sheet, we refer to [3] (or see [17]).

We are interested in the following stochastic partial differential equation:

$$
(2.1)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(t,x,u(t,x)) + \frac{\partial g}{\partial x}(t,x,u(t,x)) \\ &\quad + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}, \end{aligned}
$$

$t \in [0,T]$, $x \in \mathbb{R}$, with initial condition $u(0,x) = u_0(x)$, where $u_0 \in L^2(\mathbb{R})$. Our purpose is to introduce a notion of solution and to find a solution with paths in $C([0,T]; L^2(\mathbb{R}))$.

DEFINITION 2.1. We say that an $L^2(\mathbb{R})$-valued continuous and $\mathscr{F}_t$-adapted stochastic process $u = \{u(t), t \in [0,T]\}$ is a solution to (2.1) if for any test function $\varphi \in C_K(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}} u_0(x)\varphi(x)\,dx &+ \int_0^t \int_{\mathbb{R}} u(s,x)\varphi''(x)\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}} f(s,x,u(s,x))\varphi(x)\,dx\,ds \\ &- \int_0^t \int_{\mathbb{R}} g(s,x,u(s,x))\varphi'(x)\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma(s,x,u(s,x))\varphi(x) W(ds,dx),\end{aligned}$$

a.s., for all $t \in [0,T]$, where the last integral is an Itô integral.

We introduce the following hypotheses on the coefficients of Equation (2.1):

(A1) $f\colon [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ is a Borel function satisfying the following linear growth and Lipschitz conditions:

$$|f(t,x,r)| \le a_1(x) + K|r|, \tag{2.2}$$

$$|f(t,x,r) - f(t,x,s)| \le \big(a_2(x) + L|r| + L|s|\big)|r-s|, \tag{2.3}$$

for all $t \in [0,T]$, $x, r, s \in \mathbb{R}$, and for some constants K, L and nonnegative functions $a_1, a_2 \in L^2(\mathbb{R})$.

(A2) The function g is of the form $g(t,x,r) = g_1(t,x,r) + g_2(t,r)$, where $g_1\colon [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ and $g_2\colon [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ are Borel functions satisfying the following linear and quadratic growth conditions:

$$|g_1(t,x,r)| \le b_1(x) + b_2(x)|r|, \tag{2.4}$$

$$|g_2(t,r)| \le K|r|^2, \tag{2.5}$$

for all $t \in [0,T]$, $x, r \in \mathbb{R}$, where K is a constant, and $b_1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $b_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are nonnegative functions. Moreover, g satisfies the following Lipschitz condition:

$$|g(t,x,r) - g(t,x,s)| \le \big(b_3(x) + L|r| + L|s|\big)|r-s|, \tag{2.6}$$

for all $t \in [0,T]$, $x, r, s \in \mathbb{R}$, and for some constant L and nonnegative function $b_3 \in L^2(\mathbb{R})$.

(A3) $\sigma\colon [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ is a Borel function satisfying the following linear growth and Lipschitz conditions:

$$|\sigma(t,x,r)| \le c(x), \tag{2.7}$$

$$|\sigma(t,x,r) - \sigma(t,x,s)| \le L|r-s|, \tag{2.8}$$

for all $t \in [0, T]$, $x, r, s \in \mathbb{R}$, and for some constant L and nonnegative function $c \in L^2(\mathbb{R}) \cap L^{2q}(\mathbb{R})$, where $2 > q > 1$.

Consider the heat kernel

$$G(t, x, y) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x-y|^2}{4t} \right).$$

THEOREM 2.2. *Suppose that the coefficients* f, g *and* σ *of* (2.1) *satisfy the hypotheses* (A1), (A2) *and* (A3). *Then an* $L^2(\mathbb{R})$*-valued continuous and* $\mathscr{F}_t$*-adapted stochastic process* $u = \{u(t), t \in [0, T]\}$ *is a solution to* (2.1) *if and only if* u *satisfies the following integral equation for all* $t \in [0, T]$ *and for almost all* $x \in \mathbb{R}$,

$$\begin{aligned} u(t,x) &= \int_0^t G(t,x,y) u_0(y)\, dy + \int_0^t \int_{\mathbb{R}} G(t-s,x,y) f(s,y,u(s,y))\, dy\, ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y) g(s,y,u(s,y))\, dy\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(t-s,x,y) \sigma(s,y,u(s,y)) W(ds,dy) \quad a.s. \end{aligned} \tag{2.9}$$

THEOREM 2.3. *Suppose that the coefficients* f, g *and* σ *of Equation* (2.1) *satisfy the hypotheses* (A1), (A2) *and* (A3). *Then there is a unique solution to Equation* (2.9) *which is continuous with values in* $L^2(\mathbb{R})$ *and* $\mathscr{F}_t$*-adapted. Moreover, if* u_0 *is continuous, then the solution* $u(t, x)$ *has a modification, which is continuous in* (t, x).

3. Preliminaries. In this section we establish some L^p and uniform estimates for the convolution operators with the kernels G and $\partial G / \partial y$. These estimates are given in Lemmas 3.1 and 3.2, respectively. In Lemma 3.3 a uniform estimate for the stochastic convolution with the kernel G is obtained, and the corresponding L^p estimates are provided in Lemma 3.4.

We make use of the following estimates:

$$\left| \frac{\partial G}{\partial t}(t,x,y) \right| \le K t^{-3/2} \exp\left(-a \frac{|x-y|^2}{t} \right), \tag{3.1}$$

$$\left| \frac{\partial G}{\partial y}(t,x,y) \right| \le K t^{-1} \exp\left(-b \frac{|x-y|^2}{t} \right), \tag{3.2}$$

$$\left| \frac{\partial^2 G}{\partial y \partial t}(t,x,y) \right| \le K t^{-2} \exp\left(-c \frac{|x-y|^2}{t} \right). \tag{3.3}$$

for all $0 < t \le T$, $x, y \in \mathbb{R}$, where K, a, b, c are some positive constants. Henceforth we use the notation $|h|_p$ for the $L^p(\mathbb{R})$-norm of a real function h. Moreover, C will denote a generic constant that may be different from one formula to another.

Consider the linear operators defined by

$$(J_1 v)(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x,y) v(s,y)\, dy\, ds, \tag{3.4}$$

$$(J_2 w)(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y) w(s,y)\, dy\, ds, \tag{3.5}$$

$t \in [0,T]$, $x \in \mathbb{R}$, where v and w are functions in $L^\infty([0,T]; L^p(\mathbb{R}))$ for some $p \geq 1$.

LEMMA 3.1. *For all $p \geq 1$ and $\gamma > 1$, the operator J_1 is bounded from $L^\gamma([0,T]; L^p(\mathbb{R}))$ into $C([0,T]; L^p(\mathbb{R}))$, and the following estimates hold*:

$$|(J_1 v)(t)|_p \leq \int_0^t |v(s)|_p\, ds, \tag{3.6(i)}$$

$$|(J_1 v)(t) - (J_1 v)(s)|_p \leq C|t-s|^\alpha \left(\int_0^t |v(s)|_p^\gamma\, ds \right)^{1/\gamma}, \tag{3.7(ii)}$$

where $\alpha < 1 - 1/\gamma$ and $0 \leq s \leq t \leq T$. On the other hand, for all $p \geq 1$ and $\gamma > 2p/(2p-1)$ the operator J_1 is bounded from $L^\gamma([0,T]; L^p(\mathbb{R}))$ into $C_b([0,T] \times \mathbb{R})$, and the following estimate holds:

$$\begin{aligned} |(J_1 v)(t,x)| &\leq C \int_0^t (t-s)^{-1/(2p)} |v(s)|_p\, ds \\ &\leq C \left(\int_0^t |v(s)|_p^\gamma\, ds \right)^{1/\gamma}, \end{aligned} \tag{3.8(iii)}$$

PROOF. Using Minkowski's and Young's inequalities, we have for all $t \in [0,T]$,

$$\begin{aligned} |(J_1 v)(t)|_p &\leq \int_0^t |G(t-s,0,\cdot) * v(s,\cdot)|_p\, ds \\ &\leq \int_0^t |G(t-s,0,\cdot)|_1 |v(s)|_p\, ds = \int_0^t |v(s)|_p\, ds. \end{aligned}$$

This yields (i). In order to show the estimate (ii) we introduce the decomposition,

$$|(J_1 v)(t) - (J_1 v)(s)|_p \leq A + B,$$

where

$$A = \left| \int_s^t (G(t-r,0,\cdot) * v(r,\cdot))\, dr \right|_p$$

and

$$B = \left| \int_0^s [(G(t-r,0,\cdot) - G(s-r,0,\cdot)) * v(r,\cdot)]\, dr \right|_p.$$

Then

$$A \le \int_s^t |v(r)|_p \, dr \le |t-s|^{1-1/\gamma} \left(\int_0^t |v(s)|_p^\gamma \, ds \right)^{1/\gamma}.$$

On the other hand, by the mean value theorem, estimate (3.1) and using Young's and Hölder's inequalities, we obtain

$$\begin{aligned} B &\le C \int_0^s \left(\int_s^t (\theta - r)^{-3/2} \left| \exp\left(-a \frac{|\cdot|^2}{\theta - r} \right) * |v(r,\cdot)| \right|_p d\theta \right) dr \\ &\le C \int_0^s \left(\int_s^t (\theta - r)^{-1} \, d\theta \right) |v(r)|_p \, dr \\ &\le C|t-s|^\alpha \int_0^s \left(\int_s^t (\theta - r)^{-1/\beta} \, d\theta \right)^\beta |v(r)|_p \, dr, \end{aligned}$$

where $\alpha + \beta = 1$. Hence,

$$\begin{aligned} B &\le C|t-s|^\alpha \int_0^s (s-r)^{\beta - 1} |v(r)|_p \, dr \\ &\le C|t-s|^\alpha \left(\int_0^t |v(s)|_p^\gamma \, ds \right)^{1/\gamma}, \end{aligned}$$

provided $\alpha < 1 - 1/\gamma$. In order to prove estimate (iii) we make use of the factorization method. By virtue of the semigroup property of the kernel G and by the formula

$$\frac{\pi}{\sin(\pi\alpha)} = \int_r^t (t-s)^{\alpha-1} (s-r)^{-\alpha} \, ds, \qquad 0 < \alpha < 1,$$

we have

$$(J_1 v)(t,x) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} \int_{\mathbb{R}} G(t-s,x,y) Y(s,y) \, dy \, ds,$$

where

$$Y(s,y) = \int_0^s \int_{\mathbb{R}} (s-r)^{-\alpha} G(s-r,y,z) v(r,z) \, dz \, dr.$$

Using Hölder's, Minkowski's and Young's inequalities, we obtain

$$\begin{aligned} |(J_1 v)(t,x)| &\le C \int_0^t (t-s)^{\alpha - 1 - 1/(2p)} |Y(s)|_p \, ds \\ &\le C \int_0^t (t-s)^{\alpha - 1 - 1/(2p)} \int_0^s (s-r)^{-\alpha} |v(r)|_p \, dr \, ds \\ &= C \int_0^t (t-r)^{-1/(2p)} |v(r)|_p \, dr, \end{aligned}$$

which completes the lemma, since $(J_1 v)(t,x)$ is clearly continuous for step functions v. $\square$

In a similar way we can establish the following technical lemma.

LEMMA 3.2. *For all* $p \ge 1$ *and* $\gamma > 4p/(2p-1)$, *the operator* J_2 *is bounded from* $L^\gamma([0,T]; L^p(\mathbb{R}))$ *into* $C([0,T]; L^p(\mathbb{R}))$, *and the following estimates hold*:

$$\text{(3.9)(i)} \qquad \begin{aligned} |(J_2 w)(t)|_{2p} &\le C\int_0^t (t-s)^{-1/2-1/(4p)} |w(s)|_p \, ds \\ &\le C\int_0^t \left(\int_0^t |w(s)|_p^\gamma \, ds \right)^{1/\gamma}, \end{aligned}$$

$$\text{(3.10)(ii)} \quad |(J_2 w)(t) - (J_2 w)(s)|_{2p} \le C|t-s|^\alpha \left(\int_0^t |w(s)|_p^\gamma \, ds \right)^{1/\gamma},$$

where $\alpha + 1/\gamma < 1/2 - 1/(4p)$. *On the other hand, for all* $p > 1$ *and* $\gamma > 2p/(p-1)$, *the operator* J_2 *is bounded from* $L^\gamma([0,T]; L^p(\mathbb{R}))$ *into* $C_b([0,T] \times \mathbb{R})$, *and the following estimate holds*:

$$\text{(3.11)(iii)} \qquad \begin{aligned} |(J_2 w)(t,x)| &\le C\int_0^t (t-s)^{-1/2-1/(2p)} |w(s)|_p \, ds \\ &\le C\left(\int_0^t |w(s)|_p^\gamma \, ds \right)^{1/\gamma}. \end{aligned}$$

PROOF. Using Minkowski's and Young's inequalities and the estimate (3.2), we have for all $t \in [0,T]$,

$$\begin{aligned} |(J_2 w)(t)|_{2p} &\le K\left| \int_0^t \int_{\mathbb{R}} (t-r)^{-1} \exp\left(-b\frac{|\cdot - y|^2}{t-r} \right) w(r,y) \, dy \, dr \right|_{2p} \\ &\le K\int_0^t (t-r)^{-1} \left| \exp\left(-b\frac{|\cdot|^2}{t-r} \right) * |w(r,\cdot)| \right|_{2p} dr \\ &\le C\int_0^t (t-r)^{-1/2-1/(4p)} |w(r)|_p \, dr \\ &\le C\left(\int_0^t |w(s)|_p^\gamma \, ds \right)^{1/\gamma}. \end{aligned}$$

This shows the estimate (i). In order to show the estimate (ii), we write

$$|(J_2 w)(t) - (J_2 w)(s)|_{2p} \le C(A+B),$$

where

$$A = \left| \int_s^t \int_{\mathbb{R}} (t-r)^{-1} \exp\left(-b\frac{|\cdot - y|^2}{t-r} \right) |w(r,y)| \, dy \, dr \right|_{2p}$$

and

$$B = \int_0^s \left| \int_{\mathbb{R}} \left| \frac{\partial G}{\partial y}(t-r,\cdot,y) - \frac{\partial G}{\partial y}(s-r,\cdot,y) \right| |w(r,y)|\, dy \right|_{2p} dr.$$

For the term A we have, as in the proof of Lemma 3.1,

$$A \le C|t-s|^{1/2-1/(4p)-1/\gamma} \left(\int_0^t |w(s)|_p^\gamma \, ds \right)^{1/\gamma}.$$

On the other hand, by the mean value theorem, estimate (3.3) and using Young's and Hölder's inequalities, we obtain

$$\begin{aligned} B &\le K \int_0^s \left| \int_{\mathbb{R}} \left(\int_s^t (\theta - r)^{-2} \exp\left(-c \frac{|\cdot - y|^2}{\theta - r} \right) d\theta \right) |w(r,y)|\, dy \right|_{2p} dr \\ &\le K \int_0^s \int_s^t (\theta - r)^{-2} \left| \int_{\mathbb{R}} \exp\left(-c \frac{|\cdot - y|^2}{\theta - r} \right) |w(r,y)|\, dy \right|_{2p} d\theta\, dr \\ &\le C \int_0^s \int_s^t (\theta - r)^{-3/2-1/(4p)} |w(r)|_p \, d\theta\, dr \\ &\le C|t-s|^\alpha \int_0^s \left(\int_s^t (\theta - r)^{(-3/2-1/(4p))1/\beta} \, d\theta \right)^\beta |w(r)|_p \, dr, \end{aligned}$$

where $\alpha + \beta = 1$. Hence,

$$\begin{aligned} B &\le C|t-s|^\alpha \int_0^s (s-r)^{\beta - 3/2 - 1/(4p)} |w(r)|_p \, dr \\ &\le C|t-s|^\alpha \left(\int_0^t |v(s)|_p^\gamma \, ds \right)^{1/\gamma}, \end{aligned}$$

provided $\alpha < 1/2 - 1/(4p) - 1/\gamma$. Finally, let us show the estimate (iii). Given $1 > \alpha > 0$, we can write

$$(J_2 w)(t,x) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y) Y(s,y)\, dy\, ds,$$

where

$$Y(s,y) = \int_0^s \int_{\mathbb{R}} (s-r)^{-\alpha} G(s-r,y,z) w(r,z)\, dz\, dr.$$

Using Hölder's, Minkowski's and Young's inequalities, we obtain

$$\begin{aligned} |(J_2 w)(t,x)| &\le C \int_0^t (t-s)^{\alpha-1-1/(4p)} |Y(s)|_{2p} \, ds \\ &\le C \int_0^t (t-s)^{\alpha-1-1/(4p)} \int_0^s (s-r)^{-\alpha-1/2-1/(4p)} |w(r)|_p \, dr \\ &= C \int_0^t (t-r)^{-1/2-1/(2p)} |w(r)|_p \, dr, \end{aligned}$$

which completes the proof of the lemma. □

LEMMA 3.3. *Let* $\varphi = \{\varphi(s, y), s \in [0, T], y \in \mathbb{R}\}$ *be a progressively measurable process such that*

$$c_{p,q}(\varphi) := E\int_0^T \left(\int_{\mathbb{R}} |\varphi(s, y)|^{2q}\, dy \right)^{p/2} ds < \infty,$$

for some $q > 1$ *and* $p > \max(4q/(q-1), 2q)$. *Then*

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} G(t-s, x, y)\varphi(s, y) W(ds, dy) \right|^p \le C c_{p,q}(\varphi). \tag{3.12}$$

REMARKS. Note that the assumption of the lemma holds provided

$$E\int_0^T \int_{\mathbb{R}} |\varphi(s, y)|^r\, dy\, ds < \infty$$

for some $r > 6$, or

$$E\int_0^T \left(\int_{\mathbb{R}} |\varphi(s, y)|^r\, dy \right)^2 ds < \infty,$$

for some $r > 4$.

PROOF. Define

$$(G\varphi)(t, x) = \int_0^t \int_{\mathbb{R}} G(t-s, x, y)\varphi(s, y) W(ds, dy),$$

for $t \in [0, T]$ and $x \in \mathbb{R}$. Given $\alpha > 0$, we can write

$$(G\varphi)(t, x) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-\sigma)^{\alpha-1} \int_{\mathbb{R}} G(t-\sigma, x, z) Y(\sigma, z)\, dz\, d\sigma,$$

where

$$Y(\sigma, z) = \int_0^\sigma \int_{\mathbb{R}} (\sigma - s)^{-\alpha} G(\sigma - s, z, y)\varphi(s, y) W(ds, dy).$$

Fix $p > 1$ such that $\alpha > 3/(2p)$. Using Hölder's inequality, we obtain

$$\begin{aligned}|(G\varphi)(t, x)| &\le C\int_0^t (t-\sigma)^{\alpha - 1 - 1/(2p)} |Y(\sigma)|_p\, d\sigma \\ &\le C\left(\int_0^T |Y(\sigma)|_p^p\, d\sigma \right)^{1/p}.\end{aligned}$$

Henceforth C will denote a generic constant which may depend on the parameters T, α, p and q. As a consequence of the above computations, we obtain

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |(G\varphi)(t, x)|^p \le C\int_0^T E|Y(\sigma)|_p^p\, d\sigma.$$

By Burkholder's inequality for stochastic integrals, we can write

$$E|Y(\sigma)|_p^p = E\int_{\mathbb{R}}\left|\int_0^\sigma\int_{\mathbb{R}}(\sigma-s)^{-\alpha}G(\sigma-s,z,y)\varphi(s,y)W(ds,dy)\right|^p dz$$

$$\le CE\int_{\mathbb{R}}\left|\int_0^\sigma\int_{\mathbb{R}}(\sigma-s)^{-2\alpha}G(\sigma-s,z,y)^2\varphi(s,y)^2\,dy\,ds\right|^{p/2} dz$$

$$\le CE\left(\int_0^\sigma(\sigma-s)^{-2\alpha-1}\left|\exp\left(\frac{-|\cdot|^2}{8(\sigma-s)}\right)*\varphi(s,\cdot)^2\right|_{p/2} ds\right)^{p/2}.$$

Using Young's inequality, we have

$$\left|\exp\left(\frac{-|\cdot|^2}{8(\sigma-s)}\right)*\varphi(s,\cdot)^2\right|_{p/2} \le |\varphi(s,\cdot)^2|_q\left|\exp\left(\frac{-|\cdot|^2}{8(\sigma-s)}\right)\right|_\xi$$

$$\le C(\sigma-s)^{1/(2\xi)}|\varphi(s,\cdot)^2|_q,$$

where $2/p+1=1/q+1/\xi$. Notice that $\xi>1$ because $p>2q$. Finally, applying again Young's inequality, we get

$$E\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}|(G\varphi)(t,x)|^p \le CE\int_0^T\left(\int_0^\sigma(\sigma-s)^{-2\alpha-1+1/(2\xi)}|\varphi(s,\cdot)^2|_q\,ds\right)^{p/2}d\sigma$$

$$\le CE\int_0^T|\varphi(s,\cdot)^2|_q^{p/2}\,ds,$$

provided $-2\alpha-1+1/(2\xi)=-2\alpha-1/2+1/p-1/2q>-1$. That is, we need the following conditions on α:

$$\frac{3}{2p}<\alpha<\frac{1}{4}+\frac{1}{2p}-\frac{1}{4q},$$

and this condition can be fulfilled because $p>4q/(q-1)$. □

LEMMA 3.4. *Let $\varphi=\{\varphi(s,y),s\in[0,T],y\in\mathbb{R}\}$ be a progressively measurable process. Then the following estimates hold*:

(i) *For any $q>1$ and $p>2q$, we have*

$$E\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}}G(t-s,\cdot,y)\varphi(s,y)W(ds,dy)\right|_p^p$$

$$\le CE\int_0^T\left(\int_{\mathbb{R}}|\varphi(s,y)|^{2q}\,dy\right)^{p/2}ds.$$

(ii) *For any $p>4$, we have*

$$E\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}}G(t-s,\cdot,y)\varphi(s,y)W(ds,dy)\right|_2^p$$

$$\le CE\int_0^T\left(\int_{\mathbb{R}}|\varphi(s,y)|^2\,dy\right)^{p/2}ds.$$

PROOF. Using the same notations as in the proof of Lemma 3.3 and applying Young's inequality we obtain for any $p \geq 1$, $\gamma > 1$ and $\alpha > 1/\gamma$,

$$|(G\varphi)(t)|_p \leq C\int_0^t (t-\sigma)^{\alpha-1} |Y(\sigma)|_p \, d\sigma \leq C\left(\int_0^t |Y(\sigma)|_p^\gamma \, d\sigma\right)^{1/\gamma}.$$

Then assertion (i) follows by the same method as in the proof of Lemma 3.3 with $\gamma = p$. On the other hand, in order to show (ii), we take $p = 2$ and $\gamma = p$ and we apply Burkholder's inequality for Hilbert-valued stochastic integrals (see, e.g., in [16]). In this way we obtain

$$\begin{aligned} E \sup_{t\in[0,T]} |(G\varphi)(t)|_2^p &\leq CE\int_0^T |Y(\sigma)|_2^p \, d\sigma \\ &\leq CE\int_0^T \left(\int_0^\sigma \int_{\mathbb{R}}\int_{\mathbb{R}} (\sigma - s)^{-2\alpha} G(\sigma - s, z, y)^2 \varphi(s, y)^2 \, dy\, dz\, ds\right)^{p/2} d\sigma \\ &\leq CE\int_0^T \left(\int_0^\sigma \int_{\mathbb{R}} (\sigma - s)^{-2\alpha - 1/2} \varphi(s, y)^2 \, dy\, ds\right)^{p/2} d\sigma \\ &\leq CE\int_0^T \left(\int_{\mathbb{R}} \varphi(s, y)^2 \, dy\right)^{p/2} ds, \end{aligned}$$

provided $\alpha < 1/4$. That is, we need $p > 4$. □

4. Proofs of the main results.

PROOF OF THEOREM 2.2. Using the first estimates from Lemma 3.1, Lemma 3.2 and the estimate from Lemma 3.3, we can proof this theorem essentially in the same way as the corresponding result is proved in [9]. □

We now proceed to the proof of Theorem 2.3. Let $B(0, N)$ be the ball of radius R centered at the origin in $L^2(\mathbb{R})$. Consider the mapping $\pi_N \colon L^2(\mathbb{R}) \mapsto B(0, N)$ defined by

$$\pi_N(v) = \begin{cases} v, & \text{if } |v|_2 \leq N, \\ \dfrac{N}{|v|_2} v, & \text{if } |v|_2 > N. \end{cases}$$

Let us introduce the following integral equation for a fixed natural number N:

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}} G(t,x,y)u_0(y)\,dy \\
&\quad + \int_0^t \int_{\mathbb{R}} G(t-s,x,y) f(s,y,(\pi_N u)(s,y))\,dy\,ds \\
&\quad - \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y) g(s,y,(\pi_N u)(s,y))\,dy\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sigma(s,y,(\pi_N u)(s,y)) W(ds,dy) \quad \text{a.s.}
\end{aligned} \tag{4.1}
$$

In order to prove Theorem 2.3, we first show the uniqueness of a solution for Equation (4.1).

PROPOSITION 4.1. *Suppose that the coefficients f, g and σ satisfy the hypotheses* (A1), (A2), (A3) *and assume that $u_0 \in L^p(\mathbb{R})$. Then for any fixed $N > 0$ there is a unique solution to* (4.1) *which is an $L^2(\mathbb{R})$-valued $\mathscr{F}_t$-adapted continuous process such that $E(\sup_{t\in[0,T]} |u(t)|_2^p) < \infty$ for any $p \ge 2$.*

PROOF. The proof will be done in several steps.

Step 1. Suppose that $u = \{u(t), t \in [0,T]\}$ is an $L^2(\mathbb{R})$-valued, $\mathscr{F}_t$-adapted random process. Set

$$
\mathscr{A}u = \int_{\mathbb{R}} G(\cdot,\cdot,y)u_0(y)\,dy + \sum_{i=1}^{3} \mathscr{A}_i u, \tag{4.2}
$$

where

$$
\begin{aligned}
(\mathscr{A}_1 u)(t,x) &= \int_0^t \int_{\mathbb{R}} G(t-s,x,y) f(s,y,(\pi_N u)(s,y))\,dy\,ds, \\
(\mathscr{A}_2 u)(t,x) &= -\int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y) g(s,y,(\pi_N u)(s,y))\,dy\,ds, \\
(\mathscr{A}_3 u)(t,x) &= \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sigma(s,y,(\pi_N u)(s,y)) W(ds,dy).
\end{aligned}
$$

We claim that

$$
E\Big(\sup_{t\in[0,T]} |\mathscr{A}u(t)|_2^p\Big) < \infty \tag{4.3}
$$

for any $p \ge 2$. Indeed, by assumption (A1) and Lemma 3.1, we have

$$
\begin{aligned}
|(\mathscr{A}_1 u)(t)|_2 &\le \left| \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\big(a_1(y) + K|(\pi_N u)(s,y)|\big)\,dy\,ds \right|_2 \\
&\le t\big(|a_1|_2 + KN\big),
\end{aligned}
$$

which implies that (4.3) holds for the operator $\mathscr{A}_1$. By assumption (A2) and Lemma 3.2, we have

$$\begin{aligned}|(\mathscr{A}_2 u)(t)|_2 \leq &\left|\int_0^t \int_{\mathbb{R}} \left|\frac{\partial G}{\partial y}(t-s,x,y)\right|\right.\\ &\left.\times\left(b_1(y)+b_2(y)|(\pi_N u)(s,y)|+K|(\pi_N u)(s,y)|^2\right)dy\,ds\right|_2\\ \leq &\int_0^t (t-s)^{-3/4}\left(|b_1|_1+|b_2|_2 N+KN^2\right)ds,\end{aligned}$$

which implies that (4.3) holds for the operator $\mathscr{A}_2$. Finally, using assumption (A3) and Lemma 3.4, we get

$$\begin{aligned}E\sup_{t\in[0,T]}|(\mathscr{A}_3 u)(t)|_2^p &\leq CE\int_0^t\left(\int_{\mathbb{R}}|\sigma(s,y,(\pi_N u)(s,y))|^2\,dy\right)^{p/2} ds\\ &\leq C|c|_2^p,\end{aligned}$$

which implies that (4.3) holds for the operator $\mathscr{A}_3$.

Step 2. Fix $\lambda > 0$. Let $\mathscr{H}$ denote the Banach space of $L^2(\mathbb{R})$-valued and $\mathscr{F}_t$-adapted random processes $u = \{u(t), t \in [0,T]\}$ such that $u(0) = u_0$, with the norm

$$|u|_{\mathscr{H}}^2 = \int_0^T e^{-\lambda t} E|u(t)|_2^2\,dt < \infty.$$

Define the operator $\mathscr{A}$ on $\mathscr{H}$ by (4.2). From Step 1 it follows that $\mathscr{A}$ is an operator mapping the Banach space $\mathscr{H}$ into itself. Fix two elements $u, v \in \mathscr{H}$. By assumption (A1) and applying Young's inequality, we can write

$$\begin{aligned}&|(\mathscr{A}_1 u)(t)-(\mathscr{A}_1 v)(t)|_2\\ &\quad\leq\left|\int_0^t\int_{\mathbb{R}} G(t-s,\cdot,y)\right.\\ &\qquad\times\left(a_2(y)+L|(\pi_N u)(s,y)|\right.\\ &\qquad\left.\left.+L|(\pi_N v)(s,y)|\right)|(\pi_N u-\pi_N v)(s,y)|\,dy\,ds\right|_2\\ &\quad\leq C\int_0^t (t-s)^{-1/4}|u(s)-v(s)|_2\left(|a_2|_2^2+2N\right)ds\\ &\quad= C_N\int_0^t (t-s)^{-1/4}|u(s)-v(s)|_2.\end{aligned}$$

On the other hand, using Lemma 3.2 and the Lipschitz condition on the coefficient g, we obtain

$$\begin{aligned}&|(\mathscr{A}_2 u)(t)-(\mathscr{A}_2 v)(t)|_2^2\\ &\quad\leq C\left(\int_0^t (t-s)^{-3/4}\int_{\mathbb{R}}|g(s,y,(\pi_N u)(s,y))-g(s,y,(\pi_N v)(s,y))|\,dy\,ds\right)^2\end{aligned}$$

$$\leq C\int_0^t (t-s)^{-3/4}\bigg(\int_{\mathbb{R}} (b_3(y) + L|(\pi_N u)(s,y)| + L|(\pi_N v)(s,y)|) \times |(\pi_N u - \pi_N v)(s,y)|\,dy\bigg)^2\,ds$$

$$\leq C\int_0^t (t-s)^{-3/4}|u(s)-v(s)|_2^2\big(|b_3|_2^2 + 2N\big)\,ds$$

$$= C_N\int_0^t (t-s)^{-3/4}|u(s)-v(s)|_2^2\,ds.$$

Finally, using the Lipschitz property of σ and the isometry of the Itô stochastic integral, we deduce

$$\begin{aligned} &E|(\mathscr{A}_2 u)(t) - (\mathscr{A}_2 v)(t)|_2^2 \\ &\quad \leq L^2 E\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} G(t-s,x,y)^2|(\pi_N u - \pi_N v)(s,y)|^2\,dy\,ds\,dx \\ &\quad \leq C\int_0^t (t-s)^{-1/2} E|u(s)-v(s)|_2^2\,ds. \end{aligned}$$

From the previous estimates, we deduce

$$\begin{aligned} |\mathscr{A}u - \mathscr{A}v|_{\mathscr{H}}^2 &= \int_0^T e^{-\lambda t} E|\mathscr{A}u(t) - \mathscr{A}v(t)|_2^2\,dt \\ &\leq C\int_0^T e^{-\lambda t}\bigg(\int_0^t (t-s)^{-1/2} E|u(s)-v(s)|_2^2\,ds\bigg)\,dt \\ &\leq C\bigg(\int_0^\infty e^{-\lambda y} y^{-1/2}\,dy\bigg)|u-v|_{\mathscr{H}}^2. \end{aligned}$$

Then, taking λ in such a way that

$$C\int_0^\infty e^{-\lambda y} y^{-1/2}\,dy < 1,$$

we have that the operator $\mathscr{A}$ is a contraction on $\mathscr{H}$. Consequently, there exists a unique fixed point for this operator and this implies the existence of a unique solution for (4.1). □

Proposition 4.1 implies the uniqueness of a solution in Theorem 2.3 and a local existence of a solution. The global existence will be proved using the uniform estimate obtained in the following lemma.

LEMMA 4.2. *Let $\eta = \{\eta(t,x), t \in [0,T], x \in \mathbb{R}\}$ be a continuous and bounded function belonging to $C([0,T]; L^2(\mathbb{R}))$. Let $v \in C([0,T]; L^2(\mathbb{R}))$ be a* *solution of the integral equation,*

$$
\begin{aligned}
v(t,x) &= \int_0^t G(t,x,y)u_0(y)\,dy \\
&\quad + \int_0^t\int_{\mathbb{R}} G(t-s,x,y)f(s,y,v(s,y)+\eta(s,y))\,dy\,ds \\
&\quad - \int_0^t\int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x,y)g(s,y,v(s,y)+\eta(s,y))\,dy\,ds,
\end{aligned} \tag{4.4}
$$

where $u_0 \in L^2(\mathbb{R})$ *and* f *and* g *satisfy assumptions* (A1) *and* (A2). *Then we have*

$$
|v(t)|_2^2 \le \big(|u_0|_2^2 + C_1(1+R_1(\eta))\big)\exp\big(C_2(1+R_2(\eta))\big), \tag{4.5}
$$

where the constants C_1 *and* C_2 *depend only on* T *and on the functions* a_i, b_i, c *and the constants* K *and* L *appearing in the hypotheses* (A1) *and* (A2), *and*

$$
R_1(\eta) = \sup_{s\in[0,T]}\big(|\eta(s)|_2^2 + |\eta(s)|_4^4 + |\eta(s)|_\infty^2\big),
$$

$$
R_2(\eta) = \sup_{s\in[0,T]}\big(|\eta(s)|_\infty^2\big).
$$

PROOF. The proof will be done in several steps.

Step 1. Notice first that we can assume that $u_0 \in C_K(\mathbb{R})$. Indeed, the Lipschitz properties of the functions f and g imply that if v_1 and v_2 are the solutions to (4.4) corresponding to the initial conditions u_0^1 and u_0^2, respectively, then

$$
\begin{aligned}
|v_1(t) - v_2(t)|_2 &\le |u_0^1(t) - u_0^2(t)|_2 \\
&\quad + \bigg|\int_0^t\int_{\mathbb{R}}(a_2(y) + L|v_1(s,y)| + L|v_2(s,y)| + 2L|\eta(s,y)|) \\
&\qquad \times |v_1(s,y) - v_2(s,y)|G(t-s,\cdot,y)\,dy\,ds\bigg|_2 \\
&\quad + \bigg|\int_0^t\int_{\mathbb{R}}(b_3(y) + L|v_1(s,y)| + L|v_2(s,y)| + 2L|\eta(s,y)|) \\
&\qquad \times |v_1(s,y) - v_2(s,y)|\frac{\partial G}{\partial y}(t-s,\cdot,y)\,dy\,ds\bigg|_2 \\
&\le |u_0^1(t) - u_0^2(t)|_2 \\
&\quad + C\int_0^t (t-s)^{-1/4}|v_1(s) - v_2(s)|_2 \\
&\qquad \times \big(|a_2|_2 + L|v_1(s)|_2 + L|v_2(s)|_2 + 2L|\eta(s)|_2\big)\,ds \\
&\quad + C\int_0^t (t-s)^{-3/4}|v_1(s) - v_2(s)|_2 \\
&\qquad \times \big(|b_3|_2 + L|v_1(s)|_2 + L|v_2(s)|_2 + 2L|\eta(s)|_2\big)\,ds.
\end{aligned}
$$

Now using that v_1 and v_2 belong to $C([0,T]; L^2(\mathbb{R}))$ (this is a consequence of the growth conditions on the functions f and g), we deduce that

$$|v_1(t) - v_2(t)|_2 \leq C|u_0^1(t) - u_0^2(t)|_2.$$

Step 2. We assume in the sequel that $u_0 \in C_K(\mathbb{R})$. Suppose first that the functions f and g are uniformly bounded, globally Lipschitz in r, f has compact support in x, and satisfy the Lipschitz conditions and the conditions on growth (A1) and (A2). We will show that in this case there is a unique solution to (4.4) in $C([0,T]; L^2(\mathbb{R}))$ for which the estimate (4.5) holds. By the same argument as in the proof of Theorem 2.2 we can show that a solution $v(t,x)$ of (4.4) is a weak solution of the stochastic partial differential equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(t,x,(v+\eta)(t,x)) + \frac{\partial g}{\partial x}(t,x,(v+\eta)(t,x)), \tag{4.6}$$

with initial condition u_0.

There is a constant C such that

$$\left|\int_{\mathbb{R}} \psi'(x)\phi'(x)\,dx\right| \leq |\psi'(x)|_2 |\phi'(x)|_2, \tag{4.7}$$

$$\left|\int_{\mathbb{R}} f(t,x,(\psi+\eta)(t,x))\phi(x)\,dx\right| \leq C|\phi(x)|_2, \tag{4.8}$$

$$\left|\int_{\mathbb{R}} g(t,x,(\psi+\eta)(t,x))\phi'(x)\,dx\right| \leq C|\phi'(x)|_2, \tag{4.9}$$

for every ϕ, ψ in the Sobolev space $W^{1,2}(\mathbb{R})$. Consider the operator A defined by

$$\langle A(t,\psi),\phi\rangle = -\int_{\mathbb{R}} \psi'(x)\phi'(x)\,dx + \int_{\mathbb{R}} f(t,x,(\psi+\eta)(t,x))\phi(x)\,dx$$
$$-\int_{\mathbb{R}} g(t,x,(\eta+\eta)(t,x))\phi'(x)\,dx.$$

From the inequalities (4.7), (4.8) and (4.9) we see that $A(t,\psi)$ maps $V := W^{1,2}(\mathbb{R})$ into its dual $V^* := W^{-1,2}(\mathbb{R})$, and moreover,

$$|A(t,\psi)|_{V^*} \leq C(1 + |\psi|_V),$$

for all $t \in [0,T]$ and for all $\psi \in V$. Thus we can see that the problem (4.6) can be cast in the evolution equation

$$v(t) = u_0 + \int_0^t A(s, v(s))\,ds \tag{4.10}$$

in the triplet $V \hookrightarrow H = H^* \hookrightarrow V^*$ of spaces based on the Hilbert space $H := L^2(\mathbb{R})$. From (4.7), (4.8) and (4.9), there exist constants C_1 and C_2 such that

$$\langle A(t,\phi),\phi\rangle \leq C_1|\phi|_2^2 - \tfrac{1}{2}|\phi|_V^2 + C_2,$$

for all $\psi \in V$. This means that the operator A is coercive. By the Lipschitz conditions on the functions f and g, we obtain

$$\begin{aligned}\langle A(t,\psi) - A(t,\phi), \psi - \phi\rangle &\le -|(\psi - \phi)'|_2^2 + C_1|\psi - \phi|_2^2 \\ &\quad + C_2|\psi - \phi|_2|(\psi - \phi)'|_2 \\ &\le C|\psi - \phi|_V^2,\end{aligned}$$

for all ψ, ϕ in V, and for all $t \in [0,T]$, which means that the operator A satisfies also the monotonicity condition. Consequently, by a well-known result (see, e.g., [13]), the evolution equation (4.10) has a unique solution v in $C([0,T]; H)$ such that

$$\int_0^T |v(t)|_V^2\, dt < \infty.$$

Moreover, the energy equality

$$|v(t)|_H^2 = |u_0|_2^2 + 2\int_0^t \langle v(s), A(s, v(s))\rangle\, ds \tag{4.11}$$

holds for all $t \in [0,T]$. By uniqueness of the weak solution, we get that v is also the unique solution to (4.4). Then by (4.11) we obtain

$$|v(t)|_2^2 = |u_0|_2^2 - 2\int_0^t |v'(r)|_2^2\, dr + 2A(t) - 2B(t) - 2C(t),$$

where

$$A(t) = \int_0^t \int_{\mathbb{R}} f(s, x, v(s,x) + \eta(s,x)) v(s,x)\, dx\, ds,$$

$$B(t) = \int_0^t \int_{\mathbb{R}} g(s, x, v(s,x)) v'(s,x)\, dx\, ds,$$

$$C(t) = \int_0^t \int_{\mathbb{R}} \big(g(s, x, v(s,x) + \eta(s,x)) - g(s, x, v(s,x))\big) v'(s,x)\, dx\, ds.$$

By the linear growth condition (A1) on f, we have

$$\begin{aligned}|A(t)| &\le \int_0^t |v(s)|_2 \big(|a_1|_2 + L|v(s)|_2 + L|\eta(s)|_2\big)\, ds \\ &\le 2\int_0^t \Big(|v(s)|_2^2(1 + L^2) + |a_1|_2^2 + L^2|\eta(s)|_2^2 + 1\Big)\, ds.\end{aligned}$$

By the Lipschitz condition on the function g, we obtain

$$\begin{aligned}|C(t)| &\le \int_0^t |v'(s)|_2 \Big(|b_3|_2|\eta(s)|_\infty + L|v(s)|_2|\eta(s)|_\infty + L|\eta(s)|_4^2\Big)\, ds \\ &\le 2\int_0^t \Big(\tfrac{3}{4}|v'(s)|_2^2 + 4L^2|\eta(s)|_4^4 + 4|\eta(s)|_\infty^2|b_3|_2^2 + 4L^2|v(s)|_2^2|\eta(s)|_\infty^2\Big)\, ds.\end{aligned}$$

Clearly,

$$B(t) = B_1(t) + B_2(t),$$

where

$$B_1(t) = \int_0^t \int_{\mathbb{R}} g_1(s, x, v(s, x)) v'(s, x)\, dx\, ds,$$

$$B_2(t) = \int_0^t \int_{\mathbb{R}} g_2(s, v(s, x)) v'(s, x)\, dx\, ds.$$

By the linear growth condition on g_1, we can write

$$\begin{aligned}|B_1(t)| &\le \int_0^t |v'(s)|_2 \big(|b_1|_2 + |b_2|_\infty |v(s)|_2\big)\, ds \\ &\le 2\int_0^t \big(\tfrac{1}{4}|v'(s)|_2^2 + 8|b_1|_2^2 + 8|b_2|_\infty^2 |v(s)|_2^2\big)\, ds.\end{aligned}$$

We are going to show that $B_2(t) = 0$ for each $t \in [0, T]$. We claim that for each $t \in [0, T]$

$$\lim_{x \to \infty} v(t, x) = 0, \tag{4.12}$$

$$\lim_{x \to -\infty} v(t, x) = 0. \tag{4.13}$$

We only show (4.12) and the proof of (4.13) is analogous. The limit (4.12) is a consequence of (4.4) and the following limits:

$$\lim_{x \to \infty} \int_0^1 G(t, x, y) u_0(y)\, dy = 0, \tag{4.14}$$

$$\lim_{x \to \infty} \int_0^t \int_{\mathbb{R}} G(t - s, x, y) f(s, y, v(s, y) + \eta(s, y))\, dy\, ds = 0, \tag{4.15}$$

$$\lim_{x \to \infty} \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t - s, x, y) g(s, y, v(s, y) + \eta(s, y))\, dy\, ds = 0. \tag{4.16}$$

Equation (4.13) holds because u_0 has compact support. The convergences (4.15) and (4.16) are true by the dominated convergence theorem, taking into account that the functions $f(s, \cdot, v(s, \cdot) + \eta(s, \cdot))$ and $g(s, \cdot, v(s, \cdot) + \eta(s, \cdot))$ belong to $L^1(\mathbb{R})$.

Notice that for each $s \in [0, T]$, $g_2(s, v(s, \cdot))$ is in $L^2(\mathbb{R})$ because $g_2(s, r)$ is bounded by $C \wedge K|r|^2$, and $v(s, \cdot) \in L^2(\mathbb{R})$. So, for each $s \in [0, T]$, we have, using the convergences (4.12) and (4.13),

$$\begin{aligned}&\int_{\mathbb{R}} g_2(s, v(s, x)) v'(s, x)\, dx \\ &\quad = \lim_n \int_{-n}^n g_2(s, v(s, x)) v'(s, x)\, dx \\ &\quad = \lim_n \big[g_2(s, v(s, n)) - g_2(s, v(s, -n))\big] = g_2(s, 0) - g_2(s, 0) = 0.\end{aligned}$$

Summing up, we get constants C_1 and C_2 depending on L, $|a_1|_2$, $|b_3|_2$, $|b_1|_2$ and $|b_2|_\infty$, such that

$$|v(t)|_2^2 \le |u_0|_2^2 + C_1(1 + R_1(\eta)) + \int_0^t C_2(1 + R_2(\eta)) |v(s)|_2^2\, ds,$$

for all $t \in [0, T]$. Hence by Gronwall's lemma, we deduce the desired estimate.

Step 3. Consider the sequence of functions defined by $f_n = [(f \wedge n) \vee (-n)\varphi_n(x)] * \varepsilon_n$ and by $g_n = [(g \wedge n) \vee (-n)] * \varepsilon_n$, where ε_n is an approximation of the identity with support included in $[-1/n, 1/n]$, and $\varphi_n(x)$ is a smooth function bounded by one together with its derivative, and such that $\varphi_n(x) = 1$ if $|x| \le n$, and $\varphi_n(x) = 0$ if $|x| \ge n + 1$. Notice that f_n has compact support in x, f_n and g_n are uniformly bounded, globally Lipschitz in r and satisfy the same Lipschitz conditions and the conditions on growth as f and g, with constants independent of n. Then, if $v_n(t, x)$ denotes the solution to (4.4) with coefficients f_n and g_n, it is not difficult to show that $\lim_n |v_n(t)|_2 = |v(t)|_2$ for each $t \in [0, T]$, and then the estimate for $|v(t)|_2$ would follow from that for $|v_n(t)|_2$. $\square$

PROOF OF THEOREM 2.3. Note first that Proposition 4.1 provides a proof of the uniqueness in Theorem 2.3. In fact, suppose that u and v are two solutions to Equation (2.1). By Theorem 2.2, u and v satisfy the integral equation (2.9). For every natural number, N, we introduce the stopping time

$$\sigma_N = \inf\{t \ge 0 \colon \inf(|u(t)|_2, |v(t)|_2) \ge N\} \wedge T.$$

Set $u^N(t) = u(t \wedge \tau_N)$ and $v^N(t) = v(t \wedge \tau_N)$ for all $t \in [0, T]$. Then the processes u^N and v^N satisfy (4.1). Hence, $u^N(t) = v^N(t)$ a.s. for all $t \in [0, T]$, and letting N tend to infinity, we deduce $u(t) = v(t)$ a.s. for all $t \in [0, T]$.

In order to construct a solution to (2.9), we denote by u_N the solution to (4.1) for any $N > 0$. Consider the stopping time

$$\tau_N = \inf\{t \ge 0 \colon |u_N(t)|_2 \ge N\} \wedge T.$$

Notice that $u_M(t) = u_N(t)$ for $M \ge N$ and $t \le \tau_N$. Therefore we can set $u(t) = u_N(t)$ if $t \le \tau_N$, and in this way, taking into account Theorem 2.2, we have constructed a solution to (2.9) in the random interval $[0, \tau_\infty)$, where $\tau_\infty = \sup_N \tau_N$. Then it suffices to show that

$$P(\tau_\infty = T) = 1. \tag{4.17}$$

Define

$$\eta(t, x) = \int_0^t \int_{\mathbb{R}} G(t - s, x, y)\sigma(s, y, u(s, y))\mathbf{1}_{\{s < \tau_\infty\}} W(ds, dy). \tag{4.18}$$

It holds that

$$\eta(t, x) = \eta_N(t, x) := \int_0^t \int_{\mathbb{R}} G(t - s, x, y)\sigma(s, y, u_N(s, y))W(ds, dy),$$

if $t \le \tau_N$. Applying Lemma 3.3 and hypothesis (A3), we obtain

$$\begin{aligned} E \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |\eta(t, x)|^p &\le CE \int_0^T \left(\int_{\mathbb{R}} \mathbf{1}_{\{s < \tau_\infty\}} |\sigma(s, y, u(s, y))|^{2q} \, dy \right)^{p/2} ds \\ &\le C \left(\int_{\mathbb{R}} c(y)^{2q} \, dy \right)^{p/2}. \end{aligned}$$

Thus, if $p > \max(4q/(q-1), 2q)$ we obtain

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |\eta(t,x)|^p < \infty. \tag{4.19}$$

On the other hand, applying Lemma 3.4 we obtain

$$E \sup_{t \in [0,T]} |\eta(t)|_2^p \le C \left(\int_{\mathbb{R}} c(y)^2 \, dy \right)^{p/2}, \tag{4.20}$$

$$E \sup_{t \in [0,T]} |\eta(t)|_4^4 \le C \left(\int_{\mathbb{R}} c(y)^{2q} \, dy \right)^2, \tag{4.21}$$

provided $1 < q < 2$ and $p > 4$.

Set $v(t,x) = u(t,x) - \eta(t,x)$ for $t < \tau_\infty$. By Lemma 4.2 there exist constants C_1 and C_2 such that

$$\sup_{t \in [0,T]} \log|v(t)|_2 \le 2 \log\Big(\big(|u_0|_2^2 + C_1(1 + R_1(\eta))\big) + C_2(1 + R_2(\eta)) \Big).$$

As a consequence, taking into account the estimates (4.19), (4.20) and (4.21), we obtain

$$\sup_N E\left(\sup_{t \in [0,T]} \log|u_N(t)|_2 \right) < \infty.$$

Since

$$\begin{aligned} P(\tau_N \le T) &= P\left(\sup_{t \in [0,T]} \log|u_N(t)|_2 \ge \log N \right) \\ &\le \frac{1}{\log N} E\left(\sup_{t \in [0,T]} \log|u_N(t)|_2 \right) \le \frac{C(1 + |u_0|_2)}{\log N} \end{aligned}$$

for some constant C, we get $\tau_\infty = T$ a.s. □

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