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SLOW POINTS AND FAST POINTS OF LOCAL TIMES

BY LAURENCE MARSALLE

Université Pierre et Marie Curie

Let L be a local time. It is well known that there exist a law of the iterated logarithm and a modulus of continuity for L. Motivated by the case of real Brownian motion, we study the existence of fast points and slow points of L. We prove the existence of such points by considering the right-continuous inverse of L, which is a subordinator.

1. Introduction.

Let M be a strong Markov process started from a fixed point, say 0. According to Blumenthal and Getoor [5], we know that if 0 is regular for itself, that is,

$$\inf\{t > 0: M_t = 0\} = 0 \quad \text{a.s.},$$

then M possesses a *local time* at 0, denoted $(L_t, t \geq 0)$. Roughly speaking, the local time measures the amount of time spent by M at 0.

The right-continuous inverse X of L, defined by

$$X_t = \inf\{u > 0: L_u > t\}, \qquad t \geq 0,$$

is a *subordinator*. We denote by ϕ its Laplace exponent, which is specified by

$$\mathbb{E}[\exp - \lambda X_t] = \exp - t\phi(\lambda), \qquad t \geq 0, \lambda \geq 0.$$

Many results concerning L can be deduced from the corresponding ones for X and involve the Laplace exponent ϕ. In particular, the starting point of this work can be stated as follows.

LAW OF THE ITERATED LOGARITHM (Fristedt and Pruitt [8]). *If for some $\varepsilon > 0$, $\phi(x) \geq x^{\varepsilon}$ for x sufficiently large, then there is a positive constant c_0 such that*

$$\limsup_{s \to 0^+} \frac{L_s \phi(s^{-1} \log|\log s|)}{\log|\log s|} = c_0 \quad a.s.$$

MODULUS OF CONTINUITY (Fristedt and Pruitt [9]). *If for some $\varepsilon > 0$, $\phi(x) \geq x^{\varepsilon}$ for x sufficiently large, then there are two positive constants c' and c'' such that*

$$\liminf_{s \to 0^+} \sup_{0 \leq t \leq X_1} \frac{(L_{t+s} - L_t)\phi(s^{-1}|\log s|)}{|\log s|} = c' \quad a.s.,$$

$$\limsup_{s \to 0^+} \sup_{0 \leq t \leq X_1} \frac{(L_{t+s} - L_t)\phi(s^{-1}|\log s|)}{|\log s|} = c'' \quad a.s.$$

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When the Laplace exponent ϕ is regularly varying at infinity with index $\alpha \in (0, 1)$, we can specify the constants involved in both previous results (see [2]):

$$c_0 = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)},$$
$$c' = c'' = c_\alpha = \alpha^{(1-2\alpha)}(1-\alpha)^{-(1-\alpha)}.$$

These two results can be viewed as analogues of the well-known law of the iterated logarithm of Khintchine and the modulus of continuity of Lévy for a linear Brownian motion. The latter has led Orey and Taylor [16] to study the set of points where Brownian motion grows faster than the law of the iterated logarithm. Similarly, under the assumption that ϕ is regularly varying at infinity with index $\alpha \in (0, 1)$, using the same argument of condensation, we point out that with probability 1,

$$BA\, t \geq 0 \qquad \limsup_{s \to 0^+} \frac{(L_{t+s} - L_t)\,\phi(s^{-1}|\log s|)}{|\log s|} \leq c_\alpha,$$
$$(1) \qquad \exists\, t \geq 0 \qquad \limsup_{s \to 0^+} \frac{(L_{t+s} - L_t)\,\phi(s^{-1}|\log s|)}{|\log s|} = c_\alpha.$$

We call the latter times *fast points* of L, and adapting arguments of Orey and Taylor, we determine their Hausdorff dimension.

THEOREM A. *Suppose that ϕ is regularly varying at infinity with index $\alpha \in (0,1)$ and that $\beta > 0$. Then with probability* 1,

$$\dim\left\{ t \in (0,1) \colon \limsup_{s \to 0^+} \frac{(L_{t+s} - L_t)\,\phi(s^{-1}|\log s|)}{|\log s|} \geq \beta c_\alpha \right\} = \alpha\left(1 - \beta^{1/(1-\alpha)}\right),$$

where dim *denotes the Hausdorff dimension and with the convention that* $\dim E < 0$ *if and only if* $E = \varnothing$.

REMARK. When $\beta = 1$, the Hausdorff dimension is 0, but we know that the set is not empty thanks to (1).

A question then arises naturally: what about *slow points* of local times? This notion was first introduced for linear Brownian motion and studied by Dvoretzky [6], Kahane [12] and others. In the case of local time, we know that L presents intervals on which it is constant, which leads us to modify the notion of slow points. With this aim in mind, we introduce

$$Z^0 = \{t > 0 \colon L_{t-\varepsilon} < L_t < L_{t+\varepsilon},\ \forall\, \varepsilon > 0\},$$

the set of times t where the local time increases both to the left and to the right, and we consider the existence of exceptional times in Z^0 where the rate of growth of L is minimal. When ϕ is a power function (in that case X is a *stable* subordinator), Fristedt [7] proved the existence of slow points and, more precisely, that the minimal rate of growth is given by $s \mapsto 1/\phi(1/|s|)$.

By applying a certain transformation on the family of subordinators, we are able to extend the result of Fristedt under very mild assumptions on ϕ.

THEOREM B. *If there exists some* $\lambda > 1$ *such that*

$$1 < \liminf_{x \to +\infty} \frac{\phi(\lambda x)}{\phi(x)} \le \limsup_{x \to +\infty} \frac{\phi(\lambda x)}{\phi(x)} < \lambda,$$

one has that with probability 1,

$$\forall\, t \in Z^0 \qquad \limsup_{s \to 0} |L_{t+s} - L_t| \phi(1/|s|) > 0,$$

$$\exists\, t \in Z^0 \qquad \limsup_{s \to 0} |L_{t+s} - L_t| \phi(1/|s|) < +\infty.$$

The latter times are called *slow points* of L. Informally, they can be viewed as times where the occupation density at 0 is the smallest, whereas fast points are times where it is the greatest.

This paper is organized as follows. Preliminaries on subordinators and local times are developed in Section 2. Proofs of Theorem A and Theorem B are given in Section 3, and Section 4 is devoted to some examples.

2. Preliminaries.

2.1. *Subordinators*. A process X is called a *subordinator* if it is a right continuous, increasing process, started at $X_0 = 0$ and with independent and stationary increments. That is, for every $t \ge 0$, the shifted process $(X_{t+s} - X_t, s \ge 0)$ is independent of $(X_u, 0 \le u \le t)$ and has the same law as X. This last property implies that the Laplace transform of X can be expressed in the form

$$\mathbb{E}[\exp - \lambda X_t] = \exp - t\phi(\lambda), \qquad t \ge 0,\ \lambda \ge 0,$$

where ϕ is called the *Laplace exponent* of X. Moreover, ϕ is given by the celebrated Lévy–Khintchine formula

$$\phi(\lambda) = k + \mathbf{d}\lambda + \int_{(0,+\infty)} (1 - \exp(-\lambda x)) \pi(dx),$$

where $k = \phi(0)$ is the killing rate, $\mathbf{d} \ge 0$ is the drift coefficient and π the Lévy measure of X. Recall that a measure π can arise as the Lévy measure of some subordinator if and only if

$$\int_{(0,+\infty)} (1 \wedge x) \pi(dx) < +\infty,$$

and we denote by $\overline{\pi} = \pi((x, +\infty))$ its tail. When $k > 0$, X is identical in law with a subordinator with infinite lifetime killed at an independent exponential time with parameter k (see page 73 of [3]). Since the properties we deal with are local ones, we can deduce the behavior of X in the case $k > 0$ from the case $k = 0$. Consequently, we assume henceforth that $k = 0$. We also exclude the cases when X is a compound Poisson process or $\mathbf{d} > 0$, which correspond to a Markov process M which spends a positive time at 0 [that is, $\mathrm{Leb}(\{t: M_t = 0\}) > 0$]. This implies that ϕ is one-to-one and onto from $(0, +\infty)$ on $(0, +\infty)$, and we denote φ its inverse.

Stable subordinators form an important class of subordinators: more precisely, X is called *stable* of index $\gamma \in (0, 1)$ if $\phi(\lambda) = \phi(1)\lambda^\gamma$. An extension of that class is that of subordinators whose Laplace exponent is *regularly varying* at infinity with index $\gamma \in (0, 1)$, that is, for all $\lambda \geq 0$, $\lim_{x \to +\infty}(\phi(\lambda x)/\phi(x)) = \lambda^\gamma$.

More generally, we recall that ϕ being an increasing concave function, such that $\phi(0) = 0$, one easily gets that for every $\lambda \geq 1$, $x > 0$,

$$1 \leq \frac{\phi(\lambda x)}{\phi(x)} \leq \lambda,$$

and consequently the Laplace exponent ϕ satisfies

$$1 \leq \liminf_{x \to +\infty} \frac{\phi(\lambda x)}{\phi(x)} \leq \limsup_{x \to +\infty} \frac{\phi(\lambda x)}{\phi(x)} \leq \lambda.$$

So that the hypothesis which appears in Theorem B, that is,

(H) There exists some $\lambda > 1$ such that $1 < \liminf_{x \to +\infty}(\phi(\lambda x)/\phi(x)) \leq \limsup_{x \to +\infty}(\phi(\lambda x)/\phi(x)) < \lambda$

only excludes the limit cases and is very mild. This assumption implies that ϕ has *O-regular variation*, according to the definition given in [4], page 65. We can then restate Theorems 2.1.7 and 2.1.8 of [4] the following way.

PROPOSITION 1. *If ϕ satisfies* (H), *then there exist $0 < \beta \leq \alpha < 1$, $c \in (0, 1]$, $A \geq 0$, such that for every $y \geq x \geq A$,*

$$c(y/x)^\beta \leq \phi(y)/\phi(x) \leq c^{-1}(y/x)^\alpha.$$

Before stating another useful result about subordinators, we need to introduce some notation. We recall that two positive functions f and g are *of the same order* (denoted $f \asymp g$) if there exists some constant $c \in (0, 1]$ such that for all x,

$$cf(x) \leq g(x) \leq c^{-1}f(x).$$

In the same way, we say that f and g are of the same order near $a \in [0, +\infty]$ if

$$0 < \liminf_{x \to a} \frac{f(x)}{g(x)} \leq \limsup_{x \to a} \frac{f(x)}{g(x)} < +\infty.$$

LEMMA 2. *If X is a subordinator whose Laplace exponent satisfies* (H), *then we have*

$$\overline{\pi}(x) \asymp \phi(1/x) \quad \textit{near } 0.$$

PROOF. We define $I(x) = \int_0^x \overline{\pi}(t)\,dt$, the integrated tail. Since we have supposed that $\mathbf{d} = 0$, Proposition 1 of Chapter III of [2] reads

$$\phi(x) \asymp xI(1/x), \tag{2}$$

so that we only need to show that $I(x)/x \asymp \overline{\pi}(x)$ near 0. The fact that $\overline{\pi}$ decreases, immediately leads to $I(x)/x \geq \overline{\pi}(x)$, for all x.

On the other hand, (2) implies that there exists $c \in (0,1]$ such that for all integers $n \geq 1$,

$$I(nx)/I(x) \geq cn\phi(1/nx)/\phi(1/x),$$

and Proposition 1 gives that $\phi(1/nx)/\phi(1/x) \geq c^{-1}n^{-\alpha}$, for x small enough. Since $1-\alpha > 0$, $\lim_{n\to+\infty} n^{1-\alpha} = +\infty$, so that we can choose an integer n large enough to ensure that

$$\liminf_{x\to 0+} I(nx)/I(x) > 1. \tag{3}$$

Using again the fact that $\overline{\pi}$ decreases, we can deduce that

$$I(nx) \leq I(x) + (n-1)x\overline{\pi}(x),$$

which leads, thanks to (3), to $\liminf_{x\to 0^+} x\overline{\pi}(x)/I(x) > 0$, which completes the proof. $\square$

2.2. *Local times and subordinators.* The aim of this subsection is to give some results concerning L, the local time, and its links with its right-continuous inverse X. More precisely, we establish a useful result which relates the existence of slow points for L to the existence of some exceptional times for X. We recall that $Z^0 = \{t > 0\colon L_{t-\varepsilon} < L_t < L_{t+\varepsilon},\ \forall\ \varepsilon > 0\}$ and that φ denotes the inverse of ϕ, the Laplace exponent of X.

LEMMA 3. *We suppose that ϕ satisfies* (H). *Then the following assertions hold with probability* 1:

(i) *If there exists $t \in (0, +\infty)$ such that*

$$\liminf_{s\to 0} |X_{t+s} - X_t|\varphi(1/|s|) > 0,$$

then X is continuous at t and if we set $u = X_t$, then

$$\limsup_{v\to 0} |L_{u+v} - L_u|\phi(1/|v|) < +\infty.$$

(ii) *If for all $t \geq 0$ $\liminf_{s\to 0}|X_{t+s} - X_t|\varphi(1/|s|) < +\infty$, then for all $u \in Z^0$,*

$$\limsup_{v\to 0} |L_{u+v} - L_u|\phi(1/|v|) > 0.$$

PROOF. (i) Since ϕ satisfies (H), the hypotheses of Theorem 1 of [8] are fulfilled, so that there exists some constant $c_0 > 0$ such that

$$\liminf_{s\to 0^+} \frac{|X_{T+s} - X_T|\varphi(s^{-1}\log|\log s|)}{\log|\log s|} = c_0 \quad \text{a.s.} \tag{4}$$

for every stopping time T. This implies that $\liminf_{s \to 0^+} |X_{T+s} - X_T| \varphi(s^{-1}) = 0$ a.s. whenever T is a stopping time, since

$$\varphi(s^{-1}) = o\left(\frac{\varphi(s^{-1} \log|\log s|)}{\log|\log s|}\right) \quad \text{at } 0^+.$$

Indeed, Proposition 1 entails that there exists $c \in (0, +\infty)$ such that for $y \geq x$ large enough,

$$\frac{\varphi(x)}{\varphi(y)} \leq c\left(\frac{x}{y}\right)^{1/\alpha},$$

with $\alpha \in (0, 1)$. Consequently, for $s > 0$ small enough,

$$0 \leq \frac{\varphi(s^{-1})\log|\log s|}{\varphi(s^{-1}\log|\log s|)} \leq c(\log|\log s|)^{1-1/\alpha},$$

and since $\alpha < 1$, the upper bound actually goes to zero as $s \to 0^+$.

Finally, since the jump times of X can be described as the countable union of stopping times, (4) implies that X is continuous at whatever time t such that $\liminf_{s \to 0} |X_{t+s} - X_t| \varphi(1/|s|) > 0$.

We now set $u = X_t$ and $\eta = L_{u+v} - L_u$, so $L_{u+v} = t + \eta$. Since X is right continuous we get that

$$X_{(t+\eta)-} - X_t \leq v \leq X_{(t+\eta)} - X_t. \tag{5}$$

We suppose that $v > 0$, so that (5) yields $|v| \geq |X_{(t+\eta)-} - X_t|$ and

$$\begin{aligned} |L_{u+v} - L_u| \phi(1/|v|) &\leq |\eta| \phi\big(1/|X_{(t+\eta)-} - X_t|\big) \\ &= \frac{\phi\big(1/|X_{(t+\eta)-} - X_t|\big)}{\phi(\varphi(1/|\eta|))}. \end{aligned}$$

But since $\liminf_{s \to 0} |X_{t+s} - X_t| \varphi(1/|s|) > 0$, there exist $\varepsilon > 0$, $\xi > 0$ such that for all s with $|s| < \xi$,

$$|X_{t+s} - X_t| \geq \frac{\varepsilon}{\varphi(1/|s|)}.$$

Now let $s = u + v$, with $|u| < \xi/2$ and $-\xi/2 < v < 0$; the preceding inequality thus reads

$$|X_{(t+u)+v} - X_t| \geq \frac{\varepsilon}{\varphi(1/|u+v|)},$$

which gives, when v goes to 0,

$$|X_{(t+u)-} - X_t| \geq \frac{\varepsilon}{\varphi(1/|u|)},$$

for all $|u| < \xi/2$. This entails that

$$\liminf_{s \to 0} |X_{(t+s)-} - X_t| \varphi(1/|s|) > 0,$$

and then, using Proposition 1, that

$$\limsup_{\eta \to 0^+} \frac{\phi\big(1/|X_{(t+\eta)-} - X_t|\big)}{\phi(\varphi(1/|\eta|))} < +\infty.$$

Consequently $|L_{u+v} - L_u|\phi(1/|v|)$ is bounded.

When $v < 0$, a similar argument works with $X_{(t+\eta)-}$ replaced by $X_{t+\eta}$.

(ii) Let us suppose that there exists some $u \in Z^0$ such that

$$\limsup_{v \to 0} |L_{u+v} - L_u|\phi(1/|v|) = 0.$$

We then set $t = L_u$ and $\eta = X_{t+s} - X_t$. Since u belongs to Z^0, $X_t = u$ and consequently $t + s = L_{\eta+u}$. We thus have

$$|X_{t+s} - X_t|\varphi(1/|s|) = |\eta|\varphi\big(1/|L_{\eta+u} - L_u|\big) = \frac{\varphi\big(1/|L_{\eta+u} - L_u|\big)}{\varphi(\phi(1/|\eta|))}.$$

But since ϕ satisfies (H), the fact that the limit of $(|L_{u+v} - L_u|\phi(1/|v|))^{-1}$ is infinite implies that the limit of $|X_{t+s} - X_t|\varphi(1/|s|)$ is infinite as well. This completes the proof of the lemma. □

3. Proofs.

3.1. *Proof of Theorem A.* We introduce some notations. For $\beta \in (0,1)$ and $\gamma > 1$, we set

$$E(\beta) = \left\{ t > 0 \colon \limsup_{s \to 0^+} \frac{(L_{t+s} - L_t)\phi\big(s^{-1}|\log s|\big)}{|\log s|} \geq \beta c_\alpha \right\},$$

$$F(\gamma) = \left\{ t > 0 \colon \liminf_{s \to 0^+} \frac{(X_{t+s} - X_t)\varphi\big(s^{-1}|\log s|\big)}{|\log s|} \leq \gamma d_\alpha \right\},$$

where we recall that $c_\alpha = \alpha^{(1-2\alpha)}(1-\alpha)^{-(1-\alpha)}$ and we set $d_\alpha = \alpha(1-\alpha)^{(1-\alpha)/\alpha}$. We first show that

$$E(\beta) = X\big(F(\beta^{-1/\alpha})\big) \quad \text{a.s.}$$

We fix $\beta \in (0,1)$ and ω such that $E(\beta)(\omega) \neq \emptyset$. Let $t \in E(\beta)$, then there exist $l \in [\beta c_\alpha, c_\alpha]$ and a sequence $s_n \to 0^+$ such that

$$\lim_{n \to +\infty} \frac{(L_{t+s_n} - L_t)\phi\big(s_n^{-1}|\log s_n|\big)}{|\log s_n|} = l. \tag{6}$$

Since $l > 0$, this implies that L increases just after t, and consequently $X \circ L_t = L \circ X_t = t$. We thus set $u = L_t$ and $v_n = L_{t+s_n} - L_t$ (remark that $v_n > 0$). Since X has right-continuous increasing paths, we get that $X_{(u+v_n)-} - X_u \leq s_n$.

However (6) entails that for all $\varepsilon > 0$ there exists n_0 such that

$$\frac{(L_{t+s_n} - L_t)\phi(s_n^{-1}|\log s_n|)}{|\log s_n|} \geq (\beta c_\alpha - \varepsilon), \qquad n \geq n_0,$$

and since φ is the increasing inverse of ϕ, this inequality yields that for $n \geq n_0$,

$$\frac{v_n \phi\left(\left(X_{(u+v_n)-} - X_u\right)^{-1}|\log s_n|\right)}{|\log s_n|} \geq (\beta c_\alpha - \varepsilon),$$

$$\left(X_{(u+v_n)-} - X_u\right)^{-1}|\log s_n| \geq \varphi\left((\beta c_\alpha - \varepsilon) v_n^{-1}|\log s_n|\right),$$

$$\frac{\left(X_{(u+v_n)-} - X_u\right)\varphi\left(v_n^{-1}|\log v_n|\right)}{|\log v_n|} \leq \frac{|\log s_n|}{|\log v_n|} \times \frac{\varphi\left(v_n^{-1}|\log v_n|\right)}{\varphi\left((\beta c_\alpha - \varepsilon) v_n^{-1}|\log s_n|\right)}.$$

Now, combining the facts that $L_{t+s_n} - L_t \sim (l|\log s_n|/\phi(s_n^{-1}|\log s_n|))$ and that ϕ is regularly varying at infinity with index α, we get that

$$\lim_{n\to+\infty} \frac{|\log s_n|}{|\log v_n|} = \lim_{n\to+\infty} \frac{|\log s_n|}{\left|\log\left(L_{t+s_n} - L_t\right)\right|} = 1/\alpha.$$

Using this time that φ is regularly varying at infinity with index $1/\alpha$, we obtain that for all $\varepsilon > 0$,

$$\liminf_{v\to 0^+} \frac{\left(X_{(u+v)-} - X_u\right)\varphi\left(v^{-1}|\log v|\right)}{|\log v|} \leq \alpha^{-1}\left(\frac{\alpha}{\beta c_\alpha - \varepsilon}\right)^{1/\alpha}.$$

Since X is an increasing process, we can replace $(X_{(u+v)-} - X_u)$ by $(X_{u+v} - X_u)$. An easy computation yields that $\alpha^{-1}(\alpha/\beta c_\alpha)^{1/\alpha} = \beta^{-1/\alpha} d_\alpha$ and allows us to conclude that $u \in F(\beta^{-1/\alpha})$ and thus that $E(\beta) \subset X(F(\beta^{-1/\alpha}))$. Similar arguments are involved to prove the converse inclusion.

The equality $E(\beta) = X(F(\beta^{-1/\alpha}))$ now yields

$$\dim E(\beta) = \alpha \dim F(\beta^{-1/\alpha})$$

by using both Theorem 5.1 of [10] and the fact that, since ϕ has regular variation at infinity, the lower and upper indices of X coincide and are equal to α.

Our problem now reduces to showing that $\dim F(\gamma) = 1 - \gamma^{-\alpha/(1-\alpha)}$. To compute this Hausdorff dimension, we adapt the proof of Theorem 2 of [16], which states a similar result for linear Brownian motion. The arguments are very close, except that we use the fact that X has increasing paths instead of the continuity of Brownian paths. Moreover, the proof relies heavily on the following property of X when ϕ is regularly varying at infinity with index $\alpha \in (0, 1)$.

$$-\log \mathbb{P}\left(\frac{X_t \varphi(t^{-1}|\log t|)}{|\log t|} \leq \gamma d_\alpha\right) \sim \gamma^{-\alpha/(1-\alpha)}|\log t|$$

when $t \to 0^+$. This estimate is proved in [2], Lemma 2, and relies on results from [11]. The proof has been completely written in [15], Section 2.4.2. □

3.2. *Proof of Theorem B*. Thanks to Lemma 3, the proof of Theorem B amounts to showing that with probability 1,

$$\begin{aligned} &\exists\, t > 0 \qquad \liminf_{s\to 0} |X_{t+s} - X_t| \varphi(1/|s|) > 0, \\ &\exists\, t \geq 0 \qquad \liminf_{s\to 0} |X_{t+s} - X_t| \varphi(1/|s|) < +\infty. \end{aligned} \tag{7}$$

When S is a stable subordinator of index γ, $\varphi(x) = x^{1/\gamma}$ and (7) is obviously true, thanks to the following result in [7].

PROPOSITION 4. *If S is a stable subordinator of index $\gamma \in (0,1)$ then there exists a constant $c_\gamma \in (0, +\infty)$ such that for all intervals $I \subset (0, +\infty)$ of positive length, with probability* 1,

$$\sup_{t\in I} \liminf_{s\to 0} \frac{|S_{t+s} - S_t|}{|s|^{1/\gamma}} = c_\gamma.$$

Our task consists in reducing Theorem B to Proposition 4. To this aim, we define a transformation on subordinators the following way. For $\gamma \in (0,1)$, we consider

$$\begin{aligned} h^{(\gamma)}\colon (0, +\infty) &\to (0, +\infty), \\ x &\mapsto \left(1/\phi(1/x)\right)^{1/\gamma}, \end{aligned}$$

which is clearly one-to-one, and we denote $g^{(\gamma)}$ its inverse: $g^{(\gamma)}(x) = 1/\varphi(x^{-\gamma})$. Let ΔX be the jump process of X; that is, $\Delta X_s = X_s - X_{s-}$. Since X is a subordinator, ΔX is a Poisson point process whose characteristic measure is π. Hence the process $(h^{(\gamma)}(\Delta X_t),\ t \geq 0)$ is also a Poisson point process whose tail characteristic measure $\bar{\nu}^{(\gamma)}$ is given by

$$\bar{\nu}^{(\gamma)}(x) = \bar{\pi}\big(g^{(\gamma)}(x)\big).$$

Since $\bar{\pi}(x) \asymp \phi(1/x)$ near 0 (Lemma 2), one easily gets that

$$\bar{\nu}^{(\gamma)}(x) \asymp x^{-\gamma} \quad \text{near } 0, \tag{8}$$

and consequently,

$$\int_{(0,+\infty)} (1 \wedge x)\, \nu^{(\gamma)}(dx) < +\infty,$$

which implies that the newly defined process

$$Y_t^{(\gamma)} = \sum_{0\leq s\leq t} h^{(\gamma)}(\Delta X_s)$$

is also a subordinator. This subordinator is not a stable subordinator, but (8) suggests that its behavior might be not much different. More precisely, the following holds.

LEMMA 5. *For all* $\gamma \in (0,1)$, *there exists a constant* $d_\gamma \in (0,+\infty)$ *such that for all interval* $I \subset (0,+\infty)$ *of positive length, with probability* 1,

$$\sup_{t\in I} \liminf_{s\to 0} \frac{|Y^{(\gamma)}_{t+s} - Y^{(\gamma)}_t|}{|s|^{1/\gamma}} = d_\gamma.$$

PROOF. We fix $\gamma \in (0,1)$ and set some notations. Let S denote a stable subordinator of index γ, ν its Lévy measure. Let $a \in (0,1)$; we define

$$\begin{aligned} f_a\colon [0,+\infty) &\to \{a^n : n \in \mathbb{Z}\},\\ x &\mapsto a^{n+1} \quad \text{if } x \in [a^{n+1}, a^n) \end{aligned}$$

and $f_a(0) = 0$. If we denote by ΔS the jump process of S, then

$$S^a_t = \sum_{0\le u\le t} f_a(\Delta S_u)$$

defines a new subordinator, which jumps exactly at the same times as S, but the magnitude of the jumps is changed the following way: for all $u \ge 0$,

$$a\,\Delta S_u \le \Delta S^a_u \le \Delta S_u. \tag{9}$$

This entails that for every $t \ge 0$ and u such that $t + u \ge 0$,

$$a|S_{t+u} - S_t| \le |S^a_{t+u} - S^a_t| \le |S_{t+u} - S_t|,$$

which immediately leads to

$$\begin{aligned} a \sup_I \liminf_{u\to 0} \frac{|S_{t+u} - S_t|}{|u|^{1/\gamma}} &\le \sup_I \liminf_{u\to 0} \frac{|S^a_{t+u} - S^a_t|}{|u|^{1/\gamma}}\\ &\le \sup_I \liminf_{u\to 0} \frac{|S_{t+u} - S_t|}{|u|^{1/\gamma}}, \end{aligned}$$

for all intervals $I \subset (0,+\infty)$, and applying Proposition 4 it comes out that

$$ac_\gamma \le \sup_I \liminf_{u\to 0} \frac{|S^a_{t+u} - S^a_t|}{|u|^{1/\gamma}} \le c_\gamma.$$

Itô's representation of subordinators, and a Kolmogorov 0–1 law (see [9]) then allow us to conclude that there exists some constant $c^a_\gamma \in (0,+\infty)$ such that for all $I \subset (0,+\infty)$,

$$\sup_I \liminf_{u\to 0} \frac{|S^a_{t+u} - S^a_t|}{|u|^{1/\gamma}} = c^a_\gamma \quad \text{a.s.}$$

In an analogous way, we associate to $Y^{(\gamma)}$ a new subordinator, denoted Z, defined by

$$Z_t = \sum_{0\le u\le t} f_a\big(\Delta Y^{(\gamma)}_u\big).$$

For both S^a and Z, the Lévy measures have their supports in $\{a^n: n \in \mathbb{Z}\}$. We denote these measures by ν_a and $\nu_a^{(\gamma)}$, respectively. Thus

$$\nu_a(\{a^{n+1}\}) = \nu([a^{n+1}, a^n)) = a^{-\gamma n}(a^{-\gamma} - 1),$$
$$\nu_a^{(\gamma)}(\{a^{n+1}\}) = \nu^{(\gamma)}([a^{n+1}, a^n)).$$

However, $\bar{\nu}^{(\gamma)}(x) \asymp x^{-\gamma}$ near 0, so that there exists some constant $c \in (0,1]$ such that the following inequalities hold:

$$cx^{-\gamma} \le \bar{\nu}^{(\gamma)}(x) \le c^{-1}x^{-\gamma},$$
$$a^{-\gamma n}(ca^{-\gamma} - c^{-1}) \le \nu_a^{(\gamma)}(\{a^{n+1}\}) \le a^{-\gamma n}c^{-1}a^{-\gamma}.$$

If we choose a such that $a < c^{2/\gamma}$, there exist two positive constants k and K such that

$$k\nu_a(dx) \le \nu_a^{(\gamma)}(dx), \tag{10}$$

$$\nu_a^{(\gamma)}(dx) \le K\nu_a(dx). \tag{11}$$

Inequality (10) implies that Z has the same law as the sum of $S^a_{k\cdot}$ and of a subordinator V independent of S^a, so that

$$\sup_I \liminf_{s\to 0} \frac{|Z_{t+s} - Z_t|}{|s|^{1/\gamma}} \overset{(\text{law})}{=} \sup_I \liminf_{s\to 0} \frac{|S^a_{k(t+s)} - S^a_{kt} + V_{t+s} - V_t|}{|s|^{1/\gamma}}.$$

However, $|S^a_{k(t+s)} - S^a_{kt} + V_{t+s} - V_t| = |S^a_{k(t+s)} - S^a_{kt}| + |V_{t+s} - V_t|$, since both processes are increasing, and this leads to

$$\begin{aligned}\sup_I \liminf_{s\to 0} \frac{|S^a_{k(t+s)} - S^a_{kt} + V_{t+s} - V_t|}{|s|^{1/\gamma}} &\ge k^{1/\gamma} \sup_{kI} \liminf_{s\to 0} \frac{|S^a_{t+s} - S^a_t|}{|s|^{1/\gamma}} \\ &= k^{1/\gamma} c^a_\gamma \quad \text{a.s.}\end{aligned}$$

Similarly, (11) entails that $S^a_{K\cdot}$ has the same law as the sum of Z and of an independent subordinator W. Thus

$$\sup_I \liminf_{s\to 0} \frac{|S^a_{K(t+s)} - S^a_{Kt}|}{|s|^{1/\gamma}} \overset{(\text{law})}{=} \sup_I \liminf_{s\to 0} \frac{|Z_{t+s} - Z_t + W_{t+s} - W_t|}{|s|^{1/\gamma}}.$$

However,

$$\begin{aligned}\sup_I \liminf_{s\to 0} \frac{|S^a_{K(t+s)} - S^a_{Kt}|}{|s|^{1/\gamma}} &= K^{1/\gamma} \sup_{KI} \liminf_{s\to 0} \frac{|S^a_{t+s} - S_t|}{|s|^{1/\gamma}} \\ &= K^{1/\gamma} c^a_\gamma \quad \text{a.s.}\end{aligned}$$

On the other hand, $|Z_{t+s} - Z_t + W_{t+s} - W_t| = |Z_{t+s} - Z_t| + |W_{t+s} - W_t|$, which leads to

$$\sup_I \liminf_{s\to 0} \frac{|Z_{t+s} - Z_t|}{|s|^{1/\gamma}} \le K^{1/\gamma} c^a_\gamma \quad \text{a.s.}$$

and finally

$$k^{1/\gamma} c^a_\gamma \le \sup_I \liminf_{s\to 0} \frac{|Z_{t+s} - Z_t|}{|s|^{1/\gamma}} \le K^{1/\gamma} c^a_\gamma \quad \text{a.s.}$$

Since the lower bound is positive a.s., and the upper bound is finite a.s., using again the Kolmogorov 0–1 law, we get that there exists some constant $\delta_\gamma \in (0, +\infty)$ such that for all $I \subset (0, +\infty)$,

$$\sup_I \liminf_{s \to 0} \frac{|Z_{t+s} - Z_t|}{|s|^{1/\gamma}} = \delta_\gamma \quad \text{a.s.}$$

Now we deduce the result for $Y^{(\gamma)}$ from the result for Z the same way we deduced it from S for S^a. This completes the proof of the lemma. □

We now tackle the proof of Theorem B. The idea consists in comparing the increments of $Y^{(\gamma)}$ with the increments of X, for two well-chosen values of γ. Indeed, recalling Proposition 1, we know that

$$c(y/x)^\beta \le \phi(y)/\phi(x) \le c^{-1}(y/x)^\alpha$$

for $y \ge x \ge A$. Informally, this inequality implies that $t \mapsto h^{(\alpha)}(t)/t$ is nearly decreasing, and $t \mapsto h^{(\beta)}(t)/t$ nearly increasing. More precisely, if $t \le u \le A$, the following holds:

$$\phi(1/u)/\phi(1/t) \ge c(t/u)^\alpha,$$
$$h^{(\alpha)}(t)/t \ge c^{1/\alpha} h^{(\alpha)}(u)/u,$$

and the analogous inequalities for $h^{(\beta)}$ are obvious. These properties of approximate monotonicity entail that $h^{(\alpha)}$ is nearly subadditive and that $h^{(\beta)}$ is nearly superadditive. Indeed, if $(u_n)_{n \ge 0}$ is any sequence of positive numbers such that $\Sigma u_n \le A$, then for all $n \ge 0$,

$$h^{(\alpha)}(u_n) \ge \left(c^{1/\alpha} \frac{h^{(\alpha)}(\Sigma u_n)}{\Sigma u_n} \right) u_n,$$

and consequently $\Sigma h^{(\alpha)}(u_n) \ge c^{1/\alpha} h^{(\alpha)}(\Sigma u_n)$. Mutatis mutandi, a similar inequality holds for $h^{(\beta)}$ in reverse direction. For the sake of simplicity, we rename b_1 and b_2 the constants involved in these inequalities, so that

$$b_1 h^{(\alpha)}\big(|X_{t+s} - X_t|\big) \le |Y^{(\alpha)}_{t+s} - Y^{(\alpha)}_t|,$$
$$|Y^{(\beta)}_{t+s} - Y^{(\beta)}_t| \le b_2 h^{(\beta)}\big(|X_{t+s} - X_t|\big).$$

Now Lemma 5 implies that there exists $d_1 \in (0, +\infty)$ such that

$$\forall\ t \ge 0 \qquad \liminf_{s \to 0} h^{(\alpha)}\big(|X_{t+s} - X_t|\big)|s|^{-1/\alpha} \le d_1 \quad \text{a.s.}$$

For any fixed t, there exists a sequence $s_n \to 0$ such that

$$h^{(\alpha)}\big(|X_{t+s_n} - X_t|\big)|s_n|^{-1/\alpha} \le 2d_1, \qquad n \ge 0.$$

If one recalls that $g^{(\alpha)} = 1/\varphi(x^{-\alpha})$ is the increasing inverse of $h^{(\alpha)}$, one easily gets the following inequalities:

$$|X_{t+s_n} - X_t| \le g^{(\alpha)}\big(2d_1|s_n|^{1/\alpha}\big),$$

$$|X_{t+s_n} - X_t|\varphi(1/|s_n|) \le \frac{\varphi(1/|s_n|)}{\varphi\big((2d_1)^{-\alpha}/|s_n|\big)}.$$

However, ϕ satisfies (H), which entails that $\limsup_{s\to+\infty}(\varphi(1/|s_n|)/\varphi((2d_1)^{-\alpha}/|s_n|)$ *is finite since the ratio* $1/|s_n|/(2d_1)^{-\alpha}/|s_n|$ *is constant*, *and finally*,

$$\liminf_{s\to 0}|X_{t+s} - X_t|\varphi(1/|s|) < +\infty.$$

It only remains to show that there exists some t such that the liminf is positive. Using again Lemma 5, *we know that there exist* $d_2 > 0$, $t > 0$ *such that* $\liminf_{s\to 0}|Y^{(\beta)}_{t+s} - Y^{(\beta)}_t|\,|s|^{-1/\beta} > d_2$. *This implies that for s small enough*,

$$h^{(\beta)}(|X_{t+s} - X_t|)|s|^{-1/\beta} \ge d_2/2,$$

$$|X_{t+s} - X_t| \ge g^{(\beta)}\big(d_2|s|^{1/\beta}/2\big),$$

$$|X_{t+s} - X_t|\varphi(1/|s|) \ge \frac{\varphi(1/|s|)}{\varphi\big((d_2/2)^{-\beta}/|s|\big)},$$

and the lower bound is bounded away from zero, *again because* ϕ *satisfies* (H). *Thus*

$$\liminf_{s\to 0}|X_{t+s} - X_t|\varphi(1/|s|) > 0$$

and the proof of Theorem B is complete. □

4. Examples. To conclude this paper, we present some examples of local times for which both Theorem A and Theorem B apply.

4.1. *Stable Lévy processes*. Let M denote a stable Lévy process of index $\alpha \in (1,2]$. We thus know (see [18]) that M admits a local time at 0, L, whose right-continuous inverse X is a stable subordinator of index $\beta = 1 - 1/\alpha$. Its Laplace exponent is given by $\phi(\lambda) = c\lambda^{1-1/\alpha}$, where c denotes some constant whose value can be specified in terms of the Lévy exponent of M (see [15]). Hypotheses of Theorem A as well as of Theorem B are satisfied. We thus obtain the existence of fast points of L and for $\gamma \in (0, c^{-1}\alpha^{1/\alpha}(1 - 1/\alpha)^{(2/\alpha-1)})$, we can compute the Hausdorff dimension

$$\dim\left\{t \in (0,1)\colon \limsup_{s\to 0^+}\frac{L_{t+s} - L_t}{s^{1-1/\alpha}|\log s|^{1/\alpha}} \ge \gamma\right\}$$

$$= \left(1 - \frac{1}{\alpha}\right)\left(1 - \gamma^{\alpha}\alpha^{-1}c^{\alpha}\left(1 - \frac{1}{\alpha}\right)^{-(2-\alpha)}\right).$$

In the same way, we get the existence of slow points; that is, with probability 1,

$$\forall\, t \in Z^0 \qquad \limsup_{s \to 0} \frac{|L_{t+s} - L_t|}{|s|^{1-1/\alpha}} > 0,$$

$$\exists\, t \in Z^0, \qquad \limsup_{s \to 0} \frac{|L_{t+s} - L_t|}{|s|^{1-1/\alpha}} < +\infty.$$

4.2. *Bessel processes*. Now M is a d-dimensional Bessel process, with $0 < d < 2$. In that case there exists a jointly continuous family of local times, $(L_t^x,\ x \ge 0,\ t \ge 0)$, defined by

$$\int_0^t f(M_s)\, ds = c \int_0^{+\infty} f(x) x^{d-1} L_t^x\, dx,$$

where $f\colon [0, +\infty) \to [0, +\infty)$ is any Borelian function. If we choose

$$c = \frac{\Gamma(1 - d/2)}{\Gamma(d/2)} 2^{1-d/2}$$

according to [1], the Laplace exponent corresponding to L^0 is given by $\phi(\lambda) = \lambda^{1-d/2}$. In that case, Theorem A and B read, respectively,

$$\dim \left\{ t \in (0,1) \colon \limsup_{s \to 0^+} \frac{L_{t+s} - L_t}{s^{1-d/2} |\log s|^{d/2}} \ge \gamma \right\}$$
$$= \left(1 - \frac{d}{2}\right)\left(1 - \gamma^{2/d} \left(\frac{d}{2}\right)\left(1 - \frac{d}{2}\right)^{2/d-1}\right) \quad \text{a.s.,}$$

for $\gamma \in (0, (d/2)^{d/2}(1 - d/2)^{d-1})$, and with probability 1,

$$\forall\, t \in Z^0 \qquad \limsup_{s \to 0} \frac{|L_{t+s} - L_t|}{|s|^{1-d/2}} > 0,$$

$$\exists\, t \in Z^0 \qquad \limsup_{s \to 0} \frac{|L_{t+s} - L_t|}{|s|^{1-d/2}} < +\infty.$$

4.3. *Generalized diffusions*. We refer here to [14] and [13]. Let B be a real Brownian motion, and $(l_t^x,\ x \in \mathbb{R},\ t \ge 0)$ the jointly continuous family of its local times. Let $m\colon [0, +\infty) \to [0, +\infty)$ be an increasing, right-continuous function, such that m is regularly varying with index $\beta > 0$ at 0^+. We thus define

$$A_t = \tfrac{1}{2} \int_{[0, +\infty)} l_t^x m(dx),$$
$$\tau_s = \inf\{t > 0 \colon A_t > s\},$$
$$M_t = B_{\tau_t}.$$

The process M is a Hunt process with natural scale and speed measure dm.

Its local time at zero is defined for $t \geq 0$ by

$$L_t = \lim_{\varepsilon \to 0^+} \frac{1}{m(\varepsilon)} \int_0^t \mathbb{1}_{(M_s \leq \varepsilon)} \, ds.$$

The associated Laplace exponent, ϕ, is thus regularly varying at infinity with index $1/(1+\beta)$ (see [14]), so the hypotheses of Theorem A and B are fulfilled.

REMARK. Shieh and Taylor have studied the logarithmic multifractal spectrum of the occupation measure of a stable subordinator in [17]. In particular, they give very sharp results on two-sided "fast points" and compute their Hausdorff dimension (see Theorem 5.1). Our result concerning one-sided "fast points" for general subordinators gives the same Hausdorff dimension (Theorem A), but the methods involved are quite different.

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LABORATOIRE DE PROBABILITÉS
TOUR 56
UNIVERSITÉ PIERRE ET MARIE CURIE
4 PLACE JUSSIEU
F-75252 PARIS CEDEX 05
FRANCE
E-MAIL: marsalle@proba.jussieu.fr