The Annals of Probability
1999, Vol. 27, No. 1, 130–149

ROTATION NUMBERS FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Let $dx = \sum_{i=0}^{m} A_i x \circ dW^i$ be a linear SDE in $\mathbb{R}^d$, generating the flow Φ_t of linear isomorphisms. The multiplicative ergodic theorem asserts that every vector $v \in \mathbb{R}^d \setminus \{0\}$ possesses a Lyapunov exponent (exponential growth rate) $\lambda(v)$ under Φ_t, which is a random variable taking its values from a finite list of canonical exponents λ_i realized in the invariant Oseledets spaces E_i. We prove that, in the case of simple Lyapunov spectrum, every 2-plane p in $\mathbb{R}^d$ possesses a rotation number $\rho(p)$ under Φ_t which is defined as the linear growth rate of the cumulative infinitesimal rotations of a vector v_t inside $\Phi_t(p)$. Again, $\rho(p)$ is a random variable taking its values from a finite list of canonical rotation numbers ρ_{ij} realized in span (E_i, E_j). We give rather explicit Furstenberg–Khasminskii-type formulas for the ρ_{ij}. This carries over results of Arnold and San Martin from random to stochastic differential equations, which is made possible by utilizing anticipative calculus.

1. Introduction. Notations and preliminaries. Smooth ergodic theory is based on Oseledets's fundamental multiplicative ergodic theorem (MET) [10]. It provides us with a random substitute of linear algebra (spectral theory) and hence is at the basis of local theory of nonlinear deterministic and random dynamical systems under an invariant measure (smooth ergodic theory). See [1] for a survey and [2] for a comprehensive presentation.

The MET establishes the existence of exponential growth rates (Lyapunov exponents) for every tangent vector under the linearized flow. Lyapunov exponents are the stochastic analogue of the real parts of deterministic eigenvalues, and reduce to them in the absence of noise.

This paper aims at establishing the existence of a stochastic analogue of the imaginary parts of deterministic eigenvalues (so-called rotation numbers) for the case of linear stochastic differential equations. Again, the concept is such that rotation numbers reduce to imaginary parts of eigenvalues in the absence of noise.

The infinitesimal concept of rotation number of a 2-plane under a flow generated by a vector field on a manifold was introduced by San Martin [12]. The existence of rotation numbers for arbitrary planes was proved by Arnold and San Martin [5] for random differential equations under an invariant measure. See also [2], Section 6.5 for a systematic presentation. Rotation

Received December 1997; revised August 1998.

AMS 1991 *subject classifications*. Primary 60H10, 93E03; secondary 34D08, 34F05.

Key words and phrases. Stochastic differential equation, random dynamical system, cocycle, stochastic flow, anticipative calculus, multiplicative ergodic theorem, rotation number, Furstenberg–Khasminskii formula

numbers for stochastic differential equations were studied by Ruffino [11]. The problem has, however, defied a thorough analysis due to the appearance of quantities in the MET which are not adapted to the canonical Wiener filtration. It thus calls for the use of anticipative calculus, which is what we intend to do here.

Rotation numbers are important in stochastic bifurcation theory (see [2], Chapter 9). They also describe the rotation of invariant manifolds (see [11], Section 6).

Our basic probability space is the m-dimensional canonical Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$, enlarged such as to carry an m-dimensional "Brownian motion" indexed by $\mathbb{R}$. More precisely, $\Omega = \mathscr{C}(\mathbb{R}, \mathbb{R}^m)$ is the set of continuous functions on $\mathbb{R}$ with values in $\mathbb{R}^m$, $\mathscr{F}$ is the σ-algebra of Borel sets with respect to uniform convergence on compacts of $\mathbb{R}$, $\mathbb{P}$ the probability measure on $\mathscr{F}$ for which the "canonical Wiener process" $W_t = (W_t^1, \dots, W_t^m)$, $t \in \mathbb{R}$, makes both $(W_t)_{t \geq 0}$ and $(W_{-t})_{t \geq 0}$ usual m-dimensional Brownian motions which are independent. The natural filtration $\mathscr{F}_s^t = \sigma(W_u - W_v\colon\ s \leq u,\ v \leq t)$, $-\infty \leq s \leq t \leq \infty$, of W is assumed to be completed by the $\mathbb{P}$-completion of $\mathscr{F}$. For $t \in \mathbb{R}$, let $\theta_t\colon \Omega \to \Omega$, $\omega \mapsto \omega(t + \cdot\,) - \omega(t)$, be the "shift" of ω by t. It is well-known that θ_t preserves Wiener measure $\mathbb{P}$ for any $t \in \mathbb{R}$ and is even ergodic. Hence $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system (see [2], Appendix A).

As usual, we use a "$\circ$" to denote Stratonovich integrals with respect to the Wiener process.

We shall have to use some basic results rooted in Malliavin's calculus, for which we briefly recall the main concepts. For a more detailed treatment see [9]. For $1 \leq j \leq m$ we shall denote by D^j the derivative operator, which for a smooth random variable of the form

$$F = f\big(W_{t_1}, \dots, W_{t_n}\big), \qquad f \in \mathscr{C}_b^\infty\big((\mathbb{R}^m)^n\big),\, t_1, \dots, t_n \in \mathbb{R}_+,$$

takes the form

$$D_u^j F = \sum_{i=1}^n \frac{\partial}{\partial x_{i,j}} f\big(W_{t_1}, \dots, W_{t_n}\big) 1_{[0,t_i]}(u).$$

For $T \geq 0$ and each $p \geq 1$, $\mathbb{D}_1^p([0,T])$ will denote the Banach space of random variables on Wiener space defined as the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{p,1} := \|F\|_p + \sum_{j=1}^m \mathbb{E}\left(\left(\int_0^T |D_u^j F|^2\, du\right)^{p/2}\right)^{1/p},$$

to which D^j extends in a natural way. Malliavin's calculus will enter our treatment of rotation numbers of linear SDE via a well-known formula for Stratonovich integrals figuring in [9], page 151. We shall briefly recall it. Let $1 \leq j \leq m$ and $(u_t)_{t \geq 0}$ be a process which is smooth enough so that its

Skorokhod integral exists, along with the "traces"

$$D_t^{j+} u_t := \lim_{\varepsilon \downarrow 0} D_t^j u_{t+\varepsilon}, \qquad D_t^{j-} u_t := \lim_{\varepsilon \downarrow 0} D_t^j u_{t-\varepsilon},$$

$t \geq 0$, in the usual sense as elements of some L^p space. Then

$$\int_0^t u_s \circ dW_s^j = \int_0^t u_s \, dW_s^j + \tfrac{1}{2} \int_0^t \left(D_s^{j+} u_s + D_s^{j-} u_s \right) ds, \tag{1.1}$$

where the first integral on the right-hand side of (1.1) is a Skorokhod integral with respect to W^j. Skorokhod's stochastic integral extends Itô's integral to nonadapted integrands and shares with the latter the property of being centered. The first trace term on the right-hand side of (1.1) hides the well-known Itô–Stratonovich conversion term for adapted integrands.

We shall deal with matrices and operators on $\mathbb{R}^d$, equipped with the standard basis and standard scalar product, mainly. By I we denote the d-dimensional unit matrix (identity operator).

2. The description of rotation. We will now present the main ingredients for the explicit description of rotation numbers of 2-planes of d-dimensional linear stochastic systems. Since rotation is an infinitesimal concept; that is, is described in terms of the generator of the flow, we have to keep track of the structure of the vector fields involved. These are essentially the following: the vector fields generating the flow induced by the linear flow on the unit sphere in $\mathbb{R}^d$ and the vector fields generating rotation by $+90°$ inside a plane transported by the linear flow.

The resulting concept of rotation number of a plane coincides with the one presented in [5] and [2], Section 6.5 for random differential equations, working with the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^d$. We have, however, chosen a more "user-friendly", elementary language here, as we shall express all quantities involved in terms of $d \times d$ matrices.

Let A_i, $0 \leq i \leq m$, be $d \times d$ matrices. We consider the linear stochastic differential equation (SDE),

$$dx_t = \sum_{i=0}^{m} A_i x_t \circ dW_t^i, \tag{2.1}$$

where we use the convention $dW_t^0 = dt$ for abbreviation.

Let $(\Phi_t)_{t \in \mathbb{R}}$ be the flow of linear isomorphisms generated by (2.1). We want to describe the rotation of a vector moved by the flow with respect to a plane which is moved by the flow as well. For this purpose we start out by observing how the linear flow moves planes and lines. As has been pointed out above, we are mainly interested in the vector fields generating this transport on certain Grassmannian manifolds. We briefly recall some facts from [7]. For $1 \leq k \leq d$ and $A \in \mathbb{R}^{d \times d}$, let

$$h_A^k(p) := (I - p)Ap + pA^*(I - p), \qquad p \in G_k(d),$$

where we identify the Grassmannian manifold of k-dimensional linear subspaces of $\mathbb{R}^d$ with their orthogonal projectors, that is,

$$G_k(d) = \{p \in \mathbb{R}^{d\times d}: p^2 = p,\, p^* = p, \operatorname{rank} p = k\}.$$

If we consider the SDE on $G_k(d)$ given by

$$dp_t^k = \sum_{i=0}^m h_{A_i}^k(p_t^k) \circ dW_t^i, \tag{2.2}$$

we obtain as solution exactly the transport of k-dimensional linear subspaces generated by $(\Phi_t)_{t\in\mathbb{R}}$. More precisely, if $p^k \in G_k(d)$, then the solution p_t^k of (2.2) such that $p_0^k = p^k$ is the projector on $\Phi_t p^k(\mathbb{R}^d)$, $t \in \mathbb{R}$.

We will use (2.2) for $k = 1$ and $k = 2$ only, and denote by $(q_t)_{t\in\mathbb{R}}$ the solution of (2.2) for $k = 1$ satisfying $q_0 = q$, and by $(p_t)_{t\in\mathbb{R}}$ the solution of (2.2) for $k = 2$ satisfying $p_0 = p$. More generally, we denote by $(q_t(q)$: $t \in \mathbb{R}$, $q \in G_1(d))$, respectively, $(p_t(p)$: $t \in \mathbb{R}$, $p \in G_2(d))$, the corresponding flows (cocycles) on $G_1(d)$, respectively, $G_2(d)$. These are the flows (cocycles) generated by $(\Phi_t)_{t\in\mathbb{R}}$ on $G_1(d)$, respectively, $G_2(d)$ (see [7]).

Since we aim at obtaining rotation as a scalar, we need to know two more objects. First of all, we have to describe how vectors of length 1 are moved by the flow. This is an easy task once one knows $(q_t(q)$: $t \in \mathbb{R}$, $q \in G_1(d))$.

LEMMA 2.1. *Let $q \in G_1(d)$ and $v \in q \cap S^{d-1}$. Then the solution of the SDE*

$$dv_t = \sum_{i=0}^m (I - q_t) A_i v_t \circ dW_t^i, \qquad v_0 = v \tag{2.3}$$

satisfies

$$v_t \in q_t \cap S^{d-1}, \qquad t \in \mathbb{R}.$$

PROOF. By definition

$$\begin{aligned} d((I-q_t)v_t) &= dv_t - q_t dv_t - dq_t v_t \\ &= \sum_{i=0}^m ((I-q_t)A_i v_t - (I-q_t)A_i q_t v_t - q_t A_i^*(I-q_t)v_t) \circ dW_t^i \\ &= \sum_{i=0}^m ((I-q_t)A_i - q_t A_i^*)(I-q_t)v_t \circ dW_t^i. \end{aligned} \tag{2.4}$$

Since $(I-q)v = 0$, strong uniqueness implies that (2.4) has solution 0; that is, we have $q_t v_t = v_t$, thus $v_t \in q_t$. Moreover, obviously $d(v_t^* v_t) = 0$; that is, $v_t \in q_t \cap S^{d-1}$ for all $t \in \mathbb{R}$. □

We next want to describe rotations of planes by $+90°$. For this purpose, we consider the covering manifold of $G_2(d)$ with two leaves defined by

$$\hat{G}_2(d) := \left\{\hat{p} \in \mathbb{R}^{d\times d}: -\hat{p}^2 \in G_2(d),\, -\hat{p}^2\hat{p} = \hat{p},\, \hat{p}^* = -\hat{p}\right\},$$

and assume that $G_2(d)$ has been oriented (without making this explicit in symbols).

LEMMA 2.2. *For $p \in G_2(d)$ there exist exactly two $p^+, p^- \in \hat{G}_2(d)$ such that $p^+ = -p^-$, $-(p^+)^2 = -(p^-)^2 = p$, and such that for any positively oriented orthonormal basis (e_1, e_2) of p we have $p^+ e_1 = e_2$, $p^+ e_2 = -e_1$.*

PROOF. Let (e_1, e_2) be a positively oriented basis of p. Then the conditions

$$p^+ e_1 = e_2, \qquad p^+ e_2 = -e_1, \qquad p^+(I-p) = 0 = (I-p)p^+$$

uniquely determine an operator in $\mathbb{R}^{d\times d}$. It satisfies $-(p^+)^2 e_1 = e_1$, $-(p^+)^2 e_2 = e_2$, and $-(p^+)^2(I-p) = 0 = (I-p)(-p^+)^2$, hence $-(p^+)^2 = p$. Moreover, $pp^+ = p^+ p = p^+$ and $(p^+)^* = -p^+$, and so $p^+ \in \hat{G}_2(d)$. We take $p^- = -p^+$. Hence existence with the required properties is proved.

Now suppose $\hat{p} \in \hat{G}_2(d)$ satisfies $\hat{p}e_1 = e_2$, $\hat{p}e_2 = -e_1$ and $-\hat{p}^2 = p$. Then

$$(I-p)\hat{p} = \hat{p} + \hat{p}^3 = 0 = \hat{p}(I-p).$$

Hence

$$\ker \hat{p} = \operatorname{im}(I-p), \qquad \operatorname{im} \hat{p} = \operatorname{im} p.$$

It is thus clear that $\hat{p} = p^+$. This proves uniqueness. □

We remark that $\hat{G}_2(d)$ is indeed a covering manifold with projection π: $\hat{G}_2(d) \to G_2(d)$ given by $\hat{p} \mapsto -\hat{p}^2$.

Now given our flow of planes $(p_t(p)\colon t \in \mathbb{R},\ p \in G_2(d))$, generated by the linear flow, there is exactly one lifting onto $\hat{G}_2(d)$ to a flow of rotations of $+90°$.

LEMMA 2.3. *Let p^+ be the positive rotation by $90°$ inside $p \in G_2(d)$. Then the solution of the SDE*

$$d\hat{p}_t = \sum_{i=0}^{m} \big((I + \hat{p}_t^2) A_i \hat{p}_t + \hat{p}_t A_i^* (I + \hat{p}_t^2) \big) \circ dW_t^i, \qquad \hat{p}_0 = p^+$$

has the following property: $(-\hat{p}_t^2)_{t\in\mathbb{R}}$ *is a solution of* (2.2) *with $k = 2$ and $p_0 = p$.*

PROOF. Since $\hat{G}_2(d)$ is compact and the vector fields involved are smooth, the SDE considered has a unique strong solution. Let for $t \in \mathbb{R}$, $r_t := -\hat{p}_t^2$. Then

$$\begin{aligned} dr_t &= -d\hat{p}_t \hat{p}_t - \hat{p}_t d\hat{p}_t \\ &= \sum_{i=0}^{m} \big((I + \hat{p}_t^2) A_i r_t + r_t A_i^* (I + \hat{p}_t^2) \big) \circ dW_t^i \\ &= \sum_{i=0}^{m} h_{A_i}^2(r_t) \circ dW_t^i, \qquad r_0 = p, \end{aligned}$$

which is the desired SDE. □

We call $(\hat{p}_t)_{t \in \mathbb{R}}$ the *positive lift* of $(p_t)_{t \in \mathbb{R}}$. Of course, the positive lift of the flow (cocycle) $(p_t(p)\colon t \in \mathbb{R},\ p \in G_2(d))$ is a uniquely defined flow (cocycle) $(\hat{p}_t(\hat{p})\colon t \in \mathbb{R},\ \hat{p} \in \hat{G}_2(d))$ on the leaf $\hat{G}_2^+(d)$ of positive rotations in $\hat{G}_2(d)$.

To simplify matters, we introduce one more notation. For $t \in \mathbb{R}$ and v_t, the solution of (2.3), let

$$w_t := \hat{p}_t v_t,$$

and, to express the dependence on $q \in G_1(d)$ and $p \in G_2(d)$ or, alternatively, on $v \in S^{d-1}$ and $\hat{p} \in \hat{G}_2(d)$, we write $w_t(q, p)$ or $w_t(v, \hat{p})$ for $\hat{p}_t(p)v_t(q)$ or $\hat{p}_t(\hat{p})v_t(v)$, $t \in \mathbb{R}$. Note that (v_t, w_t) is a positively oriented orthonormal 2-frame in p_t, which connects our approach to the Stiefel manifold approach mentioned above.

After these preliminaries, we are in a position to define the rotation number of a plane. It is evident that the infinitesimal angle $d\alpha_t$ by which the vector v_t is rotated inside p_t should be measured by the length of the projection of dv_t onto w_t, that is, by $d\alpha_t = w_t^* \circ dv_t$. The cumulative rotation in the interval $[0, T]$ is hence $\alpha_T = \int_0^T w_t^* \circ dv_t$, and the rotation number of the plane should be the linear growth rate of this quantity. This leads to the following definition.

DEFINITION 2.4 (Rotation number of a plane). Let P be a random variable with values in $G_2(d)$, and let Q and V be random variables with values in $G_1(d)$ and S^{d-1}, respectively, such that $V \in Q \subset P$. Then, in case the following limit exists in probability, we call

$$\rho(P) := \lim_{T \to \infty} \frac{1}{T} \int_0^T w_t^* \circ dv_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T w_t^*(Q, P) \circ dv_t(Q) \tag{2.5}$$

the rotation number of P.

The random variable defined by (2.5), if it exists, is independent of Q and V, as is shown by Ruffino [11], Proposition 2.1 (see also [2], Section 6.5.3).

LEMMA 2.5. *We have* (*in case of existence of the limit*)

$$\rho(P) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{i=0}^m w_t^* A_i v_t \circ dW_t^i. \tag{2.6}$$

PROOF. This is a direct consequence of Lemma 2.1 and the fact that $w_t^*(I - q_t) = w_t^*$, $t \in \mathbb{R}$, which is due to the definition of $\hat{p}_t$, $t \in \mathbb{R}$. $\square$

REMARK 2.6. Due to the substitution formula for Stratonovich integrals (see, e.g., [4]), which in our case is simple since $G_1(d)$ and $G_2(d)$ are compact, the definition of $\rho(P)$ as well as the formula (2.6) make indeed sense, even if P and Q are nothing but measurable.

Our task will consist in proving that the limit in (2.6) indeed exists for any random plane (Section 4). Moreover, it turns out that the random variable $\rho(P)$ can take on only finitely many possible values ρ_{ij} which are realized as rotation numbers of canonical planes spanned by the invariant spaces of the MET (Section 3).

3. Rotation numbers for canonical planes. The linear cocycle $(\Phi_t)_{t \in \mathbb{R}}$ generated by (2.1) automatically satisfies the integrability conditions of the MET, and the underlying metric dynamical system generated by the shift on Wiener space is ergodic. Hence the MET holds and provides us with a set of r, $1 \le r \le d$, nonrandom Lyapunov exponents $\lambda_1 > \cdots > \lambda_r$ and an $\mathscr{F}_{-\infty}^{\infty}$-measurable splitting of $\mathbb{R}^d$ into Oseledets spaces,

$$\mathbb{R}^d = E_1(\cdot) \oplus \cdots \oplus E_r(\cdot),$$

where the $d_i = \dim E_i(\cdot)$ (the multiplicities of λ_i) are nonrandom with $\sum_{i=1}^r d_i = d$, $\Phi_t(\cdot)E_i(\cdot) = E_i(\theta_t \cdot)$ and

$$v \in E_i(\cdot) \setminus \{0\} \quad \Leftrightarrow \quad \lim_{t \to \pm\infty} \frac{1}{t} \log\|\Phi_t(\cdot) v\| = \lambda_i, \qquad 1 \le i \le r.$$

For all further investigations concerning rotation numbers we need to assume that the Lyapunov spectrum is simple, that is, that $r = d$. The reason for this is that we do not have enough information about how the linear flow Φ_t behaves inside a higher-dimensional Oseledets space E_i to conclude existence of rotation numbers. As examples show ([2], Example 6.5.4), there can be continuously many different rotation numbers in one higher-dimensional Oseledets space.

The following is a sufficient condition for simple Lyapunov spectrum [2] Remark 6.2.15:

(H) The subgroup G of $Gl(d, \mathbb{R})$ generated by the matrices $A_0, \dots, A_m$ equals $Gl^+(d, \mathbb{R})$ or $Sl(d, \mathbb{R})$.

This assumption is generically true, more precisely, it holds on an open and dense set of matrices $(A_0, \dots, A_m) \in (\mathbb{R}^{d \times d})^{m+1}$.

Note also that, in contrast to the deterministic case, simple Lyapunov spectrum does not preclude the existence of nonzero rotation numbers.

We shall now examine rotation numbers $\rho_{ij} := \rho(P_{ij})$ of "canonical planes" $P_{ij} := \operatorname{span}(E_i, E_j)$, $1 \le i, j \le d$, $i \ne j$. Stationarity of the size of the infinitesimal rotations will lead via the Birkhoff ergodic theorem to formulas for the ρ_{ij} of the type of Furstenberg–Khasminskii. A key role in the derivation of these formulas is played by the decomposition of general Stratonovich integrals into Skorokhod integrals and trace terms featuring Malliavin gradients of the invariant spaces. To this end we need some knowledge of the smoothness of Oseledets spaces in the sense of Malliavin calculus, which we briefly recall from [3].

Under condition (H) the law of the vector $(E_1, \dots, E_d)$ of Oseledets spaces has a C^∞ density with respect to Riemannian volume on $(G_1(d))^d$ [7], Corollary 4.1. We shall make use of the latter fact below.

The MET also yields a family $(Q_i^{\pm})_{1 \le i \le d}$ of orthogonal projectors of rank 1 such that the spectral decomposition formula

$$\lim_{t \to \pm\infty} (\Phi_t^* \Phi_t)^{1/2|t|} = \sum_{i=1}^{d} \exp(\pm \lambda_i) Q_i^{\pm}$$

is valid. The Q_i^+ are $\mathscr{F}_0^{\infty}$-measurable, and the Q_i^- are $\mathscr{F}_{-\infty}^{0}$-measurable. Let

$$P_i^+ := \sum_{k=i}^{d} Q_k^+, \qquad P_i^- := \sum_{k=1}^{i} Q_k^-, \qquad 1 \le i \le d.$$

The intersection of the spaces onto which P_i^+ and P_i^- project are the Oseledets spaces E_i. We denote the rank 1 orthogonal projector on E_i by R_i and obtain from [3], Theorem 4 for $1 \le k \le m$ and $1 \le i \le d$, the formulas

$$\begin{aligned} D_s^k R_i = {} & -(I - R_i)(I - T_i)^{-1} S_i^+ S_i^- (I - S_i^-) A_k^s R_i \\ & - R_i (A_k^s)^* (I - S_i^-) S_i^- S_i^+ (I - T_i)^{-1} (I - R_i), \end{aligned} \tag{3.1}$$

where $A_k^s := \Phi_s A_k \Phi_s^{-1}$, $s \ge 0$, $S_i^+ := P_i^+ - R_i$, $T_i := S_i^+ S_i^- S_i^+$. We also know that, under (H), $R_i \in \mathbb{D}_1^p([0,T])$ for all $T > 0$ and $1 \le p < 2$ [3], Theorem 6.

We now investigate the smoothness properties of $P := \operatorname{span}(R_i, R_j)$ (identifying, as always, E_i with R_i and P with the orthogonal projector onto P and omitting subscripts) by passing to the exterior product $\wedge^2 \mathbb{R}^d$ (for details see [2], Section 3.2.3). In this space we may identify P with $R_i \wedge R_j$. To see this, let us show that $R_i \wedge R_j$ is an orthogonal projector in $\wedge^2 \mathbb{R}^d$. Indeed, for $x, y \in \mathbb{R}^d$ we have

$$\big(R_i \wedge R_j\big)^2 (x \wedge y) = R_i^2 x \wedge R_j^2 y = R_i x \wedge R_j y = \big(R_i \wedge R_j\big)(x \wedge y)$$

and

$$\big(R_i \wedge R_j\big)^* (x \wedge y) = R_i^* x \wedge R_j^* y = R_i x \wedge R_j y = \big(R_i \wedge R_j\big)(x \wedge y).$$

Moreover, the space onto which $R_i \wedge R_j$ projects is the (one-dimensional) span of the vectors $u_i \wedge u_j$ where u_i, u_j are nontrivial vectors in R_i and R_j, respectively. This easily implies the following lemma.

LEMMA 3.1. *Assume* (H). *Then for any* $T > 0$ *and* $1 \le p < 2$ *we have* $P \in \mathbb{D}_1^p([0,T])$.

PROOF. Clearly, multilinearity allows us to apply Leibniz's rule to get the Malliavin gradients. More precisely, for $s \ge 0$ and $1 \le k \le m$,

$$D_s^k P = D_s^k R_i \wedge R_j + R_i \wedge D_s^k R_j. \tag{3.2}$$

Now use the integrability properties of $D^k R_i$, $D^k R_j$ and the simple relation

$$|x \wedge y| = \big(|x|^2 |y|^2 - \langle x, y \rangle\big)^{1/2}, \qquad x, y \in \mathbb{R}^d,$$

to obtain the desired integrability property. $\square$

The formula we would obtain for the gradient of P by using (3.2) and (3.1) is not explicit enough. So we give another derivation, and with the same method a somewhat different and slightly simpler version of (3.1).

LEMMA 3.2. *For* $1 \le i \le d$, $1 \le k \le m$ *and* $s \ge 0$, *we have*

$$\begin{aligned} D_s^k R_i = {} & -(I - R_i)\left(R_i + (I - P_i^+) + (I - P_i^-)\right)^{-1}(I - P_i^+)A_k^s R_i \\ & - R_i(A_k^s)^*(I - P_i^+)\left(R_i + (I - P_i^+) + (I - P_i^-)\right)^{-1}(I - R_i). \end{aligned} \tag{3.3}$$

PROOF. By definition of R_i we have

$$R_i P_i^+ = R_i P_i^- = R_i.$$

As the gradient of these expressions exists, Leibniz's rule yields

$$\left(D_s^k R_i\right)P_i^+ + R_i\left(D_s^k P_i^+\right) = \left(D_s^k R_i\right)P_i^- + R_i\left(D_s^k P_i^-\right) = D_s^k R_i. \tag{3.4}$$

Hence

$$\left(D_s^k R_i\right)(I - P_i^+) = R_i\left(D_s^k P_i^+\right) = -R_i P_i^+ (A_k^s)^*(I - P_i^+) \tag{3.5}$$

(see [3], Theorem 3). Since P_i^- is $\mathscr{F}_{-\infty}^0$-measurable, we have $D_s^k P_i^- = 0$, hence (3.4) yields

$$\left(D_s^k R_i\right)(I - P_i^-) = 0. \tag{3.6}$$

By adding (3.5) and (3.6) we have

$$\left(D_s^k R_i\right)(I - R_i)\left(R_i + (I - P_i^+) + (I - P_i^-)\right) = -R_i(A_k^s)^*(I - P_i^+). \tag{3.7}$$

Now observe that $R_I + (I - P_i^+) + (I - P_i^-)$ is invertible and commutes with $(I - R_i)$, and finally add the adjoint of the left-hand side of (3.7) to obtain the desired formula. □

LEMMA 3.3. *Assume* (H). *Then for* $1 \le k \le m$ *and* $s \ge 0$ *we have*

$$D_s^k P = -(I - P)X_s^* P - P X_s (I - P),$$

where

$$\begin{aligned} X_s = {} & \left((I - P) + R_i + R_j\right)^{-1} \\ & \times\Big[R_i(A_k^s)^*(I - P_i^+)\left(R_i + (I - P_i^+) + (I - P_i^-)\right) \\ & \quad + R_j(A_k^s)^*\left(I - P_j^+\right)\left(R_j + \left(I - P_j^+\right) + \left(I - P_j^-\right)\right)\Big]. \end{aligned}$$

PROOF. By the definition of P,

$$R_i P = R_i, \qquad R_j P = R_j.$$

Since by Lemma 3.1 the Malliavin gradient of P exists, we may again write

$$\big(D_s^k R_i\big)P + R_i\big(D_s^k P\big) = D_s^k R_i,$$

hence

$$\big(D_s^k R_i\big)(I-P) = R_i\big(D_s^k P\big),$$

and also

$$R_i\big(D_s^k R_i\big)(I-R_i)(I-P) = R_i P\big(D_s^k P\big)(I-P). \tag{3.8}$$

Then (3.8) and an analogous expression for j instead of i may be added to give

$$\begin{aligned} &P\big(D_s^k P\big)(I-P) \\ &\quad = \big((I-P) + R_i + R_j\big)^{-1}\Big(R_i D_s^k R_i(I-R_i) + R_j D_s^k R_j\big(I-R_j\big)\Big). \end{aligned} \tag{3.9}$$

We finally have to substitute (3.3) into (3.9) to obtain the desired result. □

Let us now come back to the task of describing the rotation number of $P = \operatorname{span}(R_i, R_j)$. For this purpose we choose $Q = R_i$ in Definition 2.4 and recall that any other choice would yield the same result. According to what we just proved both P and Q are in $\mathbb{D}_1^p([0,T])$ for $T > 0$ and $1 \le p < 2$. But to describe the spatial averages figuring in the Furstenberg–Khasminskii formulas, we also need smoothness properties of our flows $(v_t(v)\colon t \in \mathbb{R}, v \in S^{d-1})$ and $(\hat{p}_t(\hat{p})\colon t \in \mathbb{R}, \hat{p} \in \hat{G}_2^+(d))$. These are stated in the next lemma, together with the perfect cocycle property.

LEMMA 3.4. *The flows* $(v_t(v)\colon t \in \mathbb{R}, v \in S^{d-1})$ *and* $(\hat{p}_t(\hat{p})\colon t \in \mathbb{R}, \hat{p} \in \hat{G}_2^+(d))$ *possess versions which fulfill the following properties*:

(i) *The functions* $v \mapsto v_t(v)$ *and* $\hat{p} \mapsto \hat{p}_t(\hat{p})$ *are* C^∞, *with derivatives which are* p-*integrable for any* $p \ge 1$.

(ii) *Perfect cocycle property*: *For* $t, s \in \mathbb{R}$, $\omega \in \Omega$, $v \in S^{d-1}$, $\hat{p} \in \hat{G}_2^+(d)$, *we have*

$$\begin{aligned} v_{s+t}(v)(\omega) &= v_t(v_s(v)(\omega))(\theta_s\omega), \qquad v_0(v)(\omega) = v, \\ \hat{p}_{s+t}(\hat{p})(\omega) &= \hat{p}_t(\hat{p}_s(\hat{p})(\omega))(\theta_s\omega), \qquad \hat{p}_0(\hat{p})(\omega) = \hat{p}. \end{aligned}$$

PROOF. As the corresponding quantities are generated by SDE on compact manifolds with C^∞ vector fields, (i) follows from the well-known results of Kunita [8], whereas (ii) is a consequence of the perfection result of Arnold and Scheutzow [6]. □

We shall henceforth assume that versions as in Lemma 3.4 are given. In our main result we are about to formulate, the Furstenberg–Khasminskii formulas will emerge in terms of spatial averages of Malliavin gradients. Therefore, we first have to discuss these gradients.

LEMMA 3.5. *Let $v \in S^{d-1}$, $\hat{p} \in \hat{G}_2^+(d)$, $p = -\hat{p}^2$. Then we have for $1 \le k \le m$ and $s \ge 0$ [with the notation $w_s = w_s(v, \hat{p})$ and $v_s = v_s(v)$]*

$$\begin{aligned} &D_s^{k+}(w_s^* A_k v_s) \\ &\quad = w_s^*[A_k^*(I-p_s)A_k + A_k(I-p_s)A_k]v_s + w_s^* A_k v_s [w_s^* A_k w_s - v_s^* A_k v_s]. \end{aligned}$$

PROOF. Set for abbreviation for $A \in \mathbb{R}^{d\times d}$,

$$f_A(v) = (I-q)Av, \qquad v \in S^{d-1},$$

where q denotes the orthogonal projection on span(v), and

$$g_A(\hat{p}) = (I+\hat{p}^2)A\hat{p} + \hat{p}A^*(I+\hat{p}^2), \qquad \hat{p} \in \hat{G}_2^+(d).$$

Then we have

$$\begin{aligned} v_s &= v + \sum_{i=0}^m \int_0^s f_{A_i}(v_r) \circ dW_r^i, \\ \hat{p}_s &= \hat{p} + \sum_{i=0}^m \int_0^s g_{A_i}(\hat{p}_r) \circ dW_r^i \end{aligned}$$

and hence by the rules of Fréchet differentiation on Wiener space (see [9]),

$$\begin{aligned} D_s^{k+} v_s &= f_{A_k}(v_s) = (I-q_s)A_k v_s, \\ D_s^{k+}\hat{p}_s &= g_{A_k}(\hat{p}_s) = (I-p_s)A_k\hat{p}_s + \hat{p}_s A_k^*(I-p_s). \end{aligned}$$

Thus

$$D_s^{k+} w_s = \big(D_s^{k+}\hat{p}_s\big)v_s + \hat{p}_s\big(D_s^{k+}v_s\big) = (I-p_s)A_k w_s + \hat{p}_s(I-q_s)A_k v_s.$$

So we obtain, using $q = p - ww^*$, $w = \hat{p}v$ and $v = -\hat{p}w$,

$$\begin{aligned} D_s^{k+}(w_s^* A_k v_s) &= \big(D_s^{k+} w_s^*\big)A_k v_s + w_s^* A_k\big(D_s^{k+} v_s\big) \\ &= \big(w_s^* A_k^*(I-p_s) - v_s^* A_k^*(I-q_s)\hat{p}_s\big)A_k v_s + w_s^* A_k(I-q_s)A_k v_s \\ &= w_s^* A_k^*(I-p_s)A_k v_s - v_s^* A_k^*(I-q_s)\hat{p}_s v_s v_s^* A_k v_s \\ &\quad + w_s^* A_k(I-p_s)A_k v_s + w_s^* A_k w_s w_s^* A_k v_s \\ &= w_s^*\big(A_k^*(I-p_s)A_k + A_k(I-p_s)A_k\big)v_s \\ &\quad + w_s^* A_k v_s\big(w_s^* A_k w_s - v_s^* A_k v_s\big), \end{aligned}$$

which is the desired result. □

We are now in a position to formulate and prove our main result, by which rotation numbers of canonical planes are shown to exist and are explicitly described.

THEOREM 3.6 (Rotation numbers for canonical planes). *Consider the linear SDE* (2.1) *and assume that condition* (H) *is satisfied. Then the linear cocycle* $(\Phi_t)_{t\in\mathbb{R}}$ *generated by* (2.1) *has a simple Lyapunov spectrum. Denote* *for* $1 \le i \le d$ *by* R_i *the rank* 1 *orthogonal projectors on the Oseledets space* E_i *and consider for some* $1 \le i,\ j \le d,\ i \neq j$, *the canonical plane* $P = \operatorname{span}(R_i, R_j)$. *Then the rotation number of* P *exists and is given by the nonrandom scalar*

$$\rho(P) = \rho_{ij} = \mathbb{E}\left(W^* A_0 V + \frac{1}{2} \sum_{k=1}^{m} C_k(W,V) + \sum_{k=1}^{m} D_k(W,V) \right), \tag{3.10}$$

where

$$\begin{aligned} C_k(W,V) &:= W^*[A_k^*(I-P)A_k + A_k(I-P)A_k]V \\ &\quad + W^*A_kV[W^*A_kW - V^*A_kV] \end{aligned} \tag{3.11}$$

and

$$D_k(W,V) := \frac{\partial}{\partial(q,p)} (W^*(q,p)A_kV(q))\big|_{(q,p)=(Q,P)} \big(D_0^k Q, D_0^k P\big),$$

where $V(q) = v_0(q)$, $W(q,p) = w_0(q,p) = \hat{p}_0(p)v_0(q)$, $Q = R_i$, $V = V(Q)$ *and* $W = W(Q,P)$.

Proof. Let us remark first that in order to give the notation "$\partial/\partial(q,p)$" in the above formula for D_k a precise meaning in the sense of differential calculus in Euclidean space, one may as usual extend our functions from the compact manifolds on which they are defined to a tubular neighborhood of an embedding Euclidean space.

The invariance properties of our spaces P and Q yield the following equations for $t \in \mathbb{R}$:

$$v_t(Q) = v_0(Q) \circ \theta_t, \qquad \hat{p}_t(P) = \hat{p}_0(P) \circ \theta_t, \qquad w_t(Q,P) = w_0(Q,P) \circ \theta_t.$$

Hence the stochastic integrands in our definition of the rotation number become stationary processes with respect to the canonical shift on Wiener space and are given by $(w_0^*(Q,P)A_k v_0(Q)) \circ \theta_t$, $0 \le k \le m$, $t \in \mathbb{R}$. Hence Birkhoff's ergodic theorem applies and yields

$$\rho(P) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left(\int_0^t \sum_{k=0}^{m} (w_0^*(Q,P)A_k v_0(Q)) \circ \theta_s \circ dW_s^k \right).$$

We shall now compute this limit. The essential observation is that a Stratonovich integral can be decomposed according to a formula given in [9], page 151, into a Skorokhod integral and trace terms. The term for $k = 0$ is easier, however. Continuity and boundedness of the integrand yield via bounded convergence

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left(\int_0^t (w_0(Q,P)A_0 v_0(Q)) \circ \theta_s \, ds \right) = \mathbb{E}(W^* A_0 V). \tag{3.12}$$

For $1 \leq k \leq m$ we have

$$
\begin{aligned}
&\int_0^t \left[(w_0^*(Q,P) A_k v_0(Q)) \circ \theta_s \right] \circ dW_s^k \\
&\qquad = \int_0^t (w_0^*(Q,P) A_k v_0(Q)) \circ \theta_s dW_s^k \\
&\qquad\quad + \tfrac{1}{2} \int_0^t D_s^{k+} (w_0^*(Q,P) A_k v_0(Q)) \circ \theta_s, ds \\
&\qquad\quad + \tfrac{1}{2} \int_0^t D_s^{k-} (w_0^*(Q,P) A_k v_0(Q)) \circ \theta_s, ds,
\end{aligned} \tag{3.13}
$$

where the stochastic integral on the right-hand side is a Skorokhod integral. Here (3.13) makes sense since we know that by smoothness of the flows and of Q and P, the trace terms exist. We are even allowed to take expectations in (3.13) and pass to the limit $t \downarrow 0$. This is due to Lemmas 3.1 and 3.2, a corresponding smoothness result for Q and Lemma 3.4. Since Skorokhod integrals have vanishing expectation we arrive at

$$
\begin{aligned}
&\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E} \int_0^t \left[(w_0^*(Q,P) A_k v_0(Q)) \circ \theta_s \right] \circ dW_s^k \\
&\qquad = \frac{1}{2} \mathbb{E}\left(D_0^{k+} (w_0^*(Q,P) A_k v_0(Q)) \right) \\
&\qquad\quad + \frac{1}{2} \mathbb{E}\left(D_0^{k-} (w_0^*(Q,P) A_k v_0(Q)) \right).
\end{aligned} \tag{3.14}
$$

To compute the trace terms in (3.14) we apply Lemma 3.5 and the chain rule (see [9], page 47) to get

$$
\begin{aligned}
&D_0^{k+} (w_0^*(Q,P) A_k v_0(Q)) \\
&\qquad = D_0^{k+} (w_0^*(q,p) A_k v_0(q)) \big|_{(q,p)=(Q,P)} \\
&\qquad\quad + \frac{\partial}{\partial(q,p)} (w_0^*(q,p) A_k v_0(q)) \big|_{(q,p)=(Q,P)} \left(D_0^k Q, D_0^k P \right).
\end{aligned} \tag{3.15}
$$

The second term is simpler. It reads

$$
\begin{aligned}
&D_0^{k-} (w_0^*(Q,P) A_k v_0(Q)) \\
&\qquad = \frac{\partial}{\partial(q,p)} (w_0^*(q,p) A_k v_0(q)) \big|_{(q,p)=(Q,P)} \left(D_0^k Q, D_0^k P \right).
\end{aligned} \tag{3.16}
$$

An appeal to Lemma 3.5 finally gives the first term on the right-hand side of (3.15) explicitly. Now combine (3.12) to (3.16) to obtain the asserted formula. □

We now discuss several particular cases of (3.10).

First, we may take $P = \operatorname{span}(R_1, R_2)$. Then due to the above definitions we have $P = P_2^-$, $Q = P_1^-$ and obtain the corollary.

COROLLARY 3.7. *Let* $P = \operatorname{span}(R_1, R_2)$. *Then the "top" rotation number* ρ_{12} *is given by*

$$\rho(P) = \rho_{12} = \mathbb{E}\left(W^* A_0 V + \tfrac{1}{2}\sum_{k=1}^{m} C_k(W,V)\right), \tag{3.17}$$

where $C_k(W,V)$ *is given by* (3.11) *with* $V = v_0(Q)$, $W = w_0(Q,P)$, $Q = P_1^-$ *and* $P = P_2^-$.

PROOF. Since Q and P are $\mathscr{F}_{-\infty}^{0}$-measurable, we have $D_0^k Q = D_0^k P = 0$. Hence (3.10) simplifies to (3.17). □

When reversing time, we observe that the roles of P_2^- and P_1^- are taken by P_{d-1}^+ and P_d^+, respectively.

COROLLARY 3.8. *Let* $P = \operatorname{span}(R_{d-1}, R_d)$. *Then*

$$\rho(P) = \rho_{d-1,d} = \mathbb{E}\left(W^* A_0 V - \tfrac{1}{2}\sum_{k=1}^{m} C_k(W,V)\right), \tag{3.18}$$

where $C_k(W,V)$ *is given by* (3.11) *with* $V = v_0(Q)$, $W = w_0(Q,P)$, $Q = P_d^+$ *and* $P = P_{d-1}^+$.

PROOF. Since this time P and Q are $\mathscr{F}_0^{\infty}$-measurable, by definition of D^{k+}, the expression in (3.15) has to vanish. This implies

$$\begin{aligned} &D_0^{k+}(w_0^*(q,p)A_k v_0(q))|_{(q,p)=(Q,P)} \\ &\quad = -\frac{\partial}{\partial(q,p)}(w_0^*(q,p)A_k v_0(q))|_{(q,p)=(Q,P)}(D_0^k Q, D_0^k P), \end{aligned} \tag{3.19}$$

$1 \le k \le m$. The left-hand side of (3.19) is, however, just given by Lemma 3.5 and yields a term which corresponds to the second term in (3.10). This gives the desired result. □

If we specialize the results obtained in the above corollaries to the case $d = 2$, we obtain two different formulas for one and the same rotation number $\rho(\mathbb{R}^2)$. This yields additional information on the laws of Oseledets spaces.

COROLLARY 3.9. *Let* $d = 2$. *Then the rotation number of* $\mathbb{R}^2$ *exists and*

$$\rho = \rho(\mathbb{R}^2) = \mathbb{E}\left(W^* A_0 V + \tfrac{1}{2}\sum_{k=1}^{m} W^* A_k V(W^* A_k W - V^* A_k V)\right), \tag{3.20}$$

where $W = p^+ V$, $V = v_0(Q)$ *with* $Q = R_1 = P_1^-$ *and* p^+ *is rotation by* $+90°$ *of* $\mathbb{R}^2$.

Alternatively,

$$\rho = \mathbb{E}\left(W^* A_0 V - \tfrac{1}{2} \sum_{k=1}^{m} W^* A_k V (W^* A_k W - V^* A_k V) \right), \tag{3.21}$$

where $W = p^+ V$, $V = v_0(Q)$ *with* $Q = R_2 = P_2^+$.

The proof is a simplification of Corollaries 3.7 and 3.8, because $P = I$.

Formula (3.21) could also be obtained by time reversal of our linear SDE.

The expectations in (3.20) and (3.21) can be written as a mean over S^1 with respect to the distribution of V which can be found by solving a corresponding Fokker–Planck equation.

Let us also mention that it can be easily seen that for $d = 2$ the rotation number always exists; that is, condition (H) is not needed in this case.

Since Lemmas 3.2 and 3.3 give explicit descriptions of Malliavin gradients of invariant lines and planes, Theorem 3.6 gives an explicit description of the whole set of rotation numbers of canonical planes. In the following section we shall discuss rotation numbers for noncanonical planes.

4. Rotation numbers for general planes. We now consider general random planes P. One very important particular case is, of course, the case of a deterministic plane $p \in G_2(d)$ which we treat first. It turns out that, under (H), all deterministic planes rotate asymptotically as fast as the canonical plane $P = \operatorname{span}(R_1, R_2) = P_2^-$, that is, with the "top" rotation number ρ_{12} $\mathbb{P}$-a.s. This is analogous to the fact that, under a Hörmander condition on $G_1(d)$, nonrandom vectors $v \neq 0$ always grow with the top Lyapunov exponent λ_1, $\lambda(v) = \lambda_1$ $\mathbb{P}$-a.s. [2], Theorem 6.2.16.

THEOREM 4.1 (Rotation number for nonrandom plane). *Consider the linear SDE* (2.1) *and assume that condition* (H) *is satisfied. Let* $p \in G_2(d)$ *be a fixed plane. Then its rotation number* $\rho(p)$ *exists and satisfies*

$$\rho(p) = \rho_{12} \quad \mathbb{P}\text{-}a.s., \tag{4.1}$$

where $\rho_{12} = \rho(\operatorname{span}(R_1, R_2))$ *is the "top" rotation number of* (2.1) *defined by* (3.17).

PROOF. The condition (H) guarantees, as was shown in [7], that (the Lyapunov spectrum is simple and) the law of $(R_1, \dots, R_d)$ is smooth. Consequently, a fixed deterministic plane p can coincide with an Oseledets space only with probability 0.

Now choose $q \subset p$ arbitrary. According to Lemmas 6.5.11, 6.5.12 and 6.5.13 of [2], we have, by the above smoothness argument, that the indices i_0 and j_0 defined by (4.8) and (4.9) satisfy $i_0 = 1$ and $j_0 = 2$ $\mathbb{P}$-a.s. and hence

$$p_t(p) \circ \theta_t^{-1} \to P = P_2^-, \qquad q_t(q) \circ \theta_t^{-1} \to Q = P_1^-,$$

both exponentially fast. This immediately implies that, $\mathbb{P}$-a.s.,

$$\hat{p}_t(p) \circ \theta_t^{-1} \to \hat{p}_0(P), \qquad v_t(q) \circ \theta_t^{-1} \to v_0(Q),$$

both exponentially fast.

Moreover, P and Q are $\mathscr{F}_{-\infty}^{0}$-measurable. Hence the exponential convergence just stated implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (w_t^*(q,p)A_k v_t(q) - w_t^*(Q,P)A_k v_t(Q)) \circ dW_t^k = 0,$$

$0 \le k \le m$, at least in probability. Therefore, it suffices to invoke Corollary 3.7 to obtain the desired result. $\square$

We shall now consider a general random plane P and show that its rotation number exists and is given by a random variable the possible values of which are the finitely many canonical rotation numbers ρ_{ij} described by Theorem 3.6. The main difficulty consists in proving that a process which possesses a flow property and converges to 0 exponentially fast as $t \to \infty$ possesses stochastic integrals whose time averages converge to 0 as well. We shall establish this result in the following lemma.

LEMMA 4.2. *Let (S, ρ) be a separable metric space and $(X(y))_{y\in S}$ be a continuous random field with values in $\mathbb{R}^d$. Suppose $F \subset S$ is a closed set such that $X|_F = 0$ and, introducing $K_\eta(F) := \{y \in S: d(y,F) < \eta\}$, such that for any $\eta > 0$,*

$$\mathbb{E}\left(\sup_{y\in K_\eta(F)} |X(y)|^p\right) \le c\eta^q$$

with some constants $c > 0$ and $p, q \ge 1$. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables with values in S and

$$A_{m,\varepsilon} := \{\omega \in \Omega: d(Y_n, F) \le me^{-\varepsilon n}, n \in \mathbb{N}\}, \qquad m \in \mathbb{N}, \varepsilon > 0,$$
$$B_n := \{|X(Y_n)(\theta_n)| > e^{-\delta n}\}, \qquad n \in \mathbb{N}, \delta > 0.$$

Then we have

$$\mathbb{P}(A_{m,\varepsilon} \cap B_n) \le cm^q \exp(n(p\delta - q\varepsilon)), \qquad m, n \in \mathbb{N}, \varepsilon > 0.$$

If $\delta < \varepsilon q/p$, we have

$$\mathbb{P}\left(A_{m,\varepsilon} \cap \limsup_{n\to\infty} B_n\right) = 0.$$

PROOF. By definition, we have on $A_{m,\varepsilon}$ the inequality

$$|X(Y_n)| \le \sup_{y\in K_{m\exp(-\varepsilon n)}(F)} |X(y)|.$$

Hence for $m, n \in \mathbb{N}$, $\varepsilon, \delta > 0$, using that θ_n is measure preserving,

$$
\begin{aligned}
&\mathbb{P}\big(A_{m,\varepsilon} \cap \{|X(Y_n) \circ \theta_n| > \exp(-\delta n)\}\big) \\
&\quad \le \mathbb{P}\bigg(A_{m,\varepsilon} \cap \bigg\{\sup_{y \in K_{m\exp(-\varepsilon n)}(F)} |X(y)| > \exp(-\delta n)\bigg\}\bigg) \\
&\quad \le \exp(p\delta n)\mathbb{E}\bigg(\sup_{y \in K_{m\exp(-\varepsilon n)}(F)} |X(y)|^p\bigg) \\
&\quad \le \exp(p\delta n) c m^q \exp(-q\varepsilon n) \\
&\quad = c m^q \exp(n(p\delta - q\varepsilon)).
\end{aligned}
\tag{4.2}
$$

If, moreover, $\delta < \varepsilon q/p$, (4.2) yields

$$\sum_{n \in \mathbb{N}} \mathbb{P}(A_{m,\varepsilon} \cap B_n) < \infty,$$

so the Borel–Cantelli lemma applies to yield

$$\mathbb{P}\Big(A_{m,\varepsilon} \cap \limsup_{n\to\infty} B_n\Big) = 0. \qquad \square$$

Lemma 4.2 will be helpful in the following situation: let K be a compact manifold in $\mathbb{R}^{d_1}$ for some $d_1 \in \mathbb{N}$, and assume that $(u_s(x)\colon s \in \mathbb{R}, x \in K)$ is a cocycle on the manifold K, which is generated by an SDE with C^∞ vector fields. Assume further that Z_1, Z_2 are random variables with values in K such that $u_s(Z_1) - u_s(Z_2) \to 0$ as $s \to \infty$ exponentially fast; that is, there exist a random variable C and a number $\varepsilon > 0$ such that

$$|u_s(Z_1) - u_s(Z_2)| \le Ce^{-\varepsilon s}, \qquad s \ge 0. \tag{4.3}$$

LEMMA 4.3. *Let $f\colon K \to \mathbb{R}$ be a Lipschitz-continuous function, and assume that* (4.3) *holds. Then for any $1 \le k \le m$,*

$$\lim_{t\to\infty} \int_0^t \big(f(u_s(Z_1)) - f(u_s(Z_2))\big) \circ dW_s^k \quad \text{exists } \mathbb{P}\text{-}a.s.$$

In particular, if $\lim_{t\to\infty}(1/t)\int_0^t f(u_s(Z_1)) \circ dW_s^k$ exists, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(u_s(Z_1)) \circ dW_s^k = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(u_s(Z_2)) \circ dW_s^k.$$

PROOF. First, the Stratonovich integrals appearing in the statements exist, due, for example, to the results in [4]. The cocycle property of $(u_s(x)\colon s \in \mathbb{R}, x \in K)$ yields for $n \in \mathbb{N}$, $n \le t \le n+1$, $\omega \in \Omega$, $j = 1, 2$

$$
\begin{aligned}
\int_n^t f\big(u_s(Z_j)\big) \circ dW_s^k(\omega) &= \int_n^t f\big(u_{s-n}\big(u_n(Z_j)(\omega) \circ dW_{s-n}^k(\theta_n\omega)\big) \\
&= \int_0^{t-n} f\big(u_s\big(u_n(Z_j)(\omega)\big)\big) \circ dW_s^k(\theta_n\omega).
\end{aligned}
$$

Hence we may set, for $(x, y) \in K \times K$,

$$X(x, y) := \sup_{0 \le t \le 1} \left| \int_0^t \left(f(u_s(x)) - f(u_s(y)) \right) \circ dW_s^k \right|,$$

and for $n \in \mathbb{N}$,

$$Y_n = (u_n(Z_1), u_n(Z_2))$$

to obtain

$$X(Y_n) \circ \theta_n = \sup_{n \le t \le n+1} \left| \int_n^t \left(f(u_s(Z_1)) - f(u_s(Z_2)) \right) \circ dW_s^k \right|. \tag{4.4}$$

Let us now verify the hypotheses of Lemma 4.2 for X. Of course, we take $S = K \times K$ and ρ the metric induced by the Euclidean norm of $\mathbb{R}^{d_1}$. Let F be the diagonal in S. Then F is closed and $X|_F = 0$. By our hypotheses about the cocycle $(u_s(x)\colon s \in \mathbb{R},\ x \in K)$ and standard results of [8] there are for $p \ge 1$ constants c_p such that

$$\mathbb{E}|X(x, y)|^p \le c_p |x - y|^p, \qquad x, y \in K. \tag{4.5}$$

Hence the lemma of Garsia, Rodemich and Rumsey applies and yields for $p \ge 1$ and some appropriate $q = q(p)$ a constant c such that for any $\eta > 0$,

$$\mathbb{E}\left(\sup_{(x, y) \in K_\eta(F)} |X(x, y)|^p \right) \le c\eta^q. \tag{4.6}$$

So we may apply Lemma 4.2. Let $\delta < \varepsilon q/p$, $\omega \in A_{m, \varepsilon}$. Then there exists $N(\omega) \in \mathbb{N}$ such that for $n \ge N(\omega)$,

$$|X(Y_n) \circ \theta_n|(\omega) \le e^{-\delta n}.$$

Thus for $s, t \ge N(\omega)$, $s \in [n, n+1]$, $t \in [m, m+1]$ we have

$$\begin{aligned}
&\left| \int_s^t \left(f(u_r(Z_1)) - f(u_r(Z_2)) \right) \circ dW_r^k \right| (\omega) \\
&\qquad \le \sum_{j=n}^{m} \sup_{j \le t \le j+1} \left| \int_j^t \left(f(u_r(Z_1)) - f(u_r(Z_2)) \right) \circ dW_r^k \right| (\omega) \\
&\qquad = \sum_{j=n}^{m} X(Y_j) \circ \theta_j(\omega) \le \sum_{j=n}^{m} e^{-\delta j} \\
&\qquad \le e^{-\delta n} \frac{1}{1 - e^{-\delta}}.
\end{aligned} \tag{4.7}$$

Then (4.7) clearly implies that

$$\lim_{t \to \infty} \int_0^t \left(f(u_s(Z_1)) - f(u_s(Z_2)) \right) \circ dW_s^k$$

exists $\mathbb{P}$-a.s. on $A_{m, \varepsilon}$, $m \in \mathbb{N}$, with ε according to (4.3). But (4.3) also gives $A_{m, \varepsilon} \uparrow \Omega$ as $m \to \infty$. Hence the limit exists $\mathbb{P}$-a.s. on Ω. The remaining assertions now follow readily from this. □

We are now ready to prove our final multiplicative ergodic theorem for rotation numbers.

For this purpose, let P be an arbitrary random plane. Recalling the Oseledets splitting and following [2], page 361, we define for $\omega \in \Omega$,

$$i_0(\omega) := \min\{1 \le i \le d: R_i(\omega)P(\omega) \ne 0\}, \tag{4.8}$$

$$j_0(\omega) := \min\Big\{i > i_0(\omega): R_i(\omega)P(\omega)\big(I - R_{i_0(\omega)}(\omega)\big) \ne 0\Big\} \tag{4.9}$$

and

$$C(P)(\omega) := \mathrm{span}\big(R_{i_0(\omega)}(\omega), R_{j_0(\omega)}(\omega)\big). \tag{4.10}$$

Then evidently $C(P)$ is a random plane which takes its values in the set of canonical planes dealt with in Theorem 3.6.

THEOREM 4.4 (Rotation number for random plane). *Consider the linear SDE* (2.1) *and assume that condition* (H) *is satisfied. Then for any random plane P in $\mathbb{R}^d$, the rotation number $\rho(P)$ exists and satisfies*

$$\rho(P) = \rho(C(P)) = \rho_{i_0, j_0},$$

where ρ_{ij} is given by (3.10) *and i_0, j_0 and $C(P)$ are defined by* (4.8), (4.9) *and* (4.10), *respectively.*

PROOF. First, $\rho(C(P))$ exists according to Theorem 3.6 (pathwise limit), and takes on the finitely many values described there. Now let U be a random unit vector in R_i, and let $V := PU/\|PU\|$. Moreover, let $\hat{P}$ be the unique element of $\hat{G}_2^+(d)$ over P, and let $\hat{C}(P)$ be defined analogously. Then [2], Lemma 6.5.11 yields the following convergence results:

$$\begin{aligned} &p_t(P) - p_t(C(P)) \to 0, \qquad \hat{p}_t(\hat{P}) - \hat{p}_t\big(\hat{C}(P)\big) \to 0, \\ &v_t(V) - v_t(U) \to 0, \end{aligned} \tag{4.11}$$

as $t \to \infty$, exponentially fast. Define

$$u_t(v, \hat{p}) = \big(\hat{p}_t(\hat{p}), v_t(v)\big), \qquad f_k(v, \hat{p}) = v^* \hat{p}^* A_k v,$$

where $(v, \hat{p}) \in S^{d-1} \times \hat{G}_2^+(d)$, $0 \le k \le m$. Then (4.11) implies

$$u_t(V, \hat{P}) - u_t\big(U, \hat{C}(P)\big) \to 0 \tag{4.12}$$

as $t \to \infty$, exponentially fast as well, and the f_k are Lipschitz functions.

Thus for $1 \le k \le m$, Lemma 4.3 implies

$$\lim_{t\to\infty} \frac{1}{t} \left| \int_0^t f_k\big(u_s(V, \hat{P})\big) \circ dW_s^k - \int_0^t f_k\big(u_s\big(U, \hat{C}(P)\big)\big) \circ dW_s^k \right| = 0.$$

For $k = 0$ the corresponding convergence is trivial. Since

$$\rho(C(P)) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{k=0}^m f_k\big(u_s\big(U, \hat{C}(P)\big)\big) \circ dW_s^k$$

exists by Theorem 3.6, Lemma 4.3 yields, moreover, that

$$\rho(P) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{k=0}^{m} f_k\big(u_s(V, \hat{P})\big) \circ dW_s^k$$

exists as well and equals $\rho(C(P))$. This completes the proof. □

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