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THE GENEALOGY OF A CLUSTER IN THE MULTITYPE VOTER MODEL

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The genealogy of a cluster in the multitype voter model can be defined in terms of a family of dual coalescing random walks. We represent the genealogy of a cluster as a point process in a size-time plane and show that in high dimensions the genealogy of the cluster at the origin has a weak Poisson limit. The limiting point process is the same as for the genealogy of the size-biased Galton-Watson tree. Moreover, our results show that the branching mechanism and the spatial effects of the voter model can be separated on a macroscopic scale. Our proofs are based on a probabilistic construction of the genealogy of the cluster at the origin derived from Harris' graphical representation of the voter model.

1. Introduction. Consider the basic voter model $(\xi_t)_{t\geq 0}$ on the d-dimensional integer lattice $\mathbb{Z}^d$. The dynamics of $(\xi_t)_{t\geq 0}$ are simple: At any time $t \geq 0$ the voter at site x decides to change its opinion at rate one and adopts the opinion of the voter at a nearest neighbor site y with probability $(2d)^{-1}$. We will assume throughout that initially all voters have distinct opinions, $\xi_0(x) \neq \xi_0(y)$ for any $x \neq y$. [We may take the interval $(0, 1)$ for the set of possible opinions.] Let η_t^x denote the set of sites where at time t the voters (or particles) have the opinion (or type) initially at site x,

$$\eta_t^x := \{y \in \mathbb{Z}^d : \xi_t(y) = \xi_0(x)\}.$$

The size $n_t^x := |\eta_t^x|$ of this cluster is a nonnegative integer-valued martingale, and hence $P(n_t^x > 0) \to 0$ as $t \to \infty$. Clearly, those rare clusters which survive get very large. The asymptotic decay of the survival probability and the conditional distribution of the size of a certain cluster are described by the following theorem. (Here and in the sequel we abbreviate $\eta_t = \eta_t^{\mathscr{O}}$, $n_t = n_t^{\mathscr{O}}$, etc., where $\mathscr{O}$ is the origin. Note that the law of these quantities is shift invariant.)

Theorem 1.1 (Bramson and Griffeath [3]). *For any $d \geq 2$, the size of the type initially at the origin has a conditioned exponential limit law,*

$$\lim_{t\to\infty} P(\, n_t \geq\, x p_t^{-1} \mid n_t > 0) = \exp(-x),\;\; x \geq 0, \tag{1.1}$$

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where

$$p_t := P(n_t > 0) \sim \begin{cases} \dfrac{\log t}{\pi t}, & d = 2, \\ (\gamma_d t)^{-1}, & d \geq 3, \end{cases} \qquad \text{as } t \to \infty. \tag{1.2}$$

Here, γ_d is the escape probability of simple symmetric random walk on $\mathbb{Z}^d$.

The situation in $d = 1$ is different. There, p_t is asymptotically $(\pi t)^{-\frac{1}{2}}$ and the conditioned limit law is not exponential (cf. [3]).

Now let χ_t^x denote the set of all sites where, at time t, particles have the same type as the particle at site x,

$$\chi_t^x := \{y \in \mathbb{Z}^d : \xi_t(y) = \xi_t(x)\} = \eta_t^{\xi_0^{-1}(\xi_t(x))}.$$

Note that the cluster χ_t^x is always nonempty, since $x \in \chi_t^x$. Again, we write χ_t for $\chi_t^{\mathscr{O}}$, and define $N_t := |\chi_t|$ to be the size of this cluster. As was first observed by Kelly [10], N_t has the size-biased distribution of n_t,

$$P(N_t = k) = kP(n_t = k), \qquad k \geq 0. \tag{1.3}$$

The asymptotic behavior of the size of this cluster is described by the following theorem.

THEOREM 1.2 (Sawyer [13]). *For any $d \geq 2$, the size of the cluster of the type at the origin at time t has a gamma limit law with shape parameter 2,*

$$\lim_{t\to\infty} P(N_t \leq xEN_t) = \int_0^x 4y\, e^{-2y}\, dy, \qquad x \geq 0,$$

where

$$EN_t \sim 2p_t^{-1} \qquad \text{as } t \to \infty. \tag{1.4}$$

Note that Theorems 1.1 and 1.2 show that size-biasing and limiting procedures can be interchanged.

Our aim is to explore the genealogical structure of the cluster containing the origin at time t. In general, if a particle is distinguished in some branching population, then the population can be decomposed with respect to the particles' degree of relationship with the distinguished particle. Here, our branching population is the cluster of the type containing the origin at time t, and the particle at the origin is a natural candidate to be distinguished. To define the degree of relationship a particle in χ_t has with the distinguished particle, we construct our process using a percolation substructure, as introduced by Harris in [9]. This construction, given in detail in Section 2, provides a means for tracing backward in time the type of a given particle. In particular, for any $t > 0$, it yields a dual coalescing random walk system $(S_s^{x,t})_{0\leq s\leq t}$, $x \in \mathbb{Z}^d$, such that

$$\xi_t(x) = \xi_{t-s}(S_s^{x,t}) \quad \forall\, x \in \mathbb{Z}^d,\ 0 \leq s \leq t. \tag{1.5}$$

For each x and t, $(S_s^{x,t})_{0\le s\le t}$ is a continuous time, rate one simple symmetric random walk with $S_0^{x,t} = x$. For fixed $t > 0$, for $x \neq y$, $(S_s^{x,t})_{0\le s\le t}$ and $(S_s^{y,t})_{0\le s\le t}$ move independently until they collide, then the two particles merge into one and walk together. The representation (1.5) gives us the complete type history of any given particle in the voter model.

Now we can introduce the notion of a particle's branch-time. Due to our assumption that initially all sites are different, two sites have different types at time t unless the respective dual random walks have coalesced. Hence, χ_t can be expressed in terms of the family of dual random walks only,

$$\chi_t = \left\{x \in \mathbb{Z}^d : S_t^{x,t} = S_t^{\mathcal{O},t}\right\}. \tag{1.6}$$

For $x \in \chi_t$ let $\tau_t(x)$ be the time of coalescence of the two random walks started at x and the origin,

$$\tau_t(x) := \inf\left\{r \ge 0 : S_r^{x,t} = S_r^{\mathcal{O},t}\right\}.$$

For convenience, define $\tau_t(x) = \infty$ if $x \notin \chi_t$. We will refer to $t - \tau_t(x)$ as the branch-time of the particle at site $x \in \chi_t$, and write $\chi_t(s)$ for the set of sites with branch-time $t - s$,

$$\chi_t(s) := \{x \in \chi_t : \tau_t(x) = s\}, \qquad 0 \le s \le t.$$

By $N_t(s) := |\chi_t(s)|$ we denote the number of sites with branch-time $t - s$.

We may decompose the cluster χ_t with respect to the particles' branch-times. Clearly,

$$\chi_t = \bigcup_{0\le s\le t} \chi_t(s) \quad \text{and} \quad N_t = \sum_{0\le s\le t} N_t(s). \tag{1.7}$$

Keeping track of the branch-times, we represent the relationship structure of χ_t by the random set

$$\Lambda_t := \{(s, N_t(s)) : 0 < s \le t, N_t(s) > 0\}, \tag{1.8}$$

which we will refer to as the genealogy of χ_t. Whenever convenient we slightly misuse notation and identify the set Λ_t and the simple point process $\sum_{(s,z)\in\Lambda_t} \delta_{(s,z)}$. That is, we do not distinguish between the random measure and its support. A suitable rescaling of Λ_t is obtained by speeding up time by the factor t and assigning mass $(EN_t)^{-1}$ to each particle. Thus, our rescaled genealogy of χ_t is $T_t\Lambda_t$, where

$$T_t\Lambda_t := \{(t^{-1}s, (EN_t)^{-1}z) : (s, z) \in \Lambda_t\}.$$

Our main result describes the asymptotic form of the rescaled genealogy $T_t\Lambda_t$ of the type at the origin in high dimensions.

THEOREM 1.3. *For any $d \ge 3$, the sequence of random measures $(T_t\Lambda_t)_{t\ge 0}$ is tight. Any limiting point process is a simple point process on $(0, 1] \times \mathbb{R}^+$ with intensity*

$$\lambda(du\,dz) = \frac{2du}{u}\,\frac{2}{u}\,\exp\left(-\frac{2z}{u}\right)dz. \tag{1.9}$$

For $d \geq 7$,

$$T_t \Lambda_t \xrightarrow{d} \Lambda \qquad \textit{as } t \to \infty, \tag{1.10}$$

where Λ *is the simple Poisson point process with intensity* λ.

Here, $\xrightarrow{d}$ denotes convergence in distribution, which is just weak convergence of the joint distributions of $(T_t \Lambda_t(B_i),\ 1 \leq i \leq n)$ for any finite family of Borel sets B_i such that $\bigcup_{i=1}^n B_i \subset (\varepsilon, 1] \times \mathbb{R}^+$ for some $\varepsilon > 0$.

The form of the intensity λ in (1.9) is meant to suggest that the probability that some particles in χ_t have rescaled branch-time in du is asymptotically $(2/\bar{u})d\bar{u}$, $\bar{u} = 1 - u$, and the rescaled number of such particles is asymptotically exponentially distributed with mean $\bar{u}/2$.

The limit law (1.10) says that the genealogy of the voter model is asymptotically described by the critical Galton-Watson tree. To be more precise, if represented as a point process in the plane, the genealogy of the size-biased critical binary Galton-Watson tree has the same weak Poisson limit Λ (see Proposition 2.2 in [6]). There a tree with population size k at time t is k times as likely as if sampling were according to Galton-Watson measure on the space of trees and the distinguished particle is chosen purely at random among the particles alive at time t. In particular, Sawyer's theorem can be regarded as a size-biased version of Yaglom's exponential limit law for conditioned critical Galton-Watson processes (see, e.g., [2], page 20). We remark that Sawyer's result can be easily recovered from Theorem 1.3: By (1.7), summing up the mass coordinates of the points in $T_t \Lambda_t$ totals $(EN_t)^{-1} N_t$. The projection of the Poisson point process Λ on its mass coordinate is a Poisson point process on $\mathbb{R}^+$ with intensity

$$\lambda((0,1] \times dz) = 4 \int_1^\infty du\ \exp(-2zu)\, dz = 2\, \frac{e^{-2z}}{z}\, dz.$$

Having checked that summation and limiting procedures may be interchanged, Sawyer's limit law follows since $\nu_\alpha(dz) = z^{-1} e^{-\alpha z} dz$ is the Lévy measure of the gamma process with scale parameter α (see, e.g., [12]). We note, however, that our proof of Theorem 1.3 depends on Theorem 1.1, which in turn depends on Sawyer's limit law.

In the multitype voter model with mutation, particles mutate at a positive rate μ. The multitype voter model with mutation has a unique stationary distribution (see, e.g., [7]). A theorem also due to Sawyer [13] states that the rescaled size of the type at the origin in equilibrium approaches an exponential distribution as the mutation rate goes to zero. Using the limit law (1.10) this result can now be explained through a simple characterization of the exponential distribution, as is described in [6].

The independence properties of the limiting Poisson point process Λ imply that contributions to χ_t initiated at separate branch-times on the macroscopic scale t are approximately independent. The genealogy Λ_t contains no explicit information on the spatial distribution of the cluster at the origin, nor on the relationship among the sites in the clusters $\chi_t(s)$. However, as will be clear from the proof of Theorem 1.3, the branching mechanism and the spatial effects of the voter model can be separated on the scale t. Asymptotically, the contribution to χ_t initiated at time $t-s$ is seen to be a random shift of the cluster η_s. Space-time rescalings of the voter model have recently been shown to converge to super-Brownian motion in various settings (see [4], [11]).

The nearest neighbor choice assumption is essential only in our proof of the Poisson property of a limiting point process, where we explicitly use the local structure of the particle system. All other arguments can be extended to symmetric, irreducible random walks with finite variance. We believe that (1.10) is true for dimensions $d \geq 3$. In fact, Theorems 1.1 and 1.2 suggest that the Poisson limit law might also hold for a suitably rescaled genealogy of the type at the origin in $d = 2$.

The key ingredient for the proof of Theorem 1.3 is a graphical construction of the genealogy of the type at the origin for fixed time t derived from Harris' graphical representation of the voter model. This construction which works for arbitrary random walks is given in Section 2. The first step in proving Theorem 1.3 is to show convergence of the expectation measure of $T_t\Lambda_t$ toward λ, which we do in Section 3 using the graphical construction of Λ_t and the asymptotics from Theorem 1.1. The second step is to prove that any limiting point process is simple, which comes to showing that contributions initiated at distinct branch-times are negatively correlated. The third and final step is to verify the Poisson property for any limiting point process of the $T_t\Lambda_t$. This part of the proof is based on a coupling argument which works only in dimensions $d \geq 7$. Steps 2 and 3 are in Section 4. In Section 5 we put things together and prove the theorem.

2. Graphical constructions and Poisson point processes. Following Harris [9] we construct the voter model using a random space-time diagram on $\mathbb{Z}^d \times [0, \infty)$. To start, let Π be a Poisson point process on $\mathbb{Z}^d \times \mathbb{Z}^d \times [0, \infty)$ with intensity

$$\rho(\{x\} \times \{y\} \times ds) = \frac{1}{2d}\, ds \qquad \text{if } |x-y| = 1; \text{ and } 0, \text{ else.} \tag{2.1}$$

Intuitively, a point $(x, y, s) \in \Pi$ indicates that at time s the particle at site y adopts the type of the particle at x. For any point $(x, y, s) \in \Pi$ we draw an arrow from (x, s) to (y, s) in our space-time diagram. The random walk paths $(S_s^{x,t})_{0 \leq s \leq t}$, $x \in \mathbb{Z}^d$, are obtained by moving downward along vertical lines starting at (x, t), jumping to the tail of an arrow whenever we encounter its head (see Figure 1 for an example). Formally, $S_s^{x,t} = y$ if and only if there is a sequence of sites $x_0 = x, x_1, \ldots, x_n = y$ and times $0 = s_0 < s_1 < \cdots < s_n \leq s$ so that:

(i) for any $1 \leq i \leq n$ there is an arrow from x_i to x_{i-1} at time $t - s_i$;

(ii) there is no arrow with head toward x_{i-1} at times $(t - s_i, t - s_{i-1}]$, $1 \leq i \leq n$, and none toward y at times $[t - s, t - s_n]$.

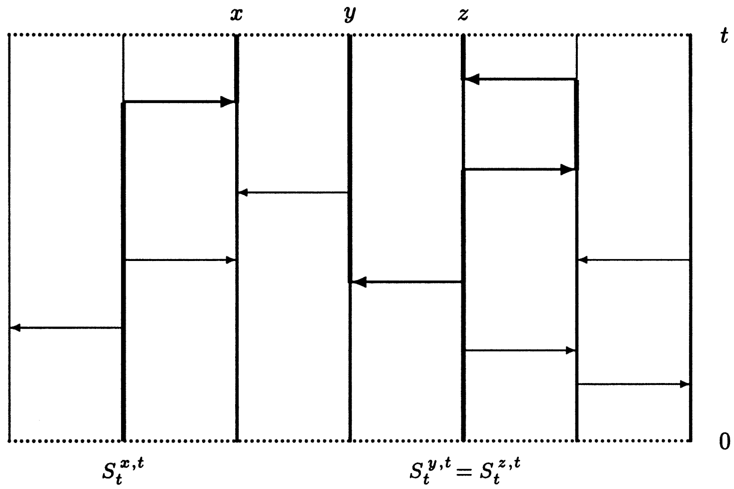

FIG. 1.

DEFINITION 2.1. *Let the interval* $(0, 1)$ *be the set of types. For any* $\xi_0 \in (0,1)^{\mathbb{Z}^d}$ *let* $\xi_t = (\xi_t(x))_{x\in\mathbb{Z}^d}$ *be defined as*

$$\xi_t(x) := \xi_0(S_t^{x,t}), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{2.2}$$

The continuous time Markov process $(\xi_t)_{t\geq 0}$ *on the space* $(0,1)^{\mathbb{Z}^d}$ *is called the nearest neighbor voter model on* $\mathbb{Z}^d$ *with initial state* ξ_0.

Fix $t > 0$ and write $\Pi_t = \Pi \cap \mathbb{Z}^d \times \mathbb{Z}^d \times [0, t]$. Let Π'_t be the random spatial shift of Π_t such that $S_t^{\mathscr{O},t}$ is mapped onto the origin,

$$\Pi'_t := \Pi_t - (S_t^{\mathscr{O},t}, S_t^{\mathscr{O},t}, 0), \tag{2.3}$$

where $A + y := \{x + y : x \in A\}$. Observe that a site which at time t has the same type as the origin is mapped onto a site which at time t has the type initially at the origin. More precisely,

$$\chi_t(\Pi_t) = \eta_t(\Pi'_t) + S_t^{\mathscr{O},t},$$

where $\chi_t(\Pi_t) = \chi_t(\Pi)$ is the cluster of the type at the origin at time t in the space-time diagram associated with Π, and $\eta_t(\Pi'_t)$ denotes the set of sites at time t with the type initially at the origin in the diagram associated with the shifted point process Π'_t. In particular, we have

$$N_t(\Pi_t) = n_t(\Pi'_t). \tag{2.4}$$

We remark that the random site $S_t^{\mathscr{O},t}$ defined in terms of the diagram associated with Π cannot be recovered from Π'_t unless $n_t(\Pi'_t) = 1$ (in fact, it can be shown that all sites in $\eta_t(\Pi'_t)$ are equally likely to be the shifted former origin). However, the point process Π'_t has a transparent structure, as can be seen from the following *probabilistic construction* of the space-time diagram associated with Π'_t.

- Take a copy of the random diagram representing $(\xi_s)_{0\le s\le t}$.
- Let $(X_s)_{0\le s\le t}$ be an independent simple random walk running forward in time, started at $X_0 = \mathscr{O}$. For each jump of the random walk add an arrow to the diagram: Draw an arrow from (X_{s^-}, s) to (X_s, s) whenever $X_s \ne X_{s^-}$.
- Delete all arrows in the diagram with heads toward the space-time path $(X_{s^-}, s)_{0\le s\le t}$.

The point process Π^*_t obtained from the construction above is formally defined as $\Pi^*_t := \Pi^* \cap \mathbb{Z}^d \times \mathbb{Z}^d \times [0,t]$, where

$$\Pi^* := (\Pi \cup A(X)) \setminus B(X). \tag{2.5}$$

Here, Π has distribution (2.1), $X = (X_s)_{s\ge 0}$ is a rate one simple symmetric random walk started at the origin which is independent of Π, and

$$A(X) := \{(X_{s^-}, X_s, s) : X_s \ne X_{s^-},\ 0 < s < \infty\},$$
$$B(X) := \{(x, X_{s^-}, s) : x \in \mathbb{Z}^d,\ 0 < s < \infty\}.$$

PROPOSITION 2.2. *For any $d \ge 1$ and $0 < t < \infty$, the two point processes Π^*_t and Π'_t agree in law. More precisely,*

$$(\Pi'_t, -S_t^{\mathscr{O},t}) \stackrel{d}{=} (\Pi^*_t, X_t).$$

PROOF. Fix $t > 0$ and define

$$X^t_s := S^{\mathscr{O},t}_{(t-s)^-} - S^{\mathscr{O},t}_t, \qquad 0 \le s \le t.$$

By duality and symmetry, $(X^t_s)_{0\le s\le t}$ is a rate one simple symmetric random walk on $\mathbb{Z}^d$ started at $X^t_0 = \mathscr{O}$. By construction, the space-time diagram Π representing $(\xi_s)_{s\ge 0}$ contains the arrows induced by the jumps of $(S^{\mathscr{O},t}_s)_{0\le s\le t}$, that is,

$$\{(S^{\mathscr{O},t}_s, S^{\mathscr{O},t}_{s^-}, t-s) : 0 \le s \le t,\ S^{\mathscr{O},t}_s \ne S^{\mathscr{O},t}_{s^-}\} \subset \Pi_t. \tag{2.6}$$

Shifting the random sets on either side of (2.6) by $S^{\mathscr{O},t}_t$ yields $A((X^t_s)_{0\le s\le t}) \subset \Pi'_t$. By construction of $(S^{\mathscr{O},t}_s)_{0\le s\le t}$ from Π, there cannot be an arrow in Π_t with head toward the path $(S^{\mathscr{O},t}_s, t-s)_{0\le s\le t}$, that is,

$$\Pi_t \cap \{(x, S^{\mathscr{O},t}_s, t-s) : x \in \mathbb{Z}^d,\ 0 \le s \le t\} = \emptyset,$$

and shifting either side by $S^{\mathscr{O},t}_t$ yields $\Pi'_t \cap B((X^t_s)_{0\le s\le t}) = \emptyset$.

Now fix any realization $\mathbf{x}$ of the random walk path up to time t, $\mathbf{x} = (x_s)_{0\le s\le t}$ with $x_0 = \mathscr{O}$, $|x_s - x_{s^-}| \le 1$, and $\mathbf{x}$ is right-continuous with left limits. The independence properties of Poisson point processes and the shift invariance of $\mathscr{L}(\Pi)$ then imply

$$\mathscr{L}(\Pi'_t \mid (X^t_s)_{0\le s\le t} = \mathbf{x}) = \mathscr{L}(\Pi_t \mid A(\mathbf{x}) \subset \Pi_t,\ B(\mathbf{x}) \cap \Pi_t = \varnothing). \tag{2.7}$$

Now recall the following elementary property of a Poisson point process N on some space $(E, \mathscr{E})$ having non-atomic intensity measure. For $A,\ B \in \mathscr{E}$ where A is countable and $A \cap B = \varnothing$,

$$\mathscr{L}(N \mid A \subset N,\ B \cap N = \varnothing) = \mathscr{L}((N \cup A) \setminus B). \tag{2.8}$$

By means of the law of total probability we obtain from (2.7) and (2.8) that

$$\Pi'_t \stackrel{d}{=} (\Pi_t \cup A((X^t_s)_{0\le s\le t})) \setminus B((X^t_s)_{0\le s\le t})$$

where $(X^t_s)_{0\le s\le t}$ is independent of Π_t. Comparison with (2.5) and the fact that $X^t_t = -S^{\mathscr{O},t}_t$ establish Proposition 2.2. $\square$

3. The expectation measure of the rescaled genealogy. The basic idea in the proof of Theorem 1.3 is to use the equality in law of Λ_t and the genealogy of the cluster $\eta^{\mathscr{O}}(\Pi^*_t)$ with the particle at X_t being distinguished. The second object is much simpler to analyze since we have the explicit probabilistic construction from the previous section at our disposal. The first and major step in the proof is to show convergence of the expectation measure λ_t.

For $t > 0$ fixed, let $0 \le \sigma_1 < \sigma_2 < \cdots$ be the jump times of the distinguished random walk path $(S^{\mathscr{O},t}_s)_{0\le s\le t}$ running backward in time and denote by $0 \le \sigma'_1 < \sigma'_2 < \cdots$ the times when in the space-time diagram arrows point away from the path $(S^{\mathscr{O},t}_s, (t-s)^-)_{0\le s\le t}$ (compare Figure 2). The independence properties of the Poisson point process Π imply that $\{\sigma_i : i \ge 1\}$

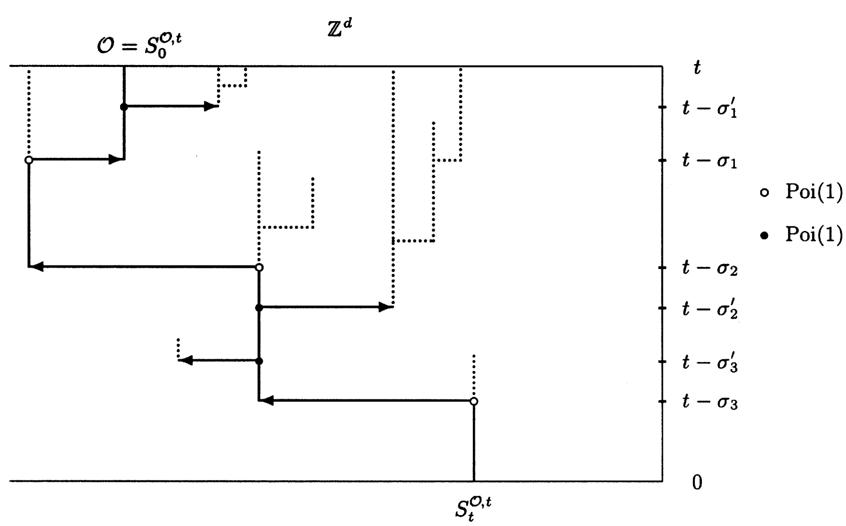

FIG. 2.

and $\{\sigma'_j : j \geq 1\}$ are independent rate one Poisson processes. Consequently, the superposition $\Sigma := \{\sigma_i : i \geq 1\} \cup \{\sigma'_j : j \geq 1\}$ is a rate two Poisson process on $[0, t]$. Note that $t - \Sigma$ is the set of potential branch-times. The idea behind the proof of (1.9) is to relate the size of a cluster $\chi_t(s)$, where $t-s$ is a potential branch-time, to the size of some fixed type at time s. To be precise, we will show

PROPOSITION 3.1. *For $d \geq 3$, as $s \leq t$ and $s \to \infty$,*

$$P(N_t(s) > 0 \mid s \in \Sigma) \sim \gamma_d \, p_s. \tag{3.1}$$

Furthermore, conditioned on non-extinction, the cluster sizes $N_t(s)$ and n_s are weakly equivalent,

$$\lim_{s \leq t,\ s \to \infty} \mathscr{L}(s^{-1} N_t(s) \mid N_t(s) > 0) = \lim_{s \to \infty} \mathscr{L}(s^{-1} n_s \mid n_s > 0). \tag{3.2}$$

Note that the asymptotic behavior of the quantities on the right-hand side of (3.1) and (3.2) is described by Theorem 1.1. Convergence of the expectation measure λ_t of $T_t \Lambda_t$ is an easy consequence. (The proof of Proposition 3.1 is temporarily deferred.)

COROLLARY 3.2. *For any $d \geq 3$ and any Borel set $B \subset (0, 1] \times \mathbb{R}^+$,*

$$\lambda_t(B) := E\, T_t \Lambda_t(B) \ \to \ \lambda(B) \qquad \text{as } \ t \to \infty. \tag{3.3}$$

PROOF. Let $B = [u_1, u_2] \times [z, \infty),\ 0 < u_1 \leq u_2 \leq 1,\ z \in \mathbb{R}^+$. Then, since Σ is a rate two Poisson process on $[0, t]$,

$$\begin{aligned} \lambda_t(B) &= E\, T_t \Lambda_t(B) \\ &= E\, |\{u_1 t \leq s \leq u_2 t : N_t(s) \geq zEN_t\}| \\ &= \int_{u_1}^{u_2} 2t\, du\, P(N_t(ut) > 0 \,|\, ut \in \Sigma)\, P(N_t(ut) \geq zEN_t \,|\, N_t(ut) > 0). \end{aligned}$$

Hence, using first (3.1) and (3.2) and then (1.1), (1.2) and (1.4), we deduce

$$\begin{aligned} \lambda_t(B) &\sim \int_{u_1}^{u_2} \frac{2\, du}{u}\, \gamma_d\, ut\, p_{ut}\, P(n_{ut} \geq zEN_t \,|\, n_{ut} > 0) \\ &\to \int_{u_1}^{u_2} \frac{2\, du}{u}\, \exp\Big(-\frac{2z}{u}\Big) = \lambda(B). \end{aligned} \tag{3.4}$$

This establishes the claim of the corollary, since the cylinder sets $[u_1, u_2] \times [z, \infty),\ 0 < u_1 \leq u_2 \leq 1,\ z \in \mathbb{R}^+$, generate the Borel σ-algebra. $\square$

We now begin preparations for the proof of Proposition 3.1. Our first goal is a more convenient representation of the set χ_t given that $s \in \Sigma$. For this purpose we need a description of the forward dynamics of the cluster process of the sites with type $\xi_t(\mathscr{O})$ initiated by a σ or σ'-event at real time $t - s$.

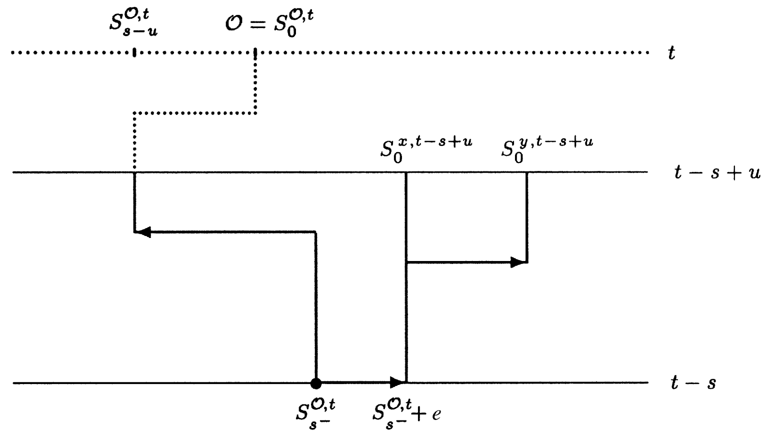

FIG. 3. *In the example above* $\zeta_u^{s,t} = \{x, y\}$ *and* $s = \sigma_i'$ *for some* i.

Formally, the cluster $\zeta_u^{s,t}$ at time $t-s+u$, $0 \le u \le s$, is defined as (compare Figure 3)

$$\zeta_u^{s,t} := \left\{ x \in \mathbb{Z}^d \;:\; \inf\left\{ r \ge 0 \;:\; S_r^{x,t-s+u} = S_r^{S^{\mathcal{O},t}_{s-u},t-s+u} \right\} = u \right\}. \tag{3.5}$$

Clearly, $(\zeta_u^{s,t})_{0\le u\le s} \equiv \emptyset$ unless $s \in \Sigma$, and also

$$\zeta_s^{s,t} = \chi_t(s). \tag{3.6}$$

For symmetry reasons we do not have to distinguish between σ and σ'-events. Also, by homogeneity of $\mathscr{L}(\Pi)$, the law of $(\zeta_u^{s,t})_{0\le u\le s}$ does not depend on t. Hence, with no loss of generality we let $t = s$ and write $\zeta_u^s := \zeta_u^{s,s}$.

From the construction of Π' in Section 2 we derive the following *probabilistic construction* of the process $(\zeta_u^s - \zeta_{0^+}^s)_{0<u\le s}$ given that $s \in \Sigma$. (For convenience we shift the process so that it starts at the origin and slightly misuse notation by identifying the set $\zeta_{0^+}^s$ and the single element in $\zeta_{0^+}^s$.)

- Start with a copy of the cluster process $(\eta_u)_{0<u\le s}$ of the type initially at the origin.
- Let $X^e = (X_u^e)_{0\le u\le s}$ be an independent random walk running forward in time, started at a nearest neighbor site e of the origin. Take the random walk to thin out the cluster process: Whenever X^e hits $(\eta_u)_{0<u\le s}$ the respective site changes its type and is removed.

Formally, the cluster process $(\vartheta_u)_{0<u\le s}$ obtained from the construction above is defined as

$$\vartheta_u := \{x \in \eta_u : S_{u-r}^{x,u} \ne X_r^e,\ 0 \le r \le u\}, \qquad 0 < u \le s. \tag{3.7}$$

The random walk X^e corresponds to the distinguished line of descent up to time t in the cluster χ_t. This immortal line of descent limits the growth of the cluster process $(\zeta_u^{s,t})_{0<u\le s}$ started at a nearest neighbor site. The thinning

procedure through the random walk X^e can be thought of as the enforcement of this limitation in growth.

PROPOSITION 3.3. *For any* $d \ge 1$ *and* $s > 0$,

$$\mathscr{L}\left((\vartheta_u)_{0<u\le s}\right) = \mathscr{L}\left((\zeta^s_u - \zeta^s_{0+})_{0<u\le s} \,|\, s \in \Sigma\right).$$

PROOF. Recall the construction of Π^*_s in (2.5). The event $\{s \in \Sigma\}$ corresponds to an arrow pointing either toward or away from the origin at time 0 in the space-time diagram associated with Π^*_s. However, the restriction of Π^*_s to $\mathbb{Z}^d \times \mathbb{Z}^d \times (0, s]$ is independent of this arrow. The construction above is therefore an immediate consequence of Proposition 2.2 and the effect of the deletion or insertion of an arrow in the diagram. Note that in contrast to the situation of Proposition 2.2 the random walk now starts at a nearest neighbor site e of the origin due to the fact that ζ^s_{0+} is shifted to the origin and $|S^{\mathscr{O},s}_{s^-} - \zeta^s_{0+}| = 1$. □

The construction above has useful consequences. For example, the process $(\vartheta_u)_{0\le u\le s}$ survives if and only if the process $(\eta_u)_{0\le u\le s}$ survives and the ancestral line of some site in η_s is not hit by the random walk X^e.

COROLLARY 3.4. *For any* $d \ge 1$ *and* $0 \le s \le t$,

$$P(N_t(s) > 0 \,|\, s \in \Sigma) = P(\exists\, x \in \eta_s \ne \varnothing :\ X^e_r \ne S^{x,s}_{s-r},\ 0 \le r \le s). \tag{3.8}$$

PROOF. Define $m_s := |\vartheta_s|$. Then, by construction of $(\vartheta_u)_{0\le u\le s}$ in (3.7),

$$\{m_s > 0\} = \{\exists\, x \in \eta_s \ne \varnothing :\ X^e_r \ne S^{x,s}_{s-r},\ 0 \le r \le s\}.$$

The claim of Corollary 3.4 follows since

$$P(N_t(s) > 0 \,|\, s \in \Sigma) = P(m_s > 0) \tag{3.9}$$

by (3.6) and Proposition 3.3. □

Before turning to the proof of Proposition 3.1, we develop several preliminary results. A property of the voter model that we will use repeatedly is a negative correlation inequality due to Arratia [1], which we state in the following general form.

LEMMA 3.5. *Let* $\eta^B_t := \bigcup_{z\in B} \eta^z_t$, $B \subset \mathbb{Z}^d$. *For any* $d \ge 1$, *finite disjoint sets* $B_1, B_2 \subset \mathbb{Z}^d$, *arbitrary sets* $A_1, A_2 \subset \mathbb{Z}^d$ *and* $t \ge 0$,

$$\begin{aligned} &P\left(\eta^{B_1}_t \cap A_1 \ne \varnothing,\ \eta^{B_2}_t \cap A_2 \ne \varnothing\right) \\ &\qquad \le P\left(\eta^{B_1}_t \cap A_1 \ne \varnothing\right) P\left(\eta^{B_2}_t \cap A_2 \ne \varnothing\right). \end{aligned} \tag{3.10}$$

Arratia proved (3.10) in the case where B_1 and B_2 are singletons and $A_1 = A_2$. However, his arguments which are based on Harris' [8] theorem on positive correlations for a monotone Markov process also work in the general case stated above. Note that, by means of the law of total probability, the sets A_1, A_2, B_1 and B_2 may be random as long as they are independent of $(\eta_t^z)_{t\geq 0}$, $z \in \mathbb{Z}^d$ and (A_1, B_1) is independent of (A_2, B_2).

Now let $(\hat{\eta}_s^t)_{0\leq s\leq t}$ be the reduced process associated with $(\eta_s)_{0\leq s\leq t}$. The set $\hat{\eta}_s^t$ consists of all particles which at time s have the type that was initially at the origin and also have a descendant at time t, that is,

$$\hat{\eta}_s^t := \{x \in \eta_s : x = S_{t-s}^{z,t} \text{ for some } z\} = \{S_{t-s}^{z,t} : z \in \eta_t\}. \tag{3.11}$$

In particular, $\hat{\eta}_0^t = \{\mathscr{O}\}$ or $\varnothing$, $\hat{\eta}_t^t = \eta_t$, and $|\hat{\eta}_r^t| \leq |\hat{\eta}_s^t|$ for $r \leq s$. Let G_t be the most recent time when a single particle was the common ancestor of all particles in η_t,

$$G_t := \sup\{s \geq 0 : |\hat{\eta}_s^t| = 1\}.$$

LEMMA 3.6. *For any* $d \geq 2$ *and* $T > 0$,

$$\lim_{\delta\to 0} \liminf_{t\to\infty} P(G_t \geq \delta t \mid n_t > 0) = 1; \tag{3.12}$$

$$\lim_{t\to\infty} E\, n_t 1\{G_t < T\} = 0. \tag{3.13}$$

PROOF. We first establish the limit statement (3.12). Note that the events $\{G_t \leq T,\ n_t > 0\}$ and $\{|\hat{\eta}_T^t| \geq 2\}$ agree. In view of (3.11) we thus have for any $0 < T \leq t$,

$$\begin{aligned} &P(G_t \leq T,\ n_t > 0)\\ &\quad = P(\exists\, x \neq y \in \eta_T : x = S_{t-T}^{z,t},\ y = S_{t-T}^{z',t} \text{ for some } z, z')\\ &\quad = \sum_{k=2}^{\infty} P\big(\exists\, x \neq y \in \eta_T : x, y \in \{S_{t-T}^{z,t} : z \in \mathbb{Z}^d\} \mid n_T = k\big)\, P(n_T = k)\\ &\quad \leq \sum_{k=2}^{\infty} k(k-1) \max_{x\neq y\in\mathbb{Z}^d} P(x, y \in \{S_{t-T}^{z,t} : z \in \mathbb{Z}^d\})\, P(n_T = k) \end{aligned} \tag{3.14}$$

where for the last inequality we used independence of $\{S_{t-T}^{z,t} : z \in \mathbb{Z}^d\}$ and η_T. Since $\{S_{t-T}^{z,t} : z \in \mathbb{Z}^d\} \overset{d}{=} \{S_{t-T}^{z,t-T} : z \in \mathbb{Z}^d\}$ by time homogeneity of Π, the duality relation (2.2) implies

$$P(x, y \in \{S_{t-T}^{z,t} : z \in \mathbb{Z}^d\}) = P(\eta_{t-T}^x \neq \varnothing,\ \eta_{t-T}^y \neq \varnothing).$$

Apply the negative correlation result (3.10) with $B_1 = \{x\}$, $B_2 = \{y\}$ and $A_1 = A_2 = \mathbb{Z}^d$ and then use (1.3) to deduce

$$\begin{aligned} P(G_t \leq T,\ n_t > 0) &\leq p_{t-T}^2 \sum_{k=1}^{\infty} k^2 P(n_T = k)\\ &= p_{t-T}^2\, EN_T, \qquad 0 \leq T \leq t. \end{aligned} \tag{3.15}$$

If we take $T = \delta t,\ 0 < \delta < 1$, then (3.15) and the asymptotics (1.2) and (1.4) imply for any $d \geq 2$,

$$\limsup_{t\to\infty} P(G_t \leq \delta t \mid n_t > 0) \leq \frac{2\delta}{(1-\delta)^2},$$

which establishes the first part of the lemma.

We now prove assertion (3.13). Using duality again, we obtain for $0 < T \leq t$,

$$\begin{aligned}
&E\,n_t 1\{G_t < T\} \\
&\quad = \sum_x P(x \in \eta_t,\ G_t < T) \\
&\quad = \sum_x \sum_z P\Big(S^{x,t}_{t-T} = z,\ S^{z,T}_T = \mathscr{O},\ G_t < T\Big) \\
&\quad = \sum_x \sum_z P\Big(S^{x,t}_{t-T} = z,\ S^{z,T}_T = \mathscr{O}, \\
&\qquad\qquad \exists\, z' \neq z :\ z' \in \{S^{y,t}_{t-T} :\ y \in \mathbb{Z}^d\},\ S^{z',T}_T = \mathscr{O}\Big) \\
&\quad \leq \sum_x \sum_z \sum_{z'\neq z} P\Big(S^{x,t}_{t-T} = z,\ S^{z,T}_T = \mathscr{O},\ z' \in \{S^{y,t}_{t-T} :\ y \in \mathbb{Z}^d\},\ S^{z',T}_T = \mathscr{O}\Big) \\
&\quad = \sum_x \sum_z \sum_{z'\neq z} P\Big(S^{x,t}_{t-T} = z,\ z' \in \{S^{y,t}_{t-T} :\ y \in \mathbb{Z}^d\}\Big) \\
&\qquad\qquad \times P\Big(S^{z,T}_T = \mathscr{O},\ S^{z',T}_T = \mathscr{O}\Big) \\
&\quad = \sum_x \sum_z \sum_{z'\neq z} P\Big(z' \in \{S^{y,t}_{t-T} :\ y \in \mathbb{Z}^d\} \mid S^{x,t}_{t-T} = z\Big) \\
&\qquad\qquad \times P(S^{x,t}_{t-T} = z)\, P(z, z' \in \eta_T).
\end{aligned} \tag{3.16}$$

By time homogeneity of Π, duality, and the correlation inequality (3.10) with $B_1 = \{z'\}$, $B_2 = \{z\}$, $A_1 = \mathbb{Z}^d$ and $A_2 = \{x\}$,

$$\begin{aligned}
P(z' \in \{S^{y,t}_{t-T} :\ y \in \mathbb{Z}^d\} \mid S^{x,t}_{t-T} = z) &= P(\eta^{z'}_{t-T} \neq \varnothing \mid x \in \eta^{z}_{t-T}) \\
&\leq P(\eta^{z'}_{t-T} \neq \varnothing) = p_{t-T}.
\end{aligned}$$

Plugging this estimate into (3.16) and using (1.3) we obtain

$$\begin{aligned}
E\,n_t 1\{G_t < T\} &\leq p_{t-T} \sum_z \sum_{z'} P(z, z' \in \eta_T) \sum_x P(X_{t-T} = z - x) \\
&= p_{t-T}\, E n_T^2 \\
&= p_{t-T}\, E N_T \ \to\ 0 \qquad \text{as } t \to \infty
\end{aligned} \tag{3.17}$$

by (1.2) and (1.4). $\square$

The next lemma states that the trunk of the reduced process asymptotically performs a simple symmetric random walk. Again we misuse notation and identify the set $\hat{\eta}^t_r$ and the single element in $\hat{\eta}^t_r$, $0 \leq r < G_t$.

LEMMA 3.7. *For any* $d \geq 2$ *and* $T > 0$,

$$\mathscr{L}((\hat{\eta}^t_r)_{0\leq r<G_t\wedge T} \mid n_t > 0) \xrightarrow{TV} \mathscr{L}((X_r)_{0\leq r<T}) \qquad \textit{as } t \to \infty, \tag{3.18}$$

where $(X_r)_{r\ge 0}$ *is rate one simple symmetric random walk started at* $X_0 = \mathscr{O}$ *and* $\xrightarrow{TV}$ *denotes convergence in total variation distance.*

PROOF. Fix $T > 0$, and assume $t > T$. Recall the following elementary properties of total variation distance $d_{TV}(\cdot,\cdot)$ on the space of signed finite measures on some Polish space $(E,\mathscr{E})$. If $\mu = \sum_i \mu_i$ and $\nu = \sum_i \nu_i$, then

$$d_{TV}(\mu,\nu) \le \sum_i d_{TV}(\mu_i,\nu_i). \tag{3.19}$$

If μ and ν are probability measures and $\alpha,\beta \in \mathbb{R}_0^+$, then

$$d_{TV}(\alpha\mu,\beta\nu) \le (\alpha\wedge\beta)\, d_{TV}(\mu,\nu) + |\alpha-\beta|. \tag{3.20}$$

The total variation distance between probability measures is at most 2. Let $\mu_{x,t} := \mathscr{L}\big((\hat\eta_r^t)_{0\le r<G_t\wedge T} \,|\, \hat\eta_T^t = x\big)$ and $\nu_x := \mathscr{L}\big((X_r)_{0\le r<T} \,|\, X_T = x\big)$. Repeatedly using (3.19) and (3.20), we have

$$\begin{aligned}
&d_{TV}\big(\mathscr{L}\big((\hat\eta_r^t)_{0\le r<G_t\wedge T} \,|\, n_t>0\big),\ \mathscr{L}\big((X_r)_{0\le r<T}\big)\big)\\
&\quad\le d_{TV}\big(\mathscr{L}\big((\hat\eta_r^t)_{0\le r<G_t\wedge T} \,|\, n_t>0\big),\ \mathscr{L}\big((\hat\eta_r^t)_{0\le r<G_t\wedge T} \,|\, |\hat\eta_T^t|=1\big)\big)\\
&\qquad + d_{TV}\big(\mathscr{L}((\hat\eta_r^t)_{0\le r<G_t\wedge T} \,|\, |\hat\eta_T^t|=1),\ \mathscr{L}\big((X_r)_{0\le r<T}\big)\big)\\
&\quad\le 2\,P(|\hat\eta_T^t|>1 \,|\, n_t>0)\\
&\qquad + \sum_{x\in\mathbb{Z}^d} d_{TV}\big(P(\hat\eta_T^t = x \,|\, |\hat\eta_T^t|=1)\,\mu_{x,t},\ P(X_T=x)\,\nu_x\big)\\
&\quad\le 2\,P(G_t\le T \,|\, n_t>0) + \sum_{x\in\mathbb{Z}^d} P(X_T=x)\, d_{TV}(\mu_{x,t},\ \nu_x)\\
&\qquad + \sum_{x\in\mathbb{Z}^d} |P(\hat\eta_T^t = x \,|\, |\hat\eta_T^t|=1) - P(X_T=x)|.
\end{aligned} \tag{3.21}$$

Now $P(G_t\le T \mid n_t>0) \to 0$ as $t\to\infty$ by Lemma 3.6. So to complete the proof of (3.18), it suffices, by the bounded convergence theorem, to show that for all $x\in\mathbb{Z}^d$,

$$\lim_{t\to\infty} d_{TV}(\mu_{x,t},\nu_x) = 0 \tag{3.22}$$

and

$$\lim_{t\to\infty} P(\hat\eta_T^t = x \mid |\hat\eta_T^t|=1) = P(X_T=x). \tag{3.23}$$

We first derive an upper bound for $d_{TV}(\mu_{x,t},\ \nu_x)$, $x\in\mathbb{Z}^d$. Let $n_{T,t}^x := |\{z\in\mathbb{Z}^d : S_{t-T}^{z,t} = x\}|$. Since

$$\{x\in\hat\eta_T^t\} = \Big\{S_T^{x,T} = \mathscr{O},\ n_{T,t}^x > 0\Big\}, \tag{3.24}$$

$(S_{T-r}^{x,T})_{0\le r\le T}$ is conditionally independent of the event $\{x\in\hat\eta_T^t\}$ given that $S_T^{x,T} = \mathscr{O}$, that is,

$$\mathscr{L}\Big((S_{T-r}^{x,T})_{0\le r<T} \,|\, x\in\hat\eta_T^t\Big) = \mathscr{L}\Big((S_{T-r}^{x,T})_{0\le r<T} \,|\, S_T^{x,T} = \mathscr{O}\Big) = \nu_x. \tag{3.25}$$

Also, observe that

$$(\hat{\eta}^t_r)_{0\le r<G_t\wedge T} = \big(S^{x,T}_{T-r}\big)_{0\le r<T} \qquad \text{on } \{\hat{\eta}^t_T = x\}. \tag{3.26}$$

Hence, using first (3.25) and (3.26) and then (3.19) and (3.20), we have

$$\begin{aligned} &d_{TV}(\mu_{x,t}, \nu_x) \\ &\quad = d_{TV}\big(\mathscr{L}\big((S^{x,T}_{T-r})_{0\le r<T} \,|\, \hat{\eta}^t_T = x\big),\ \mathscr{L}\big((S^{x,T}_{T-r})_{0\le r<T} \,|\, x \in \hat{\eta}^t_T\big)\big) \\ &\quad \le 2\, P(\hat{\eta}^t_T \ne x \,|\, x \in \hat{\eta}^t_T) = 2\, P(G_t \le T \,|\, x \in \hat{\eta}^t_T). \end{aligned} \tag{3.27}$$

Now observe that, by (3.24),

$$\begin{aligned} P(x \in \hat{\eta}^t_T \,|\, n_t > 0) &= \frac{P(S^{x,T}_T = \mathscr{O})\, P(n^x_{T,t} > 0)}{P(n_t > 0)} \\ &= \frac{p_{t-T}}{p_t}\ P(X_T = x). \end{aligned} \tag{3.28}$$

Using first (3.27) and then (3.28) and Lemma 3.6, we deduce

$$\limsup_{t\to\infty} d_{TV}(\mu_{x,t}, \nu_x) \le 2 \limsup_{t\to\infty} \frac{P(G_t \le T \,|\, n_t > 0)}{P(x \in \hat{\eta}^t_T \,|\, n_t > 0)} \ = \ 0, \tag{3.29}$$

establishing (3.22).

For (3.23), we note that

$$P(\hat{\eta}^t_T = x \,|\, |\hat{\eta}^t_T| = 1) = \frac{P(\hat{\eta}^t_T = x \,|\, n_t > 0)}{P(G_t > T \,|\, n_t > 0)} \le \frac{P(x \in \hat{\eta}^t_T \,|\, n_t > 0)}{P(G_t > T \,|\, n_t > 0)} \tag{3.30}$$

and

$$\begin{aligned} P(\hat{\eta}^t_T = x \,|\, |\hat{\eta}^t_T| = 1) &\ge P(\hat{\eta}^t_T = x \,|\, n_t > 0) \\ &\ge P(x \in \hat{\eta}^t_T \,|\, n_t > 0) - P(G_t \le T \,|\, n_t > 0). \end{aligned} \tag{3.31}$$

If we pass to the limit $t \to \infty$ in (3.30) and (3.31), respectively, then (3.28), (1.2) and Lemma 3.6 show that

$$\lim_{t\to\infty} P(\hat{\eta}^t_T = x \,|\, |\hat{\eta}^t_T| = 1) = P(X_T = x),$$

which completes our proof. $\square$

Let $A_{T,t}$ be the event that the trunk of the reduced process and the immortal line do not collide until time T,

$$A_{T,t} := \{X^e_r \ne \hat{\eta}^t_r,\ 0 \le r < G_t \wedge T\}. \tag{3.32}$$

Then, for $d \ge 3$,

$$\lim_{T\to\infty} \lim_{t\to\infty} P(A_{T,t} \,|\, n_t > 0) = \gamma_d. \tag{3.33}$$

Indeed, if R denotes the first hitting time of the origin of a rate two simple symmetric random walk started at a nearest neighbor site e, then Lemma 3.7 implies

$$\begin{aligned}\lim_{t\to\infty} P(A_{T,t}\,|\,n_t>0) &= P(X_r^e \neq X_r,\ 0\le r<T)\\ &= P(R\ge T)\to \gamma_d \qquad \text{as } T\to\infty,\end{aligned}\tag{3.34}$$

where for the second equality we have used independence of $(X_r^e)_{0\le r\le t}$ and $(\hat\eta_r^t)_{0\le r\le t}$. The following lemma states that the thinning of the reduced process through the immortal line is substantial only if the trunk of the reduced process is hit by the immortal line ("all or nothing") and that the conditioned limit law of n_t is the same whether the trunk of the reduced process is hit or not.

LEMMA 3.8. *For $d\ge 3$,*

$$\lim_{T\to\infty}\limsup_{t\to\infty} E(t^{-1}|n_t-m_t|\,|\,n_t>0, A_{T,t})=0.\tag{3.35}$$

Furthermore, for any $T>0$,

$$\lim_{t\to\infty} d_{TV}\big(\mathscr{L}(n_t\,|\,n_t>0, A_{T,t}),\ \mathscr{L}(n_{t-T}\,|\,n_{t-T}>0)\big)=0,\tag{3.36}$$

so that, in particular,

$$\lim_{t\to\infty}\mathscr{L}(t^{-1}n_t\,|\,n_t>0, A_{T,t})=\lim_{t\to\infty}\mathscr{L}(t^{-1}n_t\,|\,n_t>0).\tag{3.37}$$

PROOF. We first prove (3.36) following similar lines as in the proof of Lemma 3.7. Fix $T>0$, and assume $t>T$. Let $\bar\mu_{x,t} := \mathscr{L}(n_t\,|\,\hat\eta_T^t = x, A_{T,t})$, then, using (3.19) and (3.20), we have

$$\begin{aligned}
&d_{TV}\big(\mathscr{L}(n_t\,|\,n_t>0, A_{T,t}),\ \mathscr{L}(n_{t-T}\,|\,n_{t-T}>0)\big)\\
&\quad\le d_{TV}\big(\mathscr{L}(n_t\,|\,n_t>0, A_{T,t}),\ \mathscr{L}(n_t\,|\,n_t>0, G_t>T, A_{T,t})\big)\\
&\qquad + d_{TV}\big(\mathscr{L}(n_t\,|\,|\hat\eta_T^t|=1, A_{T,t}),\ \mathscr{L}(n_{t-T}\,|\,n_{t-T}>0)\big)\\
&\quad\le 2\,P(G_t\le T\,|\,n_t>0, A_{T,t})\\
&\qquad+\sum_{x\in\mathbb{Z}^d} P\left(\hat\eta_T^t = x\,|\,|\hat\eta_T^t|=1, A_{T,t}\right) d_{TV}\big(\bar\mu_{x,t},\ \mathscr{L}(n_{t-T}\,|\,n_{t-T}>0)\big).
\end{aligned}\tag{3.38}$$

By (3.12) and (3.34), as $t\to\infty$,

$$P(G_t\le T\,|\,n_t>0, A_{T,t})\to 0$$

and

$$P(|\hat\eta_T^t|=1,\ A_{T,t}\,|\,n_t>0)\to P(R\ge T).$$

Use the second statement and apply Lemma 3.7 to conclude that the sequence $\mathscr{L}(\hat{\eta}_T^t \,|\, |\hat{\eta}_T^t| = 1, A_{T,t})$, $t > T$, is tight. So to establish (3.36), it suffices, by the bounded convergence theorem, to show that for all $x \in \mathbb{Z}^d$,

$$\lim_{t\to\infty} d_{TV}\big(\bar{\mu}_{x,t},\ \mathscr{L}(n_{t-T} \,|\, n_{t-T} > 0)\big) = 0. \tag{3.39}$$

Fix $x \in \mathbb{Z}^d$ and recall the definition of $n_{T,t}^x$ before (3.24). Note that

$$n_{T,t}^x \stackrel{d}{=} n_{t-T}, \tag{3.40}$$

and that $n_{T,t}^x$ is independent of the event $\{S_T^{x,T} = \mathscr{O}\} \cap A_T^x$, where $A_T^x := \{X_r^e \neq S_{T-r}^{x,T},\ 0 \le r < T\}$. In view of (3.24) we thus have

$$\mathscr{L}(n_{T,t}^x \,|\, x \in \hat{\eta}_T^t,\ A_T^x) = \mathscr{L}(n_{t-T} \,|\, n_{t-T} > 0). \tag{3.41}$$

Also, observe that

$$n_t = n_{T,t}^x \qquad \text{on } \{\hat{\eta}_T^t = x\} \tag{3.42}$$

and

$$\{\hat{\eta}_T^t = x\} \cap A_{T,t} = \{\hat{\eta}_T^t = x\} \cap A_T^x. \tag{3.43}$$

Putting together (3.41), (3.42) and (3.43) yields

$$\begin{aligned} &d_{TV}\big(\bar{\mu}_{x,t},\ \mathscr{L}(n_{t-T} \,|\, n_{t-T} > 0)\big) \\ &\quad = d_{TV}\big(\mathscr{L}(n_{T,t}^x \,|\, \hat{\eta}_T^t = x,\ A_T^x), \\ &\qquad \mathscr{L}(n_{T,t}^x \,|\, x \in \hat{\eta}_T^t,\ A_T^x)\big) \le 2\,P(G_t \le T \,|\, x \in \hat{\eta}_T^t,\ A_T^x). \end{aligned} \tag{3.44}$$

By Lemma 3.6, the right-hand side of (3.44) tends to 0 provided that

$$\liminf_{t\to\infty} P(x \in \hat{\eta}_T^t,\ A_T^x \,|\, n_t > 0) > 0. \tag{3.45}$$

To verify (3.45), note that (3.24) implies

$$P(x \in \hat{\eta}_T^t,\ A_T^x \,|\, n_t > 0) = \frac{p_{t-T}}{p_t}\, P(S_T^{x,T} = \mathscr{O},\ A_T^x),$$

where the factor depending on t is greater than 1. This establishes (3.39) and completes our proof of (3.36). The weak equivalence assertion (3.37) is an immediate consequence of (3.36).

We now turn to the proof of (3.35). By construction of ϑ_t in (3.7), $n_t - m_t$ is a nonnegative quantity and

$$\eta_t \setminus \vartheta_t = \{x \in \eta_t :\ X_r^e = S_{t-r}^{x,t} \text{ for some } 0 \le r \le t\}.$$

Also, note that given $A_{T,t}$, we have $X_r^e \neq S_{t-r}^{x,t}$ for all $0 \le r < G_t \wedge T$, $x \in \eta_t$. Consequently,

$$\begin{aligned}
&E(|n_t - m_t| \,|\, n_t > 0,\ A_{T,t})\\
&\quad = E\big(|\{x \in \eta_t : X_r^e = S_{t-r}^{x,t} \text{ for some } G_t \wedge T \le r \le t\}| \,|\, n_t > 0,\ A_{T,t}\big)\\
&\quad \le P(A_{T,t} | n_t > 0)^{-1}\\
&\qquad \times E\big(|\{x \in \eta_t :\ X_r^e = S_{t-r}^{x,t} \text{ for some } G_t \wedge T \le r \le t\}| \,|\, n_t > 0\big)\\
&\quad = P(A_{T,t} | n_t > 0)^{-1} \sum_x P(x \in \eta_t,\ X_r^e = S_{t-r}^{x,t}\\
&\qquad\qquad \text{for some } G_t \wedge T \le r \le t \,|\, n_t > 0)\\
&\quad = P(n_t > 0,\ A_{T,t})^{-1} \sum_x P(x \in \eta_t,\ X_r^e = S_{t-r}^{x,t} \text{ for some } G_t \wedge T \le r \le t)
\end{aligned}$$

$$\begin{aligned}
&\quad \le P(n_t > 0,\ A_{T,t})^{-1} \sum_x \big(P(x \in \eta_t,\ G_t < T)\\
&\qquad\qquad + P(x \in \eta_t,\ X_r^e = S_{t-r}^{x,t} \text{ for some } T \le r \le t)\big)
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
&\quad = P(n_t > 0,\ A_{T,t})^{-1} \Big(E\, n_t 1\{G_t < T\}\\
&\qquad + \sum_x P(X_r^e = S_{t-r}^{x,t} \text{ for some } T \le r \le t \,|\, S_t^{x,t} = \mathscr{O})\, P(S_t^{x,t} = \mathscr{O})\Big)\\
&\quad = P(n_t > 0,\ A_{T,t})^{-1} \Big(E\, n_t 1\{G_t < T\}\\
&\qquad\qquad + \sum_x P(X_r^e = X_r \text{ for some } T \le r \le t \,|\, X_t = x)\, P(X_t = x)\Big)\\
&\quad = P(n_t > 0,\ A_{T,t})^{-1} \big(E\, n_t 1\{G_t < T\} + P(X_r^e = X_r \text{ for some } T \le r \le t)\big).
\end{aligned}$$

Therefore, by (3.17),

$$\begin{aligned}
&t^{-1} E(|n_t - m_t| \,|\, n_t > 0,\ A_{T,t})\\
&\qquad \le \frac{p_{t-T} E N_T + P(X_r^e = X_r \text{ for some } r \ge T)}{t p_t P(A_{T,t} \mid n_t > 0)}.
\end{aligned}$$

By (1.2) and (3.34), as $t \to \infty$, the right-hand side above tends to

$$P(X_r^e = X_r \text{ for some } r \ge T) = P(X_r^e = \mathscr{O} \text{ for some } r \ge 2T).$$

Since simple symmetric random walk is transient in $d \ge 3$, letting $T \to \infty$ completes the proof of (3.35). □

PROOF OF PROPOSITION 3.1. Recall from Proposition 3.3 and (3.6) that $\mathscr{L}(m_s) = \mathscr{L}(N_t(s) \,|\, s \in \Sigma)$ for any $0 \le s \le t$. In particular, $\mathscr{L}(N_t(s) \,|\, s \in \Sigma)$ does not depend on t and we may let $s = t$. By construction of ϑ_t and $(\hat{\eta}_s^t)_{0 \le s \le t}$ in (3.7) and (3.11), we have

$$m_t \;=\; m_t 1\{A_{T,t}\} \le n_t 1\{A_{T,t}\}, \quad 0 \le T \le t. \tag{3.47}$$

In particular,

$$\frac{P(m_t > 0)}{P(n_t > 0)} = P(m_t > 0, A_{T,t} \,|\, n_t > 0)$$
$$= P(A_{T,t} \,|\, n_t > 0) - P(m_t = 0, A_{T,t} \,|\, n_t > 0).$$

Hence, for any $T \geq 0$,

$$\limsup_{t\to\infty} \left| \frac{P(m_t > 0)}{P(n_t > 0)} - \gamma_d \right| \leq \limsup_{t\to\infty} |P(A_{T,t} \,|\, n_t > 0) - \gamma_d| + \limsup_{t\to\infty} P(m_t = 0 \,|\, n_t > 0, A_{T,t}). \tag{3.48}$$

By (3.34), the first term on the right-hand side of (3.48) is bounded above by $P(T < R < \infty)$, which tends to 0 as $T \to \infty$. For the second term, note that for any $\varepsilon > 0$,

$$\begin{aligned} &\limsup_{t\to\infty} P(m_t = 0 \,|\, n_t > 0, A_{T,t}) \\ &\quad \leq \limsup_{t\to\infty} P(n_t \leq \varepsilon t \,|\, n_t > 0, A_{T,t}) \\ &\qquad + \limsup_{t\to\infty} P(|n_t - m_t| \geq \varepsilon t \,|\, n_t > 0, A_{T,t}). \end{aligned} \tag{3.49}$$

By (3.35), we have

$$\limsup_{t\to\infty} P(|n_t - m_t| \geq \varepsilon t \,|\, n_t > 0,\ A_{T,t}) \to 0 \qquad \text{as } T \to \infty. \tag{3.50}$$

For the other term in (3.49), note that the asymptotic equivalence (3.37) and the limit law (1.1) imply

$$\lim_{t\to\infty} P(n_t \leq \varepsilon t \,|\, n_t > 0, A_{T,t}) = 1 - \exp(\varepsilon \gamma_d^{-1}) \ \to\ 0 \qquad \text{as } \varepsilon \to 0. \tag{3.51}$$

Since the left-hand side of (3.49) does not depend on ε, the estimates (3.50) and (3.51) imply

$$\lim_{T\to\infty} \limsup_{t\to\infty} P(m_t = 0 \,|\, n_t > 0, A_{T,t}) = 0. \tag{3.52}$$

Since the quantity on the left-hand side of (3.48) does not depend on T, the limit statement (3.52) shows

$$\lim_{t\to\infty} \frac{P(m_t > 0)}{P(n_t > 0)} = \gamma_d,$$

which establishes assertion (3.1).

We now turn to the asymptotic equivalence (3.2). As previously noted,

$$\mathscr{L}(s^{-1} N_t(s) | N_t(s) > 0) = \mathscr{L}(s^{-1} m_s \,|\, m_s > 0),$$

and we may set $s = t$. The fact that $\{m_t > 0\} \subset \{n_t > 0\} \cap A_{T,t}$ for any $0 \leq T \leq t$ (see (3.47)) implies that $\mathscr{L}(t^{-1} m_t \,|\, m_t > 0)$ is stochastically larger than $\mathscr{L}(t^{-1}m_t \mid n_t > 0,\ A_{T,t})$. Consequently, using first (3.50) and then (3.37), we obtain for any $x > 0$,

$$\begin{aligned}\liminf_{t\to\infty} P(m_t > xt \mid m_t > 0) &\geq \limsup_{T\to\infty} \liminf_{t\to\infty} P(m_t > xt \mid n_t > 0,\ A_{T,t})\\ &= \limsup_{T\to\infty} \liminf_{t\to\infty} P(n_t > xt \mid n_t > 0,\ A_{T,t})\\ &= \lim_{t\to\infty} P(n_t > xt \mid n_t > 0).\end{aligned} \tag{3.53}$$

For the upper bound in (3.2), use $P(A|B) \leq P(A|C)P(B|C)^{-1}$, $B \subset C$, to deduce that

$$P(m_t > xt \mid m_t > 0) \leq \frac{P(m_t > xt \mid n_t > 0,\ A_{T,t})}{P(m_t > 0 \mid n_t > 0,\ A_{T,t})}. \tag{3.54}$$

It follows from (3.37) and (3.50) that

$$\lim_{T\to\infty} \limsup_{t\to\infty} P(m_t > xt \mid n_t > 0,\ A_{T,t}) = \lim_{t\to\infty} P(n_t > xt \mid n_t > 0),$$

and from (3.52) that

$$\lim_{T\to\infty} \liminf_{t\to\infty} P(m_t > 0 \mid n_t > 0, A_{T,t}) = 1.$$

Plugging these estimates into (3.54), we get that

$$\limsup_{t\to\infty} P(m_t > xt \mid m_t > 0) \leq \lim_{t\to\infty} P(n_t > xt \mid n_t > 0),$$

which combined with the estimate (3.53) establishes the second claim of Proposition 3.1. $\square$

4. Properties of a limiting point process. The key to deduce properties as uniform integrability of $(T_t\Lambda_T)$ and simplicity of a limiting point process is the correlation inequality (3.10). For $0 \leq r \leq t$ and $x \in \mathbb{Z}^d$ write $(\eta_u^{x,r})_{0\leq u\leq t}$ for the cluster initiated at the space-time point (x, r) in the random diagram associated with Π (here, u stands for real time),

$$\eta_u^{x,r} := \begin{cases} \varnothing, & 0 \leq u < r,\\ \eta_{u-r}^x(\theta_r\Pi), & r \leq u \leq t,\end{cases} \tag{4.1}$$

where θ_r denotes the time shift of Π by r restricted to $\mathbb{Z}^d \times \mathbb{Z}^d \times (0,\infty)$,

$$\theta_r\Pi := \{(x, y, s-r) : (x, y, s) \in \Pi,\ s > r\}.$$

The correlation inequality (3.10) yields the following upper bound for the probability that the two processes $(\eta_u^{x,r})_{0\leq u\leq t}$ and $(\eta_u^{y,s})_{0\leq u\leq t}$ survive disjointly. For later reference we state it as

LEMMA 4.1. *For any $d \geq 1$, $x, y \in \mathbb{Z}^d$ and $0 \leq r, s \leq t$,*

$$P(\eta_t^{x,r} \neq \varnothing,\ \eta_t^{y,s} \neq \varnothing,\ \eta_t^{x,r} \neq \eta_t^{y,s}) \leq p_{t-r}p_{t-s}. \tag{4.2}$$

PROOF. Suppose $0 \le r \le s \le t$ and let $\bar\eta^x_{s-r} \stackrel{d}{=} \eta^x_{s-r}$, independent of Π. By time homogeneity of Π,

$$P\left(\eta_t^{x,r} \neq \varnothing,\ \eta_t^{y,s} \neq \varnothing,\ \eta_t^{x,r} \neq \eta_t^{y,s}\right) = P\left(\eta_{t-s}^{\bar\eta^x_{s-r}\setminus\{y\}} \neq \varnothing,\ \eta^y_{t-s} \neq \varnothing\right).$$

We apply (3.10) in the case $B_1 = \bar\eta^x_{s-r} \setminus \{y\}$, $B_2 = \{y\}$ and $A_1 = A_2 = \mathbb{Z}^d$ to deduce (see the remark following Lemma 3.5 regarding the fact that B_1 is random)

$$P\left(\eta_t^{x,r} \neq \varnothing,\ \eta_t^{y,s} \neq \varnothing,\ \eta_t^{x,r} \neq \eta_t^{y,s}\right) \le p_{t-s}\, P\left(\eta_{t-s}^{\bar\eta^x_{s-r}\setminus\{y\}} \neq \varnothing\right).$$

Finally, note that

$$P\left(\eta_{t-s}^{\bar\eta^x_{s-r}\setminus\{y\}} \neq \varnothing\right) \le P\left(\eta_{t-s}^{\bar\eta^x_{s-r}} \neq \varnothing\right) = P\left(\eta^x_{t-r} \neq \varnothing\right) = p_{t-r}. \qquad \square$$

We use Lemma 4.1 to obtain an upper bound for the probability that contributions to the cluster at the origin occur at distinct potential branch-times $t-r$ and $t-s$.

LEMMA 4.2. *For any $d \ge 1$ and $0 \le r < s \le t$,*

$$P(N_t(r) > 0,\, N_t(s) > 0 \mid r, s \in \Sigma) \le p_r p_s. \tag{4.3}$$

PROOF. Recall the graphical construction Π_t^* of the shifted point process Π_t' in (2.5). Let Σ^* be the set of times $u \le t$ such that either the random walk X has a jump at time $t-u$ or an arrow points away from X_{t-u} in the space-time diagram associated with Π,

$$\Sigma^* := \left\{0 \le u \le t:\ X_{t-u} \neq X_{(t-u)^-} \text{ or } (X_{t-u}, z, t-u) \in \Pi \text{ for some } z\right\}.$$

For $u \in \Sigma^*$ let Y_u be the site where the potential contribution to $\eta_t^{\mathscr{O}}(\Pi_t^*)$ with branch-time $t-u$ is initiated,

$$Y_u = z \ :\Leftrightarrow\ X_{t-u} \neq X_{(t-u)^-} = z \text{ or } (X_{t-u}, z, t-u) \in \Pi, \tag{4.4}$$

and let $D_{r,x,s,y}$ be the event that potential contributions initiate at space-time points $(x, t-r)$ and $(y, t-s)$,

$$D_{r,x,s,y} := \{r, s \in \Sigma^*,\ Y_r = x, Y_s = y\}. \tag{4.5}$$

Let us write $\eta_u^{*x,r} := \eta_u^{x,r}(\Pi_t^*)$ for the cluster at time u initiated at the space-time point (x, r) in the random diagram associated with the point process Π_t^*. Proposition 2.2 implies

$$\{(u, \chi_t(u)):\ u \in \Sigma\} \stackrel{d}{=} \left\{(u, \eta_t^{*Y_u, t-u} - X_t):\ u \in \Sigma^*\right\}. \tag{4.6}$$

Consequently,

$$
\begin{aligned}
&P(N_t(r) > 0, N_t(s) > 0 \mid r, s \in \Sigma) \\
&\qquad = P\left(\eta_t^{*Y_r, t-r} \neq \varnothing,\ \eta_t^{*Y_s, t-s} \neq \varnothing \mid r, s \in \Sigma^*\right) \\
&\qquad \leq \max_{x, y \in \mathbb{Z}^d} P\left(\eta_t^{*x, t-r} \neq \varnothing,\ \eta_t^{*y, t-s} \neq \varnothing \mid D_{r,x,s,y}\right).
\end{aligned} \tag{4.7}
$$

Note that on the event $D_{r,x,s,y}$ the cluster $\eta_t^{*x,t-r}$ is obtained from $\eta_t^{x,t-r} = \eta_t^{x,t-r}(\Pi)$ by a pruning procedure through X, as is $\eta_t^{*y,t-s}$ from $\eta_t^{y,t-s}$, that is,

$$
\eta_t^{*x,t-r} \subset \eta_t^{x,t-r}, \quad \eta_t^{*y,t-s} \subset \eta_t^{y,t-s} \ \text{ on } D_{r,x,s,y}. \tag{4.8}
$$

Also, we claim that the clusters $\eta_t^{x,t-r}$ and $\eta_t^{*y,t-s}$ are disjoint if $r, s \in \Sigma^*$ and $Y_r = x$, $Y_s = y$. This is because a particle starting in $\eta_t^{x,t-r}$ has time of coalescence (in reversed time and with respect to Π_t^*) with the distinguished path $(X_{t-u})_{u\geq 0}$ at most r, while for a particle starting in $\eta_t^{*y,t-s}$ this coalescence time equals $s > r$. Consequently, by (4.8),

$$
\eta_t^{x,t-r} \neq \eta_t^{y,t-s} \ \text{ on } D_{r,x,s,y} \cap \{\eta_t^{*y,t-s} \neq \varnothing\}. \tag{4.9}
$$

Combining (4.7) with (4.8) and (4.9) we obtain

$$
\begin{aligned}
&P(N_t(r) > 0, N_t(s) > 0 \mid r, s \in \Sigma) \\
&\qquad \leq \max_{x, y \in \mathbb{Z}^d} P\left(\eta_t^{x,t-r} \neq \varnothing,\ \eta_t^{y,t-s} \neq \varnothing,\ \eta_t^{x,t-r} \neq \eta_t^{y,t-s} \mid D_{r,x,s,y}\right).
\end{aligned} \tag{4.10}
$$

The claim of Lemma 4.2 will follow by applying Lemma 4.1, if we show that the events $D_{r,x,s,y}$ and $\{\eta_t^{x,t-r} \neq \varnothing,\ \eta_t^{y,t-s} \neq \varnothing,\ \eta_t^{x,t-r} \neq \eta_t^{y,t-s}\}$ are independent. Note that it is sufficient to show independence of the latter event of disjoint survival and $\{(x', x, t-r) \in \Pi\}$, $|x'-x| = 1$. This is by independence of the random walk X and Π and since we have assumed $r < s$ [recall from (4.1) that $(\eta_u^{x,t-r})_{0\leq u\leq t}$ and $(\eta_u^{y,t-s})_{0\leq u\leq t}$ are defined in terms of $\Pi \cap \mathbb{Z}^d \times \mathbb{Z}^d \times (t-s, t]$ only]. Now observe that the event $\{\eta_t^{x,t-r} \neq \varnothing,\ \eta_t^{y,t-s} \neq \varnothing,\ \eta_t^{x,t-r} \neq \eta_t^{y,t-s}\}$ is the same as

$$
\bigcup_{z: z \neq x} \left\{\eta_t^{x,t-r} \neq \varnothing, \eta_t^{z,t-r} \neq \varnothing, S_{(s-r)^-}^{z,t-r} = y\right\}.
$$

This representation shows that the event of disjoint survival does not depend on the existence of arrows at time $t - r$ pointing toward x from any site. Now recall from (2.8) that

$$
\mathscr{L}(\Pi \mid (x', x, t-r) \in \Pi) = \mathscr{L}(\Pi \cup \{(x', x, t-r)\}), \tag{4.11}
$$

to conclude the desired independence and complete the proof of Lemma 4.2. □

A simple consequence of Lemma 4.2 is the following asymptotic upper bound for the variance of the total mass of restrictions of $T_t \Lambda_t$.

COROLLARY 4.3. *For* $d \geq 3$ *and any* $B = [u_1, u_2] \times \mathbb{R}^+$, $0 < u_1 \leq u_2 \leq 1$,

$$\limsup_{t\to\infty} E\, T_t\Lambda_t(B)^2 \leq \lambda(B) + \gamma_d^{-2}\lambda(B)^2. \tag{4.12}$$

PROOF. Recall that Σ is a rate two Poisson process. Hence,

$$\begin{aligned} E\, T_t\Lambda_t(B)^2 &= E\,|\{u_1 t \leq s \leq u_2 t:\ N_t(s) > 0\}|^2 \\ &= \lambda_t(B) + E|\{r \neq s \in [u_1 t, u_2 t] : N_t(r) > 0, N_t(s) > 0\}| \\ &= \lambda_t(B) + \int_{u_1 t}^{u_2 t}\int_{u_1 t}^{u_2 t} P(N_t(r) > 0, N_t(s) > 0 \mid r, s \in \Sigma)\, 4\, dr ds. \end{aligned}$$

Using first (4.3) and then (3.1) we deduce

$$\begin{aligned} E\, T_t\Lambda_t(B)^2 &\leq \lambda_t(B) + \int_{u_1 t}^{u_2 t}\int_{u_1 t}^{u_2 t} p_r p_s 4\, dr ds \\ &\sim \lambda_t(B) + \gamma_d^{-2}\lambda_t(B)^2. \end{aligned} \tag{4.13}$$

The claim of Corollary 4.3 follows from (4.13) and (3.3) as $t \to \infty$. $\square$

Immediate from Corollary 4.3 is the fact that the points of any limiting point process already differ by their time coordinate. In particular, any limiting point process of $(T_t\Lambda_t)$ is simple.

COROLLARY 4.4. *Suppose that* Λ *is the weak limit of* $(T_{t_i}\Lambda_{t_i})$ *along some sequence* $t_i \to \infty$. *Then for* $d \geq 3$,

$$P(\Lambda(\{u\} \times \mathbb{R}^+) \geq 2 \text{ for some } 0 < u \leq 1) = 0. \tag{4.14}$$

PROOF. Fix $\varepsilon > 0$ and partition $(\varepsilon, 1]$ into disjoint intervals I_j, $1 \leq j \leq k_\varepsilon$. Use $P(X \geq 2) \leq EX(X-1)$ for a nonnegative integer-valued random variable X to deduce

$$\begin{aligned} &P(\Lambda(\{u\} \times \mathbb{R}^+) \geq 2 \text{ for some } \varepsilon < u \leq 1) \\ &\quad\leq P(\Lambda(I_j \times \mathbb{R}^+) \geq 2 \text{ for some } 1 \leq j \leq k_\varepsilon) \\ &\quad\leq \sum_{j=1}^{k_\varepsilon} P(\Lambda(I_j \times \mathbb{R}^+) \geq 2) \\ &\quad= \sum_{j=1}^{k_\varepsilon} \lim_{i\to\infty} P(T_{t_i}\Lambda_{t_i}(I_j \times \mathbb{R}^+) \geq 2) \\ &\quad\leq \sum_{j=1}^{k_\varepsilon} \limsup_{i\to\infty} (E\, T_{t_i}\Lambda_{t_i}(I_j \times \mathbb{R}^+)^2 - \lambda_{t_i}(I_j \times \mathbb{R}^+)) \\ &\quad\leq \sum_{j=1}^{k_\varepsilon} \gamma_d^{-2}\lambda(I_j \times \mathbb{R}^+)^2 = \gamma_d^{-2}\sum_{j=1}^{k_\varepsilon}\left(\int_{I_j}\frac{2du}{u}\right)^2, \end{aligned} \tag{4.15}$$

where for the last inequality we applied Corollaries 3.2 and 4.3. Since $\int_\varepsilon^1 u^{-1}du < \infty$, the sum on the right-hand side of (4.15) vanishes as the maximum length of the I_j tends to 0. Hence,

$$P(\Lambda(\{u\} \times \mathbb{R}^+) \geq 2 \text{ for some } \varepsilon < u \leq 1) = 0.$$

The claim of Corollary 4.4 follows as $\varepsilon \to 0$. $\square$

Before presenting our final proposition that shows (in high dimensions) that any limiting point process of $T_t\Lambda_t$ satisfies a strong independence property, we introduce a construction of Griffeath ([7], proof of Theorem II.2.6). This construction makes precise the notion that $(\eta_u^{x,r})_{u\geq 0}$ and $(\eta_u^{y,s})_{u\geq 0}$ evolve independently up until the time the distance between the two clusters reaches one. For $B, C \subset \mathbb{Z}^d$ let $d(B,C) := \min_{z_1\in B, z_2\in C} |z_1 - z_2|$. Let $\bar{\Pi}$ be an independent copy of Π, and let $(\bar{\eta}_u^{y,s})_{u\geq 0}$, $y \in \mathbb{Z}^d$ and $s \geq 0$, denote the cluster initiated at the space-time point (y, s) in the diagram associated with $\bar{\Pi}$. Fix $0 \leq r \leq s \leq t$ and $x, y \in \mathbb{Z}^d$, and define $\tau := \inf\{u : d(\eta_u^{x,r}, \bar{\eta}_u^{y,s}) \leq 1\}$ and

$$\check{\eta}_u^{y,s} := \begin{cases} \bar{\eta}_u^{y,s}, & u \leq \tau, \\ \bigcup_{z\in\bar{\eta}_\tau^{y,s}} \eta_u^{z,\tau}, & u > \tau. \end{cases} \tag{4.16}$$

Note that $(\eta_u^{x,r}, \check{\eta}_u^{y,s})_{u\geq 0}$ has the same law as $(\eta_u^{x,r}, \eta_u^{y,s})_{u\geq 0}$.

Let $G(z) := \int_0^\infty P(X_u = z)\,du$ be the expected occupation time at z of the simple random walk X started at $X_0 = \mathscr{O}$.

LEMMA 4.5. *For any $d \geq 3$, there is a finite constant c such that for all $x, y \in \mathbb{Z}^d$ and $r, s \geq 0$,*

$$P(\, d(\eta_u^{x,r}, \eta_u^{y,s}) \leq 1 \textit{ for some } u \geq 0) \leq c\, G(y - x). \tag{4.17}$$

NOTE. The estimate is useful only if $|y - x|$ is large compared to $|r - s|$ since $P(n_{|r-s|} > 0) \sim (\gamma_d|r - s|)^{-1}$ is a trivial upper bound for the left-hand side of (4.17). $\square$

PROOF. We may suppose that $r \leq s$. The left-hand side of (4.17) is not changed if we replace $\eta_u^{y,s}$ with $\check{\eta}_u^{y,s}$, and then replace the latter with $\bar{\eta}_u^{y,s}$. Let us consider now $P(\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing$ for some $0 \leq u \leq t + 1)$. Given that $d(\eta_u^{x,r}, \bar{\eta}_u^{y,s}) \leq 1$, the probability that the clusters will intersect within one time unit is bounded below by a positive constant c_1 not depending on u. This is because the conditional probability of intersection is minimal if $\eta_u^{x,r}$ and $\bar{\eta}_u^{y,s}$ are distinct singletons. Thus, by the Markov property,

$$\begin{aligned} &P(\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing \text{ for some } 0 \leq u \leq t + 1) \\ &\qquad \geq c_1 P(d(\eta_u^{x,r}, \bar{\eta}_u^{y,s}) \leq 1 \text{ for some } 0 \leq u \leq t). \end{aligned}$$

By similar reasoning, given that $\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing$, the probability that the clusters will have a nonempty intersection over a time interval of length one is bounded below by a positive constant c_2. Thus,

$$P\left(\int_0^{t+2} 1\{\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing\}\, du \geq 1\right)$$
$$\geq c_2\, P(\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing \text{ for some } 0 \leq u \leq t+1).$$

On account of these estimates, there is a finite c such that

$$\begin{aligned} &P(d(\eta_u^{x,r}, \bar{\eta}_u^{y,s}) \leq 1 \text{ for some } 0 \leq u \leq t) \\ &\quad \leq c\, E\int_0^{t+2} 1\{\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing\} du. \end{aligned} \tag{4.18}$$

By independence of $(\eta_u^{x,r})_{u\geq 0}$ and $(\bar{\eta}_u^{y,s})_{u\geq 0}$, and symmetry of the random walk, we have, for $r \leq s \leq u$,

$$\begin{aligned} E|\eta_u^{x,r} \cap \bar{\eta}_u^{y,s}| &= E \sum_{z\in\mathbb{Z}^d} 1\{z \in \eta_u^{x,r},\ z \in \bar{\eta}_u^{y,s}\} \\ &= \sum_{z\in\mathbb{Z}^d} P(z \in \eta_u^{x,r})\, P(z \in \bar{\eta}_u^{y,s}) \\ &= \sum_{z\in\mathbb{Z}^d} P(S_{u-r}^{z,u} = x)\, P(S_{u-s}^{z,u} = y) \\ &= \sum_{z\in\mathbb{Z}^d} P(X_{u-r} = z - x)\, P(X_{u-s} = y - z) \\ &= P(X_{2u-r-s} = y - x)\,. \end{aligned} \tag{4.19}$$

The estimates (4.18) and (4.19) imply

$$\begin{aligned} &P(d(\eta_u^{x,r}, \bar{\eta}_u^{y,s}) \leq 1 \text{ for some } 0 \leq u \leq t) \\ &\quad \leq c\, E\int_0^{t+2} 1\{\eta_u^{x,r} \cap \bar{\eta}_u^{y,s} \neq \varnothing\}\, du \\ &\quad \leq c\, E\int_0^{t+2} |\eta_u^{x,r} \cap \bar{\eta}_u^{y,s}|\, du \\ &\quad = c\int_s^{t+2} P(X_{2u-r-s} = y - x)\, du \ \leq c\, G(y-x), \end{aligned}$$

which completes our proof. □

PROPOSITION 4.6. *Suppose $d \geq 7$, then for any finite family of disjoint Borel sets $B_1, \dots, B_n \subset (0,1]$, arbitrary intervals $I_1, \dots, I_n \subset \mathbb{R}^+$ and $k_1, \dots, k_n \in \mathbb{N}$,*

$$\Big| P(T_t\Lambda_t(B_i \times I_i) = k_i, 1 \leq i \leq n) - \prod_{i=1}^n P(T_t\Lambda_t(B_i \times I_i) = k_i) \Big| \to 0 \tag{4.20}$$

as $t \to \infty$.

REMARK. The asymptotic independence stated in (4.20) carries over to any finite family of disjoint Borel sets $C_1, \dots, C_n \subset (0,1] \times \mathbb{R}^+$ as will be explained in the proof of Theorem 1.3 in Section 5.

PROOF OF PROPOSITION 4.6. We first prove (4.20) for sets $B_1 = [u_1, u_2]$ and $B_2 = [u_3, u_4]$, $0 < u_1 \le u_2 < u_3 \le u_4 \le 1$. The idea is to show that, as $t \to \infty$, the clusters initiated at potential branch-times in $[(1-u_4)t, (1-u_3)t]$ and those initiated at times in $[(1-u_2)t, (1-u_1)t]$ evolve on disjoint parts of the Poisson point process Π used in the graphical construction of Π_t^* in (2.5).

Let $E_{a,b} = E_{a,b}(t)$ be the set of points (x, r) such that $at \le r \le bt$ and a potential contribution to $\eta_t^{\mathscr{O}}(\Pi_t^*)$ with branch-time $t - r$ is initiated at site x,

$$E_{a,b} := \{(Y_u, u) : u \in \Sigma^* \cap [at, bt]\}, \qquad 0 \le a \le b \le 1, \tag{4.21}$$

with Y_u and Σ^* as defined in (4.4). Recall from (4.6) that

$$\begin{aligned}\Lambda_t &\overset{d}{=} \left\{\left(u, |\eta_t^{*Y_u, t-u}|\right) : u \in \Sigma^*, \ |\eta_t^{*Y_u, t-u}| > 0\right\} \\ &= \left\{(r, |\eta_t^{*x, t-r}|) : (x, r) \in E_{0,1}, \ |\eta_t^{*x,t-r}| > 0\right\}.\end{aligned} \tag{4.22}$$

The distributional identity (4.22) shows that assertion (4.20) for sets $B_1 = [u_1, u_2]$ and $B_2 = [u_3, u_4]$ is equivalent to the asymptotic independence of the random sets

$$\begin{aligned}&T_t \left\{(r, |\eta_t^{*x,t-r}|) : (x, r) \in E_{u_1,u_2}, \ |\eta_t^{*x,t-r}| > 0\right\} \ \text{ and} \\ &T_t \left\{(s, |\eta_t^{*y,t-s}|) : (y, s) \in E_{u_3,u_4}, \ |\eta_t^{*y,t-s}| > 0\right\}.\end{aligned} \tag{4.23}$$

To prove this independence, we introduce $(\tilde{\eta}_u^{*y,t-s})_{u \ge 0}$, the cluster initiated at the space-time point $(y, t-s)$ in the diagram associated with $\widetilde{\Pi} \overset{d}{=} \Pi$ after the pruning procedure through an independent random walk $\widetilde{X}$, where $\widetilde{\Pi}$ and $\widetilde{X}$ are independent of Π and X, and the set $\widetilde{E}_{a,b}$, which is the analog of $E_{a,b}$ defined in terms of $\widetilde{\Pi}$ and $\widetilde{X}$. To prove the asymptotic independence of the sets in (4.23), and hence (4.20) for the sets $B_1 = [u_1, u_2]$ and $B_2 = [u_3, u_4]$, we will prove that the joint distribution of the sets in (4.23) and the joint distribution of

$$\begin{aligned}&T_t \left\{(r, |\eta_t^{*x,t-r}|) : (x, r) \in E_{u_1,u_2}, \ |\eta_t^{*x,t-r}| > 0\right\} \ \text{ and} \\ &T_t \left\{(s, |\tilde{\eta}_t^{*y,t-s}|) : (y, s) \in \widetilde{E}_{u_3,u_4}, \ |\tilde{\eta}_t^{*y,t-s}| > 0\right\}\end{aligned} \tag{4.24}$$

asymptotically agree in the sense of (4.20). This is enough, since the two sets in (4.24) are independent by construction.

The key step in establishing this asymptotic equality in law is to show that the total variation distance α_t, say, between

$$\mathscr{L}(\{(\eta_u^{x,t-r})_{0 \le u \le t} : (x, r) \in E_{u_1,u_2} \cup E_{u_3,u_4}\}, (X_u)_{0 \le u \le t})$$

and

$$\mathscr{L}(\{(\eta_u^{x,t-r})_{0\le u\le t} : (x,r)\in E_{u_1,u_2}\} \cup\{(\bar\eta_u^{y,t-s})_{0\le u\le t} : (y,s)\in \bar E_{u_3,u_4}\},\ (X_u)_{0\le u\le t}) \tag{4.25}$$

tends to 0 as $t\to\infty$. Here, $(\bar\eta_u^{y,t-s})_{u\ge 0}$ is the cluster initiated at the space-time point $(y,t-s)$ in the diagram associated with $\bar\Pi$, where $\bar\Pi \stackrel{d}{=} \Pi$ is independent of Π and X. $\bar E_{a,b}$ is the analog of $E_{a,b}$ defined in terms of the same (!) random walk X, but Π replaced with $\bar\Pi$. We first show how $\lim_{t\to\infty}\alpha_t = 0$ implies the asymptotic equality in law of the sets in (4.23) and (4.24), and then prove the indicated limit.

Suppose $\lim_{t\to\infty}\alpha_t = 0$. To make use of this, we introduce yet another modification of our basic process, for which we suspend the pruning procedure of the clusters with potential branch-times in $[(1-u_4)t,(1-u_3)t]$ through the random walk X from time $(1-u_2)t$ to t, but keep on pruning the clusters initiated at other potential branch-times. For $(y,s)\in E_{u_3,u_4}$, respectively $\bar E_{u_3,u_4}$, let $\eta_t^{\circ y,t-s}$, respectively $\bar\eta_t^{\circ y,t-s}$, denote the cluster at time t initiated at $(y,t-s)$ with the pruning mechanism through X suspended during the time interval $[(1-u_2)t,\ t]$. Note that

$$\eta_t^{*y,t-s}\subset \eta_t^{\circ y,t-s}\subset \eta_t^{y,t-s}. \tag{4.26}$$

Recall that for random variables Y and Y', and a measurable function f, the total variation distance between $\mathscr{L}(f(Y))$ and $\mathscr{L}(f(Y'))$ is at most the distance between $\mathscr{L}(Y)$ and $\mathscr{L}(Y')$. Hence, by (4.25), the total variation distance between

$$\begin{aligned}&\mathscr{L}\Big(\{(r,|\eta_t^{*x,t-r}|) : (x,r)\in E_{u_1,u_2}\}\cup\{(s,|\eta_t^{\circ y,t-s}|) : (y,s)\in E_{u_3,u_4}\}\Big)\\ &\text{and}\\ &\mathscr{L}\Big(\{(r,|\eta_t^{*x,t-r}|) : (x,r)\in E_{u_1,u_2}\}\cup\{(s,|\bar\eta_t^{\circ y,t-s}|) : (y,s)\in \bar E_{u_3,u_4}\}\Big)\end{aligned} \tag{4.27}$$

is at most α_t.

Now observe that $\{(r,\eta_t^{*x,t-r}) : (x,r)\in E_{u_1,u_2}\}$ and $\{(s,\bar\eta_t^{\circ y,t-s}) : (y,s)\in \bar E_{u_3,u_4}\}$ are conditionally independent given $X_{(1-u_2)t}$ and that the distribution of $\{(r,|\eta_t^{*x,t-r}|) : (x,r)\in E_{u_1,u_2}\}$ does not depend on $X_{(1-u_2)t}$. Hence, the latter distribution in (4.27) is not changed if for the pruning of the clusters $(\bar\eta_u^{y,t-s})_{0\le u\le t}$ and in the definition of $\bar E_{u_3,u_4}$ we take a random walk independent of X. In other words, the total variation distance between

$$\begin{aligned}&\mathscr{L}\Big(\{(r,|\eta_t^{*x,t-r}|) : (x,r)\in E_{u_1,u_2}\}\cup\{(s,|\eta_t^{\circ y,t-s}|) : (y,s)\in E_{u_3,u_4}\}\Big)\\ &\text{and}\\ &\mathscr{L}\Big(\{(r,|\eta_t^{*x,t-r}|) : (x,r)\in E_{u_1,u_2}\}\cup\{(s,|\tilde\eta_t^{\circ y,t-s}|) : (y,s)\in \widetilde E_{u_3,u_4}\}\Big)\end{aligned} \tag{4.28}$$

is at most α_t, where $(\tilde\eta_u^{\circ y,t-s})_{0\le u\le t}$ is the cluster associated with $\widetilde\Pi$ with the pruning mechanism through $\widetilde X$ suspended from time $(1-u_2)t$ to t.

To get from this fact to the asymptotic equality in law of the sets in (4.23) and (4.24), we have to show that the suspension of the pruning mechanism from time $(1-u_2)t$ to t is asymptotically negligible for clusters with branch-time in $[(1-u_4)t, (1-u_3)t]$. More precisely, we must show that, as $t \to \infty$, the signed measure [compare the remark following (1.8)]

$$\begin{aligned} &T_t\{(s, |\eta_t^{*y,t-s}|) : (y,s) \in E_{u_3,u_4}, |\eta_t^{*y,t-s}| > 0\} \\ &\quad - T_t\{(s, |\eta_t^{\circ y,t-s}|) : (y,s) \in E_{u_3,u_4}, |\eta_t^{\circ y,t-s}| > 0\} \end{aligned} \tag{4.29}$$

converges to the zero measure on $(0,1] \times \mathbb{R}^+$ in the sense of (4.20). That is, if we let μ_t^1 denote the first random measure in (4.29), and μ_t^2 the second, we must show that, for any Borel set $B \subset \mathbb{R}^+$, any interval I and $k \in \mathbb{N}$,

$$P(\mu_t^1(B \times I) = k) - P(\mu_t^2(B \times I) = k) \to 0 \tag{4.30}$$

as $t \to \infty$. Note that the law of the signed random measure in (4.29) is not changed if we replace $\eta_t^{*y,t-s}$, $\eta_t^{\circ y,t-s}$ and E_{u_3,u_4} by $\tilde{\eta}_t^{*y,t-s}$, $\tilde{\eta}_t^{\circ y,t-s}$ and $\widetilde{E}_{u_3,u_4}$. Consequently, given that $\lim_{t\to\infty} \alpha_t = 0$, verification of (4.30) implies the asymptotic equality in law of (4.23) and (4.24), and thus (4.20).

To prove assertion (4.29) note that, by (4.22) and Corollary 3.2, any limiting point process of the first measure in (4.29) has intensity λ restricted to $[u_3, u_4] \times \mathbb{R}^+$. Now use (4.26) and the fact that both clusters $\eta_t^{*y,t-s}$ and $\eta_t^{\circ y,t-s}$ are empty if the trunk of the reduced process associated with $(\eta_u^{y,t-s})_{u\ge 0}$ is hit before real time $(1-u_2)t$, to deduce

$$\begin{aligned} &\limsup_{t\to\infty} E(t^{-1}||\eta_t^{\circ y,t-s}| - |\eta_t^{*y,t-s}|| \,|\, (y,s) \in E_{u_3,u_4}, |\eta_t^{y,t-s}| > 0) \\ &\quad \le \limsup_{t\to\infty} \sup_{u_3 t \le s \le u_4 t} E(t^{-1}|n_s - m_s| \,|\, n_s > 0, A_{T,s}) \end{aligned} \tag{4.31}$$

for any $T > 0$. By (3.35), the right-hand side of (4.31) tends to 0 as $T \to \infty$, which shows that the left-hand side of (4.31) is 0. Combining this observation with the fact that the intensity measure λ is absolutely continuous implies (4.29). Hence, in the case $B_1 = [u_1, u_2]$, $B_2 = [u_3, u_4]$, the asymptotic independence (4.20) follows once we show that $\alpha_t \to 0$ as $t \to \infty$.

From the construction of Griffeath [compare (4.16)] we see that α_t is bounded above by $2P(\Gamma_t)$ where

$$\Gamma_t = \left\{ d\left(\bigcup_{(x,r)\in E_{u_1,u_2}} \eta_u^{x,t-r}, \bigcup_{(y,s)\in E_{u_3,u_4}} \eta_u^{y,t-s} \right) \le 1 \text{ for some } 0 \le u \le t \right\}.$$

Now recall the definition of the event $D_{r,x,s,y}$ in (4.5). We claim that for any $0 \le r < s \le t$,

$$\begin{aligned} &P(d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t \,|\, D_{r,x,s,y}) \\ &\quad \le P(d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t). \end{aligned} \tag{4.32}$$

Using the same arguments as following (4.10) one easily sees that it is sufficient to prove (4.32) with $D_{r,x,s,y}$ replaced by the event $\{(x', x, t-r) \in \Pi\}$, $|x'-x| = 1$. Now suppose that $(x', x, t-r) \in \Pi$. Deleting the arrow from x'

to x at time $t-r$ in the space-time diagram representing Π does not affect the process $(\eta_u^{x,t-r})_{u\ge 0}$ at all. It affects $(\eta_u^{y,t-s})_{u\ge 0}$ only if $x' \in \eta_{t-r}^{y,t-s}$ and $x \notin \eta_{(t-r)^-}^{y,t-s}$ or if $x' \notin \eta_{t-r}^{y,t-s}$ and $x \in \eta_{(t-r)^-}^{y,t-s}$. In the first case $d(\eta_{t-r}^{x,t-r}, \eta_{t-r}^{y,t-s}) \le |x-x'| = 1$, no matter if the arrow is deleted or not. In the second case deleting the arrow would only enlarge the cluster $\eta_u^{y,t-s}$, $u \ge t-r$. Recall from (4.11) that deleting the arrow corresponds to removing the conditioning. Hence, our arguments show that

$$\begin{aligned} &P(\, d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t \,|\, (x', x, t-r) \in \Pi) \\ &\quad \le \; P(\, d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t). \end{aligned}$$

We now derive an upper bound for the infinitesimal probability that $E_{0,1}$ contains the distinct points (x, r) and (y, s). Note that $|X_{(t-r)^-} - x|, |X_{(t-s)^-} - y| \le 1$, if $(x, r), (y, s) \in E_{0,1}$. Using the fact that Σ^* is a rate two Poisson process we have

$$\begin{aligned} &P((x,u),(y,v) \in E_{0,1} \text{ for some } u \in [r, r+dr),\ v \in [s, s+ds)) \\ &\quad \le 4P(|X_{t-s} - y| \le 1)\, P(|X_{s-r} - (x-y)| \le 2)\, dr\,ds \qquad (4.33) \\ &\quad \le 4\,(2d)^5\, \bar{c}^2\, P(X_{t-s+1} = y)\, P(X_{s-r+1} = x-y)\, dr\,ds, \end{aligned}$$

where

$$\bar{c} = \sup_{u \ge 0} \sup_{|z_1 - z_2| \le 2} \frac{P(X_u = z_1)}{P(X_{u+1} = z_2)} < \infty.$$

For the factor $(2d)^5$ note that there are at most $(2d)^k$ distinct points z with $|z| = k$. To see that $\bar{c}$ is finite, we note that $P(X_{u+1} = z_2) \ge P(X_u = z_1) P(X_1 = z_2 - z_1)$ for any $u \ge 0$ and $z_1, z_2 \in \mathbb{Z}^d$.

Using first (4.33), then (4.32) and Lemma 4.5, and letting c denote a finite positive constant whose value may change from line to line, we have

$$\begin{aligned} P(\Gamma_t) &\le E \sum_{\substack{(x,r)\in E_{u_1,u_2},\\ (y,s)\in E_{u_3,u_4}}} 1\{d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t\} \\ &\le c \sum_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \int_{u_3 t}^{u_4 t} \int_{u_1 t}^{u_2 t} P(X_{t-s+1} = y)\, P(X_{s-r+1} = x-y) \\ &\qquad\qquad \times P(d(\eta_u^{x,t-r}, \eta_u^{y,t-s}) \le 1 \text{ for some } 0 \le u \le t \,|\, D_{r,x,s,y}) dr\,ds \qquad (4.34) \\ &\le c \sum_{z\in\mathbb{Z}^d} G(z) \int_{u_3 t}^{u_4 t} \int_{u_1 t}^{u_2 t} P(X_{s-r+1} = z) dr\,ds \\ &\le c\,t \sum_{z\in\mathbb{Z}^d} G(z) \int_{(u_3-u_2)t}^{(u_4-u_1)t} P(X_{s+1} = z)\, ds \\ &= c\,t \int_{(u_3-u_2)t}^{(u_4-u_1)t} EG(X_{s+1})\, ds. \end{aligned}$$

By symmetry and the Markov property of the simple random walk X, we have

$$\begin{aligned} EG(X_s) &= \sum_{z\in\mathbb{Z}^d} P(X_s = z)\; E\left(\int_0^\infty 1\{X_u = z\}\,du\right) \\ &= \sum_{z\in\mathbb{Z}^d} P(X_s = z)\, E\left(\int_s^\infty 1\{X_u = \mathscr{O}\}\,du \,\middle|\, X_s = z\right) \\ &= \int_s^\infty P(X_u = \mathscr{O})\,du\,. \end{aligned}$$

Using a local limit theorem for the random walk X we obtain

$$\sup_{(u_3-u_2)t\le s\le(u_4-u_1)t} EG(X_{s+1}) \le c\int_{(u_3-u_2)t}^\infty u^{-\frac{d}{2}}\,du \le c(u_2,u_3)\,t^{1-\frac{d}{2}}. \tag{4.35}$$

By (4.35) and (4.34), for $d \ge 7$ and $0 < u_1 \le u_2 < u_3 \le u_4 \le 1$,

$$\lim_{t\to\infty} P(\Gamma_t) = 0, \tag{4.36}$$

and, consequently, $\lim_{t\to\infty}\alpha_t = 0$ which proves (4.20) in the case $B_1 = [u_1, u_2]$ and $B_2 = [u_3, u_4]$.

Clearly, the argument works for finite n and, taking limits, for open and half-open disjoint intervals which shows that (4.20) holds for any finite family of disjoint Borel sets $B_1, \dots, B_n \subset (0, 1]$. □

5. Proof of Theorem 1.3. Using Chebyshev's inequality we obtain from (1.9) and (3.3) that

$$\lim_{k\to\infty}\sup_{t\ge 0} P(T_t\Lambda_t(B) \ge k) = 0$$

for any compact B, which is equivalent to tightness of the sequence of random measures $(T_t\Lambda_t)_{t\ge0}$ (see [5] for a general reference on point processes). Suppose Λ is the weak limit of $(T_{t_i}\Lambda_{t_i})$ along some sequence $t_i \to \infty$, then

$$T_{t_i}\Lambda_{t_i}(B) \xrightarrow{d} \Lambda(B)$$

for any bounded stochastic continuity set B for Λ. Clearly, (1.9) and (3.3) imply that the intensity of Λ is absolutely continuous with respect to Lebesgue measure. Hence,

$$E\Lambda(\partial B) = 0$$

for any bounded convex set B with $B \subset (\varepsilon, 1]\times\mathbb{R}^+$ for some $\varepsilon > 0$, that is, any such set is a continuity set for Λ. Since Corollary 4.3 shows that the random variables $T_{t_i}\Lambda_{t_i}(B)$ are uniformly integrable, relation (3.3) implies

$$E\Lambda(B) = \lim_{i\to\infty} E\,T_{t_i}\Lambda_{t_i}(B) = \lambda(B)$$

for any such B and hence, for any Borel set $B \subset (0, 1] \times \mathbb{R}^+$. The existence of multiple points in the limiting point process Λ is ruled out by Corollary 4.4. This completes the proof of the first part of Theorem 1.3.

For the convergence result (1.10) note that Proposition 4.6 implies that in dimension $d \geq 7$ any limiting point process Λ satisfies

$$P(\Lambda(B_i \times \mathbb{R}^+) = k_i,\ 1 \leq i \leq n) = \prod_{i=1}^{n} P(\Lambda(B_i \times \mathbb{R}^+) = k_i) \tag{5.1}$$

for any $k_1, \ldots, k_n \in \mathbb{N}$ and finite family $B_1, \ldots, B_n$ of disjoint Borel sets $\subset (0, 1]$. Corollary 4.4 and (5.1) show that the projection of Λ on $(0, 1]$ is a simple Poisson point process with intensity $\bar{\lambda}(du) = \lambda(du \times \mathbb{R}^+)$. Hence, Λ can be generated as follows: First, choose the time coordinates of the points in Λ according to a simple Poisson point process $\bar{\Lambda}$, say, on $(0, 1]$ with intensity $\bar{\lambda}$. Given $u \in \bar{\Lambda}$ choose the mass coordinate Z_u of the point in Λ from the distribution

$$P(Z_u \geq z) = \frac{\lambda(du \times [z, \infty))}{\lambda(du \times \mathbb{R}^+)} = \exp\left(-\frac{2}{u} z\right), \quad z \geq 0.$$

By Proposition 4.6 this can be done independently for any $u \in \bar{\Lambda}$ which characterizes Λ as the simple Poisson point process on $(0, 1] \times \mathbb{R}^+$ with intensity λ. $\square$

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