The Annals of Probability
2000, Vol. 28, No. 2, 725–758

INTERSECTING RANDOM HALF-SPACES: TOWARD THE GARDNER–DERRIDA FORMULA[1]

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Gardner and Derrida have introduced a natural version of the problem of the capacity of the binary perceptron "with temperature," and they proposed (based on "physical" methods) remarkable formulas for this model. We give a complete rigorous proof that these formulas are correct at sufficiently high temperature for a much larger class of models.

1. Introduction. Throughout the paper, N denotes a (very large) integer, and Σ_N denotes the discrete cube $\{-1, 1\}^N$. Consider M independent random half-spaces through the origin. How large should M be so that their typical intersection with Σ_N be empty? This problem arises from the theory of neural networks, where it is known as the problem of the capacity of the binary perceptron. More generally, what is the "typical size" of the intersection of Σ_N with these half-spaces? The point here is that "typical size" is very different from "average size" (and usually much smaller). These important geometric questions motivate the present paper. While they originated in the theory of neural networks, we find that the appeal of these problems go well beyond their origins, and the present paper certainly assumes no knowledge whatsoever about neural networks. (The reader is referred to [5] for an introduction to these.) This paper is (at least in principle) self-contained. The approach and its motivation will be briefly outlined below. The paper is, however, part of a larger program that is described in [11].

Random half-spaces involve random directions. These are modeled by a sequence $(\boldsymbol{\xi}^k)_{k\le M}$ independently uniformly distributed over Σ_N. Equivalently, $\boldsymbol{\xi}^k = (\xi_i^k)_{i\le N}$, where the variables ξ_i^k are independent Bernoulli random variables, that is, take values 1 or -1 with equal probability 1/2. Corresponding random half-spaces are then obtained as

$$\left\{\boldsymbol{x} \in \mathbb{R}^N; \boldsymbol{\xi}^k \cdot \boldsymbol{x} = \sum_{i\le N} \xi_i^k x_i \ge 0\right\}.$$

This definition of "random directions" is motivated by the origin of the question (the binary perceptron). From the point of view of geometry, it would be more natural to set $\boldsymbol{\xi}^k = (g_i^k)_{i\le N}$ where (g_i^k) are independent $N(0,1)$. For the purpose of the present discussion, let us refer to this situation as the "Gaussian model" in contrast with the previously defined "Bernoulli model" that will be considered in this paper. It turns out that for the problems we will consider,

Received April 1999; revised September 1999.
[1]Supported in part by an NSF Grant.
AMS 1991 *subject classifications*. Primary 82A57; secondary 60G99, 60F99.
Key words and phrases. Spin glasses, cavity method, replica-symmetric formula, perceptron.

the answers are the same in the Gaussian model and in the Bernoulli model, but the Bernoulli model is technically harder. Thus the reader interested in the Gaussian model should have no problem in adapting our results to this case.

Rather than simply trying to compute the typical size of the intersection of Σ_N with random half-spaces, it turns out to be very helpful to consider a more general problem that we describe now. Consider a function u from $\mathbb{R}$ to $\mathbb{R}$, and the random function

$$H_{N,M}(\boldsymbol{\sigma}) = \sum_{k\le M} u\left(\frac{\boldsymbol{\xi}^k\cdot\boldsymbol{\sigma}}{\sqrt{N}}\right) \tag{1.1}$$

on Σ_N. If we denote by μ_N the uniform probability on Σ_N, we can then consider the random probability $G = G_{N,M}$ on Σ_N given by

$$G_{N,M}(\{\boldsymbol{\sigma}\}) = \frac{\exp(H_{N,M}(\boldsymbol{\sigma}))}{Z} \tag{1.2}$$

where $Z = Z_{N,M}$ is the normalizing factor

$$Z = \int \exp(H_{N,M}(\boldsymbol{\sigma}))\, d\mu_N(\boldsymbol{\sigma}). \tag{1.3}$$

Thus, in the "limiting case,"

$$e^{u(x)} = 1_{\{x\ge\kappa\}} \tag{1.4}$$

then

$$Z_{N,M} = \mu_N(H(\kappa)),$$

where

$$H(\kappa) = \left\{\boldsymbol{\sigma}\in\Sigma_N,\ \forall\, k\le M,\ \boldsymbol{\xi}^k\cdot\boldsymbol{\sigma}\ge\kappa\sqrt{N}\right\}$$

is the intersection of Σ_N with M random half-spaces at distance κ from the origin, and $G_{N,M}$ is the uniform probability on $H(\kappa)$.

It turns out that the interesting case is when M is a proportion of N. Given a number α, and writing αN for $\lfloor\alpha N\rfloor$, we are interested in computing

$$\lim_{N\to\infty}\frac{1}{N} E\log Z_{N,\alpha N}. \tag{1.5}$$

It turns out from general principles [8] that the random variable

$$\frac{1}{N}\log Z_{N,M}$$

is nearly constant for large M, so that (1.5) somehow amounts to studying the "typical value" of $Z_{N,M}$.

In summary of this discussion, if we know how to compute (1.5) in the case (1.4) [or at least in cases from which (1.4) could be recovered by a limiting argument], we would know the typical value of $N^{-1}\log\mu_N(H(\kappa))$. Unfortunately, we do not know how to do this in general. The main result of the paper is the computation of (1.5) under conditions on α, u that amount to a "high temperature hypothesis" in statistical mechanics. The reader who is familiar with statistical mechanics would expect, rather than (1.3), to read

$$Z = \int \exp\bigl(-\beta H_{N,M}(\boldsymbol{\sigma})\bigr)\, d\mu_N(\boldsymbol{\sigma}), \tag{1.6}$$

where β is a parameter that represents the inverse of the temperature. However, all our constructions will be made at fixed temperature, and the conditions will be expressed in term of u, so that the study of (1.6) reduces to the study of (1.5) by replacing u by $-\beta u$.

Since this is a domain where work of rather different nature to our own has been done, we must comment on this even before we state our results.

The case that has been considered in the literature is

$$e^{u(x)} = e^{-\beta} + (1 - e^{-\beta})1_{\{x \geq \kappa\}} \tag{1.7}$$

for some parameters β, κ, of which (1.4) is itself the "limiting case" $\beta = \infty$. Equivalently to (1.7) (since adding a constant to u changes nothing) is the case

$$u(x) = -\beta 1_{\{x \geq \kappa\}}. \tag{1.8}$$

In cases (1.4) and (1.7), Gardner [3] and Gardner and Derrida [2] have proposed formulas for (1.5). These formulas are derived using the replica method. While replica formalism is certainly an impressive way to guess magic formulas, it is currently far from being a rigorous method. It is, in fact, to shed light upon the magic formula of [3] that Mézard [7] proposed an alternative approach to the problem using the "cavity method." Mézard's paper (which had a considerable influence upon the present work) explains the mysteries of Gardner's formula, but makes no attempt to be rigorous. It seems to us that replacing Mézard's "physical" arguments by provable estimates is not a trivial task, and that, in fact, his approach has to be considerably modified for this purpose (perhaps beyond recognition). Most important, Mézard derives his results from a condition that will be called here "the replica symmetric condition" (the RS condition) and that we will explain below. This condition physically means that "the system governed by the Gibbs measure (1.2) has only one state" (to be precise, this statement is only part (1.10) of the condition RS to be considered below) and is accepted by Mézard on physical grounds. Our aim, on the other hand, is to *prove* that this condition holds. Doing this rigorously requires a work of a different magnitude from just deriving the value of (1.5) from this expression.

Let us now explain what the RS condition is and some of our notation. Averages with respect to the Gibbs measure (1.2) will be denoted by $\langle\cdot\rangle$. Expectation and probability with respect to the variables $\boldsymbol{\xi}^k$ (that we will call the disorder) will be denoted by E and P, respectively. The idea of replicas simply consists in taking products of the probability space (Σ_N, G). Of importance in particular is the function

$$(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \to \frac{1}{N}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'$$

on Σ_N^2. The RS condition simply means that this function is essentially constant, and more precisely,

(1.9) There is a' number q such that $\lim_{N\to\infty} E\left\langle\left(\frac{1}{N}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - q\right)^2\right\rangle = 0.$

There, of course, the bracket $\langle\cdot\rangle$ denotes a double integration

$$\iint\left(\frac{1}{N}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - q\right)^2 dG(\boldsymbol{\sigma})\,dG(\boldsymbol{\sigma}').$$

The limit in (1.9) is taken at $M = \alpha N$. It will turn out to be convenient to break (1.9) into two parts, namely,

$$\lim_{N\to\infty} E\left\langle\left(\frac{1}{N}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - \frac{1}{N}\|\langle\boldsymbol{\sigma}\rangle\|^2\right)^2\right\rangle = 0, \tag{1.10}$$

$$\lim_{N\to\infty} E\left(\frac{1}{N}\|\langle\boldsymbol{\sigma}\rangle\|^2 - q\right)^2 = 0. \tag{1.11}$$

Condition (1.10) physically means that "the system has one single pure state" as is explained in detail in [11]. Condition (1.11) means that the important parameter $(1/N)\langle\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\rangle = (1/N)\|\langle\boldsymbol{\sigma}\rangle\|^2 = (1/N)\sum_{i\le N}\langle\sigma_i\rangle^2$ of the system is essentially nonrandom.

When we consider the general case rather than simply the case (1.7), what conditions is it reasonable to impose upon u? A case of particular interest is when $u(x) = \beta x^2/2$, the case of the famous Hopfield model [4]. A specific feature of this case is that when $\boldsymbol{\sigma} = \boldsymbol{\xi}^k$, the contribution of the corresponding term in (1.1) is already of order N, so that there is considerable "attraction" of the system towards $\boldsymbol{\xi}^k$ (an attraction that results in the system breaking into a number of different states; see, e.g., [1], [9]). The case of (1.7) shows that here we are not interested in this type of phenomenon, but rather in the case where u is bounded.

We now formally state our results. Throughout the paper we assume

$$\forall\ x, |u(x)| \le D. \tag{1.12}$$

THEOREM 1.1. *There is a number $\alpha_0(D) > 0$ with the following property. Consider a function u: $\mathbb{R} \to \mathbb{R}$, that satisfies* (1.12), *and that is continuous except possibly at finitely many points. Consider $\alpha \le \alpha_0(D)$, and the system of equations*

$$q = E\mathrm{th}^2\big(z\sqrt{\hat q}\big), \tag{1.13}$$

$$\hat q = \alpha E\Phi^2\big(z\sqrt{q}, q\big), \tag{1.14}$$

where, for $x \in \mathbb{R}$, $y \in [0, 1)$,

$$\Phi(x, y) = \frac{Eh\exp u(x + h\sqrt{1-y})}{\sqrt{1-y}E\exp u(x + h\sqrt{1-y})} \tag{1.15}$$

and where h, z are $N(0,1)$ random variables. Then this system has a unique solution $q = q(\alpha)$, $\hat{q} = \hat{q}(\alpha)$. Consider the function

$$\begin{aligned}\mathrm{RS}(\alpha) = \tfrac{1}{2}\hat{q}(1-q) + E\log\big(2\operatorname{ch} z\sqrt{\hat{q}}\big)\\ + \alpha E\log E_h \exp u\big(z\sqrt{q} + h\sqrt{1-q}\big),\end{aligned} \tag{1.16}$$

where E_h denotes expectation in h only. Then for $\alpha \le \alpha_0(D)$, $M = \lfloor \alpha N \rfloor$ we have

$$\lim_{N\to\infty} \frac{1}{N} E\log Z_{N,M} = RS(\alpha). \tag{1.17}$$

In this theorem, we have not tried to reach the weakest possible regularity conditions on u. Conditions (1.12) could be replaced by the condition

$$|u(x)| \le D(1 + |x|^\theta)$$

for $\theta < 2$. It is however rather straightforward to do this, and certainly the problem is difficult enough that there is no need to introduce secondary complications. We do not know what is the best possible type of dependence of $\alpha_0(D)$ upon D. Our arguments currently give an estimate $\alpha_0(D) \ge \exp(-LD)$ where L is a number.

The reader observes that Theorem 1.1 does *not* say that (1.9) holds. In a case where u is not smooth, such as (1.7), we do not know whether this is the case, even when α is very small. In fact, it even seems to us that in this case, the heuristic arguments of [7] (that do not attempt to prove (1.9) but attempt only the easier task of drawing conclusions from this condition) rely upon a mathematically unjustified inversion of limits. The way Theorem 1.1 will be proved will be by first assuming that u is smooth enough (a condition that allows power expansions) and then by proving that (1.9) holds for $\alpha \le \alpha_0(D)$. (A crucial difficulty there is that one has to reach a value of α that does *not* depend upon the smoothness of u). After (1.9) has been proved, it is rather easy to prove (1.17), and the case where u need not be smooth is then recovered by an approximation procedure.

When u is smooth enough, we can rather precisely describe the structure of Gibbs measure. There is "decoupling of chaos." This means (in a sense that the reader should have no problem in making formally precise) that given any n, the law of $(\sigma_1, \dots, \sigma_n)$ under Gibbs measure is asymptotically a product measure. It is thus determined by $(\langle\sigma_1\rangle, \dots, \langle\sigma_n\rangle)$. The law of $\langle\sigma_1\rangle, \dots, \langle\sigma_n\rangle$ is asymptotically i.i.d. (with an explicit distribution). Decoupling of chaos seems to be an automatic consequence of (1.9) and of the tools we have developed for "computations in a pure state" (see, e.g., [11]) so we will not do the routine job of proving this. But we do not know what happens without a smoothness assumption on u.

There is a related model of interest, the "spherical model," where Σ_N is replaced by $S_N = \{\boldsymbol{\sigma} \in \mathbb{R}^N; \sum_{i\le N} \sigma_i^2 = N\}$ and μ_N is replaced by the homogeneous measure on S_N. Physicists believe that the spherical model is simpler than the Ising model [because it is in a pure state for large β in the case (1.7), while this is apparently not true for the Ising model]. Unfortunately, to study the spherical model mathematically one has to show first that Gibbs measure is essentially carried by the configurations for which $\max_{i\le N}|\sigma_i|^2$ is $o(N)$, which seems difficult and is better left for further study.

2. Overview of proof. The paper relies on the "cavity method." In our previous work [8–11], this refers to induction upon N. This is how our approach starts: the first step is to show that we can estimate averages with respect to $G_{N,M}$ provided we know how to estimate averages for $G_{N-1,M}$. This is done in Section 5. This is where the smoothness of u is useful. The main tool is a Taylor expansion, and identification of the leading terms. The remaining terms become of lower order as $N\to\infty$, but how large N has to be taken seems to depend not only upon D, but also upon the size of the first few derivatives of u (so that we face a severe problem of interversion of limits when trying to obtain results without a smoothness assumption on u). These calculations (that are tedious and predictable) bring to light the importance of the vector $\mathbf{A}$ given by

$$\mathbf{A}=\left(\frac{u'(S_k)}{\sqrt{N}}\right)_{k\le M},$$

where $S_k=N^{-1/2}\sum_{i\le N}\xi_i^k\sigma_i$. Here we already realize how dangerous the situation is because there is no possible way that the quantity $N^{-1}\sum_{k\le M}u'(S_k)^2$ remains bounded (say, in expected thermal average) only in function of D, and this makes it a priori difficult to reach values of α that depend only upon D. What makes this at all possible is that, in certain respects, there is some boundedness in $\mathbf{A}$. More specifically, if $\mathbf{A}'$ is an independent copy of $\mathbf{A}$, $\mathbf{A}\cdot\mathbf{A}'$ does remain bounded in function of D only. Of course, this cannot be proved through induction upon N because we have no chance to make any reasonable estimates this way before some type of boundedness has been proved on $\mathbf{A}$; but fortunately, we have succeeded in obtaining a priori estimates, using essentially induction upon M. Quite logically, these should be presented before any use of the cavity method, in Section 4. These estimates in turn rely upon a few simple observations (making use of elementary theory of Gaussian processes) that are amazingly effective, and crucial at several places. These basic tools are thus presented in Section 3.

We find it convenient to consider three replicas, denoted $\boldsymbol{\sigma}^l$ ($l\le 3$) and to consider the quantity

$$C_{N,M}=E\left\langle\left(\frac{(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2)\cdot\boldsymbol{\sigma}^3}{N}\right)^2\right\rangle, \tag{2.1}$$

where the bracket now represents an average for $G_{N,M}^{\otimes 3}$. A basic idea there is, of course, to replace the uninviting centering term in (1.10) by symmetrization. The specific choice of $C_{N,M}$ is motivated in [11], where the reader might find a useful overview of our approach to similar problems.

While trying to compute $C_{N,M}$ in function of $C_{N-1,M}$, we meet the quantity $D_{N-1,M}$, where

$$D_{N,M} = E\left\langle\left(\frac{(\mathbf{A}^1 - \mathbf{A}^2)\cdot\mathbf{A}^3}{N}\right)^2\right\rangle. \tag{2.2}$$

A similar situation was faced in the case of the Hopfield model by taking advantage of the fact that $u'(x)$ is particularly simple in that case. Possibly this approach could also be used here (using Taylor expansion) but instead, following Mézard [7], we will use induction upon M, relating now $G_{N,M}$ to $G_{N,M-1}$. The energy spent having to deal with a nonexplicit function such as u was well used, because it brought to light some simple general facts that allow simplifying computations even in the case where u is explicit. It is in relating $D_{N,M}$ that a crucial factor α appears to $C_{N,M-1}$. This factor is related to the fact that $\mathbf{A}$ has M components, while there is a denominator N in (2.2). The induction upon M is developed in Section 6; combining the results of Sections 5 and 6 we obtain a relation $C_{N,M} \le \alpha K(D) C_{N-1,M-1} + o(1)$ where $\lim_{N\to\infty} o(1) = 0$, and this implies (1.10) for α small. As explained at length in [1], this is the crucial step. To prove (1.11), one has to prove that the variance of $N^{-1}\sum_{i\le N}\langle\sigma_i\rangle^2$ goes to zero. The proof of this is similar to the proof of (1.10) but easier and also uses two stages. This is done in Section 7 and the proof of Theorem 1.1 is finished in Section 8.

3. Basic lemmas. Consider an integer Q and a function f from $\mathbb{R}^Q$ to $\mathbb{R}$. A recurring task we will face will be the estimation of $Ef(\mathbf{g})$ where $\mathbf{g} = (g_l)_{l\le Q}$ is a jointly Gaussian family. The basic idea will be to replace the family $\mathbf{g}$ by a simpler family $\mathbf{g}' = (g'_l)_{l\le Q}$ for which the estimation is easier, and to estimate the error made when replacing $\mathbf{g}$ by $\mathbf{g}'$. The basic principle is due to Kahane [6] and was used in [10] in the same circle of ideas.

LEMMA 3.1. *Assume that f is twice differentiable* (*and of moderate growth*). *Assume that the Gaussian families $\mathbf{g}', \mathbf{g}$ are independent of each other. Consider, for $0 \le t \le 1$, the function*

$$\varphi(t) = Ef(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t}).$$

Then, for $0 < t < 1$ we have

$$\varphi'(t) = \frac{1}{2}\sum_{l,m\le Q}\big(E(g_l g_m) - E(g'_l g'_m)\big)E\frac{\partial^2 f}{\partial x_l \partial x_m}(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t}). \tag{3.1}$$

PROOF. We write

$$\varphi'(t) = \frac{1}{2}\sum_{l\le Q} E\left(\left(\frac{g_l}{\sqrt{t}} - \frac{g'_l}{\sqrt{1-t}}\right)\frac{\partial f}{\partial x_l}(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t})\right).$$

We will use the elementary integration by parts formula,

$$E(gh(g)) = Eg^2 E(h'(g)), \tag{3.2}$$

where h is smooth (of moderate growth) and g is Gaussian. For this we write $g_m = \bar{g}_m + a_{l,m} g_l$ where $a_{l,m} = E(g_l g_m)/E(g_l^2)$, and where $\bar{g}_m$ is independent of g_l. We then use (5.2) conditionally upon $(\bar{g}_m)_{m\le Q}$ and $\mathbf{g}'$ to get

$$E g_l \frac{\partial f}{\partial x_l}(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t}) = \sqrt{t} \sum_{m\le Q} E(g_l g_m) E \frac{\partial^2 f}{\partial x_l \partial x_m}(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t})$$

and we proceed similarly for the second term. □

We will combine (3.1) with the estimate

$$|\varphi(1) - \varphi(0)| \le \int_0^1 |\varphi'(t)|\, dt \tag{3.3}$$

so that we see the need to control $|E(\partial^2 f/\partial x_l \partial x_m)(\mathbf{g}\sqrt{t} + \mathbf{g}'\sqrt{1-t})|$. A remarkable fact is that under certain conditions, it will be possible to integrate by parts again and to bound these quantities in function of $\|f\|_\infty = \sup|f|$ only. This is the purpose of the next lemma.

LEMMA 3.2. *There exists a number L with the following property. Consider integers $Q \le 6$, $Q' \le 6$, $Q'' = Q + Q'$, $(n_l)_{l\le Q''}$, $(m_l)_{l\le Q''}$ with $n = \sum_{l\le Q''} n_l$, $m = \sum_{l\le Q''} m_l$, $n, m \le 6$. Consider jointly Gaussian r.v. $(g_l)_{l\le Q''}$ with $Eg_l^2 \le 1$, and assume that*

$$\inf\left(E\left(\sum_{l\le Q} t_l g_l\right)^2;\ \sum_{l\le Q} t_l^2 = 1\right) \ge 1/4. \tag{3.4}$$

Then for any smooth functions f on $\mathbb{R}^Q$, θ on $\mathbb{R}^{Q'}$, we have

$$\left|E g_1^{n_1} \cdots g_{Q''}^{n_{Q''}} \frac{\partial^m}{\partial x_1^{m_1} \cdots \partial x_{Q''}^{m_{Q''}}} (f(g_1, \dots, g_Q)\theta(g_{Q+1}, \dots, g_{Q''}))\right| \le L\|f\|_\infty \|\theta\|_{\infty,m},$$

where

$$\|\theta\|_{\infty,m} = \max\left(\left\|\frac{\partial^p \theta}{\partial x_1^{p_1} \cdots \partial x_{Q'}^{p_{Q'}}}\right\|_\infty;\ p = \sum_{l\le Q'} p_l \le m\right).$$

Comment. There exists a number $L > 0$ such that (3.4) holds provided $Eg_l^2 = 1$ for $l \le Q$ and $|Eg_l g_{l'}| \le 1/L$ for $1 \le l < l' \le Q$.

PROOF. If $\mathbf{g} = (g_1, \dots, g_Q)$, there is a rotation R of $\mathbb{R}^Q$ such that the Gaussian vector $\mathbf{h} = R(\mathbf{g})$ has independent coordinates; moreover, for $l \le Q$, we have $Eh_l^2 \ge 1/4$ by (3.4). We consider the function $\bar{f}(\mathbf{x}) = f(R^{-1}(\mathbf{x}))$ on $\mathbb{R}^Q$, so that $f(\mathbf{y}) = f(R(\mathbf{y}))$, and a partial derivative of f is a linear combination (with coefficients bounded by 1) of partial derivatives of $\bar{f}$ of the same order. As a consequence we see that it is enough to prove the lemma when $g_1, \dots, g_Q$ are independent. However, all we have to do is perform successive integrations by parts in $g_1, \dots, g_Q$, using the fact that if g is Gaussian, v, w are smooth functions, then

$$E(v'(g)w(g)) = E\left(\frac{g}{Eg^2}v(g)w(g) - v(g)w'(g)\right). \qquad \square$$

The following (obvious) simple observation will help to check (3.4).

LEMMA 3.3. *Consider two independent jointly Gaussian sequences $(g_l)_{l\le Q}$, $(g'_l)_{l\le Q}$. If*

$$\sum_{l\le Q} t_l^2 = 1 \Rightarrow E\left(\sum_{l\le Q} t_l g_l\right)^2 \ge \tfrac{1}{4}; E\left(\sum_{l\le Q} t_l g'_l\right)^2 \ge \tfrac{1}{4},$$

then for $0 \le t \le 1$,

$$\sum_{l\le Q} t_l^2 = 1 \Rightarrow E\left(\sum_{l\le Q} t_l(g_l\sqrt{t} + g'_l\sqrt{1-t})\right)^2 \ge \tfrac{1}{4}.$$

4. A priori estimates. We start with a very simple (yet crucial) observation.

LEMMA 4.1. *For $t \ge 0$, we have*

$$G^2(\{\boldsymbol{\sigma}, \boldsymbol{\sigma}'; |\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'| \ge tN\}) \le 2\exp\left(4MD - \frac{Nt^2}{2}\right). \tag{4.1}$$

In particular we have

$$EG^2(\{\boldsymbol{\sigma}, \boldsymbol{\sigma}'; |\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'| \ge 4N\sqrt{\alpha D}\}) \le \exp\left(-\frac{N}{K}\right). \tag{4.2}$$

There, as in the rest of the paper, K denotes a constant that does not depend upon N, but that might depend upon α, D, etc.

Comment. This means in particular that in practice $|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'/N|$ is always less than or equal to 1/2 if $L\alpha D \le 1$.

PROOF. Since $|H_{N,M}| \le MD$, we have $Z \ge 2^N \exp(-MD)$. Now

$$\mathrm{card}\{(\boldsymbol{\sigma}, \boldsymbol{\sigma}'); |\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'| \ge tN\} \le 2^{2N+1}\exp -Nt^2/2$$

(a well-known bound on the tails of the binomial law) and thus

$$\frac{1}{Z^2}\sum \exp(H_{N,M}(\boldsymbol{\sigma}) + H_{N,M}(\boldsymbol{\sigma}')) \le 2\exp\left(4MD - \frac{Nt^2}{2}\right),$$

where the summation is over the couples $(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$ with $|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'| \ge tN$. $\square$

From now on we assume that u is five times differentiable, and that for $0 \le l \le 5$, we have

$$\forall\ x, |U^{(l)}(x)| \le D'. \tag{4.3}$$

There is no loss of generality to assume $D' \ge D$. Our estimates will involve both D and D', but the terms containing D' will go to zero as $N \to \infty$. We write

$$S_k = \frac{1}{\sqrt{N}} \sum_{i \le N} \xi_i^k \sigma_i, \qquad S'_k = \frac{1}{\sqrt{N}} \sum_{i \le N} \xi_i^k \sigma'_i,$$

$$\mathbf{A} = \big(u'(S_k)\big)_{k \le M}, \qquad \mathbf{A}' = \big(u'(S'_k)\big)_{k \le M}.$$

LEMMA 4.2. *Assume $L\alpha D \le 1$. Then there is a number $N_0 = N_0(D', \alpha)$ such that if $|t| D'^4 \le 1$ and $N \ge N_0$ we have*

$$E\langle \exp tN\mathbf{A} \cdot \mathbf{A}' \rangle \le \exp(Lt(M+1) \exp 4D). \tag{4.4}$$

Using Chebyshev inequality, we get the following.

COROLLARY 4.3. *If $L\alpha \exp 4D \le 1$, for N large enough we have*

$$EG_n^2\big(\{|\mathbf{A} \cdot \mathbf{A}'| \ge L\alpha \exp 4D\}\big) \le \exp -\frac{N}{K}. \tag{4.5}$$

As pointed out in Section 2, the importance of (4.5) is that it is a kind of boundedness property of $\mathbf{A}$, under a smallness condition of α that depends only upon D, while $\|\mathbf{A}\|^2$ is not governed by D, but can be very large.

Even though it would require less work to prove weaker statements than (4.5) that are still suitable for our purposes, we decided to prove the cleaner exponential inequality (4.5). The following simple observation will be crucial.

LEMMA 4.4. *Consider random variables $X, (Y_k)_{k \le M}$ on a probability space, and $(X'; (Y'_k)_{k \le M})$ an independent copy of the family $(X; (Y_k)_{k \le M})$. Then for each $n \ge 0$, we have*

$$EXX' \left(\sum_{k \le M} Y_k Y'_k \right)^n \ge 0.$$

PROOF. We have

$$\begin{aligned} EXX' \left(\sum_{k \le M} Y_k Y'_k \right)^n &= \sum_{k_1, \dots, k_n} EXX' Y_{k_1} Y'_{k_1}, \dots, Y_{k_n} Y'_{k_n} \\ &= \sum_{k_1, \dots, k_n} \big(EXY_{k_1} \cdots Y_{k_n}\big)^2 \ge 0 \end{aligned}$$

by independence, where the sum is over all choices of $k_1, \dots, k_n$. $\square$

PROOF OF LEMMA 4.2. We have

$$\begin{aligned}\langle \exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle &= \sum_{n\geq 0}\frac{(Nt)^n}{n!}\langle(\mathbf{A}\cdot\mathbf{A}')^n\rangle\\ &\leq \sum_{n\geq 0}\frac{(N|t|)^n}{n!}\langle(\mathbf{A}\cdot\mathbf{A}')^n\rangle\\ &\leq \langle \exp N|t|\mathbf{A}\cdot\mathbf{A}'\rangle\end{aligned}$$

since (using Lemma 4.4 for the Gibbs measure with $X = 1$) we know that $\langle(\mathbf{A}\cdot\mathbf{A}')^n\rangle \geq 0$.

We consider the function

$$f_{N,M}(t) = E\langle \exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle$$

so that

$$\begin{aligned}f'_{N,M}(t) &= E\langle N\mathbf{A}\cdot\mathbf{A}' \exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle\\ &= ME\langle u'(S_M)u'(S'_M)\exp Nt\mathbf{A}\cdot\mathbf{A}\rangle,\end{aligned}\tag{4.6}$$

using the symmetry in $k \leq M$. We now denote by $\langle\cdot\rangle_{N,M-1}$ the Gibbs measure relative to the Hamiltonian $H_{N,M-1} = \sum_{k\leq M-1} u(S_k)$. It should be obvious that we have

$$\begin{aligned}&\langle u'(S_M)u'(S'_M)\exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle\\ &\quad= \frac{\langle u'(S_M)u'(S'_M)\exp(u(S_M)+u(S'_M))\exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle_{N,M-1}}{\langle \exp(u(S_M)+u(S'_M))\rangle_{N,M-1}}.\end{aligned}\tag{4.7}$$

This formula is the first occurrence of "induction upon M," a recurring theme of the paper.

If we use Lemma 4.5 for $G_{N,M-1}$ [with $X = u'(S_M)\exp u(S_M)$] we see (expanding $\exp Nt\mathbf{A}\cdot\mathbf{A}'$ in a power series) that the numerator of the right-hand side of (4.7) is greater than or equal to 0. Thus we have

$$\begin{aligned}&\langle u'(S_M)u'(S'_M)\exp Nt\mathbf{A}\cdot\mathbf{A}'\rangle\\ &\quad\leq \exp 2D\langle u'(S_M)u'(S'_M)\exp(u(S_M)+u(S'_M))\\ &\qquad\qquad \times \exp tu'(S_M)u'(S'_M)U\rangle_{N,M-1},\end{aligned}\tag{4.8}$$

where

$$U = \exp t\sum_{k\leq M-1} u'(S_k)u'(S'_k).$$

To get rid of the term $\exp tu'(S_M)u'(S'_M)$, we use the fact that if $tD'^2 \leq 1$ we have

$$|\exp tu'(S_M)u'(S_M) - 1| \leq 3tD'^2\tag{4.9}$$

so that if $tD'^4 \le 1$ we have

$$f'_{N,M}(t) \le M\Big(3e^{2D}E\langle U\rangle_{N,M-1} + E\big\langle E_M\big(u'(S_M)u'(S'_M)\exp(u(S_M) + u(S'_M))U\big)\big\rangle_{N,M-1}\Big), \tag{4.10}$$

where E_M denotes expectation in the variables $(\xi_i^M)_{i\le N}$ only.

To evaluate the term starting with E_M, we use the tools of Section 3. The variables S_M, S'_M are not Gaussian; they would be Gaussian, however, if the variables $(\xi_i^M)_{i\le M}$ were Gaussian. Thus the natural approach is to replace successively, one at a time, the variables ξ_i^M by i.i.d. standard normal r.v., a procedure known as Trotter's method, which is described in detail in [9]. The procedure here implies

$$E_M\big(u'(S_M)u'(S'_M)\exp(u(S_M)+u(S'_M))\big) \le \frac{K}{N} + E(u'(g)u'(g')\exp(u(g)+u(g'))), \tag{4.11}$$

where the variables g, g' are jointly Gaussian, and satisfy $E(g'^2) = E(g^2) = 1$, $Egg' = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'/N$. Trotter's method will always produce error terms that vanish as $N \to \infty$, and essentially we will be able in the rest of the paper to pretend that the variables $(\xi_i^M)_{i\le N}$ are i.i.d. $N(0,1)$ [and independent of the (ξ_i^k), $k < M$]. (The use of Trotter's method is of course not required in the case of the Gaussian model.)

We now appeal to Lemma 3.2 (with $Q = 2$, $Q' = 0$, $n = 0$, $m_1 = m_2 = 1$) with the function $f(x_1, x_2) = \exp(u(x_1) + u(x_2))$ to see that

$$|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'| \le N/2 \quad \Rightarrow \quad \left|E\frac{\partial^2}{\partial x_1\partial x_2}f(g,g')\right| \le L\exp 2D.$$

The use of the condition $|Egg'| = |\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'/N| \le 1/2$ is of course to obtain (3.4). Thus we have

$$E\big\langle E_M(u'(S_M)u'(S'_M))\exp(u(S_M)+u(S'_M))U1_{\{|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'|\le N/2\}}\big\rangle_{N,M-1} \le \left\langle\frac{K}{N} + L\exp 2D\right\rangle E\langle U\rangle_{N,M-1}. \tag{4.12}$$

Using (4.1) for $t = 1/2$, we see by trivial bounds that if $N \ge K$ we have

$$E\big\langle E_M(u'(S_M)u'(S'_M))\exp(u(S_M)+u(S'_M))U1_{\{|\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'|\ge N/2\}}\big\rangle_{N,M-1} \le 2D'^2\exp\left(2D + tMD'^2 + 4MD - \frac{N}{16}\right). \tag{4.13}$$

Combining (4.10) and (4.13), if $N \ge N_0 = N_0(D')$, $100\alpha D \le 1$, we have

$$\begin{aligned} f'_{N,M}(t) &\le M(L\exp 2D)E\langle U\rangle_{N,M-1} \\ &= M(D\exp 2D)f_{N,M-1}(t) \end{aligned}$$

from which the result follows by induction upon M. □

We now need a result of the same nature as Lemma 4.2, but where u' is replaced by

$$w(x) = u''(x) + u'(x)^2 - xu'(x). \tag{4.14}$$

The special form of w is rather important, in particular the fact that u'' and u'^2 have the same coefficient. This will be used through the identity

$$\left(e^{u(x)}\right)'' = \left(u''(x) + u'^2(x)\right)e^{u(x)}. \tag{4.15}$$

The interesting point is that the terms u'' and u'^2 in (4.14) will occur for different reasons, yet they match perfectly.

We consider

$$\mathbf{W} = \left(w(S_k)\right)_{k\le M}; \qquad \mathbf{W}' = \left(w(S'_k)\right)_{k\le M}.$$

LEMMA 4.5. *There exist numbers $t_0 = t_0(D'), N_0 = N_0(D')$ such that if $0 \le |t| \le t_0$, $N \ge N_0$ we have*

$$E\langle \exp t\mathbf{W}\cdot\mathbf{W}'\rangle \le \exp(tL(M+1)\exp 4D). \tag{4.16}$$

PROOF. It is very similar to that of Lemma 6.2, with only the extra difficulty that w is not uniformly bounded.

Rather than a uniform bound such as (4.9), one uses that

$$E_M(\exp tw(S_N)w(S'_N) - 1)^2 \le 1 \tag{4.17}$$

for $|t| \le t_0(D')$. The details are left to the reader. □

COROLLARY 4.6. *We have*

$$EG\left(\left\{\left|\frac{1}{N}\sum_{k\le M} w(S_k)\right| \ge L\alpha \exp 2D\right\}\right) \le \exp\left(-\frac{N}{K}\right). \tag{4.18}$$

PROOF. We compute

$$\begin{aligned}\left\langle\left(\frac{1}{N}\sum_{k\le M} w(S_k)\right)^n\right\rangle &= \frac{1}{N^n}\sum_{k_1,\dots,k_n}\langle w(S_{k_1})\cdots w(S_{k_n})\rangle \\ &\le \frac{M^{n/2}}{N^n}\left(\sum_{k_1,\dots,k_n}\langle w(S_{k_1})\cdots w(S_{k_n})\rangle^2\right)^{1/2} \\ &= \frac{\alpha^{n/2}}{N^{n/2}}\langle(\mathbf{W}\cdot\mathbf{W}')^n\rangle^{1/2},\end{aligned} \tag{4.19}$$

where we have used Cauchy–Schwarz in the second line.

By Chebyshev inequality we have, using (4.19),

$$EG\left(\left\{\left|\frac{1}{N}\sum_{k\le M} w(S_k)\right| \ge t\right\}\right) \le t^{-n} E\left\langle\left(\frac{1}{N}\sum w(S_k)\right)^n\right\rangle \tag{4.20}$$
$$\le \left(\left(\frac{\alpha}{Nt^2}\right)^n E\langle(\mathbf{W}\cdot\mathbf{W}')^n\rangle\right)^{1/2}.$$

It follows from Lemma 4.4 that for each n we have $\langle(\mathbf{W}\cdot\mathbf{W}')^n\rangle \ge 0$, and thus, for each n, and each $t_0 > 0$, we have

$$\langle(\mathbf{W}\cdot\mathbf{W}')^n\rangle \le \frac{n!}{t_0^n}\langle\exp t_0 \mathbf{W}\cdot\mathbf{W}'\rangle$$

so that if we take for t_0 the number of Lemma 4.4 we get that the right-hand side of (4.20) is at most

$$\left[\left(\frac{\alpha n}{N t_0 t^2}\right)^n \exp(t_0 ML \exp 4D)\right]^{1/2}$$

and the result follows by taking $n \simeq N t_0 t^2/e\alpha$ and $t = L\alpha \exp 2D$. $\square$

The following result looks strange a priori, but the need for it will actually occur quite naturally.

We consider four replicas, and $S_k^l = N^{-1/2}\sum_{i\le N}\xi_i^k\sigma_i^l$ for $l \le 4$.

LEMMA 4.7. *If $L\alpha \exp LD \le 1$, for N large enough we have*

$$EG_N^4\left(\left\{\left|\frac{1}{M}\sum_{k\le M}\prod_{l\le 4} u'(S_k^l)\right| \ge L\exp 8D\right\}\right) \le \exp -\frac{N}{K}. \tag{4.21}$$

The proof is identical to that of Corollary 4.6 (proving first a suitable version of Lemma 4.5).

5. Relating an N-spin system with an $(N-1)$-spin system. For $\boldsymbol{\sigma}$ in Σ_N, we write $\boldsymbol{\eta} = (\sigma_i)_{i\le N-1} \in \Sigma_{N-1}$. If we define

$$S_k = \sum_{i\le N}\frac{\xi_i^k\sigma_i}{\sqrt{N}}, \qquad s_k = \sum_{i\le N-1}\frac{\xi_i^k\sigma_i}{\sqrt{N-1}},$$

we then have

$$S_k = s_k + x, \tag{5.1}$$

where

$$x = s_k\left(\sqrt{1-\frac{1}{N}} - 1\right) + \frac{\xi_N^k\sigma_N}{\sqrt{N}}. \tag{5.2}$$

Using (4.3), we perform a Taylor expansion

$$u(S_k) = u(s_k) + xu'(s_k) + \frac{x^2}{2}u''(s_k) + R_{1,k}, \tag{5.3}$$

where

$$|R_{1,k}| \le D'|x|^3. \tag{5.4}$$

We use that

$$\left|\sqrt{1-\frac{1}{N}} - 1 + \frac{1}{2N}\right| \le \frac{1}{N^2}, \qquad \left|\sqrt{1-\frac{1}{N}} - 1\right| \le \frac{1}{N}$$

and we see that, since $|s_k| \le \sqrt{N}$,

$$\left|xu'(s_k) - \left(-\frac{1}{2N}s_k u'(s_k) + \frac{\xi_N^k \sigma_N}{\sqrt{N}}u'(s_k)\right)\right| \le D'\frac{|s_k|}{N^2},$$

$$\left|\frac{x^2}{2}u''(s_k) - \frac{u''(s_k)}{2N}\right| \le D'\left(\frac{s_k^2}{N^2} + \frac{2|s_k|}{N^{3/2}}\right) \le 3D'\frac{|s_k|}{N^{3/2}},$$

$$|x^3| \le L\left(\frac{|s_k|^3}{N^3} + \frac{1}{N^{3/2}}\right) \le L\left(\frac{1}{N^{3/2}} + \frac{|s_k|}{N^2}\right).$$

Summation yields the following.

LEMMA 5.1. *We have*

$$\begin{aligned}\sum_{k\le M} u(S_k) &= \sum_{k\le M} u(s_k) + \sigma_N\left(\sum_{k\le M}\frac{\xi_M^k u'(s_k)}{\sqrt{N}}\right)\\ &\quad + \frac{1}{2N}\sum_{k\le M}(u''(s_k) - s_k u'(s_k)) + R,\end{aligned} \tag{5.5}$$

where $R = R(\boldsymbol{\sigma})$ *satisfies*

$$|R| \le LD'\left(\frac{\alpha}{\sqrt{N}} + \frac{1}{N^{3/2}}\sum_{k\le M}|s_k|\right) \tag{5.6}$$

The following elementary fact is proved in particular in the Appendix of [9].

LEMMA 5.2. *There exists an event* Ω_0 (*in the variables* $\boldsymbol{\xi}^k$) *such that*

$$P(\Omega_0) \ge 1 - \exp -N/L \tag{5.7}$$

while, on Ω_0,

$$\forall \boldsymbol{\eta} \in \Sigma_{N-1}, \qquad \sum_{k\le M} s_k^2 \le LN. \tag{5.8}$$

For clarity, we will neglect exponentially small sets, so we will pretend that (5.8) always hold. We then see from (5.6) that $|R| \le LD'/\sqrt{N}$.

THEOREM 5.3. *Consider a function f on $\{-1,1\}^N$. Then we have*

$$\left|\langle f(\boldsymbol{\sigma})\rangle - \frac{1}{Z}\mathrm{Av}(\langle f(\sigma_1,\dots,\sigma_N)\mathscr{E}\rangle_0)\right| \le \frac{K(\|f\|_\infty, D')}{\sqrt{N}}, \tag{5.9}$$

where

$$\mathscr{E} = \mathscr{E}(\boldsymbol{\eta},\sigma_N) = \exp\left(\sigma_N \sum_{k\le M}\frac{\xi_N^k u'(s_k)}{\sqrt{N}} + \frac{1}{2N}\sum_{k\le M}(u''(s_k) - s_k u'(s_k))\right) \tag{5.10}$$

and

$$Z = \mathrm{Av}\langle\mathscr{E}\rangle_0. \tag{5.11}$$

There, Av *denotes average in $\sigma_N = \pm 1$, $\langle\cdot\rangle$ denotes average with respect to the Gibbs measure relative to the Hamiltonian $H_{N,M}$ given by* (1.1), *while $\langle\cdot\rangle_0$ denotes average with respect to the Gibbs measure relative to the Hamiltonian $H_{N-1,M}(\boldsymbol{\eta}) = \sum_{k\le M} u(s_k)$, for $s_k = s_k(\boldsymbol{\eta}) = \sum_{i\le N-1}\xi_i^k\eta_i/\sqrt{N-1}$.*

PROOF. If $\mathscr{D} = H_{N,M}(\boldsymbol{\sigma}) - H_{N-1,M}(\boldsymbol{\eta})$, then the relation

$$\langle f(\boldsymbol{\sigma})\rangle = \frac{1}{Z}\mathrm{Av}\langle f(\sigma_1,\dots,\sigma_N)\exp\mathscr{D}\rangle_0$$

for $Z = \mathrm{Av}\langle\exp\mathscr{D}\rangle_0$, is an algebraic identity. In the beginning of this section we proved that $|\mathscr{D} - \log\mathscr{E}| \le LD'/\sqrt{N}$ (on Ω_0). It then follows that

$$|\exp\mathscr{D} - \mathscr{E}| \le \frac{K(D')}{\sqrt{N}}\exp\mathscr{D},$$

where $K(D')$ depends upon D' only. This implies (5.9) through elementary estimates. □

We should point out that even though Theorem 5.3 is formally similar to the corresponding result in the case of the Hopfield model, a new source of difficulty is that it is much less obvious how to bound Z from below.

6. The cavity method: increasing N. This section contains a series of estimates that culminate in Theorem 6.6. Throughout the paper, we write $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$, $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2$, etc.

LEMMA 6.1. *We have*

$$C_{N,M} = E\left\langle\left(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\right)^2\right\rangle \le \frac{K}{\sqrt{N}} + E\frac{1}{Z}\left\langle\frac{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3}{N}\mathrm{Av}\,\tilde{\sigma}_N\sigma_N^3\mathscr{E}\right\rangle_0,$$

where $Z = \langle\mathrm{Av}\,\mathscr{E}\rangle_0^3$,

$$\mathscr{E} = \exp\left(\sum_{l\le 3}\sigma_N^l\boldsymbol{\xi}\cdot\mathbf{A}^l + \sum_{l\le 3}B^l\right),$$

$$\mathbf{A}^l = \left(\frac{1}{\sqrt{N}}u'(s_k^l)\right)_{k\le M}, \qquad B^l = \frac{1}{2N}\sum_{k\le M}\big(u''(s_k^l) - s_k^l u'(s_k^l)\big),$$

$$s_k^l = \frac{1}{\sqrt{N-1}}\sum_{i\le N}\xi_i^k\sigma_i^l.$$

Of course, in this lemma, $\langle\cdot\rangle$ denotes average with respect to $G^{\otimes 3}_{N,M}$ and $\langle\cdot\rangle_0$ with respect to $G^{\otimes 3}_{N-1,M}$, $\boldsymbol{\xi} = (\xi_k)_{k\le M}$ is a sequence with $P(\xi_k = 1) = P(\xi_k = -1) = 1/2$, independent of all other sequences, and $\boldsymbol{\xi}\cdot\mathbf{A}^l = N^{-1/2}\sum_{k\le M}\xi_k u'(s_k^l)$. The notation $\mathbf{A}^l$ is slightly abusive, since it relates to the quantity denoted by $\mathbf{A}$ in the previous sections by a change of the factor $N^{-1/2}$ into $(N+1)^{-1/2}$ and a replacement of N by $N-1$. This abuse of notation hopefully has no consequences.

PROOF. We use that

$$\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3 = \sum_{i\le N}\tilde{\sigma}_i\sigma_i^3$$

and the symmetry between sites to see that

$$\begin{aligned} C_{N,M} &= E\left\langle \tilde{\sigma}_N\sigma_N^3\left(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\right)\right\rangle \\ &\le \frac{4}{N} + E\left\langle \tilde{\sigma}_N\sigma_N^3\left(\frac{\boldsymbol{\eta}\cdot\boldsymbol{\eta}^3}{N}\right)\right\rangle. \end{aligned}$$

We conclude by applying (the obvious extension to 3-replicas of) Theorem 5.3. $\square$

We now write

$$U = \left\langle \frac{\boldsymbol{\eta}\cdot\boldsymbol{\eta}^3}{N}\operatorname{Av}(\tilde{\sigma}_N\sigma_N^3\mathscr{E})\right\rangle_0, \tag{6.1}$$

$$\widehat{Z} = \big(a\operatorname{ch}(\boldsymbol{\xi}\cdot\mathbf{C})\big)^3, \tag{6.2}$$

where for any $l \le 3$,

$$\mathbf{C} = \langle\mathbf{A}^l\rangle, \qquad a = \left\langle \exp\left(\frac{\|\mathbf{A}^l\|^2}{2} - \frac{\|\mathbf{C}\|^2}{2} + B^l\right)\right\rangle_0. \tag{6.3}$$

The idea behind $\widehat{Z}$ is that this constitutes a fairly good candidate for an approximation of Z. Since we will in the end assume that $\alpha\exp LD$ is small, by Corollaries 4.3 and 4.6, we can assume that $\|\mathbf{C}\|^2 \le 1$, $|\frac{1}{2}\|\mathbf{A}^l\|^2 + B^l| \le 1$ (neglecting exponentially small sets). Thus, for instance,

$$a, a^{-1} \le L, \qquad \frac{1}{\widehat{Z}} \le L. \tag{6.4}$$

LEMMA 6.2. *We have*

$$C_{N,M} \le E\frac{U}{\widehat{Z}} + L\left(E\left(\frac{U}{Z}\right)^2\right)^{1/2}\big(E(Z-\widehat{Z})^2\big)^{1/2} + \frac{K}{\sqrt{N}}. \tag{6.5}$$

PROOF. We use Lemma 6.1 and we write

$$\frac{U}{Z} = \frac{U}{\widehat{Z}} + \frac{U}{Z}\frac{(\widehat{Z} - Z)}{\widehat{Z}}. \tag{6.6}$$

We use Cauchy–Schwarz on the last term and (6.4) to eliminate $\widehat{Z}$ from the denominator. □

The approach above will offer the advantage of avoiding the problem of bounding Z from below.

LEMMA 6.3. *We have*

$$E\frac{U}{\widehat{Z}} \le LC_{N-1,\,M}^{1/2} D_{N-1,\,M}^{1/2}, \tag{6.7}$$

where $D_{N,\,M}$ *is given by* (2.2).

PROOF. We denote by E_ξ expectation in $\boldsymbol{\xi}$ only, and we write, using (6.4),

$$E\frac{U}{\widehat{Z}} \le L\left(\mathscr{E}_1\left|\frac{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3}{N}\right|\left|E_\xi\frac{\mathrm{Av}\,\tilde{\sigma}_N\sigma_N^3\mathscr{E}_0}{\mathrm{ch}^3\boldsymbol{\xi}\cdot\mathbf{C}}\right|\right)_0, \tag{6.8}$$

where

$$\mathscr{E}_1 = \exp\sum_{l\le 3} B^l, \qquad \mathscr{E}_0 = \exp\sum_{l\le 3}\sigma_N^l\boldsymbol{\xi}\cdot\mathbf{A}^l.$$

The main task is the estimation of $E_\xi(\mathrm{Av}\,\tilde{\sigma}_N\sigma_N^3\mathscr{E}_0\mathrm{ch}^{-3}\boldsymbol{\xi}\cdot\mathbf{C})$. As already mentioned, using Trotter's method we can assume that the variables $(\xi_k)_{k\le M}$ are in fact i.i.d. $N(0,1)$, creating only an error K/N.

We first observe that

$$\sigma_N^1 - \sigma_N^2 \ne 0 \quad\Rightarrow\quad \sigma_N^1 = -\sigma_N^2$$

and thus

$$\mathrm{Av}\,\tilde{\sigma}_N\sigma_N^3\mathscr{E}_0 = \mathrm{Av}\,\varepsilon_1\varepsilon_2\exp\big(\varepsilon_1\boldsymbol{\xi}\cdot\widetilde{\mathbf{A}} + \varepsilon_2\boldsymbol{\xi}\cdot\mathbf{A}^3\big), \tag{6.9}$$

where on the right, the average is over $\varepsilon_1, \varepsilon_2 = \pm 1$.

Consider the function

$$f(x_1, x_2, x_3) = \mathrm{Av}\,\varepsilon_1\varepsilon_2\frac{\exp(\varepsilon_1 x_1 + \varepsilon_2 x_2)}{\mathrm{ch}^3(x_3)}, \tag{6.10}$$

where the average is again over $\varepsilon_1, \varepsilon_2 = \pm 1$.

Using (6.9), we have

$$E_\xi\frac{\mathrm{Av}\,\tilde{\sigma}_{\mathrm{N}}\sigma_{\mathrm{N}}^3\mathscr{E}_0}{\mathrm{ch}^3\boldsymbol{\xi}\cdot C} = E_\xi f(g_1, g_2, g_3), \tag{6.11}$$

where

$$g_1 = \boldsymbol{\xi}\cdot\widetilde{\mathbf{A}}, \qquad g_2 = \boldsymbol{\xi}\cdot\mathbf{A}^3, \qquad g_3 = \boldsymbol{\xi}\cdot\mathbf{C}.$$

Conditionally upon the r.v. involved in $G_{N-1,\,M}$, and given $(\mathbf{A}^l)_{l\le 3}$, we define a jointly Gaussian family $(g'_l)_{l\le 3}$ (independent of the g_l) such that

$$\begin{aligned}
&\forall\, l \le 3, \qquad E_\xi g_l'^2 = E_\xi g_l^2,\\
&\forall\, l = 2, 3, \qquad E_\xi g'_1 g'_l = 0,\\
&2 \le l, l' \le 3 \quad \Rightarrow \quad E_\xi g'_l g'_{l'} = E_\xi g_l g_{l'}.
\end{aligned}$$

[In other words, we make g_1 independent of the couple (g_2, g_3).] There, E_ξ denotes conditional expectation at all the r.v. occurring in $G_{N-1,\,M}$ fixed.

Consider the function

$$\varphi(t) = E_\xi f\big((g_l\sqrt{t} + g'_l\sqrt{1-t})_{l\le 4}\big).$$

We will use the bound

$$|\varphi(1) - \varphi(0)| \le \int_0^1 |\varphi'(t)|\, dt. \tag{6.12}$$

We show first that

$$\varphi(0) = 0. \tag{6.13}$$

To see this, we integrate in g'_1 first. Since g'_1 is independent of g'_2, g'_3, we have

$$\varphi(0) = \operatorname{Av} \varepsilon_1\varepsilon_2 E_\xi \frac{\exp\big(\frac{1}{2}\|\widetilde{\mathbf{A}}_l\|^2 + \varepsilon_2 g'_2\big)}{\operatorname{ch}^3 g'_3} = 0$$

because $\operatorname{Av}\varepsilon_1 = 0$.

To control $\varphi'(t)$, we observe that for $l, l' \le 3$ we have

$$\left|\frac{\partial^2 f}{\partial x_l \partial x_{l'}}\right| \le L \operatorname{Av} \exp \sum_{l\le 2} \varepsilon_l x_l$$

so that by Lemma 3.1 we have

$$\begin{aligned}
|\varphi'(t)| \le &\sum_{l,\, l'\le 3} \big|E_\xi(g_l g_{l'}) - E_\xi(g'_l g'_{l'})\big|\\
&\times E_\xi \operatorname{Av} \exp \sum_{l\le 2} \varepsilon_l (g_l\sqrt{t} + g'_l\sqrt{1-t}).
\end{aligned} \tag{6.14}$$

Now

$$\begin{aligned}
&E_\xi \exp \sum_{l\le 2} \varepsilon_l (g_l\sqrt{t} + g'_l\sqrt{1-t})\\
&\quad = \exp\left(\frac{1}{2}\sum_{l\le 3} \|\mathbf{A}^l\|^2 + t\varepsilon_1\varepsilon_2 \widetilde{\mathbf{A}}\cdot\mathbf{A}^3 - t\mathbf{A}^1\cdot\mathbf{A}^2\right).
\end{aligned}$$

Since, by Corollary 4.3 we can pretend that $|\mathbf{A}^l \cdot \mathbf{A}^{l'}| \le 1$ for $l \ne l'$, we have from (6.12), (6.14) that

$$|\varphi(1)| \le L \sum_{l,\, l' \le 3} \bigl|E_\xi(g_l g_{l'}) - E_\xi(g'_l g'_{l'})\bigr| \exp\Biggl(\frac{1}{2}\sum_{l\le 3}\|\mathbf{A}^l\|^2\Biggr) \tag{6.15}$$

and thus, recalling the values of $E(g_l g_{l'})$, $E(g'_l g'_{l'})$ and going back to (6.8),

$$E\frac{U}{\widehat{Z}} \le L\biggl\langle\biggl|\frac{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3}{N}\biggr|(|\widetilde{\mathbf{A}}\cdot\mathbf{A}^3| + |\widetilde{\mathbf{A}}\cdot\mathbf{C}|)\mathscr{E}_2\biggr\rangle_0, \tag{6.16}$$

where

$$\mathscr{E}_2 = \exp\Biggl(\frac{1}{2}\sum_{l\le 3}\|\mathbf{A}^l\|^2 + B^l\Biggr).$$

The term $u'^2(s_k^l)$ coming from $\|\mathbf{A}^l\|^2$ nicely combines with the term $u''(s_k^l)$ coming from B^l; we have

$$\mathscr{E}_2 = \exp\sum_{l\le 3}\frac{1}{2N}\sum_{k\le M} w\bigl(s_k^l\bigr),$$

so that by Corollary 4.6 we can pretend that $\mathscr{E}_2 \le L$. We then use Cauchy–Schwarz, together with the elementary observation that

$$\bigl\langle(\widetilde{\mathbf{A}}\cdot\mathbf{C})^2\bigr\rangle_0 \le \bigl\langle(\widetilde{\mathbf{A}}\cdot\mathbf{A}^3)^2\bigr\rangle_0. \qquad \square$$

LEMMA 6.4. *We have*

$$E\frac{U^2}{Z^2} \le \frac{K}{\sqrt{N}} + 4C_{N,\,M}.$$

PROOF. We have, since $|\tilde{\sigma}_N\sigma_N^3| \le 2$,

$$\begin{aligned}\biggl|\frac{U}{Z}\biggr| \le 2\frac{\langle \mathrm{Av}|\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}^3|\mathscr{E}\rangle_0}{N\langle \mathrm{Av}\,\mathscr{E}\rangle_0} &\le \frac{4}{N} + 2\frac{\langle \mathrm{Av}|\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3|\mathscr{E}\rangle_0}{N\langle \mathrm{Av}\,\mathscr{E}\rangle_0}\\ &\le \frac{K}{\sqrt{N}} + 2\biggl\langle\biggl|\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr|\biggr\rangle\\ &\le \frac{K}{\sqrt{N}} + 2\biggl\langle\biggl(\frac{\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma}^3}{N}\biggr)^2\biggr\rangle^{1/2},\end{aligned}$$

using (5.9) in the second line. $\square$

Despite the somewhat optimistic description of Section 2, we are required, besides $C_{N,\,M}$ and $D_{N,\,M}$, to consider a new quantity, namely,

$$E_{N,\,M} = E\biggl\langle\biggl(\frac{1}{N}\sum_{k\le M}\bigl(w(S_k) - w(S'_k)\bigr)\biggr)^2\biggr\rangle. \tag{6.17}$$

LEMMA 6.5. *We have*

$$E(Z-\widehat{Z})^2 \le L(D_{N-1,M}+E_{N-1,M}). \tag{6.18}$$

PROOF. We have

$$Z^2 = \Big\langle \mathrm{Av}\exp\sum_{l\le 6}\big(\sigma_N^l \boldsymbol{\xi}\cdot\mathbf{A}^l + B^l\big)\Big\rangle_0,$$

so that

$$E_\xi Z^2 = \mathrm{Av}\bigg\langle \exp\bigg[\bigg(\sum_{l\le 6}\frac{\|\mathbf{A}^l\|^2}{2}+B^l\bigg)+\sum_{1\le l<l'\le 6}\sigma_N^l\sigma_N^{l'}\mathbf{A}^l\cdot\mathbf{A}^{l'}\bigg]\bigg\rangle_0.$$

We have

$$\begin{aligned}
E_\xi \widehat{Z}^2 &= a^6 E_\xi \mathrm{Av}\exp\sum_{l\le 6}\sigma_N^l\boldsymbol{\xi}\cdot\mathbf{C}\\
&= a^6\mathrm{Av}\exp\bigg(3\|\mathbf{C}\|^2+\sum_{1\le l<l'\le 6}\sigma_N^l\sigma_N^{l'}\|\mathbf{C}\|^2\bigg).
\end{aligned}$$

Thus

$$|E_\xi Z^2 - E_\xi\widehat{Z}^2| \le \mathrm{Av}\Big|\langle(\exp X)(\exp Y-\exp Y')\rangle_0\Big|, \tag{6.19}$$

where

$$\begin{aligned}
X &= \sum_{l\le 6}\frac{\|\mathbf{A}^l\|^2}{2}+B^l,\\
Y &= \sum_{1\le l<l'\le 6}\sigma_N^l\sigma_N^{l'}\mathbf{A}^l\cdot\mathbf{A}^{l'},\\
Y' &= \sum_{1\le l<l'\le 6}\sigma_N^l\sigma_N^{l'}\|\mathbf{C}\|^2.
\end{aligned}$$

We observe (Corollaries 4.3 and 4.6) that we can pretend that $|X|, |Y'|, |Y| \le L$. We then write

$$|\exp Y - \exp Y' - (Y-Y')\exp Y'| \le L(Y-Y')^2$$

and

$$|e^X(Y-Y')\exp Y' - e^{\langle X\rangle}(Y-Y')\exp Y'| \le L|X-\langle X\rangle|\,|Y-Y'|.$$

Since $\langle Y-Y'\rangle_0 = 0$, we deduce from (6.19) that

$$\begin{aligned}
\big|E_\xi Z^2 - E_\xi\widehat{Z}^2\big| &\le L\big(\langle(Y-Y')^2\rangle_0 + \langle(X-\langle X\rangle_0)^2\rangle_0\big)\\
&\le L\langle(\widetilde{\mathbf{A}}\cdot\mathbf{A}^3)^2\rangle_0 + L\bigg\langle\bigg(\frac{1}{N}\sum_{k\le M}\big(w(s_k^1)-w(s_k^2)\big)\bigg)^2\bigg\rangle_0 \qquad (6.20)\\
&\le L\big(D_{N-1,M}+E_{N-1,M}\big).
\end{aligned}$$

A similar bound for $|E_\xi Z\widehat{Z} - E_\xi\widehat{Z}^2|$ then completes the proof. □

THEOREM 6.6. *We have*

$$C_{N,M} \le \frac{1}{2} C_{N-1,M} + L\big(D_{N-1,M} + E_{N-1,M}\big) + \frac{K}{\sqrt{N}}.$$

PROOF. Combining the previous estimates, we have

$$\begin{aligned} C_{N,M} \le \frac{K}{\sqrt{N}} &+ L\big(C_{N-1,M} D_{N-1,M}\big)^{1/2} \\ &+ L D_{N-1,M} + L E_{N-1,M}. \end{aligned}$$ □

7. The Cavity method: increasing *M*. The purpose of this section is to establish the bounds of Theorems 7.6 and 7.7. These are then combined with Theorem 6.6 to prove (1.10). We start with the study of $D_{N-1,M}$. The first lemma holds no surprise.

LEMMA 7.1. *We have*

$$D_{N-1,M} \le \frac{K}{N} + \alpha E \frac{U}{Z}, \tag{7.1}$$

where

$$U = \Big\langle \tilde{v}_M\, v_M^3 \tilde{\mathbf{a}} \cdot \mathbf{a}^3 \exp \sum_{l \le 3} u(s_M^l) \Big\rangle_1,$$

$$Z = \Big\langle \exp \sum_{l \le 3} u(s_M^l) \Big\rangle_1,$$

for $\tilde{\mathbf{a}} = \mathbf{a}^1 - \mathbf{a}^2$, $\mathbf{a}^l = (u'(s_k^l))_{k \le M-1}$, $v_M^l = v'(s_M^l)$, $\tilde{v}_M = v_M^1 - v_M^2$ *and where* $\langle \cdot \rangle_1$ *denotes integration with respect to* $G_{N-1,M-1}$.

PROOF. We have

$$\begin{aligned} D_{N-1,M} &= E \frac{1}{N-1} \Big\langle \sum_{k \le M} \tilde{v}_k v_k^3\, \widetilde{\mathbf{A}} \cdot \mathbf{A}^3 \Big\rangle_0 \\ &= \frac{M}{M-1} E \Big\langle \tilde{v}_M\, v_M^3\, \widetilde{\mathbf{A}} \cdot \mathbf{A}^3 \Big\rangle_0 \\ &\le \frac{K}{N} + \alpha E \Big\langle \tilde{v}_M\, v_M^3\, \tilde{\mathbf{a}} \cdot \mathbf{a}^3 \Big\rangle_0 \end{aligned}$$

One then relates $\langle \cdot \rangle_0$ to $\langle \cdot \rangle_1$ in the obvious manner. □

We consider

$$\mathbf{b} = (b_i)_{i \le N-1}, \qquad b_i = \frac{1}{\sqrt{N-1}} \langle \sigma_i \rangle_1,$$

and $\|\mathbf{b}\|^2 = \sum_{i \le N-1} b_i^2$. We write $\boldsymbol{\xi} = (\xi_i^M)_{i \le N-1}$. (Thus the precise meaning of $\boldsymbol{\xi}$ in this section is different from its meaning in Section 6, but this should not be confusing because the roles these two quantities play are identical.)

We consider three independent $N(0,1)$ r.v. $(h_l)_{l\le 3}$, independent of all the other r.v. considered, and we define

$$\widehat{Z} = E_h \exp \sum_{l\le 3} u\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h^l\sqrt{1-\|\mathbf{b}\|^2}\Big), \tag{7.2}$$

where of course $\boldsymbol{\xi}\cdot\mathbf{b} = \sum_{i\le N-1} \xi_i^M b_i$.

LEMMA 7.2. *We have*

$$E\frac{U}{Z} \le E\frac{U}{\widehat{Z}} + \exp 3D\bigg(E\Big(\frac{U}{Z}\Big)^2\bigg)^{1/2}\Big(E(Z-\widehat{Z})^2\Big)^{1/2}. \tag{7.3}$$

PROOF. As (6.5), using now that $\widehat{Z} \ge \exp -3D$. $\square$

The reader has already guessed that the methods of this section will resemble those of Section 6. This is true, but there will be significant differences.

LEMMA 7.3. *We have*

$$E\frac{U}{\widehat{Z}} \le L \exp LD(C_{N-1,\,M-1}D_{N-1,\,M-1})^{1/2}. \tag{7.4}$$

PROOF. We will write E_ξ for expectation in $\boldsymbol{\xi}$ (even though this was denoted E_M in Section 4). We will use techniques of Section 3 to estimate

$$E_\xi\frac{U}{\widehat{Z}} = \bigg(\tilde{\mathbf{a}}\cdot\mathbf{a}^3 E_\xi\bigg(\frac{\tilde{v}_M\, v_M^3 \exp\sum_{l\le 3} u(s_M^l)}{\widehat{Z}}\bigg)\bigg)_1.$$

We consider the function

$$f(x_1,x_2,x_3,x_4) = \frac{(u'(x_1)-u'(x_2))u'(x_3)\exp\sum_{l\le 3}u(x_l)}{\widehat{Z}(x_4)},$$

where

$$\widehat{Z}(x_4) = E_h \exp\sum_{l\le 3} u\Big(x_4 + h^l\sqrt{1-\|\mathbf{b}\|^2}\Big).$$

Using Trotter's method, we can assume that the variables $(\xi_i^M)_{i\le N-1}$ are i.i.d. $N(0,1)$. We then have to evaluate $E_\xi f(g_1,g_2,g_3,g_4)$, where

$$g_l = \boldsymbol{\xi}\cdot\boldsymbol{\eta}^l/\sqrt{N-1} \quad \text{for } l\le 3,\ g_4 = \boldsymbol{\xi}\cdot\mathbf{b}.$$

conditionally upon the r.v. occurring in $G_{N-1,\,M-1}$, and given $(\boldsymbol{\eta}^l)_{l\le 3}$, we introduce a new jointly Gaussian family $(g_l')_{l\le 4}$, where

$$E_\xi g_l'^2 = 1 \quad \text{if } l\le 3, \qquad E_\xi g_l' g_{l'}' = \|\mathbf{b}\|^2 \quad \text{if } \ l<l' \text{ or } l'=4.$$

We consider

$$\varphi(t) = E_\xi f\big((g_l\sqrt{t} + g_l'\sqrt{1-t})_{l\le 4}\big)$$

and will again use (6.12).

By symmetry between x_1 and x_2, it should be obvious that $\varphi(0) = 0$. To control $\varphi'(t)$, we use Lemma 3.1 to see that

$$|\varphi'(t)| \le \sum_{1\le l\le l'\le 4} \left| E_\xi(g_l g_{l'}) - E_\xi(g'_l g'_{l'}) \right| \times \left| E_\xi \frac{\partial^2}{\partial x_l\, \partial x_{l'}} f\big((g_l\sqrt{t} + g'_l\sqrt{1-t})_{l\le 4}\big) \right|. \tag{7.5}$$

We will use Lemmas 3.3, 3.2 to prove that for $(\boldsymbol{\eta}^l)_{l\le 3}$ outside an exponentially small set [depending upon the r.v. $(\boldsymbol{\xi}_i^k)_{i\le N-1,\, k\le M-1}$] we have

$$\left| E_\xi \frac{\partial^2}{\partial x_l\, \partial x_{l'}} f\big((g_l\sqrt{t} + g'_l\sqrt{1-t})_{l\le 4}\big) \right| \le L \exp LD. \tag{7.6}$$

The crucial observation is that

$$f(x_1, x_2, x_3, x_4) = \left(\frac{\partial^2}{\partial x_1\, \partial x_2} - \frac{\partial^2}{\partial x_2\, \partial x_3} \right) \big(f_0(x_1, x_2, x_3)\theta(x_4)\big),$$

where

$$f_0(x_1, x_1, x_3) = \exp \sum_{l\le 3} u(x_l),$$

$$\theta(x_4) = \frac{1}{\widehat{Z}(x_4)}.$$

We will use Lemma 3.2 with $Q = 3$, $Q' = 1$. We observe that by (6.1) $|E(g_l g_{l'})| = |\boldsymbol{\eta}^l \cdot \boldsymbol{\eta}^{l'}|/(N-1)$ can be assumed less than or equal to 10^{-2} outside an exponentially small set, so that (3.4) will hold. Thus, to apply Lemma 3.2, all we need is to control $\|\theta\|_{\infty,4}$. Since $\exp -D < \widehat{Z} < \exp D$, it suffices to show that $|\widehat{Z}^{(k)}(x)| \le \exp LD$ for all x and for $k \le 4$. But this follows from integration by parts. For example,

$$\begin{aligned}
\widehat{Z}'(x) &= E_h\Big(\sum_{l\le 3} u'(x + h^l\sqrt{1-\|\mathbf{b}\|^2}) \Big) \exp \sum_{l\le 3} u\Big(x + h^l\sqrt{1-\|\mathbf{b}\|^2}\Big) \\
&= \frac{1}{\sqrt{1-\|\mathbf{b}\|^2}} E_h\Big(\sum_{l\le 3} h^l \Big) \exp \sum_{l\le 3} u\Big(x + h^l\sqrt{1-\|\mathbf{b}\|^2}\Big)
\end{aligned}$$

and thus $|\widehat{Z}'(x)| \le L \exp 3D$. This proves (7.6).

Combining (7.5) and (7.6), we then have

$$\begin{aligned}
&\left\langle |\tilde{\mathbf{a}}\cdot \mathbf{a}^3| \left| E_\xi\left(\frac{\tilde{v}_M\, v_M^3 \exp \sum_{l\le 3} u(s_M^l)}{\widehat{Z}} \right) \right| \right\rangle_1 \\
&\qquad \le L \exp 3D \left\langle |\tilde{\mathbf{a}}\cdot \mathbf{a}^3| \left(\sum_{l<l'} \left| \frac{\boldsymbol{\eta}^l \cdot \boldsymbol{\eta}^{l'}}{N-1} - \|\mathbf{b}\|^2 \right| + \sum_l \left| \frac{\boldsymbol{\eta}^l \cdot \mathbf{b}}{\sqrt{N-1}} - \|\mathbf{b}\|^2 \right| \right) \right\rangle_1
\end{aligned}$$

from which the result follows by Cauchy–Schwarz. □

LEMMA 7.4. *We have*

$$E\frac{U^2}{Z^2} \le (L\exp LD)D_{N-1,\,M} + \frac{K}{N^2}. \tag{7.7}$$

PROOF. We have

$$\left|\frac{U}{Z} - \frac{1}{Z}\langle \tilde{v}_M\, v_M^3 \widetilde{\mathbf{A}}\cdot\mathbf{A}^3 \exp\sum_{l\le 3} u(s_M^l)\rangle_1\right| \le \frac{K}{N}$$

or, equivalently,

$$\left|\frac{U}{Z} - \langle \tilde{v}_M\, v_M^3 \widetilde{\mathbf{A}}\cdot\mathbf{A}^3\rangle_0\right| \le \frac{K}{N}.$$

Using replicas,

$$\frac{U^2}{Z^2} \le \frac{K}{N^2} + 2\big\langle \tilde{v}_M\, v_M^3 v_M^* v_M^6 \widetilde{\mathbf{A}}\cdot\mathbf{A}^3\mathbf{A}^*\cdot\mathbf{A}^6\big\rangle_0, \tag{7.8}$$

where $v_M^* = u'(s_M^4) - u'(s_M^5)$, and $\mathbf{A}^*$ is defined similarly. Thus, by symmetry between the values of $k \le M$,

$$E\frac{U^2}{Z^2} \le \frac{K}{N^2} + 2E\left\langle\left(\frac{1}{M}\sum_{k\le M} \tilde{v}_k v_k^3 v_k^* v_k^6\right)\widetilde{\mathbf{A}}\cdot\mathbf{A}^3\mathbf{A}^*\cdot\mathbf{A}^6\right\rangle_0.$$

We appeal to Lemma 4.7 to get

$$\begin{aligned} E\frac{U^2}{Z^2} &\le \frac{K}{N^2} + 2\exp LDE\big\langle\big|\widetilde{\mathbf{A}}\cdot\mathbf{A}^3\big|\big\rangle_0^2 \\ &\le \frac{K}{N^2} + 2\exp LDE\big\langle\big|\widetilde{\mathbf{A}}\cdot\mathbf{A}^3\big|^2\big\rangle_0. \end{aligned}$$ □

LEMMA 7.5. *We have*

$$E(Z - \widehat{Z})^2 \le (L\exp LD)C_{N-1,\,M-1} + \frac{K}{N}.$$

PROOF. We will prove such a bound for $|EZ^2 - E\widehat{Z}^2|$, $|EZ\widehat{Z} - E\widehat{Z}^2|$. We consider only the first quantity; the case of the second is similar. We have

$$EZ^2 = E\langle E_\xi f((g_l)_{l\le 6})\rangle_1,$$

where $f((x_l)_{l\le 6}) = \exp\sum_{l\le 6} u(x_l)$, and $g_l = \boldsymbol{\xi}\cdot\boldsymbol{\eta}^l/\sqrt{N-1}$. We consider another jointly Gaussian family $(g_l')_{l\le 6}$ with $E_\xi g_l'^2 = 1$, $E_\xi g_l' g_{l'}' = \|\mathbf{b}\|^2$ if $l < l'$, and the function

$$\varphi(t) = E_\xi f\Big(\big(g_l\sqrt{t} + g_l'\sqrt{1-t}\big)_{l\le 6}\Big).$$

We will now use the bound

$$\big|\varphi(1) - \varphi(0) - \varphi'(0)\big| \le \int_0^1 |\varphi''(t)|\,dt. \tag{7.9}$$

First, we note that $\varphi(0) = E\widehat{Z}^2$. The reason for this is simply that if g, $(h_l)_{l\le 6}$ are i.i.d. $N(0,1)$ and $g''_l = g\|\mathbf{b}\| + h_l\sqrt{1-\|\mathbf{b}\|^2}$, then the family (g''_l) has the same distribution as (g'_l), so that

$$\varphi(0) = E_g\biggl(E_h \exp\sum_{l\le 6} u\Bigl(g\|\mathbf{b}\| + h_l\sqrt{1-\|\mathbf{b}\|^2}\Bigr)\biggr)$$

$$= E_g\biggl(\Bigl(E_h \exp\sum_{l\le 3} u\Bigl(g\|\mathbf{b}\| + h_l\sqrt{1-\|\mathbf{b}\|^2}\Bigr)\Bigr)^2\biggr) = E\widehat{Z}^2.$$

Next, we show that $\langle\varphi'(0)\rangle_1 = 0$. To see this, we use (3.1). We observe that $Eg_lg_{l'} - Eg'_lg'_{l'} = 0$ if $l = l'$ and $= (N-1)^{-1}\boldsymbol{\eta}^l\cdot\boldsymbol{\eta}^{l'} - \|\mathbf{b}\|^2$ if $l \ne l'$, so that $\langle Eg_lg_{l'} - Eg'_lg'_{l'}\rangle_1 = 0$; and we observe that $(\partial^2/\partial x_l\,\partial x_{l'})f((g'_l))$ does not depent upon the quantities $\boldsymbol{\eta}^l$.

Thus, we now know from (7.9) that

$$\bigl|EZ^2 - E\widehat{Z}^2\bigr| \le E\biggl\langle \int_0^1 |\varphi''(t)|\,dt\biggr\rangle_1. \tag{7.10}$$

To estimate $\varphi''(t)$ we apply (3.1) twice. Then (using Cauchy–Schwarz) we see that the issue is to prove that (possibly outside an exponentially small set) we have

$$\left|E_\xi \frac{\partial^4}{\partial x_{l_1}\partial x_{l_2}\,\partial x_{l_3}\partial x_{l_4}} f\Bigl(\bigl(g_l\sqrt{t} + g'_l\sqrt{1-t}\bigr)_{l\le 6}\Bigr)\right| \le L\exp LD.$$

This, however, follows from Lemmas 3.2, with $Q = 6$, $Q' = 0$ and Lemma 3.3. □

Combining the previous lemmas, we have the following theorem.

THEOREM 7.6. *We have*

$$D_{N-1,M} \le \frac{K}{\sqrt{N}} + L\alpha\exp LD(C_{N-1,M-1} + D_{N-1,M-1} + D_{N-1,M}).$$

We need a similar estimate for $E_{N-1,M}$.

THEOREM 7.7. *We have*

$$E_{N-1,M} \le \frac{K}{\sqrt{N}} + L\alpha\exp LD(C_{N-1,M-1} + E_{N-1,M-1} + E_{N-1,M}).$$

The proof is almost identical to the proof of Theorem 7.6 (with obvious changes, such as use of Corollary 4.6 rather than Lemma 4.7) so it is better left to the reader.

We now reach our goal, the proof of (1.10).

THEOREM 7.8. *There is a constant L such that if $L\alpha \exp LD \le 1$, then for $M = \alpha N$,*

$$\lim_{N\to\infty} C_{N,M} = 0, \qquad \lim_{N\to\infty} D_{N,M} = 0, \qquad \lim_{N\to\infty} E_{N,M} = 0. \tag{7.11}$$

PROOF. We then get from Theorems 7.6 and 7.7 that

$$D_{N-1,M} = \frac{K}{\sqrt{N}} + L\alpha \exp LD(C_{N-1,M-1} + D_{N-1,M-1}),$$
$$E_{N-1,M} = \frac{K}{\sqrt{N}} + L\alpha \exp LD(C_{N-1,M-1} + E_{N-1,M-1}).$$

Together with Theorem 6.6 this implies

$$C_{N,M} \le \frac{K}{\sqrt{N}} + \frac{1}{2}C_{N-1,M} + L\alpha \exp LD(C_{N-1,M-1} + D_{N-1,M-1} + E_{N-1,M-1}).$$

When $\alpha \exp LD$ is small enough, we have

$$C_{N,M} \le \frac{K}{\sqrt{N}} + \frac{1}{2}C_{N-1,M} + 10^{-2}(C_{N-1,M-1} + D_{N-1,M-1} + E_{N-1,M-1}),$$
$$D_{N-1,M} \le \frac{K}{\sqrt{N}} + 10^{-2}(C_{N-1,M-1} + D_{N-1,M-1}),$$
$$E_{N-1,M} \le \frac{K}{\sqrt{N}} + 10^{-2}(C_{N-1,M-1} + E_{N-1,M-1}).$$

We observe that $C_{N,M}$, $D_{N,M}$, $E_{N,M}$ are bounded independently of N (as long as, say, $M \le N$). It should then be obvious that the result follows by iteration of the previous relations. □

8. Proof of Theorem 1.1.

Now we have proved (1.10), we will be able to do all kinds of computations that will readily lead us to (1.11) and to the proof of Theorem 1.1.

The following is a special case of a general principle that is left to the reader to formulate. We keep the notation of Section 6.

PROPOSITION 8.1. *Consider a bounded function h on Σ_N^4 that does not depend upon (σ_N^l), $l \le 4$. Then*

$$E\langle \sigma_N^1 \sigma_N^2 h\rangle = E(\langle h\rangle_0 \operatorname{th}^2(\boldsymbol{\xi}\cdot \mathbf{C})) + o(1). \tag{8.1}$$

In the rest of the paper, $o(1)$ denotes a quantity that goes to zero as $N \to \infty$. It is always understood that $M = \alpha N$, where α is small enough that Theorem 7.8 holds.

PROOF OF PROPOSITION 8.1. We use Theorem 5.3 to see that

$$E\langle f\rangle = E\frac{\langle h \operatorname{Av} \sigma_N^1 \sigma_N^2 \mathscr{E}\rangle_0}{Z} + o(1),$$

where

$$\mathscr{E} = \exp \sum_{l \le 4} (\sigma_N^l \, \boldsymbol{\xi} \cdot \mathbf{A}^l + B^l)$$

and $Z = \langle \mathrm{Av}\, \mathscr{E} \rangle_0$. Consider

$$\widehat{Z} = \mathrm{ch}^4 \, \boldsymbol{\xi} \cdot \mathbf{C} \Big\langle \exp \sum_{l \le 4} \Big(\frac{\|\mathbf{A}^l\|}{2} + B^l - \frac{\|\mathbf{C}\|^2}{2} \Big) \Big\rangle_0 .$$

Looking at Lemma 7.5, how we proved (7.3) and Theorem 7.8 we see that

$$E \frac{\langle h \,\mathrm{Av}\, \sigma_N^1 \sigma_N^2 \mathscr{E} \rangle_0}{Z} = E \frac{\langle h \,\mathrm{Av}\, \sigma_N^1 \sigma_N^2 \mathscr{E} \rangle_0}{\widehat{Z}} + o(1).$$

If we apply the method of (6.10) and (6.14), it should be obvious that

$$\begin{aligned} &E \frac{\langle h \,\mathrm{Av}\, \sigma_N^1 \sigma_N^2 \mathscr{E} \rangle_0}{\widehat{Z}} \\ &\quad = E \bigg(\mathrm{sh}^2 \, \boldsymbol{\xi} \cdot \mathbf{C} \, \mathrm{ch}^2 \, \boldsymbol{\xi} \cdot \mathbf{C} \frac{\langle h \exp \sum_{l \le 4} \big(\|\mathbf{A}^l\|^2/2 + B^l - \|\mathbf{C}\|^2/2 \big) \rangle_0}{\widehat{Z}} \bigg) + o(1), \end{aligned}$$

because, thanks to Theorem 7.8, we know that the terms containing a factor $(\mathbf{A}^l \cdot \mathbf{A}^{l'} - \|\mathbf{C}\|^2)$ are vanishing. Finally we use that $\lim_{N \to \infty} E_{N-1,\,M} = 0$, meaning that the function $\sum_{l \le 4} \big(\|\mathbf{A}^l\|^2/2 + B^l \big)$ is essentially constant, to obtain the result. $\square$

We now consider

$$q_{N,\,M} = \frac{1}{N} \sum_{i \le N} \langle \sigma_i \rangle^2 = \frac{1}{N} \| \langle \boldsymbol{\sigma} \rangle \|^2 = \Big\langle \frac{\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2}{N} \Big\rangle. \tag{8.2}$$

LEMMA 8.2. *We have*

$$E q_{N,\,M} = E \, \mathrm{th}^2 \, (\boldsymbol{\xi} \cdot \mathbf{C}) + o(1), \tag{8.3}$$

$$E q_{N,\,M}^2 = E \, (q_{N-1,\,M} \mathrm{th}^2 \, (\boldsymbol{\xi} \cdot \mathbf{C})) + o(1). \tag{8.4}$$

There of course $\mathbf{C} = (\langle u'(s_k) \rangle_0)_{k \le M}$.

PROOF OF LEMMA 8.2. To prove (8.3) we write

$$E q_{N,\,M} = E \Big\langle \frac{\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2}{N} \Big\rangle = E \langle \sigma_N^1 \sigma_N^2 \rangle$$

and we use (8.1) for $h = 1$. To prove (8.4) we write

$$\begin{aligned} E q_{N,\,M}^2 = E \langle \sigma_N \rangle^2 q_{N,\,M} &= E \Big\langle \sigma_N^1 \sigma_N^2 \frac{\boldsymbol{\sigma}^3 \cdot \boldsymbol{\sigma}^4}{N} \Big\rangle \\ &= o(1) + E \Big\langle \sigma_N^1 \sigma_N^2 \frac{\boldsymbol{\eta}^3 \cdot \boldsymbol{\eta}^4}{N} \Big\rangle \end{aligned}$$

and we use (8.1) for $h = \boldsymbol{\eta}^3 \cdot \boldsymbol{\eta}^4 / N$. $\square$

We now introduce

$$\hat{q}_{N,M} = \frac{1}{N} \sum_{k \le M} \langle u'(s_k) \rangle^2. \tag{8.5}$$

LEMMA 8.3. *We have*

$$\operatorname{Var} q_{N,M} \le L(\operatorname{Var} q_{N-1,M} \operatorname{Var} \hat{q}_{N-1,M})^{1/2} + o(1). \tag{8.6}$$

PROOF. We first observe that

$$Eq_{N,M} = Eq_{N-1,M} + o(1). \tag{8.7}$$

To see this, we note that

$$Eq_{N,M} = E(\sigma_N^1 \sigma_N^2) = E\left\langle \frac{\boldsymbol{\eta}^1 \cdot \boldsymbol{\eta}^2}{N-1} \right\rangle$$

and we use an obvious adaptation of Proposition 8.1. Now,

$$\begin{aligned} &E\big((q_{N,M} - Eq_{N,M})(\operatorname{th}^2 \boldsymbol{\xi} \cdot \mathbf{C} - E \operatorname{th}^2 \boldsymbol{\xi} \cdot \mathbf{C})\big) \\ &\quad = E(q_{N-1,M} \operatorname{th}^2 \boldsymbol{\xi} \cdot \mathbf{C}) - Eq_{N-1,M} E \operatorname{th}^2 \boldsymbol{\xi} \cdot \mathbf{C} = \operatorname{Var} q_{N,M} + o(1) \end{aligned} \tag{8.8}$$

by (8.3), (8.4), (8.7).

Since ξ is independent of all the other r.v., considering the function

$$\varphi(x) = E_g \operatorname{th}^2 (g\sqrt{x}) \tag{8.9}$$

we see that the left-hand side of (8.8) is

$$E\Big((q_{N-1,M} - Eq_{N-1,M})\big(\varphi(\hat{q}_{N-1,M}) - E\varphi(\hat{q}_{N-1,M})\big)\Big)$$

and by Cauchy–Schwarz, this is at most

$$(\operatorname{Var} q_{N-1,M})^{1/2} (\operatorname{Var} \varphi(\hat{q}_{N-1,M}))^{1/2}.$$

Now integration by parts shows that φ' is bounded, so that $\operatorname{Var} \varphi(\hat{q}_{N-1,M}) \le L \operatorname{Var} \hat{q}_{N-1,M}$. (Note that $2 \operatorname{Var} X = E(X-Y)^2$ where Y is an independent copy of X.) The result then follows from (8.8). □

At this point the reader guesses that we will have to use the "induction upon M" version of what we did up to this point in this section. We start by a principle similar to Proposition 8.1.

PROPOSITION 8.4. *Consider an integer p, a bounded function $\overline{f}$ on Σ_{N-1}^p, that does not depend upon the variables ξ_i^M. Then, for each collection of smooth bounded functions $(v_l)_{l \le p}$ we have*

$$E\left\langle \overline{f} \prod_{l \le p} v_l(s_M^l) \right\rangle_1 = E\left(\langle \overline{f} \rangle \prod_{l \le p} E_h v_l\Big(\boldsymbol{\xi} \cdot \mathbf{b} + h\sqrt{1 - \|\mathbf{b}\|^2}\Big) \right) + o(1),$$

where $\mathbf{b} = (\langle \sigma_i \rangle_1 / \sqrt{N-1})_{i \le N-1}$ and where h is $N(0,1)$ independent of $\boldsymbol{\xi}$.

PROOF. We write

$$E\left\langle \overline{f} \prod_{l\le p} v(s_M^l)\right\rangle_1 = E\left\langle \overline{f} E_\xi \prod_{l\le p} v_l\Big(\boldsymbol{\xi}\cdot \boldsymbol{\eta}^l/\sqrt{N-1}\Big)\right\rangle_1.$$

Now

$$E_\xi \prod_{l\le p} v_l\Big(\boldsymbol{\xi}\cdot \boldsymbol{\eta}^l/\sqrt{N-1}\Big) = E_g \theta((g_l)_{l\le p}),$$

where $g_l = \boldsymbol{\xi}\cdot\boldsymbol{\eta}^l/\sqrt{N-1}$ and $\theta((x_l)_{l\le p}) = \prod_{l\le p} v_l(x_l)$. Consider a jointly Gaussian family $(g'_l)_{l\le p}$ given by $g'_l = \boldsymbol{\xi}\cdot\mathbf{b} + h_l\sqrt{1-\|\mathbf{b}\|^2}$ where $(h_l)_{l\le p}$ are i.i.d. $N(0,1)$. Thus one sees that

$$E_g\theta((g'_l)_{l\le p}) = E_\xi \prod_{l\le p}\Big(E_h v_l\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h\sqrt{1-\|\mathbf{b}\|^2}\Big)\Big).$$

To conclude, we note that by (3.1), and since $Eg_l'^2 = 1$, $Eg'_l g'_{l'} = \|\mathbf{b}\|^2$ if $l \ne l'$, we have

$$\big|E_g\theta((g_l)) - E_g\theta((g'_l))\big| \le K(f)\sum_{l<l'}\left|\frac{\boldsymbol{\eta}^l\cdot\boldsymbol{\eta}^{l'}}{N-1} - \|\mathbf{b}\|^2\right|$$

and we use Theorem 7.8. □

COROLLARY 8.5. *We have* $E|X-Y| \to 0$ *where*

$$X = \left\langle \overline{f}\prod_{l\le p} v_l(s_M^l)\right\rangle_1,$$

$$Y = \langle\overline{f}\rangle_1 \prod_{l\le p} E_h v_l\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h\sqrt{1-\|\mathbf{b}\|^2}\Big).$$

PROOF. Proposition 8.4 implies $EX = EY + o(1)$, and, using replicas, that $EX^2 = EY^2 + o(1)$, $EXY = EY^2 + o(1)$. □

COROLLARY 8.6. *We have, if* $p' \le p$,

$$E\left\langle \overline{f}\prod_{l\le p'} v(s_M^l)\right\rangle_0 = E(\langle\overline{f}\rangle_1 R^{p'}) + o(1)$$

for

$$R = \frac{E_h v\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h\sqrt{1-\|\mathbf{b}\|^2}\Big)\exp u\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h\sqrt{1-\|\mathbf{b}\|^2}\Big)}{E_h \exp u\Big(\boldsymbol{\xi}\cdot\mathbf{b} + h\sqrt{1-\|\mathbf{b}\|^2}\Big)}.$$

PROOF. We write

$$\left\langle \overline{f} \prod_{l \le p'} v(s_M^l) \right\rangle_0 = \frac{\left\langle \overline{f} \prod_{l \le p'} v(s_M^l) \exp\big(u(s_M^l)\big) \prod_{p' < l \le p} \exp\big(u(s_M^l)\big) \right\rangle_1}{\left\langle \prod_{l \le p} \exp\big(u(s_M^l)\big) \right\rangle_1}$$

and we apply Corollary 8.5 to both numerator and denominator. □

We recall the function $\Phi(x, y)$ given by (1.15). Since we assume u differentiable, we can integrate by parts the numerator and get

$$\Phi(x, y) = \frac{E_h u'(x + h\sqrt{1-y}) \exp u(x + h\sqrt{1-y})}{E_h \exp u(x + h\sqrt{1-y})}. \tag{8.10}$$

LEMMA 8.7. *We have*

$$E\hat{q}_{N-1, M} = \alpha E \Phi^2 (\boldsymbol{\xi} \cdot \mathbf{b}, \|\mathbf{b}\|) + o(1), \tag{8.11}$$

$$E\hat{q}^2_{N-1, M} = \alpha E \left(\hat{q}_{N-1, M} \Phi^2 (\boldsymbol{\xi} \cdot \mathbf{b}, \|\mathbf{b}\|)\right) + o(1). \tag{8.12}$$

PROOF. It is very similar to the proof of Lemma 8.2. We have

$$E\hat{q}_{N-1, M} = \frac{1}{N-1} E \sum_{k \le M} \langle u'(s_k) \rangle_0^2 = \alpha E \langle u'(s_M^1) u'(s_M^2) \rangle_0 + o(1)$$

and we use Corollary 8.5 with $p = 2$, $\overline{f} = 1$ to obtain (8.11). To obtain (8.12), we proceed similarly, starting with the relation

$$E\hat{q}^2_{N-1, M} = \alpha E \langle u'(s_M^1) u'(s_M^2) \mathbf{A}^3 \cdot \mathbf{A}^4 \rangle_0 + o(1). \qquad \square$$

We now consider a new $N(0, 1)$ variable z and the function

$$\psi(t) = E_z \Phi^2 (z\sqrt{t}, t). \tag{8.13}$$

LEMMA 8.7. *We have*

$$\operatorname{Var} \hat{q}_{N-1, M} \le \alpha \big(\operatorname{Var} \hat{q}_{N-1, M-1} \operatorname{Var} \psi(\|\mathbf{b}\|)\big)^{1/2} + o(1).$$

The proof is similar to the proof of Lemma 8.3.

LEMMA 8.8. *If $t \le 1/2$ we have $\psi'(t) \le L \exp 2D$.*

PROOF. This is straight computation. We start with

$$\begin{aligned} \psi'(t) &= E_z \left(\frac{z}{2\sqrt{t}} \partial_1 \Phi^2 (z\sqrt{t}, t) + \partial_2 \Phi^2 (z\sqrt{t}, t) \right) \\ &= E_z \left(\frac{1}{2} \partial_1^2 \Phi^2 (z\sqrt{t}, t) + \partial_2 \Phi^2 (z\sqrt{t}, t) \right), \end{aligned}$$

which we compute using (1.15) and integrating by parts. □

We know by Lemma 3.1 that (essentially) $\|\mathbf{b}\| \leq 1/2$ (for $L\alpha D \leq 1$). It then follows from Lemma 8.7 that

$$\operatorname{Var} \hat{q}_{N-1,M} \leq L\alpha \exp 2D(\operatorname{Var} \hat{q}_{N-1,M-1} \operatorname{Var} q_{N-1,M-1})^{1/2} + o(1). \tag{8.14}$$

THEOREM 8.9. *If u is five times differentiable and if $L\alpha \exp LD \leq 1$, then the following holds*:

$$\lim_{N\to\infty} \operatorname{Var} q_{N,M} = \lim_{N\to\infty} \operatorname{Var} \hat{q}_{N,M} = 0. \tag{8.15}$$

The system of equations (1.13), (1.14) *has a unique solution* $q, \hat{q}$ *and*

$$\lim_{N\to\infty} Eq_{N,M} = q, \qquad \lim_{N\to\infty} E\hat{q}_{N,M} = \hat{q}. \tag{8.16}$$

PROOF. It should be obvious that (8.15) follows from (8.6), (8.14).

The system of equations (1.13), (1.14) has a unique solution because the equation $q = \varphi(\alpha\psi(q))$ has a unique solution [where φ is given by (8.9)], since $x \to \varphi(\alpha\psi(x))$ is a contraction. Moreover (8.15) and (8.3) imply

$$Eq_{N,M} = \varphi(E\hat{q}_{N-1,M}) + o(1),$$

while (8.15) and (8.11) imply

$$E\hat{q}_{N-1,M} = \alpha\psi(Eq_{N-1,M-1}) + o(1)$$

so that

$$Eq_{N,M} = \varphi(\alpha\psi(Eq_{N-1,M-1})) + o(1)$$

from which the result follows by iteration. □

We now prove (1.17) under the conditions of Theorem 8.9. If we denote by $RS(\alpha, q, \hat{q})$ the right-hand side of (1.16), a simple computation (integration by parts) shows that (1.13) and (1.14), respectively, mean that

$$\frac{\partial RS}{\partial q}(\alpha, q, \hat{q}) = 0, \qquad \frac{\partial RS}{\partial \hat{q}}(\alpha, q, \hat{q}) = 0$$

so that, even through $q, \hat{q}$ depend upon α, we have

$$\frac{\partial RS}{\partial \alpha}(\alpha, q, \hat{q}) = E \log E_h \exp u(z\sqrt{q} + h\sqrt{1-q}).$$

Thus to prove (1.16), it suffices to prove that

$$\begin{aligned} &\lim_{N\to\infty} E \log Z_{N-1,M} - E \log Z_{N-1,M-1} \\ &\qquad = E \log E_h \exp u(z\sqrt{q} + h\sqrt{1-q}). \end{aligned} \tag{8.17}$$

(The index $N-1$ rather than N is for consistency with our previous choices.) The right-hand side of (8.17) is

$$E \log \langle \exp u(s_M) \rangle_1$$

and (8.17) follows from Corollary 8.5 (for $p = 1$, $\overline{f} = 1$, $v_1 = \exp u$) and (8.15), (8.16). □

We can look to (1.16) as saying that

$$E\log Z_{N,M} \simeq N\Big(\tfrac{1}{2}\hat{q}(1-q) + E\log\big(2\operatorname{ch} z\sqrt{\hat{q}}\big)\Big) + ME\log E_h \exp u\big(z\sqrt{q} + h\sqrt{1-q}\big).$$

Thereby one should expect that, as $N \to \infty$,

$$E\log Z_{N,M} - E\log Z_{N-1,M} \to \tfrac{1}{2}\hat{q}(1-q) + E\log\big(2\operatorname{ch} z\sqrt{\hat{q}}\big). \tag{8.18}$$

The interesting fact is that this is true (of course) but far from being obvious. To me, there is some algebraic mystery in formula (1.16), and if any reader understands it, I would be grateful to have it explained to me. The left-hand side of (1.18) is

$$E\log\big(2\big\langle \operatorname{Av} \exp(\sigma_N \boldsymbol{\xi}\cdot\mathbf{A} + B)\big\rangle_0\big). \tag{8.19}$$

The reader should have no problem showing that, following the idea of Corollary 8.5,

$$\big\langle \operatorname{Av} \exp(\sigma_N \boldsymbol{\xi}\cdot\mathbf{A} + B)\big\rangle_0 \simeq \operatorname{ch} \boldsymbol{\xi}\cdot\mathbf{C} \exp\Big(\tfrac{1}{2}\langle\|\mathbf{A}\|^2\rangle_0 + \langle B\rangle_0 - \tfrac{1}{2}\|\mathbf{C}\|^2\Big)$$

and this yields that (8.19) is about

$$E\log(2\operatorname{ch} z\sqrt{q}) + \frac{\alpha}{2}\bigg(\frac{E_h(u'^2(x)e^{u(x)})}{D} - \frac{\big(E_h(u'(x)e^{u(x)})\big)^2}{D^2} + \frac{E_h((u''(x) - xu'(x))e^{u(x)})}{D^2}\bigg), \tag{8.20}$$

where $x = z\sqrt{q} + h\sqrt{1-q}$, $D = E_h e^{u(x)}$. It can be checked through integration by parts that the coefficient of α is indeed equal to $\hat{q}(1-q)$, but this quite miraculous computation is not really satisfying.

It now remains to prove Theorem 1.1; that is, we have to show that (1.16) holds even when u is not five times differentiable, but has only finitely many discontinuities. The natural regularity hypothesis of u is in fact to assume that u is Riemann-measurable; that is, that the set of points of discontinuity of u has measure zero. In this case, it is elementary to see that we can find two sequences $(u_n), (v_n)$ of $\mathscr{C}^\infty$ functions

$$v_1 \le \cdots v_n \le v_{n+1} \le \cdots \le u \le \cdots \le u_{n+1} \le u_n \le \cdots \le u_1,$$

such that $|v_1| \le 2D$, $|u_1| \le 2D$ and $v_n(x), u_n(x) \to u(x)$ for x outside a set of measure zero. With obvious notation, we have

$$Z_{N,M}(v_n) \le Z_{N,M}(u) \le Z_{N,M}(u_n),$$

so that all have to prove is that $RS(v_n, \alpha) - RS(u_n, \alpha) \to 0$. The function $\Phi_u(x, y)$ was only used in the proof for values of $y \le 1/2$. In this domain, integration by parts shows that the partial derivatives of all orders of $\Phi_{u_n}(x, y) - \Phi_{v_n}(x, y)$ converge uniformly to zero, so that the values of $q_{N,M}, \hat{q}_{N,M}$ corresponding to u_n, v_n, respectively, have a common limit. The conclusion is then obvious.

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