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# CRITICAL EXPONENTS FOR A REVERSIBLE NEAREST PARTICLE SYSTEM ON THE BINARY TREE[1]

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The uniform model is a reversible interacting particle system that evolves on the homogeneous tree. Occupied sites become vacant at rate one provided the number of occupied neighbors does not exceed one. Vacant sites become occupied at rate $\beta$ times the number of occupied neighbors. On the binary tree, it has been shown that the survival threshold $\beta_c$ is 1/4. In particular, for $\beta \le 1/4$, the expected extinction time is finite. Otherwise, the uniform model survives locally. We show that the survival probability decays faster than a quadratic near $\beta_c$. This contrasts with the behavior of the survival probability for the contact process on homogeneous trees, which decays linearly. We also provide a lower bound that implies that the rate of decay is slower than a cubic. Tools associated with reversibility, for example, the Dirichlet principle and Thompson's principle, are used to prove this result.

**1. Introduction.** In this paper an interacting particle system called the *uniform model* is studied. The process evolves on the homogeneous tree $\mathbb{T}^d$ in which each vertex has degree $d+1$. Each site (or vertex) of the tree is said to be either occupied or vacant, and the process evolves in the following manner. A vacant site becomes occupied at rate $\beta$ times the number of occupied sites within distance one. One can regard this transition mechanism as occupied sites giving birth onto each vacant neighboring site at rate $\beta$. Thus, $\beta$ is called the *birth parameter*. Occupied sites become vacant at rate one if there are at least $d$ neighboring sites that are vacant. So having two or more occupied neighboring sites insulates a given particle from death. This has the effect that connected components of occupied sites remain connected until absorption into the empty set, or until coalescing with another connected component.

Formally, the state space for this Markov process is $X = \{0,1\}^{\mathbb{T}^d}$. An element $\eta \in X$ is called a configuration and can be viewed as a function $\eta$: $\mathbb{T}^d \to \{0,1\}$ with $\eta(x) = 1$ having the interpretation that the site $x$ is occupied. Other times $\eta \in X$ is thought of as a subset of $\mathbb{T}^d$, the set of occupied

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sites. The rate $c(x, \eta)$ at which $\eta(x)$ flips to $1 - \eta(x)$ is given by

$$c(x, \eta) = \begin{cases} \beta \sum_{\{y:\ \|x-y\|=1\}} \eta(y), & \text{if } \eta(x) = 0, \\ 1, & \text{if } \eta(x) = 1 \text{ and } \sum_{\{y:\ \|x-y\|=1\}} \eta(y) \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Here, $\|x - y\|$ is length of the shortest path connecting the sites $x$ and $y$. Since $c(x, \cdot)$ depends only on the values of $\eta$ within distance one of $x$, these rate functions determine the generator of a strong Markov process. For a complete construction of the process, the reader is referred to [6].

The uniform model was first introduced in [12]. There it was proved that the system undergoes a phase transition as the parameter $\beta$ increases. In fact, several potential phase transitions were considered in that paper. Let us focus on the most obvious transition to look for. Since particles must be present in order to create new particles, the empty set ($\eta \equiv 0$, also denoted by $\eta = \varnothing$) is an absorbing state. Therefore, it is natural to ask whether or not an initial state with a finite number of particles hits the empty set in finite time. Let

$$s(\beta) = P^O(\eta_t \neq \varnothing \,\forall\, t)$$

be the probability that the uniform model starting from a single particle avoids absorption into the empty set. Here, $\eta_t$ is the state of the process at time $t$, $O$ is a distinguished vertex called the root, and $P^O$ is the probability measure for the evolution of the process when $\eta_0 = O$ almost surely. If $s(\beta) > 0$ is positive, we say that the process survives. In turn $s(\beta)$ is called the *survival probability*. It is clear that $s(\beta)$ is increasing in $\beta$ and therefore it is natural to define

$$\beta_c = \inf\{\beta\colon s(\beta) > 0\}.$$

In [12], it is shown that $\beta_c$ is not trivial (it is neither 0 nor $\infty$) and moreover that

$$\frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1} \le \beta_c(d) \le \frac{d}{2(d-1)^2}. \tag{1.2}$$

It is also shown that the lower bound in (1.2) is the correct value for $\beta_c$ on the binary tree: $\beta_c(2) = 1/4$. The lower bound in (1.2) is obtained by considering the critical value

$$\beta_f = \inf\{\beta\colon \mathbb{E}^O \tau = \infty\},$$

where $\tau = \inf\{t\colon \eta_t = \varnothing\}$ and $\mathbb{E}^O$ is the expected value with respect to $P^O$. It is immediate that $\beta_f \le \beta_c$. By comparing the uniform model with related Markov chains, $\beta_f$ was computed explicitly and was shown to be equal to the lower bound in (1.2): $\beta_f(d) = (1/d)((d-1)/d)^{d-1}$. Thus the result $\beta_c(2) = 1/4$ is the same as to say that $\beta_f(2) = \beta_c(2)$. In [12], it is conjectured that $\beta_f = \beta_c$ for $d \ge 3$.

The focus of this paper is the behavior of the survival probability, the expected extinction time and the susceptibility (also called the total space–time occupation measure) as functions as $\beta$. Of particular interest will be the behavior of these functions near the critical value(s). The core of the paper is concerned with bounding the survival probability from above and below. Such bounds are used to describe the rate at which the survival probability tends to zero as $\beta$ decreases to $\beta_c$ on the binary tree.

THEOREM 1.1. *On the binary tree,*

$$\limsup_{\beta \searrow \beta_c} \frac{P^O(\eta_t \neq \varnothing \,\forall\, t)}{(\beta - \beta_c)^{5/2}} < \infty \tag{1.3}$$

*and*

$$\liminf_{\beta \searrow \beta_c} \frac{P^O(\eta_t \neq \varnothing \,\forall\, t)}{(\beta - \beta_c)^{1+\sqrt{13}/2}} > 0. \tag{1.4}$$

To contrast this result with what has been proved for related interacting particle systems, let us consider the *contact process*. Here the birth mechanism is the same as that of the uniform model, while particles die at rate one independent of the location and number of other particles in the tree. So the uniform model stochastically dominates the contact process. If

$$\lim_{\beta \searrow \beta_c} \frac{\log P^O(\eta_t \neq \varnothing \,\forall\, t)}{\log(\beta - \beta_c)^b} = 1,$$

then the survival probability decays like a power law with exponent $b$. This exponent $b$ is said to be the *critical exponent* of the survival probability. For the contact process on the homogeneous tree, Barsky and Wu [1] showed that if a condition called the triangle condition holds, then the exponent $b$ exists and takes its mean field value, which is one. Wu [17] verified that this condition holds for $d \geq 5$. Later Schonmann [14] completed the story by verifying that this condition holds for $d \geq 2$. Theorem 1.1 implies that the critical exponent $b$ for the uniform model is not one, and moreover that $b$ lies in the interval $[5/2, 1 + \sqrt{13}/2]$ (if it exists).

The techniques used to obtain Theorem 1.1 rely on a property of the uniform model that it does not share with the contact process, reversibility. An interacting particle system in which the initial configuration has a finite number of particles is said to be *reversible* if there exists a measure $\pi$ supported on the states with finitely many occupied individuals such that

$$\pi(A)c(x, A) = \pi(A \cup x)c(x, A \cup x) \tag{1.5}$$

for all configurations $A$ (except possibly a single absorbing configuration) with finitely many occupied sites and all $x \notin A$. The equations in (1.5) are known as the detailed balance equations. If there is no exceptional state, then the detailed balance equations are equivalent to self-adjointness of the semigroup operator with respect to the measure $\pi$. It is easily verified that the measure

$\pi(A) = \beta^{|A|}$, where $|A|$ is the number of occupied sites in configuration $A$, satisfies (1.5) on the state space of finite, connected, nonempty subsets of $\mathbb{T}^d$ in case $c(x, A)$ is defined by (1.1). Reversibility admits tools in the form of the Dirichlet principle and Thompson's principle that apply in this setting. As we will see in Section 3, these tools can be used to generate upper and lower bounds on the survival probability. Section 2 contains the statements of and a brief discussion about these principles.

The Dirichlet principle has been used to estimate the survival probability for a class of particles systems known as *reversible nearest particle systems*. These systems are defined only on the one-dimensional integer lattice $\mathbb{Z}$. Nearest particle systems are generalizations of the contact process due to Spitzer [15] in which the rate at which vacant sites become occupied depends on the distances to the nearest occupied sites to the left and to the right of the vacant site. In the reversible setting, the rate at which a vacant site becomes occupied takes a particularly nice form,

$$c(x, \eta) = \lambda \frac{\beta(l_x(\eta))\beta(r_x(\eta))}{\beta(l_x(\eta) + r_x(\eta))},$$

where $\lambda$ is a nonnegative parameter, $\beta(\cdot)$ is a strictly positive probability density function on $\{1, 2, \ldots\}$ and $l_x(\eta)$ [resp. $r_x(\eta)$] is the distance to the nearest occupied site to the left (resp. right) of $x$ in configuration $\eta$. Using the Dirichlet principle, Griffeath and Liggett [5] showed that

$$\frac{\lambda - 1}{\lambda} \leq P^0(\eta_t \neq \varnothing \, \forall \, t) \leq \left| \lambda \log\left( \frac{\lambda - 1}{\lambda} \right) \right|^{-1} \quad \text{for } \lambda > 1.$$

This has the consequence that $s(\lambda) > 0$ if and only if $\lambda > 1$, that is, $\lambda_c = 1$, and provides estimates on the rate at which $s(\lambda)$ decreases to zero. Under the additional assumption that $\beta(\cdot)$ has a finite second moment, it was shown in [8] that

$$P^0(\eta_t \neq \varnothing \, \forall \, t) \leq c(\lambda - 1) \quad \text{for } \lambda > 1,$$

where $c$ is a constant depending on $\beta(\cdot)$, establishing that the critical exponent is one.

The main connection between the uniform model and reversible nearest particle systems on $\mathbb{Z}$ lies in the reversibility property. An important distinction is that the uniform model is finite range; that is to say, the rate function $c(x, \eta)$ depends only on values of $\eta$ at sites within distance one of $x$. For reversible nearest particle systems this is not the case. Moreover, in a classical reversible nearest particle system, every particle dies at rate one independent of the spin values in the neighborhood, which is not true of the uniform model. It is not obvious how to generalize reversible nearest particle systems to graphs besides $\mathbb{Z}$ because it is not so clear what the rates should depend on. For example, what is the appropriate analog of the nearest particle to the left and to right on $\mathbb{Z}^2$? This was one of the original motivations for introducing the uniform model on $\mathbb{T}^d$. Others studies of reversible models on graphs besides $\mathbb{Z}$ include [2], [3], [7] and [9], all of which exploit the Dirichlet principle as a means for obtaining estimates on the critical value.

One consequence of Theorem 1.1 is that on the binary tree $s(\beta)$ is continuous at $\beta_c$ that is, $s(\beta_c) = 0$. This can also be established without obtaining actual estimates on $s(\beta)$. In fact, by adapting technology developed for the analysis of the contact process on the homogeneous tree to the setting of the uniform model, it can be shown that $s(\beta_c) = 0$ for $d \geq 2$. The idea is to consider the sequence

$$u(n) = P^O(O_n \in \eta_t \text{ some } t),$$

where $O_n$ is a fixed site that satisfies $\|O_n - O\| = n$. Then $\log u(n)$ is subadditive, and consequently one can define the *growth parameter*

$$g(\beta) = \lim_{n\to\infty} u(n)^{1/n}.$$

An important theorem in [13] (stated for the contact process but also valid for the uniform model by essentially the same proof) says that if $g(\beta) > 1/\sqrt{d}$, then the probability that the origin is occupied at time $t$ is uniformly bounded below. This combined with left continuity of $g(\beta)$ implies that $g(\beta_c) \leq 1/\sqrt{d}$. But $g(\beta) = 1$ turns out to be equivalent to $s(\beta) > 0$ for the uniform model. Indeed, as was shown in [12], $s(\beta) = P^O(O \in \eta_t \text{ for unbounded } t)$, and thus $s(\beta_c) = 0$ for $d \geq 2$. See [10] for a full account of these ideas in the setting of the contact process.

With regard to the subcritical approach to critical, there are quantities that typically diverge as $\beta$ increases to $\beta_f$. For example, the expected extinction time is often infinite at the critical point. In the case of the uniform model, such quantities do not diverge because, as was proved in [12], a related Markov chain called the shape chain exhibits positive recurrent behavior at $\beta_f$. Instead, they approach some constant. Due to reversibility, more information than simply the rate at which these quantities approach some constant can be provided. In fact, explicit formulas for the expected extinction time and the susceptibility are obtainable.

THEOREM 1.2. *For $\beta \leq \beta_f$,*

$$\text{(a)} \quad \mathbb{E}^O(\tau) = \frac{1}{\beta}\int_0^\beta \frac{C^2(x) - C(x)}{x}\,dx$$

*and*

$$\text{(b)} \quad \mathbb{E} = \int_0^\infty |\eta_t^O|\,dt = \frac{C^2(\beta) - C(\beta)}{\beta},$$

*where $C(\beta) = \sum_{n=0}^\infty c(n)\beta^n$ with $c(n) = \binom{dn}{n}/((d-1)n+1)$.*

The proof of Theorem 1.2 can be found in [11]. It is based on coupling the uniform model and the shape chain. The coupling is described in Section 3, and the analysis proceeds by considering appropriate quantities for the shape chain.

The approach taken to prove Theorem 1.1 is a two-pronged attack taking advantage of both the Dirichlet principle and Thompson's principle. The principles themselves are stated in Section 2, and the scheme for how these tools are used is outlined in Section 3. The bounds that lead to proofs of statements (1.3) and (1.4) are derived in Sections 4 and 5, respectively. In Section 4, a statement analogous to (1.3) is proved for all $d \geq 2$, except that $\beta$ decreases to $\beta_{\mathrm{f}}$, not $\beta_{\mathrm{c}}$. Statement (1.3) follows from these bounds and the fact that $\beta_{\mathrm{c}} = \beta_{\mathrm{f}}$ on the binary tree. As noted earlier, it is believed, but not proved, that $\beta_{\mathrm{c}} = \beta_{\mathrm{f}}$ for $d \geq 3$. A proof of this fact would immediately extend the result in (1.3) to all homogeneous trees. Section 5 takes advantage of the work done in [12] to show that $\beta_{\mathrm{c}} = \beta_{\mathrm{f}}$ when $d = 2$. This work is constructive in nature and new ideas are needed in order to extend the construction to $d \geq 3$. In particular, it would be sufficient to prove that the hypothesis of Lemma 15 in [12] holds for $d \geq 3$. A verification of this would also immediately lead to an extension of (1.4) to $d \geq 3$ expect that the exponent would be given by 7/2, with the possibility of improvement.

**2. The Dirichlet and Thompson's principles.** The Dirichlet principle and Thompson's principle provide powerful tools for describing the behavior of the survival probability. These principles apply in the setting of a reversible Markov chain. The Dirichlet principle states that the probability that the Markov chain escapes from some fixed subset of the state space before returning to the initial state is expressible as an infimum of a certain variational functional over all functions in some class. Likewise, Thompson's principle expresses this same probability as a supremum of an energy functional over all functions in some class. Furthermore, there is a unique function that optimizes each of these functionals. The precise statements of these principles are as follows.

Let $X_t$ be an irreducible, reversible Markov chain with state space $S$, stationary measure $\pi$, and transition rates $q(y, z)$. For any subset $R$ of the state space $S$, let

$$\tau_R = \inf\{t\colon X_t \in R\} \quad \text{and} \quad \tau_R^+ = \inf\{t > \tau_{R^c}\colon X_t \in R\}.$$

Given a function $h\colon S \to [0, 1]$, let $\Phi(h)$ be the *Dirichlet form* evaluated at $h$,

$$\Phi(h) = \tfrac{1}{2} \sum_{y,z} \pi(y) q(y, z)(h(z) - h(y))^2.$$

Given a subset $R$ of the state space $S$ and $x \in S \setminus R$, let

$$\mathscr{H}_x^R = \{h\colon S \to [0, 1]\colon h(x) = 0, h(y) = 1 \text{ for all } y \in R\}.$$

THEOREM 2.1 (The Dirichlet principle). *Provided* $P^x(\tau_R < \infty) = 1$,

$$\pi(x) q(x) P^x(\tau_R \leq \tau_x^+) = \inf_{h \in \mathscr{H}_x^R} \Phi(h),$$

*where* $q(x) = \sum_{y \neq x} q(x, y)$. *Furthermore,*

$$h(y) = P^y(\tau_R \leq \tau_x) \tag{2.1}$$

*attains the infimum.*

The optimal function (2.1) satisfies a certain averaging property known as harmonicity. Recall that a function is said to be harmonic on some subset of the state space $U$ if

$$h(y) = \sum_{x \neq y} \frac{q(y, z)}{q(y)} h(z) \quad \text{for all } y \in U.$$

In particular, $h(y)$ is an average over values in the neighborhood. For an irreducible subset $U$ of the state space, this averaging property implies that if the function $h$ attains its maximum or minimum value in $U$, then $h$ is constant on $U$. Consequently, specifying the values of $h$ on $U$ complement and requiring harmonicity on $U$ determines $h$, provided the Markov chain hits $U$ complement with probability 1. It turns out that $\Phi(h)$ is minimal on $\mathscr{H}_x^R$ if and only if $h$ is harmonic on $S \setminus (R \cup x)$. The function defined by (2.1) is in fact the unique harmonic function with the stated boundary conditions. A proof of the Dirichlet principle can be found in [6].

The Dirichlet principle can be stated in a dual form known as Thompson's principle. Given an antisymmetric function $w$: $S \times S \to \mathbb{R}$, let $\mathscr{K}(w)$ denote the *kinetic energy* of $w$:

$$\mathscr{K}(w) = \tfrac{1}{2} \sum_{y, z} \frac{w^2(y, z)}{\pi(y) q(y, z)}.$$

Given a subset $R$ of the state space $S$ and $x \in S \setminus R$, let

$$\begin{aligned} \mathscr{W}_x^R = \{w\colon S \times S \to \mathbb{R}\colon & w(y, z) = -w(z, y),\ \ y, z \in S; \\ & \sum_z w(x, z) = 1; \text{ and } \sum_z w(y, z) = 0,\ \ y \notin R \cup x\}. \end{aligned}$$

Such a function $w$ is said to be a *unit flow* from $x$ to $R$.

THEOREM 2.2 (Thompson's principle). *Provided* $P^x(\tau_R < \infty) = 1$,

$$\sup_{w \in \mathscr{W}_x^R} \frac{1}{\mathscr{K}(w)} = \pi(x) q(x) P^x(\tau_R \leq \tau_x^+),$$

*where* $q(x) = \sum_{y \neq x} q(x, y)$. *Furthermore, the unit flow given by*

$$\begin{aligned} w(y, z) = {} & \mathbb{E}^x \,(\textit{number of one-step transitions from } y \textit{ to } z \textit{ before time } \tau_R) \\ & -\mathbb{E}^x \,(\textit{number of one-step transitions from } z \textit{ to } y \textit{ before time } \tau_R). \end{aligned}$$

*attains the supremum.*

Not surprisingly, the optimal unit flow is related to the harmonic function that appears in the Dirichlet principle. To see this, define a path from $y \in S$ to $z \in S$ to be a sequence $\{y_i\}_{i=0}^m$ of states in the Markov chain such that $y_0 = y$, $q(y_i, y_{i+1}) > 0$, and $y_m = z$. A path from $y$ to $R$ is defined similarly except that $\{y_i\}_{i=0}^m \cap R = y_m$. It turns out that the optimal flow satisfies.

$$\sum_{i=0}^{n-1} \frac{w(y_i, y_{i+1})}{\pi(y_i)q(y_i, y_{i+1})} = \sum_{i=0}^{m-1} \frac{w(z_i, z_{i+1})}{\pi(z_i)q(z_i, z_{i+1})},$$

for all pairs of paths to $R$ such that $y_0 = z_0$. Therefore, one can define a function $h$: $S \to \mathbb{R}_+$ as follows: Choose an arbitrary path from $x$ to $y \in S \setminus (R \cup x)$, or an arbitrary path from $x$ to $R$ in case $y \in R$, and set

$$h(y) = \sum_{i=0}^{k-1} \frac{w(y_i, y_{i+1})}{\pi(y_i)q(y_i, y_{i+1})}.$$

The fact that $\sum_z w(y, z) = 0$ for all $y \in S \setminus (R \cup x)$ implies that $h$ is harmonic. After normalizing $h$ so that it takes the value one on $R$, we see that the optimal flow is related to the harmonic function from the Dirichlet principle by the equation

$$w(y, z) = \frac{\pi(y)q(y, z)(h(z) - h(y))}{\pi(x)q(x)P^x(\tau_R \le \tau_x^+)}.$$

For more background on the Dirichlet and Thompson's principle, the reader is referred to [4].

**3. The shape chain.** In order to apply the Dirichlet principle and Thompson's principle to the uniform model, a related reversible Markov chain called the *shape chain* is introduced. As motivation for the definition of this Markov chain, observe that the issue of whether or not the uniform model avoids absorption into the empty set is independent of the location of the occupied set. Furthermore, the evolution of the uniform model depends only on the "shape" of the occupied set. So, it seems reasonable to identify isomorphic occupied sets and record the shape rather than the location of the occupied set. This allows a transition from the empty set to the singleton to be introduced while preserving reversibility.

More formally, the shape chain is defined as follows. An automorphism of a graph $G = (V, E)$ is a bijection $\phi$: $V \to V$ of the vertices of the graph such that there is an edge $e_1 \in E$ between the vertices $v$ and $w$ if and only if there is an edge $e_2 \in E$ between the vertices $\phi(v)$ and $\phi(w)$. Let $\mathrm{Aut}(\mathbb{T}^d)$ be the set of all automorphisms of $\mathbb{T}^d$. Configurations $A$ and $B$ are said to be equivalent if there exists $\phi \in \mathrm{Aut}(\mathbb{T}^d)$ such that $\phi(B) = A$. We write $A \sim B$ to indicate that $A$ and $B$ are equivalent. The relation $\sim$ defines an equivalence relation on the set of all configurations. Let $\hat{A} = \{B\colon B \sim A\}$ and

$$\hat{\mathscr{S}} = \hat{\varnothing} \cup \{\hat{A}\colon A \text{ is a finite connected subset of } \mathbb{T}^d\}.$$

Roughly speaking, $\hat{\mathscr{S}}$ denotes the set of all finite connected shapes that can be embedded into $\mathbb{T}^d$. It will be convenient to consider the Markov chain $\hat{A}_t$ induced on $\hat{\mathscr{S}}$ by the dynamics of the uniform model. In particular, for $\hat{A} \neq \hat{\varnothing}$,

$$\hat{q}(\hat{A}, \hat{B}) = \sum_{\{x:\ A, \triangle x \sim B\}} c(x, A),$$

where $A \in \hat{A}$ is fixed and $A \triangle x$ is the symmetric difference of the sets $A$ and $x$. It is given by $A \cup x$ if $x \notin A$ and $A \setminus x$ if $x \in A$. Since $\hat{q}(\hat{A}, \cdot)$ depends on $A$ only through its equivalence class, the transition rates are well defined. We refer to $\hat{A}_t$ as the shape chain. In order to make the shape chain irreducible, a transition from $\hat{\varnothing}$ to the singleton $\hat{O}$ is introduced at rate $\beta$.

The shape chain is reversible with respect to the measure

$$\hat{\pi}(\hat{A}) = \frac{M(\hat{A})\beta^{|\hat{A}|}}{|\hat{A}|},$$

where $M(\hat{A}) = |\{A \in \hat{A}\colon O \in A\}|$ and $|\hat{A}|$ is the number of vertices in $A \in \hat{A}$, that is, $|\hat{A}| = |A|$. In order to prove this, it suffices to show that the detailed balance equations hold,

$$\hat{\pi}(\hat{A})\hat{q}(\hat{A}, \hat{B}) = \hat{\pi}(\hat{B})\hat{q}(\hat{B}, \hat{A}) \tag{3.1}$$

for all $\hat{A}, \hat{B} \in \hat{\mathscr{S}}$. Without loss of generality, $|\hat{B}| \geq |\hat{A}|$. If either the left- or right-hand side of (3.1) is nonzero, then there exist $A \in \hat{A}$ and $B \in \hat{B}$ such that $A \cup x = B$ for some $x \in \mathbb{T}^d$. Thus, proving that (3.1) holds is equivalent to proving that

$$\frac{M(\hat{A})|\{D \sim B\colon D \supset A\}|}{|\hat{A}|} = \frac{M(\hat{B})|\{C \sim A\colon C \subset B\}|}{|\hat{B}|} \tag{3.2}$$

for all finite, connected subsets $A$ and $B$ containing $O$ such that $A \cup x = B$. Liggett has proved that (3.2) holds for all finite (not necessarily connected) subsets of $\mathbb{T}^d$: see (3.8) in [7].

By definition, the shape chain starting from the singleton $\hat{A}_t^{\hat{O}}$ and the uniform model starting from the origin $\eta_t^O$ can be coupled such that $\eta_t^O \in \hat{A}_t^{\hat{O}}$ for all times $t \leq \tau_{\hat{\varnothing}}$. Thus, the problem of determining the asymptotic behavior of $P(\eta_t^O \neq \varnothing\, \forall\, t)$ as $\beta$ decreases to $\beta_c$ is equivalent to determining the asymptotic behavior of $P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty)$. Also, note that $P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty)$ can be expressed as a limit of escape probabilities. To see this, fix a sequence $\{\hat{\mathscr{S}}_N\}$ of subsets of $\hat{\mathscr{S}}$ that has the properties that $\hat{\mathscr{S}} \setminus \hat{\mathscr{S}}_N$ increases to $\hat{\mathscr{S}}$, $\hat{\varnothing} \notin \hat{\mathscr{S}}_N$, and $P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} < \infty) = 1$ for each $N \in \mathbb{N}$. Since $P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \leq \tau_{\hat{\varnothing}}^+) = P^{\hat{O}}(\tau_{\hat{\mathscr{S}}_N} \leq \tau_{\hat{\varnothing}})$,

$$P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty) = \lim_{N\to\infty} P^{\hat{O}}(\tau_{\hat{\mathscr{S}}_N} \leq \tau_{\hat{\varnothing}}) = \lim_{N\to\infty} P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \leq \tau_{\hat{\varnothing}}^+).$$

Therefore,

$$P^O(\eta_t \neq \varnothing\, \forall\, t) = \lim_{N\to\infty} P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \leq \tau_{\hat{\varnothing}}^+).$$

Since the probabilities $P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \le \tau^+_{\hat{\varnothing}})$ can be expressed in terms of the Dirichlet principle and Thompson's principle, this framework provides a strategy for estimating the survival probability.

**4. Upper bounds on the survival probability.** This section is devoted to obtaining upper bounds on the survival probability via the Dirichlet principle.

THEOREM 4.1. *For the shape chain with* $\beta > \beta_{\mathrm{f}}$,

$$P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty) \le C\beta\left(\frac{\beta - \beta_{\mathrm{f}}}{\beta}\right)^{5/2} \tag{4.1}$$

*for some constant* $0 < C < \infty$.

Before proving Theorem 4.1, we pause to give a brief outline of the proof and to explain the origins of the functions that are used in the proof. We begin by fixing a sequence $\{\hat{\mathscr{S}}_N\}$ of subsets of $\hat{\mathscr{S}}$ that have the properties that $\hat{\mathscr{S}} \setminus \hat{\mathscr{S}}_N$ increases to $\hat{\mathscr{S}}$, $\hat{\varnothing} \notin \hat{\mathscr{S}}_N$, and $P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} < \infty) = 1$ for each $N \in \mathbb{N}$. Then, a function $h_N \in \mathscr{H}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}$ is selected for each $N \in \mathbb{N}$. By the Dirichlet principle, $\beta P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \le \tau^+_{\hat{\varnothing}}) \le \Phi(h_N)$. Therefore,

$$\beta P^{\hat{\varnothing}}(\tau_{\hat{\varnothing}} = \infty) \le \liminf_{N\to\infty} \Phi(h_N).$$

The idea is to choose $h_N$ so that the lim inf is as small as possible. Here $h_N$ is chosen to be the minimizer of the Dirichlet form over all functions in $\mathscr{H}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}$ that depend only on cardinality. In spite of the fact that the functions $h_N$ take almost none of the structure of the sets into account, this choice of $h_N$ provides a lower bound on $\beta_{\mathrm{c}}$ that turns out to be equal to $\beta_{\mathrm{c}}$ on the binary tree. Even more remarkably, these nondiscriminating functions seem to provide the correct order of magnitude for the rate of decay of the survival probability. This is suggested by simulations of Tretyakov and Konno [16] that predict the decay rate to be $2.3 \pm 0.2$ on the binary tree. Note that (4.1) implies that the rate of decay is at least 2.5. Thus, we conjecture that critical exponent is given by 2.5.

The next proposition is used in the proof of Theorem 4.1. It is an immediate consequence of Stirling's formula, which says that $n! \sim n^n e^{-n}\sqrt{2\pi n}$ where $\sim$ means that the ratio tends to 1, and the fact that $\beta_{\mathrm{f}} = (1/d)((d-1)/d))^{d-1}$.

PROPOSITION 4.2. *For each* $d \ge 2$, *there exist constants* $0 < K_1$, $K_2 < \infty$ *such that*

$$\frac{K_2}{\sqrt{j}\beta_t^j} \le \binom{dj}{j} \le \frac{K_1}{\sqrt{j}\beta_t^j}$$

*for each* $j \ge 1$.

When it is necessary to emphasize which $d$ is being considered, we write $K_1(d)$ [resp. $K_2(d)$] for $K_1$ (resp. $K_2$).

Another fact that will be needed in the proof is that the rate at which the shape chain increases in cardinality is simply a function of the cardinality, not the "shape." To see this let $\mathscr{N}(\hat{A}) = \{\hat{B}\colon \hat{q}(\hat{A}, \hat{B}) > 0, |\hat{A}| < |\hat{B}|\}$. Using connectedness, it is easy to inductively show that

$$\sum_{\hat{B} \in \mathscr{N}(\hat{A}) \in \mathscr{N}_2(|)} \{B \in \hat{B}\colon A \subset B\}| = (d-1)|\hat{A}| + 2. \tag{4.2}$$

Proof of Theorem 4.1. Let $\hat{\mathscr{S}}_N = \{\hat{A} \in \hat{\mathscr{S}}\colon |\hat{A}| \geq N\}$. Also, let $g\colon \{1, 2, \dots\} \to \mathbb{R}$ be given by

$$g(k+1) = \begin{cases} 1, & \text{if } k = 0, \\ g(k) + \dfrac{k}{((d-1)k+2)t(k)\beta^k}, & \text{otherwise,} \end{cases}$$

where $t(k)$ is the number of connected subsets of $\mathbb{T}^d$ of size $k$ containing $O$. By convention, $t(0) = 1$. Also, define

$$h_N(\hat{A}) = \begin{cases} 0, & \text{if } \hat{A} = \hat{\varnothing}, \\ \dfrac{g(|\hat{A}|)}{g(N)}, & \text{if } 1 \leq |\hat{A}| \leq N, \\ 1, & \text{otherwise.} \end{cases}$$

Note that $h_N \in \mathscr{H}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}$.

To obtain some insight into the definitions of $g$ and $h_N$, recall that our strategy was to choose $h_N$ to be the minimizer of $\Phi(\cdot)$ over functions in $\mathscr{H}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}$ that depend only on cardinality. If $h$ is such a function, then $h(A) = f(|A|)$ for some function $f\colon \{0, 1, \dots\} \to \mathbb{R}$ with $f(0) = 0$ and $f(n) = 1$ for $n \geq N$. Substituting such an $h$ into $\Phi(\cdot)$ gives

$$\begin{aligned} \Phi(h) &= \frac{1}{2} \sum_{\hat{A}, \hat{B}} \hat{\pi}(\hat{A}) \hat{q}(\hat{A}, \hat{B}) \big(h(\hat{B}) - h(\hat{A})\big)^2 \\ &= \beta \big(h(\hat{O}) - h(\hat{\varnothing})\big)^2 \\ &\quad + \sum_{n=1}^{\infty} \sum_{|\hat{A}|=n} \sum_{\hat{B} \in \mathscr{N}(\hat{A})} \frac{M(\hat{A})\beta^{n+1}}{n} \big|\{B \in \hat{B}\colon A \subset B\}\big| \big(h(\hat{B}) - h(\hat{A})\big)^2 \\ &= \beta(f(1) - f(0))^2 + \sum_{n=1}^{N-1} \frac{t(n)}{n} \beta^{n+1} ((d-1)n + 2)(f(n+1) - f(n))^2, \end{aligned}$$

by (4.2) and the fact that $\sum_{|\hat{A}|=n} M(\hat{A}) = t(n)$. A little calculus and some algebra reveals that one should choose $h_N$ as defined above.

By evaluating $\Phi(\cdot)$ at $h_N$, one obtains

$$\begin{aligned}\Phi(h_N) &= \frac{\beta}{g^2(N)}\left(1+\sum_{n=1}^{N-1}\frac{t(n)((d-1)n+2)\beta^n}{n}\left(\frac{n}{((d-1)n+2)t(n)\beta^n}\right)^2\right)\\ &= \frac{\beta}{g^2(N)}\left(g(1)+\sum_{n=1}^{N-1}(g(n+1)-g(n))\right)\\ &= \frac{\beta}{g(N)}.\end{aligned}$$

By the Dirichlet principle,

$$P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \le \tau^+_{\hat{\varnothing}}) \le \frac{1}{g(N)}.$$

Since $P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \le \tau^+_{\hat{\varnothing}}) = P^{\hat{O}}(\tau_{\hat{\mathscr{S}}_N} \le \tau_{\hat{\varnothing}})$, it follows that

$$P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty) \le \liminf_{N\to\infty}\frac{1}{g(N)} \le \frac{1}{\sum_{n=1}^{\infty} n/(((d-1)n+2)t(n)\beta^n))}. \tag{4.3}$$

This gives an upper bound on the survival probability in the form of the reciprocal of a power series. The next objective is to show that this series blows up at the appropriate rate.

In order to determine the asymptotics of the upper bound in (4.3), we need to determine the asymptotics of $t(n)$. For this purpose, it will be convenient to make comparisons with similar quantities on a single branch of the tree. Let $\mathbb{B}^d$ be the homogeneous tree in which each vertex has degree $d+1$ except the root, which has degree $d$. Also, let $c(n)$ be the number of connected subsets of $\mathbb{B}^d$ of size $n$ containing the root. Again, $c(0) = 1$ by convention. It was shown in [11], Lemma 2.5.1, that

$$c(n) = \frac{1}{(d-1)n+1}\binom{dn}{n}. \tag{4.4}$$

The quantities $c(n)$ and $t(n)$ are related via the recursion

$$t(n) = \sum_{k=0}^{n-1} c(k+1)c(n-1-k) \tag{4.5}$$

for $n \ge 1$. To see this, take two copies of $\mathbb{B}^d$ and add an edge between the two roots to obtain of $\mathbb{T}^d$. By identifying the root of one copy with the root of $\mathbb{T}^d$, one obtains (4.5). By a similar argument,

$$c(n) \sum_{k_1+\cdots+k_d=n-1} c(k_1)\cdots c(k_d) \tag{4.6}$$

for $n \ge 1$. By (4.5) and (4.6), $t(n) \le c(n+1)$. Combining this with the definition of $c(j)$ gives

$$((d-1)j+2)t(j) \le ((d-1)j+2)c(j+1) \le \binom{d(j+1)}{j+1}.$$

By Proposition 4.2, it follows that

$$((d-1)j+2)t(j) \le \frac{K_1(d)}{\sqrt{j+1}\beta_{\mathrm{f}}^{j+1}}. \tag{4.7}$$

Using the fact that $c(n) \le t(n)$, one could argue that the opposite inequality holds with a different a constant, of course. Therefore, this estimate is asymptotically precise.

Returning to the upper bound in (4.3) and using (4.7), one obtains

$$P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty) \le \frac{K_1(d)}{\beta_{\mathrm{f}} \sum_{j=1}^{\infty} j\sqrt{j+1}(\beta_{\mathrm{f}}/\beta)^j}. \tag{4.8}$$

Making the transformation $s = \beta_{\mathrm{f}}/\beta$, it suffices to obtain an appropriate lower bound on the series

$$\sum_{j=0}^{\infty} \sqrt{j+2}(j+1)s^j$$

for $0 \le s < 1$. By expanding in a power series about zero,

$$\frac{1}{(1-s)^{5/2}} = \frac{2}{3}\sum_{k=0}^{\infty} \frac{(2k+3)(k+1)}{4^{k+1}} \binom{2(k+1)}{k+1} s^k.$$

Using Proposition 4.2,

$$\frac{(2n+3)(n+1)}{4^{n+1}} \binom{2(n+1)}{n+1} \le K_1(2)(2n+3)\sqrt{n+1} \le K_1(2)3(n+1)\sqrt{n+2}$$

for $n \ge 0$. Thus,

$$\frac{1}{(1-s)^{5/2}} \le 2K_1(2)\sum_{j=0}^{\infty} \sqrt{j+2}(j+1)s^j.$$

Substituting $s = \beta_{\mathrm{f}}/\beta$ and using (4.8), it follows that

$$P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty) \le \frac{2K_1(2)K_1(d)\beta}{\beta_{\mathrm{f}}^2} \left(\frac{\beta - \beta_{\mathrm{f}}}{\beta}\right)^{5/2}. \qquad \square$$

Since the uniform model and the shape chain $\hat{A}_t^{\hat{O}}$ can be coupled such that $\eta_t^O \in \hat{A}_t^{\hat{O}}$ for all $0 \le t \le \tau_{\hat{\varnothing}}$, Theorem 4.1 has the following corollary.

COROLLARY 4.3. *For $d \ge 2$, the survival probability satisfies*

$$\limsup_{\beta \searrow \beta_{\mathrm{f}}} \frac{P(\eta_t^O \ne \varnothing \,\forall\, t)}{(\beta - \beta_{\mathrm{f}})^{5/2}} < \infty.$$

Statement (1.3) of Theorem 1.1 follows immediately from Corollary 4.3 and the fact that $\beta_{\mathrm{c}} = \beta_{\mathrm{f}} = 1/4$ on the binary tree.

**5. Lower bounds on the survival probability in $d = 2$.** This section is devoted to obtaining lower bounds on the survival probability via Thompson's principle. In order to do this, consider functions $\hat{w}$: $\hat{\mathscr{S}} \times \hat{\mathscr{S}} \to \mathbb{R}$ such that $\hat{w} \in \hat{\mathscr{W}}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}$ for all $N \in \mathbb{N}$. Such a function is said to be a unit flow from $\hat{\varnothing}$ to infinity, or simply a *flow*. If $\hat{w}$ is a flow, then by Thompson's principle,

$$\frac{1}{\mathscr{K}(\hat{w})} \le \sup_{\hat{w} \in \mathscr{W}_{\hat{\varnothing}}^{\hat{\mathscr{S}}_N}} \frac{1}{\mathscr{K}(\hat{w})} = \beta P^{\hat{\varnothing}}(\tau_{\hat{\mathscr{S}}_N} \le \tau_{\hat{\varnothing}}^{+}) = \beta P^{\hat{O}}(\tau_{\hat{\mathscr{S}}_N} \le \tau_{\hat{\varnothing}}).$$

By letting $N$ tend to infinity,

$$\frac{1}{\mathscr{K}(\hat{w})} \le \beta P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty).$$

Therefore, in order to obtain lower bounds on the survival probability, it suffices to construct a flow on $\hat{\mathscr{S}}$ and to estimate the energy.

The flow analyzed here was constructed in [12] for a related Markov chain called the rooted chain. That flow is lifted to the state space of the shape chain providing lower bounds on the probability of survival and upper bounds on the critical exponent. Unfortunately, that flow was only completely constructed for $d = 2$ which explains the specialization to the binary tree in this section. The contribution here is the estimate on the energy. Presumably, the techniques used to estimate the energy can be executed more generally provided that the flow can be constructed. This is discussed more fully at the end of this section.

Before proceeding to define the lift, we review the definition of the rooted chain and the flow that was constructed in [12]. Given a vertex $x \in \mathbb{T}^d$, define the branch $\mathbb{B}(x)$ with root $x$ to be $\{y \in \mathbb{T}^d\colon \|O - x\| + \|x - y\| = \|O - y\|\}$. In other words, the shortest path connecting $y$ to $O$ passes through $x$. Let $\{x_1, \dots, x_{d+1}\}$ denote the $d + 1$ vertices adjacent to the root $O$ and set $\mathbb{B}^d = \mathbb{T}^d \backslash \mathbb{B}(x_{d+1})$. Consider a uniform model with initial configuration $\eta_o = \mathbb{B}(x_{d+1})$. By connectedness, $\eta_0 \subseteq \eta_t$ for all $t \ge 0$ so that it suffices to keep track of the intersection with $\mathbb{B}^d$, namely $A_t = \eta_t \cap \mathbb{B}^d$. The process $A_t$ is a Markov chain with state space $\mathscr{C}_d = \{\text{finite, connected } A \subset \mathbb{B}^d \text{ containing } O\} \cup \varnothing$ and rates

$$q(A, A \triangle x) = c(x, \eta_0 \cup A).$$

It is easy to verify that $A_t$ is reversible with stationary measure $\pi(A) = \beta^{|A|}$, where $|A|$ is number of vertices in $A$. Since $O \in A_t$ whenever $A_t \ne \varnothing$, we refer to $A_t$ as the rooted chain on $\mathbb{B}^d$.

The advantage of constructing flows for the rooted chain is that its state space allows flows to be constructed recursively by taking advantage of self-similarity properties of $\mathbb{B}^d$, as we will see. Since the construction is only valid for $d = 2$, attention is restricted to binary tree henceforth. For each $n \ge 1$, let

$$\alpha(n, k) = \frac{(k+1)(2k+1)(3n-2k)}{n(n+1)(2n+1)} \tag{5.1}$$

for $0 \le k \le n-1$. Note that $\alpha(n,k) \ge 0$ and that $\alpha(n,k)+\alpha(n,n-1-k)=1$. Given a set $A \in \mathscr{C}_2$ such that $|A| \ge 1$, let $A_i = A \cap \mathbb{B}(x_i)$, $i=1,2$. For each $A \in \mathscr{C}_2$, define the map $r(A,\cdot)$ with domain $\mathscr{N}_2(A) = \{B \in \mathscr{C}_2\colon q(A,B)=\beta\}$ by

$$r(A,B) = \begin{cases} 1, & \text{if } (A,B) = (\varnothing, O), \\ \alpha(|A|,|A_i|)r(A_i,B_i), & \text{if } A \neq \varnothing \text{ and } B_i \in \mathscr{N}_2(A_i), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $r(A,\cdot) \ge 0$ and $\sum_{B \in \mathscr{N}_2(A)} r(A,B) = 1$ for each $A$. Finally,

$$w(A,B) = \begin{cases} r(A,B)/c(n), & \text{if } |A| = n \text{ and } B \in \mathscr{N}_2(A), \\ -w(B,A), & \text{if } A \in \mathscr{N}_2(B), \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 15 and (33) in [12],

$$\sum_{\{B \in \mathscr{C}_2\colon\ A \in \mathscr{N}_2(B)\}} w(B,A) = \frac{1}{c(n)} \tag{5.2}$$

for each $A \in \mathscr{C}_2$ such that $|A| = n$ and for each $n \in \mathbb{N}$. The quantity on the left side of (5.2) can be viewed as the net flow into $A$ from below. From (5.2), we see that this choice of $\alpha(n,\cdot)$ apportions an equal amount of fluid to each set of size $n$, and therefore this flow is called the uniform distributed flow. The functions $\alpha(n,\cdot)$ were chosen especially to insure that (5.2) holds. For $B \in \mathscr{N}_2(A)$, $r(A,B)$ is the proportion of fluid routed from $A$ to $B$. With this interpretation, it is easy to see that (5.2) implies that $w$ satisfies the requirement

$$\sum_B w(A,B) = 0 \quad \text{for all } A \neq \varnothing.$$

The next objective is to lift the uniformly distributed flow on $\mathscr{C}_2$ to the state space of the shape chain $\hat{\mathscr{S}}$. For this purpose define an antisymmetric function on $\hat{\mathscr{S}} \times \hat{\mathscr{S}}$ by

$$\hat{w}(\hat{A},\hat{B}) = \sum_{\{A \in \hat{A}\colon\ A \in \mathscr{C}_2\}} \sum_{\{B \in \hat{B}\colon\ B \in \mathscr{C}_2\}} w(A,B) \tag{5.3}$$

for $\hat{A},\hat{B} \in \hat{\mathscr{S}}$. It is immediate that $\hat{w}$ is a flow on $\hat{\mathscr{S}}$. The energy of this flow is given by

$$\begin{aligned} \mathscr{K}(\hat{w}) &= \sum_{n=0}^{\infty} \sum_{|\hat{A}|=n} \sum_{\hat{B} \in \mathscr{N}_2(\hat{A})} \frac{\hat{w}^2(\hat{A},\hat{B})}{\hat{\pi}(\hat{A})\hat{q}(\hat{A},\hat{B})} \\ &= \frac{1}{\beta} + \sum_{n=1}^{\infty} \frac{n}{\beta^{n+1}} \sum_{|\hat{A}|=n} \sum_{\hat{B} \in \mathscr{N}_2(\hat{A})} \frac{\left(\sum_{\{A \in \hat{A}\colon\ A \in \mathscr{C}_2\}} \sum_{\{B \in \hat{B}\colon\ B \in \mathscr{C}_2\}} w(A,B)\right)^2}{M(\hat{A})|\{B \in \hat{B}\colon A \subset B\}|}. \end{aligned}$$

By the Cauchy–Schwarz inequality and the fact that for each pair $\hat{A}$ and $\hat{B}$, the number of terms that appears in the numerator is at most

$M(\hat{A})|\{B \in \hat{B}\colon A \subset B\}|$,

$$K(\hat{w}) \le \frac{1}{\beta} + \sum_{n=1}^{\infty} \frac{n}{\beta^{n+1}} \sum_{|\hat{A}|=n} \sum_{\hat{B}\in\mathscr{N}_2(\hat{A})} \sum_{\{A\in\hat{A}:\ A\in\mathscr{C}_2\}} \sum_{\{B\in\hat{B}:\ B\in\mathscr{C}_2\}} w^2(A,B)$$

(5.4)

$$= \frac{1}{\beta} + \sum_{n=1}^{\infty} \frac{n}{\beta^{n+1}} \sum_{A\in\mathscr{C}_2(n)} \sum_{B\in\mathscr{N}_2(A)} w^2(A,B),$$

where $\mathscr{C}_2(n) = \{A \in \mathscr{C}_2\colon |A| = n\}$. Using the definition of $w$ and inequality (5.4),

$$\mathscr{K}(\hat{w}) \le \frac{1}{\beta} + \sum_{n=1}^{\infty} \frac{n}{c(n)^2\beta^{n+1}} \sum_{A\in\mathscr{C}_2(n)} \sum_{B\in\mathscr{N}_2(A)} r^2(A,B) \tag{5.5}$$

This gives an upper bound on the energy for which we proceed to determine the asymptotic behavior.

Since the asymptotic behavior of $c(n)$ is known, it would suffice to determine the asymptotic behavior of

$$g(n) = \sum_{A\in\mathscr{C}_2(n)} \sum_{B\in\mathscr{N}_2(A)} r^2(A,B).$$

However, it turns out to be more manageable to determine the asymptotic behavior of the series itself. By (4.4) and Proposition 4.2 with $d = 2$,

$$\sum_{n=1}^{\infty} \frac{ng(n)}{c(n)^2\beta^{n+1}} \le \frac{1}{K_2^2(2)\beta} \sum_{n=1}^{\infty} \frac{g(n)n^2(n+1)^2}{(16\beta)^n}. \tag{5.6}$$

Making the substitution $s = 1/16\beta$, the series of interest becomes

$$\sum_{n=1}^{\infty} g(n)n^2(n+1)^2 s^n \tag{5.7}$$

as $s$ increases to $1/4$, because $\beta_c = 1/4$ and $\beta$ decreases to $1/4$ as $s$ increases to $1/4$.

We begin by showing that $g(n)$ satisfies a certain recursion. This recursion implies that a series similar to series (5.7) is a solution of an ordinary differential equation. The ordinary differential equation takes a particularly nice form. In fact, fairly elementary techniques allow one to exhibit the general solution of the ordinary differential equation. This provides an alternative representation of the series solution. This alternative representation readily reveals the asymptotic behavior of the series solution as $s$ increases to $1/4$. Relating the series solution of the ordinary differential equation to series (5.7) gives a lower bound on the survival probability that implies inequality (1.4) in Theorem 1.1.

PROPOSITION 5.1. *For $n \ge 1$,*

$$g(n) = 2\sum_{k=0}^{n-1} c(n-1-k)g(k)\alpha^2(n,k).$$

PROOF. By definition of $g(n)$ and $r(A, B)$,

$$g(n) = \sum_{|A|=n} \sum_{B \in \mathscr{N}_2(A)} r^2(A, B)$$
$$= \sum_{|A|=n} \left( \alpha^2(n, |A_1|) \sum_{B_1 \in \mathscr{N}_2(A_1)} r^2(A_1, B_1) + \alpha^2(n, |A_2|) \sum_{B_2 \in \mathscr{N}_2(A_2)} r^2(A_2, B_2) \right)$$

for $n \geq 1$. By conditioning on the size of $A_1$.

$$g(n) = \sum_{k=0}^{n-1} \sum_{\{|A|=n,\, |A_1|=k\}} \alpha^2(n, k) \sum_{B_1 \in \mathscr{N}_2(A_1)} r^2(A_1, B_1)$$
$$+ \sum_{k=0}^{n-1} \sum_{\{|A|=n,\, |A_1|=k\}} \alpha^2(n, n-1-k) \sum_{B_2 \in \mathscr{N}_2(A_2)} r^2(A_2, B_2)$$
$$= 2 \sum_{k=0}^{n-1} \sum_{\{|A|=n,\, |A_1|=k\}} \alpha^2(n, k) \sum_{B_1 \in \mathscr{N}_2(A_1)} r^2(A_1, B_1).$$

Using the fact that $\alpha^2(n, k) \sum_{B_1 \in \mathscr{N}_2(A_1)} r^2(A_1, B_1)$ is independent of $A_2$,

$$g(n) = 2 \sum_{k=0}^{n-1} \sum_{\{|A_1|=k\}} c(n-1-k) \alpha^2(n, k) \sum_{B_1 \in \mathscr{N}_2(A_1)} r^2(A_1, B_1)$$
$$= 2 \sum_{k=0}^{n-1} c(n-1-k) \alpha^2(n, k) g(k). \qquad \square$$

As a consequence of Proposition 5.1, we obtain the next lemma.

LEMMA 5.2. *Let* $G(s) = \sum_{n=0}^{\infty} g(n)(n+1)^2(2n+1)^2 s^n$. *Then* $G(0) = 1$, $G(s)$ *converges for* $|s| < 1/4$, *and* $G(s)$ *is a solution of*

$$s(1-4s)^2 G''(s) + (1-16s)(1-4s) G'(s) - 18(1-2s) G(s) = 0 \tag{5.8}$$

*for* $|s| < 1/4$.

PROOF. By the Cauchy–Schwarz inequality,

$$\frac{1}{n+1} \leq \sum_{B \in \mathscr{N}_2(A)} r^2(A, B) \leq 1 \quad \text{for } |A| = n.$$

Thus,

$$\frac{c(n)}{n+1} \leq g(n) \leq c(n).$$

This, together with (4.4), Proposition 4.2 and the fact that $\beta_c = 1/4$, implies that the series defining $G(s)$ converges if and only if $|s| < 1/4$.

By Proposition 5.1 and (5.1),

$$\begin{aligned} & g(n)(n+1)^2(2n+1)^2n^2 \\ & \quad = 2\sum_{k=0}^{n-1} c(n-1-k)g(k)(k+1)^2(2k+1)^2(3(n-k)+k)^2 \end{aligned}$$

for $n \geq 1$. Expanding the factor $(3(n-k)+k)^2$ as $9(n-k)^2+6(n-k)k+k^2$, multiplying by $s^{n-1}$, and taking the sum from $n=1$ to infinity, implies that

$$\begin{aligned} G'(s)+sG''(s) &= 18(C(s)+3sC'(s)+s^2C''(s))G(s) \\ &\quad +12(C(s)+sC'(s))sG'(s)+2C(s)(sG'(s)+s^2G''(s)) \end{aligned}$$

where $C(s)=\sum_{n=0}^{\infty} c(n)s^n$. Multiplying recursion (4.6) by $s^{n-1}$ and taking the sum from $n=1$ to infinity gives

$$C(s)=\frac{1-\sqrt{1-4s}}{2s} \quad \text{for } 0 \leq s \leq 1/4.$$

Using this explicit expression for $C(s)$,

$$s(1-4s)^2G''(s)+(1-16s)(1-4s)G'(s)-18(1-2s)G(s)=0,\ G(0)=1. \quad \square$$

We will show that the ordinary differential equation determines the rate at which $G(s)$ tends to infinity as $s$ increases to $1/4$. Since the coefficient of $G''(s)$ in the ordinary differential equation has a factor of $s$, it follows that there are solutions to the ordinary differential equation that also blow up as $s$ tends to zero. On $(0,1/4)$, the general solution to the ordinary differential equation is of the form

$$c_1(1-4s)^{r_1}H_1(s)+c_2(1-4s)^{r_2}H_2(s), \tag{5.9}$$

where $c_i$, $i=1,2$, are arbitrary constants, $r_1=-1-\sqrt{13}/2$, $r_2=-1+\sqrt{13}/2$, and $H_i(s)=\sum_{n=0}^{\infty} h_i(n)(1-4s)^n$ with $h_i(0)=1$, $i=1,2$, and

$$h_i(n)=\frac{2n^2+(2+4r_i)n+5-2r_i}{2n^2+(4+4r_i)n}h_i(n-1), \tag{5.10}$$

for $n \geq 1$.

In order to see that expression (5.9) is the general solution, let

$$H(s)=\sum_{n=0}^{\infty} h(n)(1-4s)^{n+r},$$

where $h(n)$ is defined as in (5.10) except with $r_i$ replaced by $r$. We have

$$\begin{aligned} H(s) &= \sum_{n=0}^{\infty} h(n)(1-4s)^{n+r}, \\ (1-4s)H'(s) &= -4\sum_{n=0}^{\infty} h(n)(n+r)(1-4s)^{n+r}, \\ (1-4s)^2H''(s) &= 16\sum_{n=0}^{\infty} h(n)(n+r)(n+r-1)(1-4s)^{n+r}. \end{aligned}$$

Also, expressing the coefficients of $(1-4s)^n G^{(n)}(s)$ in (5.8) as linear combinations of $\{(1-4s)^m\}_{m\in\mathbb{N}}$ gives

$$-18(1-2s) = -9 - 9(1-4s), \qquad (1-16s) = -3 + 4(1-4s)$$

and

$$s = 1/4 - (1-4s)/4.$$

Therefore,

$$-18(1-2s)H(s) = -9h(0) - 9\sum_{n=1}^{\infty}(h(n)+h(n-1))(1-4s)^{n+r},$$

$$\begin{aligned}(1-16s)(1-4s)H'(s) = 12h(0)r + \sum_{n=1}^{\infty}&(12h(n)(n+r)\\ &-16h(n-1)(n+r-1))(1-4s)^{n+r}\end{aligned}$$

and

$$\begin{aligned}s(1-4s)^2H''(s) = 4h(0)r(r-1) + 4\sum_{n=1}^{\infty}&(h(n)(n+r)(n+r-1)\\ &-h(n-1)(n+r-1)(n+r-2))(1-4s)^{n+r}.\end{aligned}$$

Adding these three expressions and combining like terms shows that $H(s)$ is a solution if and only if

$$h(0)(-9+8r+4r^2) = 0,$$
$$h(n)(4n^2+(8+8r)n) = h(n-1)(4n^2+(4+8r)n+10-4r),$$

$n \geq 1$. In particular, $(1-4s)^{r_i}H_i(s)$, $i = 1, 2$, are two linearly independent solutions to the ordinary differential equation.

Since all solutions on $(0, 1/4)$ are given by expression (5.9), there exists a choice of $c_i$, $i = 1, 2$, such that

$$G(s) = c_1(1-4s)^{r_1}H_1(s) + c_2(1-4s)^{r_2}H_2(s)$$

on $(0, 1/4)$. The fact that $r_2 > 0$ implies that $c_2(1-4s)^{r_2}H_2(s)$ tends to zero as $s$ increases to $1/4$. since $G(s)$ tends to infinity as $s$ increases to $1/4$, it follows that $c_1 > 0$ and

$$G(s) \sim c_1(1-4s)^{r_1} \quad \text{as } s \nearrow 1/4.$$

Equivalently,

$$G(1/16\beta) \sim c_1\left(\frac{\beta - 1/4}{\beta}\right)^{r_1} \quad \text{as } \beta \searrow 1/4. \tag{5.11}$$

THEOREM 5.3. *For the shape chain on the binary tree,*

$$0 < \liminf_{\beta\searrow 1/4} \frac{P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty)}{(\beta - 1/4)^{1+\sqrt{13}/2}}. \tag{5.12}$$

PROOF. By definition of $G(s)$ and the fact that $n \le (2n+1)/2$,

$$\frac{G(s)}{4} \ge \sum_{n=1}^{\infty} g(n) n^2 (n+1)^2 s^n. \tag{5.13}$$

This together with (5.6) gives

$$\frac{G(1/16\beta)}{4} \ge K_2^2(2)\beta \sum_{n=1}^{\infty} \frac{ng(n)}{c(n)^2 \beta^{n+1}}.$$

Combining this with (5.5) and Thompson's principle results in

$$\frac{1}{1+\frac{1}{4K_2^2(2)}G(1/16\beta)} \le \frac{1}{\beta \mathscr{K}(\hat{w})} \le P^{\hat{O}}(\tau_{\varnothing} = \infty).$$

By asymptotic relation (5.11),

$$\frac{K_2^2(2)}{c_1 4^{\sqrt{13}/2}} \le \liminf_{\beta \searrow 1/4} \frac{P^{\hat{O}}(\tau_{\hat{\varnothing}} = \infty)}{(\beta - 1/4)^{1+\sqrt{13}/2}}. \qquad \square$$

As before, statement (1.4) of Theorem 1.1 follows immediately from Theorem 5.3, the coupling with the shape chain, and the fact that $\beta_c = 1/4$ on the binary tree. Together statements (1.3) and (1.4) imply that if the critical exponent exists, then on the binary tree it is lies in the interval $[5/2, 1+\sqrt{13}/2]$. If one could show that the hypothesis of Lemma 15 in [12] holds for $d \ge 3$, then the analog of g(n) satisfies $g(n) \le c(n)$ which gives an upper bound of $7/2$ on the critical exponent on the $d$-ary tree. Furthermore, the analog of the recursion in Proposition 5.1 will hold;

$$g(n) = d \sum_{k=0}^{n-1} \sum_{j_1+\cdots+j_{d-1}=n-1-k} c(j_1)\cdots c(j_{d-1})\alpha^2(n;(k, j_1, \dots, j_{d-1}))g(k).$$

If one provides more information about $\alpha(n;\cdot)$ in proving that the hypothesis of Lemma 15 in [12] holds, then it may be possible to improve the exponent to $1+\sqrt{13}/2$. However, as indicated in Section 4, we believe that the actual rate of decay is 2.5 due to the simulations in [16].

To prove that the rate of decay is 2.5, one needs to construct a different flow. This is true because the estimates used here are not generous. In fact, (5.4), and thus (5.5), actually holds with equality since $w(A,B)$ is constant on $\{(A,B)\colon A \in \hat{A}, B \in \hat{B} \text{ and } B \in \mathscr{N}_2(A)\}$. Furthermore, (5.6) holds in the opposite direction with a different constant due to Proposition 4.2, as does (5.13). Therefore, a rate of $1+\sqrt{13}/2$ is the best that this flow achieves. Nevertheless, it would still be worthwhile to show that the construction of the uniformly distributed flow can be carried out for $d \ge 3$. Even though the extension is likely not to give the correct decay rate, it would still have the consequence that $\beta_f = \beta_c$ for $d \ge 3$.

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**Note added in Proof.** Since the submission of this paper, Liggett has proved that $\beta_{\mathrm{f}} = \beta_{\mathrm{c}}$ for $d \geq 3$ by verifying that the hypothesis of Puha's Lemma 15 in [12] holds for $d \geq 3$: see [18]. As previously mentioned, this immediately extends (1.3) to all $d$-ary trees. Moreover, statement (1.4) holds for $d \geq 3$ with an exponent of 7/2 (rather than $1 + \sqrt{13}/2$). This follows by comparing the terms in the energy series with the terms in the power series expansion for $(1-s)^{-7/2}$ in a manner similar to that in the proof of Theorem 4.1 of the present paper.

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