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EQUILIBRIUM FLUCTUATIONS FOR $\nabla\varphi$ INTERFACE MODEL

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We study the large scale space–time fluctuations of an interface which is modeled by a massless scalar field with reversible Langevin dynamics. For a strictly convex interaction potential we prove that on a large space–time scale these fluctuations are governed by an infinite-dimensional Ornstein–Uhlenbeck process. Its effective diffusion type covariance matrix is characterized through a variational formula.

1. Introduction. It is a common phenomenon that at low temperature two pure thermodynamic phases spatially coexist and are separated by an interface, which is very sharp with a width of a few atomic distances. In thermal equilibrium such an interface is planar and local deformations will relax back diffusively in order to minimize surface tension. To build a statistical mechanics model for the interface, one assumes that transverse deviations from the perfectly flat interface are given through a scalar field φ, that is, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ with $\varphi \equiv 0$ corresponding to the flat interface. To have a mathematically well-defined model we discretize $\mathbb{R}^2$. We also generalize to arbitrary dimension. Then $\varphi\colon \mathbb{Z}^d \to \mathbb{R}$ with φ_x is the height of the interface at site $x \in \mathbb{Z}^d$. Neighboring heights are connected through an elastic potential V. While we need more stringent assumptions later on, at this point V should be bounded from below and increase sufficiently rapidly for large arguments. To each configuration φ we associate the (elastic) energy

$$H(\varphi) = \frac{1}{2} \sum_{|x-y|=1} V(\varphi_x - \varphi_y). \tag{1.1}$$

The formal equilibrium measure is given by

$$Z^{-1} \exp[-H(\varphi)] \prod_x d\varphi_x, \tag{1.2}$$

where for simplicity we have absorbed the inverse temperature in V.

If the order parameter of the pure phases is not conserved, then there is no constraint on the interface. Thus it is natural to assume a Langevin dynamics reversible with respect to (1.2) which is governed by

$$d\varphi_x(t) = \sum_{|e|=1} V'\big(\varphi_{x+e}(t) - \varphi_x(t)\big)\, dt + \sqrt{2}dB_x(t), \qquad x \in \mathbb{Z}^d. \tag{1.3}$$

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Here $\{B_x(t), x \in \mathbb{Z}^d\}$ is a collection of independent standard Brownian motions. In the physics literature (1.2) is called the static and (1.3) the dynamic Ginzburg–Landau $\nabla\varphi$ model since in the continuum approximation the energy is given by $H(\varphi) = \int V(\underline{\partial}\varphi(x))\,dx$. We have used the notation $\underline{\partial}$ for the gradient in $\mathbb{R}^d$.

In [13], macroscopic deviations from the average equilibrium profile were studied and, for a strictly convex V with bounded second derivative, were shown to relax according to the nonlinear diffusion equation

$$\frac{\partial}{\partial t}h(x,t) = \sum_{\alpha,\beta=1}^{d} \sigma_{\alpha\beta}(\underline{\partial}h)\frac{\partial^2}{\partial x_\alpha \partial x_\beta}h(x,t). \tag{1.4}$$

Here $h(\cdot,t)\colon \mathbb{R}^d \to \mathbb{R}$ is the height at time t on the macroscopic scale and ∇h is the spatial gradient. $\sigma(u)$ is the surface tension at tilt u, which is defined through

$$\sigma(u) = \lim_{\Lambda \nearrow \mathbb{Z}^d} -\frac{1}{|\Lambda|}\log\int e^{-H(\varphi)}\prod_{x\in\Lambda} d\varphi_x \prod_{y\in\partial\Lambda}\delta(\varphi_y - u\cdot y) \tag{1.5}$$

for a sequence of boxes Λ tending to $\mathbb{Z}^d$ and

$$\sigma_{\alpha\beta}(u) = \frac{\partial^2}{\partial u_\alpha \partial u_\beta}\sigma(u). \tag{1.6}$$

Note that in (1.5) we fixed the boundary heights, φ_y, to enforce a definite tilt u. We warn the reader regarding (1.5) that up to now there is no proof that $\sigma\colon \mathbb{R}^d \to \mathbb{R}$ is twice differentiable (the best result is that $\sigma \in C^1$ with Lipschitz derivatives, cf. [13]), even if it is expected to be C^∞. Therefore (1.6) and (1.4) are formal, and (1.4) should be intended in a weak form (see [13] for a precise formulation).

In this paper we will study small (central limit type) fluctuations. Such fluctuations should relax according to the linearized version of (1.4) and, according to (1.3) should be perturbed by white noise. Thus if we fix the average slope u and denote the fluctuations relative to the average by ζ, they should be governed by

$$\frac{\partial}{\partial t}\zeta = \sum_{\alpha,\beta=1}^{d}\sigma_{\alpha\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\zeta + \sqrt{2}\dot{W} \tag{1.7}$$

with $\dot{W}$ normalized Gaussian space–time white noise.

Our main result will be a proof of (1.7) with some strictly positive diffusion matrix $q_{\alpha\beta}$. Unfortunately, we still miss the identity $q_{\alpha\beta} = \sigma_{\alpha\beta}$ although it does hold at finite volume; compare Appendix A. In our proof we rely on the beautiful observation of Naddaf and Spencer [20] that the Helffer–Sjöstrand [14] elliptic PDE representation translates the fluctuation problem for (1.2) into a massless homogenization problem. This observation naturally extends to dynamic correlations and we reinterpret the PDE representation in a probabilistic way: equilibrium expectations will be expressed in terms of an auxiliary random walk in a dynamic random environment (see also [8], Section 2). If $V(\eta) = (\eta)^2$ for every $\eta \in \mathbb{R}$, then the equilibrium measure (1.2) is Gaussian, the diffusion process (1.3) is linear, and the auxiliary random walk is homogeneous. Here V is strictly convex but not quadratic. The transition rates of the auxiliary random walk then become random and one is led to understand this homogenization problem. While in [20] functional analytic methods were used, we will fully exploit the probabilistic structure.

At first glance, our results look rather similar to the hydrodynamic fluctuation theory for stochastic particle systems. (We refer to the recent book by Kipnis and Landim [16] where these notions are discussed and reference to previous work can be found. For general background on stochastic particle systems see [18].) There is, however, one important distinction. For particle systems, the Laplacian in the drift of (1.7) comes from the conservation of the number of particles. In our case there is no local conservation law and the Laplacian has its origin in the long-range correlations of the equilibrium measure.

We close our introduction with a few remarks to motivate the mathematical set-up of our problem. If in (1.2) we fix the heights to be zero at the boundary of a box, Λ, centered at the origin, then

$$\langle \varphi_0^2 \rangle_\Lambda = \begin{cases} \mathscr{O}(|\Lambda|), & \text{in} \quad d = 1, \\ \mathscr{O}(\log|\Lambda|), & \text{in} \quad d = 2, \\ \mathscr{O}(1), & \text{in} \quad d \geq 3 \end{cases} \tag{1.8}$$

for large $|\Lambda|$ [4]. Thus, in dimension $d = 1, 2$, the infinite volume Gibbs measure does not exist, whereas in $d \geq 3$ the average height will be determined through the boundary conditions. To avoid such inessential complications and to have a unified framework it is more convenient to go over the gradient field $\eta_{(x,y)} = \varphi_y - \varphi_x$, $|x - y| = 1$, which by construction is constrained to have zero curl. Clearly, (1.3) can be read as a dynamics for the gradient field. In equilibrium the gradient field is stationary in space and time. Prescribing the tilt means to fix the average value of $\eta_{(x,y)}(t)$. Although the scalar field φ is more intuitive, it has completely dropped out of the picture. Of course, given say φ_0, it can be reconstructed from η.

We note that in one dimension, the dynamics of the η's corresponds to a usual Ginzburg–Landau model with a conserved order parameter. The hydrodynamic fluctuations have been studied both for equilibrium [25] and nonequilibrium [6]. In the former case no convexity assumption is needed.

A word on notation: as already seen in this introduction, we reserve the symbol ∇ for discrete gradients. The continuum gradients are denoted by $\underline{\partial}$ and its components are denoted, according to the context, by $\partial/\partial x_i$, $\partial/\partial u_\alpha$, and so on ($x$ and u are vectors and i and α are natural numbers). Sometimes we will also use the short-cut notation ∂_i meaning the derivative with respect to the ith component. We anticipate, however, that later on in the text the notation ∂_x, $x \in \mathbb{Z}^d$ will be introduced and it should not be confused with ∂_i. Finally, by Δ we will always mean the Laplacian on $\mathbb{R}^d$; that is $-\underline{\partial} \cdot \underline{\partial}$. The Laplacian on $\mathbb{Z}^d$ (see below) will be denoted by Δ_1.

2. Main Results.

2.1. *The model.*

2.1.1. *Configurations.* At each site x of the d-dimensional lattice $\mathbb{Z}^d$, there is a real random variable φ_x to be interpreted as the height of the interface. We regard φ as a real-valued function on $\mathbb{Z}^d$. Let us denote by e_α, $\alpha = 1, \dots, d$ the unit vector in the direction α; that is, $(e_\alpha)_\beta = \delta_{\alpha\beta}$. $\mathbb{Z}^{d*}$ is the set of *positively directed bonds* $b = (x, x + e_\alpha)$ for some x and α. We will also use the notation $b = (x, y) = (x_b, y_b)$ and $-b$ for the negatively directed bond, $-b = (y, x)$. The basic objects are the increments $\eta_b = \varphi_y - \varphi_x$ with $b = (x, y)$. We set $\eta_{x,\alpha} = \eta_b$ for $b = (x, x + e_\alpha)$. Notationally it will be convenient to define also $\eta_{-b} = -\eta_b$, but η_{-b} is not regarded as an independent variable. Finally, given a function $f\colon \mathbb{Z}^d \to \mathbb{R}$ we set $\nabla_\alpha f(x) = f(x + e_\alpha) - f(x)$, $\nabla_\alpha^* f(x) = f(x - e_\alpha) - f(x)$ and $\Delta_1 = \sum_{\alpha=1}^d \nabla_\alpha^* \nabla_\alpha$.

As a vector field, η_b has zero curl. This means

$$\sum_{b\in\mathscr{C}} \eta_b = 0 \tag{2.1}$$

for every closed loop $\mathscr{C}$, that is, the bonds (x_i, x_{i+1}) of the finite sequence $\{x_i\}_{i=1,\dots,n}$ of points in $\mathbb{Z}^d$ such that $x_1 = x_n$ and $|x_i - x_{i+1}| = 1$ for $i = 1, \dots, n-1$. Conversely, given a vector field η_b which satisfies (2.1) for all $\mathscr{C}$, then up to a constant (say the value of φ_0), there exists a unique scalar field φ such that $\nabla_\alpha \varphi_x = \eta_{x,\alpha} = \eta_{(x,x+e_\alpha)}$. We denote by χ the subset of $\mathbb{R}^{\mathbb{Z}^{d*}}$ which has zero curl in the sense of (2.1). $\mathbb{R}^{\mathbb{Z}^d}$ and χ are equipped with the local product topology unless otherwise stated. Given a measurable space $(E, \mathscr{B}(E))$ we will denote by $\mathscr{P}(E)$ the set of probability measures over E.

2.1.2. *Equilibrium measures.* For the interaction potential $V\colon \mathbb{R} \to \mathbb{R}$ we assume that:

1. $V \in C^2(\mathbb{R})$ and it is even.
2. There exists $C_- > 0$ and $C_+ > 0$ such that

$$C_- \le V''(\eta) \le C_+ \qquad \text{for all } \eta \in \mathbb{R}. \tag{2.2}$$

The strict convexity is crucial for the proof. Even for $V(\eta) = a\eta^2 + b\eta^4$ we miss several steps.

To write down the Gibbs measure for the gradient field, we have to explain how to set up the boundary conditions. Let $\Lambda \subset \mathbb{Z}^d$ a finite set of sites. Then Λ^* is the set of positively directed bonds with at least one endpoint in Λ. We set $(\eta \vee \xi)_b = \eta_b$ if $b \in \Lambda^*$ and $(\eta \vee \xi)_b = \xi_b$ otherwise. The set of configurations with boundary condition ξ is then

$$\chi_{\Lambda,\xi} \equiv \{\eta \in \mathbb{R}^{\Lambda^*} | \eta \vee \xi \in \chi\}. \tag{2.3}$$

Note that $\chi_{\Lambda,\xi}$ depends on ξ only through those ξ_b where b borders Λ^*. The finite volume Gibbs measure with b.c.'s ξ is defined as

$$\mu_{\Lambda,\xi}(d\eta) = Z_{\Lambda,\xi}^{-1} \exp\Big\{-\sum_{b\in\Lambda^*} V(\eta_b)\Big\} d\eta_{\Lambda,\xi} \in \mathscr{P}(\chi_{\Lambda,\xi}), \tag{2.4}$$

where $d\eta_{\Lambda,\xi}$ is the uniform measure over $\chi_{\Lambda,\xi}$ and $Z_{\Lambda,\xi}$ is the normalization. Infinite volume Gibbs measures are defined via the DLR equations.

DEFINITION 2.1 (Gibbs measures). A measure $\mu \in \mathscr{P}(\chi)$ is an infinite volume Gibbs state if $\int(\eta_b)^2\,d\mu < \infty$ for all $b \in \mathbb{Z}^{d^*}$ and

$$\mu(\,\cdot\,|\mathscr{F}_{\mathbb{Z}^{d*}\setminus\Lambda^*})(\xi) = \mu_{\Lambda,\xi}(\cdot), \tag{2.5}$$

for any finite $\Lambda \subset \mathbb{Z}^d$ and μ-a.e. ξ. We used also the notation $\mathscr{F}_A \equiv \sigma(\eta(b), b \in A)$ with $A \subset \mathbb{Z}^{d*}$.

Let us denote by τ_x, $x \in \mathbb{Z}^d$, the shift in χ by x; that is, $\tau_x\eta_b = \eta_{b+x}$, with $b + x = (x_b + x, y_b + x)$. We set $\tau_\alpha = \tau_{e_\alpha}$, $\alpha = 1, \dots, d$. The set of all shift invariant Gibbs measures is denoted by $\mathscr{G}^\tau$. Then, given any $u \in \mathbb{R}^d$, there exists a unique shift ergodic μ_u; that is, μ_u is an extremal point of $\mathscr{G}^\tau$, such that

$$\int_\chi \eta_{0,\alpha}\,d\mu_u(\eta) = u_\alpha, \tag{2.6}$$

for every $\alpha = 1, \dots, d$ (cf. [13], Theorems 3.1 and 3.2). Here u is the *tilt* and μ_u the *u-tilted* measure. Finally we will use the following exponential bound, which is a consequence of the Brascamp–Lieb inequality (cf. [3])

$$\int \exp\left\{\lambda\left[\sum_x f(x)(\eta_{x,\alpha} - u_\alpha)\right]^2\right\}d\mu_u(\eta) \le \left[1 - \frac{2\lambda^2}{C_-}\sum_{x\in\mathbb{Z}^d}(f(x))^2\right]^{-1/2} \tag{2.7}$$

for some $\lambda > 0$ small enough. Here f is a bounded function on $\mathbb{Z}^d$ with compact support.

2.1.3. *Reversible dynamics.* We require the dynamics to be reversible with respect to the Gibbs measure of Definition 2.1. If the noise does not depend on the process, then the φ field is governed by (1.3), which implies that the gradient field is governed by the SDEs

$$d\eta_{x,\alpha}(t) = -\sum_{\beta=1}^d \nabla_\alpha\nabla_\beta^* V'(\eta_{x,\beta})\,dt + \sqrt{2}\,\nabla_\alpha\,dB_x(t). \tag{2.8}$$

Under the growth condition (2.2), it is by now standard to build the dynamics for a large set of initial conditions $\eta(0)$ [9, 12, 13, 22]. In particular if for $r > 0$ we define

$$\chi_r = \left\{\eta \in \chi\colon \|\eta\|_r^2 \equiv \sum_{b\in\mathbb{Z}^{d^*}} |\eta_b|^2 e^{-2r|x_b|} < \infty\right\}, \tag{2.9}$$

and we equip it with the norm $\|\cdot\|_r$, we have that ([13], Lemma 2.2) for each $\eta \in \chi_r$, $r > 0$ (2.8) has a unique $C^0(\mathbb{R}^+;\chi_r)$ solution starting at $\eta(0) = \eta$. We denote by $\mathbb{P}_\eta$ the law of $\{\eta(t)\}_{t\ge 0}$ (and by $\mathbb{E}_\eta$ the corresponding expectation). It is immediate to verify that if η is distributed according to μ_u, any $u \in \mathbb{R}^d$, there exists μ_u-a.s. a unique solution to (2.8) and μ_u is an invariant reversible measure for the evolution ([13], Proposition 3.1). The law of such an evolution will be denoted by $\mathbb{P}_{\mu_u}$ (and $\mathbb{E}_{\mu_u}$ the corresponding expectation).

2.2. *Convergence to an infinite dimensional Ornstein–Uhlenbeck process.* The large scale fluctuations of the interface will be studied by looking at a suitable empirical field. For the moment we view the fluctuation field as taking values in $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^d)$, the Schwartz distributions. For the duality between $\mathscr{S}$ and $\mathscr{S}'$ we use the notation $(\cdot, \cdot)_{\mathscr{S}}$.

We fix the tilt $u \in \mathbb{R}^d$ and start the $\eta(t)$ process in the Gibbs measure μ_u, which implies that $\eta(t)$ is stationary in space and time.

DEFINITION 2.2 (Fluctuation fields). For any $\varepsilon > 0$, $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$, $\alpha = 1, \dots, d$, and $t \geq 0$ let us define

$$\xi_\alpha^\varepsilon(f, t) = \varepsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(\varepsilon x)\big[\eta_{x,\alpha}(\varepsilon^{-2} t) - u_\alpha\big]. \tag{2.10}$$

The fluctuation field $\xi^\varepsilon = \{\xi_\alpha^\varepsilon(t)\}_{\alpha, t} \in C^0(\mathbb{R}^+; (\mathscr{S}')^d)$ is defined by $(\xi_\alpha^\varepsilon(t), f)_{\mathscr{S}} \equiv \xi_\alpha^\varepsilon(f, t)$.

DEFINITION 2.3 (Limit Gaussian field). Let us call $\xi = \{\xi_\alpha(t)\}_{\alpha, t}$ a continuous $(\mathscr{S}')^d$-valued version of the centered Gaussian stochastic process, of law $\mathbb{P}^0$ with covariance given by

$$\mathbb{E}^0(\xi_\alpha(f, t)\xi_\beta(g, s)) = \int k_\alpha k_\beta (k \cdot qk)^{-1} e^{-(k \cdot qk)|t-s|} \hat{f}(k)^* \hat{g}(k)\, dk, \tag{2.11}$$

where $\xi_\alpha(f, t) = (f, \xi_\alpha(t))_{\mathscr{S}}$, $f, g \in \mathscr{S}$ and $\hat{f}$ denotes the Fourier transform of f ($\hat{f}(k) = (2\pi)^{-d/2} \int e^{ik \cdot x} f(x)\, dx$, $k \in \mathbb{R}^d$). In (2.11) q is a $d \times d$ strictly positive definite (symmetric) matrix.

We note that $\xi_\alpha(x, t) = \partial \zeta(x, t)/\partial x_\alpha$ (in the distributional sense) where ζ is governed by (1.7) with $\sigma_{\alpha\beta} = q_{\alpha\beta}$. Our limit procedure will select one special matrix $q = q_u$, which can be characterized through the variational formula for its quadratic form: for every $v \in \mathbb{R}^d$,

$$v \cdot q_u v \equiv 2 \inf_\psi \left\{ \sum_{\alpha=1}^d \left\langle \big(v_\alpha - D_\alpha \psi(\eta)\big)^2 V''(\eta_{0,\alpha}) \right\rangle_{\mu_u} + \sum_{x \in \mathbb{Z}^d} \left\langle \big(\partial_x \psi(\eta)\big)^2 \right\rangle_{\mu_u} \right\}, \tag{2.12}$$

where $\langle \cdot \rangle_{\mu_u}$ denotes the expectation with respect to μ_u, $D_\alpha \psi(\eta) = \psi(\tau_\alpha \eta) - \psi(\eta)$ and the infimum is taken over smooth, bounded, local functions ψ: $\chi \to \mathbb{R}$. The derivative ∂_x, and consequently the notion of smooth function from $\chi \to \mathbb{R}$, is defined as follows. In Section 2.1 it has been pointed out that for every $\varphi \in \mathbb{R}^{\mathbb{Z}^d}$ there is a unique *increment configuration* $\eta \in \chi$; we therefore define an application η^*: $\mathbb{R}^{\mathbb{Z}^d} \to \chi$ such that $\eta^*(\varphi) = \eta$. A local function F: $\chi \to \mathbb{R}$ is said to be differentiable if the local function $F(\eta^*(\cdot))$: $\mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ is differentiable as a function of its finitely many arguments. In this case we define $\partial_x F(\eta) = \partial F(\eta^*(\varphi))/\partial \varphi_x$ and this is a good definition since the right-hand side evaluated at $\varphi = \varphi'$ coincides with the same expression evaluated at $\varphi = \varphi''$ if $\eta^*(\varphi') = \eta^*(\varphi'')$. Observe moreover that (2.12) is well posed, since, as explained in Section 4, in the sense of matrices

$$2C_-\mathbb{I} \le q_u \le 2C_+\mathbb{I}, \tag{2.13}$$

where $\mathbb{I}$ is the $d \times d$ identity matrix and $C_\pm$ are the constants from (2.2). A slightly better bound is given in (4.13).

For our purposes $\mathscr{S}'$ is unnecessarily large and we introduce a more restricted family of distribution spaces. Let $\mathscr{H}_\kappa$ be the Sobolev space associated to the scalar product

$$(f, g)_\kappa = \int_{\mathbb{R}^d} f(r)(|r|^2 - \Delta)^\kappa g(r)\, dr, \tag{2.14}$$

and let $\mathscr{H}_{-\kappa}$ be its dual space (with respect to the L^2 scalar product).

Our main result is the following.

THEOREM 2.1. *Let* $\kappa > 1 + d$.

(i) *For every* $\varepsilon > 0$, *we have that the process* $\xi^\varepsilon \in C^0(\mathbb{R}^+; (\mathscr{H}_{-\kappa})^d)$, $\mathbb{P}_{\mu_u}$-*a.s.*, *and* $\xi \in C^0(\mathbb{R}^+; (\mathscr{H}_{-\kappa})^d)$, $\mathbb{P}$-*a.s.*

(ii) *The law of* ξ^ε *on* $C^0([0, T]; (\mathscr{H}_{-\kappa})^d)$, $T > 0$, *converges, as* $\varepsilon \to 0$, *to the law of the Gaussian process* ξ *with covariance specified by* (2.11), *where* $q = q_u$ *as given by the variational formula* (2.12).

Theorem 2.1 is proved in Section 5. A direct corollary of Theorem 2.1 is a central limit theorem for the equilibrium measure μ_u.

COROLLARY 2.2. (i) *The* $(\mathscr{H}_{-\kappa})^d$-*valued random variable* $\xi^\varepsilon(0)$ *converges in law to the centered Gaussian field* $\xi(0)$ *with covariance*

$$\mathbb{E}^0\big[\xi_\alpha(f, 0)\xi_\beta(g, 0)\big] = \int_{\mathbb{R}^d} \frac{k_\alpha k_\beta}{(k, qk)} \hat{f}(k)^* \hat{g}(k)\, dk, \tag{2.15}$$

for all test functions f *and* g *and every* $\alpha, \beta \in \{1, \dots, d\}$.

(ii)

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu_u}[\exp(\xi^\varepsilon_\alpha(f, 0))] = \exp\left(\frac{1}{2} \int_{\mathbb{R}^d} \frac{k_\alpha^2}{(k, qk)} |\hat{f}(k)|^2\, dk\right), \tag{2.16}$$

The exponential moment convergence (2.16) is a direct consequence of (2.15) and (2.7).

2.3. *Strategy of the proof.* We start proving an *infinite-dimensional* version of the identity of Helffer and Sjöstrand [14] to express static correlations as the solution of suitable elliptic PDEs. Rewriting them in terms of diffusion processes, we obtain a random walk in a dynamic random environment. This environment turns out to be governed by the SDEs (2.8), which in particular explains that dynamical correlations can be handled in the same fashion. The Helffer–Sjöstrand (H–S) identity in the probabilistic form is proved easily at finite volume (see [8], where one can find a detailed treatment of this case for general graphs, including various bounds that can be extracted from this representation). However, the extension to infinite volume is nontrivial since it requires time-mixing of the Langevin dynamics (2.8) with respect to the measure μ_u, which we prove by exploiting the coupling in [13].

Once the infinite volume H–S representation is established, it translates the original fluctuation problem into a problem involving an invariance principle for a certain random walk in a dynamic random environment; that is, we have to prove the convergence of this random walk to a Brownian motion in $\mathbb{R}^d$ with diffusion matrix q_u. This will follow directly by a well-known result of Kipnis and Varadhan [15], which is applicable to general reversible Markov processes (Section 4). In essence only the existence of the dynamics and some ergodic properties are required.

From the homogenization step we would like to conclude that the dynamic covariance has a limit as $\varepsilon \to 0$ as given by (2.11). This is, however, not immediate, because the H–S identity contains an integration over all $t \geq 0$, whereas the homogenization yields the convergence of the integrand only in bounded time intervals. To overcome this difficulty we need sufficiently sharp bounds on the kernel of the semigroup generated by divergence form operator with time-dependent coefficients. For $d \geq 3$ the basic Nash inequality suffices. However, for $d = 2$ the sharper Nash continuity estimate is required. Such an estimate can be found in the literature in the generality we need only in the case of diffusions in $\mathbb{R}^d$, for example, in [10]. In $\mathbb{Z}^d$ the situation is technically more involved and the needed results have been developed only in the case of operators which are independent of time [24].

In Section 5 we lift the proof from the covariance to the convergence of the process ξ^ε by first establishing its tightness and then by identifying the limit. This identification is based on showing that any limit point satisfies a martingale problem. The martingale part will converge to (the gradient of a) Gaussian white noise, while with the help of the reversibility of the process and the information on the limit dynamic covariance (2.11) we will prove that the drift is the diffusion operator in (1.7) with $\sigma_{\alpha\beta} = q_{\alpha\beta}$. The martingale problem that arises has a unique solution, up to the initial conditions, which is the infinite-dimensional Ornstein–Uhlenbeck process (1.7). To have ξ stationary in time, the field at $t = 0$ must be distributed as the space derivative of a massless Gaussian field and this will allow us to conclude the proof.

3. The random walk representation. In this section we will establish the validity of a formula which represents space–time correlations of functionals of our process in terms of expectations over *a random walk in a dynamic random environment*. Our formula can be understood as the *nonlinear* analog of the standard random walk representation of the spatial correlations for the Gaussian massless field. In that case we would just be dealing with a simple random walk (see, e.g., [4, 11]). Our representation here is an infinite dimensional probabilistic version of the (H–S) PDE representation [14]. While the finite volume case is already contained in [14, 8], the infinite volume case requires some extra work.

For the rest of the paper we fix the tilt u once for all and therefore drop it from the expressions (in particular $\mu = \mu_u$). Moreover we will sometimes use the shorthand $\langle \cdot \rangle$, which stands for $\langle \cdot \rangle_{\mu_u}$.

3.1. *The evolution semigroup.* To introduce the generator of the dynamics we make use of the operator ∂_x defined right after (2.12). Here we observe that if F is the restriction to χ of a differentiable local function G: $\mathbb{R}^{\mathbb{Z}^{d^*}} \to \mathbb{R}$ then for $\eta \in \chi$,

$$\partial_x F(\eta) = \sum_{\alpha=1}^{d} \nabla_\alpha^* \frac{\partial G}{\partial \eta'_{x,\alpha}}(\eta')\bigg|_{\eta'=\eta} = \sum_{\alpha=1}^{d} \left(\frac{\partial G}{\partial \eta'_{x-e_\alpha,\alpha}}(\eta') - \frac{\partial G}{\partial \eta'_{x,\alpha}}(\eta') \right)\bigg|_{\eta'=\eta}. \tag{3.1}$$

As made explicit in (3.1), ∇ acts on the x variable in the expression to follow.

The pregenerator of the η-dynamics is defined as

$$LF(\eta) = -\sum_{x\in\mathbb{Z}^d} \left[\partial_x^2 F(\eta) - \left(\sum_{\alpha=1}^{d} \nabla_\alpha^* V'(\eta_{x,\alpha}) \right) \partial_x F(\eta) \right], \tag{3.2}$$

for F: $\chi \to \mathbb{R}$ local, bounded and smooth such that both $\partial_x F$ and $\partial_x^2 F$ are bounded and smooth for every x (call $\mathscr{D}_0$ this set of functions). By using the DLR equations for μ one can easily verify that for $F, G \in \mathscr{D}_0$

$$\mathscr{E}(F, G) \equiv \langle FLG \rangle = \sum_{x\in\mathbb{Z}^d} \langle \partial_x F \partial_x G \rangle \tag{3.3}$$

and therefore L is symmetric in $\mathscr{D}_0$. We extend L to a closed self-adjoint operator on $L^2(\chi; \mu)$ [with domain $\mathscr{D}(L)$]. We denote this extension still by L and by $\{e^{-Lt}\}_{t\geq 0}$ the $L^2(\mu)$-semigroup generated by L. Note that the adjoint ∂_x^* of ∂_x in $L^2(\mu)$ is $-\partial_x + \sum_{\alpha=1}^{d} \nabla_\alpha V'(\eta_{x-e_\alpha,\alpha})$. The equality in (3.3) still holds for F and G in $\mathscr{D}(L)$ which extends the definition of the Dirichlet form $\mathscr{E}$, provided we interpret the derivatives ∂_x in the weak sense: $F \in L^2(\mu)$ is weakly differentiable if for every $x \in \mathbb{Z}^d$ there exists a function $\partial_x F \in L^2(\mu)$ such that $\langle \partial_x F, G \rangle = \langle F, \partial_z^* G \rangle$ for every $G \in \mathscr{D}_0$.

We have already remarked in the Introduction that the evolution equation (2.8), with initial datum in χ_r [cf. (2.9)] has a unique solution in $C^0(\mathbb{R}^+; \chi_r)$. Since a trajectory typical for μ is in χ_r for any $r > 0$, the evolution is well posed when the initial datum is μ-typical. The construction of the associated probability translation semigroup in $L^2(\mu)$, $\{P_t\}_{t\geq 0}$, is carried out, for example, in [12], Section 3 and Theorem 4.2. In particular the two semigroups P_t and e^{-Lt} coincide.

3.2. *The random walk process.* A basic object for our analysis is a Markov process on $\mathbb{Z}^d \times \chi$ which we denote by $\{X(t), \eta(t)\}_{t\geq 0}$. It is easily constructed in the following way: given the process $\{\eta(t)\}_{t\geq 0}$, with initial condition $\eta(0) \in \chi_r$

for some $r > 0$, $\{X(t)\}_{t\geq 0}$ is the random walk which performs nearest neighbor jumps starting from $X(0) = x \in \mathbb{Z}^d$ with time dependent rates given by

$$V''(\eta_b(t)), \qquad b \in \mathbb{Z}^{d*}. \tag{3.4}$$

Since (2.2) is assumed, this is nondegenerate random walk with bounded rates and its construction is straightforward. We will denote the law of this process by $\mathbf{P}_{x,n}$ with (x, η) the initial condition (and by $\mathbf{E}_{x,\eta}$ the corresponding expectation).

The (pre)generator of this process is given by

$$[\mathscr{L} f](x, \eta) = [Lf(x, \cdot)](\eta) + \sum_{\alpha=1}^{d} [\nabla_\alpha^*(V''(\eta_{x,\alpha})\nabla_\alpha f)](x, \eta), \tag{3.5}$$

where $f(x, \eta)$ has compact support in x and is local and smooth in η. As in the case of L, we observe that $\mathscr{L}$ is symmetric with respect to $L^2(\Sigma \otimes \mu)$, where Σ is the counting measure on $\mathbb{Z}^d$. We keep the notation $\mathscr{L}$ for its closed self-adjoint extension.

We are now ready to state the main result of this section. We say that $F \in C^k_{\mathrm{loc}}(\chi)$, $k \in \mathbb{Z}^+$, if F is local and it is C^k as a function of its (finitely many) entries. We also set $(\partial F)(x, \eta) \equiv \partial_x F(\eta)$.

PROPOSITION 3.1. *For any F and G in $C^2_{\mathrm{loc}} \cap L^2(\chi; \mu)$, such that both LF and $LG \in L^2(\chi; \mu)$, and any $\tau \geq 0$,*

$$\begin{aligned} &\mathbb{E}_\mu[F(\eta(0))G(\eta(\tau))] - \mathbb{E}_\mu[F(\eta(0))]\mathbb{E}_\mu[G(\eta(\tau))] \\ &\qquad = \int_0^\infty \sum_{x\in\mathbb{Z}^d} \langle (\partial F)(x, \eta)\mathbf{E}_{x,\eta}[(\partial G)(X(t+\tau), \eta(t+\tau))]\rangle_\mu \, dt. \end{aligned} \tag{3.6}$$

We will use the following lemma.

LEMMA 3.2. *For any $F \in C^2_{\mathrm{loc}} \cap L^2(\chi; \mu)$, such that $LF \in L^2(\chi; \mu)$ and such that $\langle F\rangle_\mu = 0$ we have that*

$$\lim_{t\to\infty} \mathbb{E}_\mu[F(\eta(0))F(\eta(t))] = 0. \tag{3.7}$$

PROOF. First of all we claim that it is sufficient to prove the statement for F uniformly Lipschitz with respect to its finitely many arguments. In fact, for general $F \in C^2_{\mathrm{loc}} \cap L^2(\chi; \mu)$, let us define $F_l(\eta) = F(\eta)\chi_l(\eta) - \langle F\chi_l\rangle$ where $l > 0$ and χ_l is a suitable smooth function equal to 1 if $\sum_\alpha \sum_{\|x\|\leq R} \eta^2_{x,\alpha} \leq l$ and equal to 0 if the same sum is larger than $2l$, with R larger than the diameter of the support of F. Since F is locally Lipschitz, F_l is uniformly Lipschitz, and since $\lim_{l\to\infty} F_l = F$ in $L^2(\chi, \mu)$, the claim is proved.

Then let us set $u(\eta, t) = \mathbb{E}_\eta F(\eta(t))$. Consider (as in [13], Proposition 2.1) the coupled process $\{\eta(t), \bar\eta(t)\}_{t\geq 0}$, where η and $\bar\eta$ are two solutions of (2.8). We take $\eta(0)$ and $\bar\eta(0)$ to be independent and distributed according to μ. The law of this coupled process will be denoted by $\mathbb{P}^{\otimes 2}$. By applying twice Jensen's inequality we obtain that

$$\begin{aligned} \operatorname{var}_\mu(u(\cdot,t)) &\le \int\int [u(\eta,t)-u(\bar\eta,t)]^2\,d\mu(\eta)\,d\mu(\bar\eta) \\ &\ge \mathbb{E}^{\otimes 2}[(F(\eta(t))-F(\bar\eta(t)))^2] \\ &\le C\max_{x,\alpha}\mathbb{E}^{\otimes 2}\big[(\eta_{x,\alpha}(t)-\bar\eta_{x,\alpha}(t))^2\big], \end{aligned} \tag{3.8}$$

where C is the Lipschitz constant of F and the maximum is over a finite number of bonds. Since by [13], Proposition 2.1, for all x and α,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}^{\otimes 2}\big[(\eta_{x,\alpha}(t)-\bar\eta_{x,\alpha}(t))^2\big]\,dt = 0, \tag{3.9}$$

and since, by (3.3) and the fact that $u(\cdot,t)\in\mathscr{D}(L)$ [because $F\in\mathscr{D}(L)$], for any $t>0$,

$$\frac{d}{dt}\operatorname{var}_\mu(u(\cdot,t)) = -\langle u(\cdot,t)Lu(\cdot,t)\rangle \le 0, \tag{3.10}$$

we conclude that

$$\lim_{t\to\infty}\operatorname{var}_\mu(u(\cdot,t)) = 0. \tag{3.11}$$

Therefore by conditioning and by using the Cauchy–Schwarz inequality, we obtain (3.7). □

PROOF OF PROPOSITION 3.1. It is sufficient to prove the result for $F=G$ and $\langle F\rangle = 0$. By applying Lemma 2.2 we obtain that

$$\mathbb{E}_\mu[F(\eta(0))F(\eta(t))] = \int_0^\infty \langle FLe^{-(t+s)L}F\rangle\,dt, \tag{3.12}$$

and using (3.3) we can rewrite

$$\langle FLe^{-(t+s)L}F\rangle = \sum_{x\in\mathbb{Z}^d}\langle \partial_x F(\eta)\partial_x e^{-(t+s)L}F(\eta)\rangle. \tag{3.13}$$

The aim is to *commute* ∂_x and the semigroup e^{-tL} guided by the observation that for $G\in C^3_{\mathrm{loc}}(\chi)$,

$$\partial_x LG = L\partial_x G + \sum_{\alpha=1}^d \nabla^*_\alpha[V''(\eta_{x,\alpha})\nabla_\alpha\partial_x G], \tag{3.14}$$

where the operators L and ∂ act on η and ∇ acts on x; that is,

$$\partial_x LG(\eta) = (\mathscr{L}(\partial G))(x,\eta). \tag{3.15}$$

More precisely we will show that

$$\mathbb{E}_\mu[F(\eta(0))F(\eta(t))] = \int_0^\infty \sum_{x\in\mathbb{Z}^d}\big\langle(\partial F)(x,\eta)\big[e^{-(t+\tau)\mathscr{L}}(\partial F)\big](x,\eta)\big\rangle\,d\tau. \tag{3.16}$$

To prove (3.16) it is sufficient to show that for every $t > 0$,

$$\partial e^{-tL} F = e^{-t\mathscr{L}}(\partial F), \tag{3.17}$$

where the terms in both sides of (3.17) are clearly in $L^2(\Sigma \otimes \mu)$. Equation (3.17) is a consequence of the following lemma.

LEMMA 3.3. *For any $\lambda > 0$ and F as above*,

$$\partial_x(\lambda + L)^{-1} F(\eta) = (\lambda + \mathscr{L})^{-1}(\partial F)(x, \eta). \tag{3.18}$$

Given this lemma, the proof of Proposition 3.1 is as follows. Equation (3.18) can be rewritten as

$$\partial_x \int_0^\infty e^{-\lambda t} e^{-tL} F(\eta)\, dt = \int_0^\infty e^{-\lambda t} e^{-t\mathscr{L}}(\partial F)(x, \eta)\, dt. \tag{3.19}$$

Observe that, since $F \in \mathscr{D}(L)$, $\partial e^{tL} F \in L^2(\Sigma \otimes \mu)$ for every $t \geq 0$. We claim also that if $G \in \mathscr{D}(L)$, then $\partial_x^h G$, defined for $h > 0$ as $[G(\eta + h\xi^x) - G(\eta)]/h$, $\xi_b^x = \nabla_b \varphi^{\{x\}}$ with $\varphi_y^{\{x\}} = \mathbf{1}_x(y)$, converges in $L^2(\mu)$ to $\partial_x G$. This follows because for μ-a.e. $\eta \partial_x^h G(n) = (1/h) \int_0^h \partial_x G(\eta + s\xi^x)\, ds$ and therefore, by Jensen's inequality, we have that

$$\langle [\partial_x^h G - \partial_x G]^2 \rangle \leq \frac{1}{h} \int_0^h \langle [\partial_x G(\cdot + s\xi^x) - \partial_x G(\cdot)]^2 \rangle\, ds. \tag{3.20}$$

However, the integrand in the right-hand side of (3.20) vanishes as $s \to 0$, by continuity of the L^2 norm, and the claim is proved. Therefore, by *approximating* ∂_x with ∂_x^h and by taking the limit [in $L^2(\mu)$] as $h \to 0$ one verifies that for every x,

$$\partial_x \int_0^\infty e^{-\lambda t} e^{-tL} F(\eta)\, dt = \int_0^\infty e^{-\lambda t} \partial_x e^{-tL} F(\eta)\, dt. \tag{3.21}$$

By (3.19), (3.21) and by applying the Cauchy–Schwarz inequality and the contractivity of the semigroup to justify the exchange of the order of integration, we obtain that for every $\lambda > 0$,

$$\begin{aligned} &\int_0^\infty \sum_x \langle \partial_x F(\eta) e^{-\lambda t} \partial_x e^{-tL} F(\eta) \rangle\, dt \\ &\qquad = \int_0^\infty \sum_x \langle \partial F(x, \eta) e^{-\lambda t} e^{-t\mathscr{L}}(\partial F)(x, \eta) \rangle\, dt \end{aligned} \tag{3.22}$$

and by using the fact that positive finite measures are identified by their Laplace transform on $\mathbb{R}^+$ combined with the fact that the left-hand side for $\lambda = 0$ is equal to $\langle F^2 \rangle$ and therefore it is finite, we obtain that for every $t \geq 0$,

$$\sum_x \langle \partial_x F(\eta) \partial_x e^{-tL} F(\eta) \rangle = \sum_x \langle \partial F(x, \eta) e^{-t\mathscr{L}}(\partial F)(x, \eta) \rangle. \tag{3.23}$$

By polarization we conclude that (3.17) holds and the proof of Proposition 3.1 is complete. □

PROOF OF LEMMA 3.3. Let u_λ be solution of the resolvent equation $\lambda u_\lambda + Lu_\lambda = F$. Then u_λ is in the domain of L, so $\partial_x u_\lambda \in L^2(\mu)$ exists for every x. Since $F \in \mathscr{D}(L)$, we have that $\partial_x L u_\lambda$ exists for every x too. We can therefore write $\lambda \partial_x u_\lambda + \partial_x L u_\lambda = \partial_x F$ and if we show that (3.14) holds with $G = u_\lambda$ we are done, since this establishes that $\{\partial u_\lambda\}_{\lambda > 0}$ is the resolvent of $\mathscr{L}$. This is once again established via approximation: choose $h > 0$ and set $\eta^{x,h} = \eta + h\xi^x$. We obtain

$$\partial_x^h L u_\lambda = L \partial_x^h u_\lambda + \sum_{\alpha=1}^d \nabla_\alpha^* [V'^{,h}(\eta_{x,\alpha}) \nabla_\alpha \partial_x u_\lambda(\eta^{x,h})], \tag{3.24}$$

where $V'^{,h}(\cdot) = [V'(\cdot + h) - V'(\cdot)]/h$. Since $\lim_{h\to 0} \sup_{\eta \in \mathbb{R}} |V'^{,h}(\eta) - V''(\eta)| = 0$ and $\partial_x u_\lambda(\eta^{x,h}) - \partial_x u_\lambda(\eta) \to 0$ in $L^2(\mu)$ as $h \to 0$, the last term in (3.24) converges in $L^2(\mu)$ to

$$\sum_{\alpha=1}^d \nabla_\alpha^* [V''(\eta_{x,\alpha}) \nabla_\alpha \partial_\alpha u_\lambda(\eta)]. \tag{3.25}$$

However, by (3.20), also the left-hand side as well as $\partial_x^h u_\lambda$ converge in $L^2(\mu)$, respectively, to $\partial_x L u_\lambda$ and to $\partial_x u_\lambda$. Since L is a closed operator, (3.14) is established if $G = u_\lambda$ and the proof of Lemma 3.3 is complete. □

4. Homogenization and effective diffusivity.

For any $\varepsilon > 0$ we set

$$X^\varepsilon(t) \equiv \varepsilon X(\varepsilon^{-2} t), \qquad t \geq 0. \tag{4.1}$$

Moreover, we will denote by $\{Y(t)\}_{t \in \geq 0}$ the $\mathbb{R}^d$-valued Brownian process with covariance matrix $q_u = q > 0$, as given by the variational formula (2.12), that is, the centered continuous Gaussian process with covariance

$$\mathbf{E}_0^q(Y_\alpha(t) Y_\beta(s)) = (t \wedge s) q_{\alpha\beta}, \qquad \alpha, \beta = 1, \dots, d \tag{4.2}$$

and $t, s \geq 0$. By $\mathbf{P}_z^q$ we will denote the same process starting from $z \in \mathbb{R}^d$ and by $\mathbf{E}_z^q$ the corresponding expectation.

The main result of this section is the following.

PROPOSITION 4.1. *As ε tends to zero, $X^\varepsilon(t)$ converges to $Y(t)$ weakly in the Skorohod space $D([0, T]; \mathbb{R}^d)$, for any $T \geq 0$.*

We follow the Kipnis–Varadhan approach [15, 7, 21]. Consider the process describing the environment as seen from the position of the random walk,

$$\tilde{\eta}(t) = \tau_{-X(t)} \eta(t). \tag{4.3}$$

We obtain in this way a process on χ. The pregenerator of this process, acting on $F \in \mathscr{D}_0$, is given by

$$\widetilde{\mathscr{L}} F(\eta) = LF(\eta) + \sum_{\alpha=1}^d D_\alpha^* [\widetilde{V}_\alpha''(\eta) D_\alpha F](\eta), \tag{4.4}$$

where $D_\alpha f(\eta) = F(\tau_\alpha \eta) - F(\eta)$ and $\widetilde{V}''_\alpha\colon \chi \to \mathbb{R}$ is defined by

$$\widetilde{V}''_\alpha(\eta) = \widetilde{V}''(\eta_{0,\alpha}). \tag{4.5}$$

As a consequence of the fact that L is self-adjoint and that the jump intensity is bounded, it is easy to see that $\widetilde{\mathscr{L}}$ is essentially self-adjoint in $L^2(\chi;\mu)$. The process is therefore reversible with respect to μ. Once again we keep the notation $\widetilde{\mathscr{L}}$ for its self-adjoint extension.

The process $\{\tilde{\eta}(t)\}_{t\geq 0}$ will always be considered at equilibrium; that is, $\tilde{\eta}(0)$ is distributed according to μ. The law of the process will be denoted by $\widetilde{\mathbb{P}}$.

An important ingredient for the Kipnis–Varadhan approach is the following lemma.

LEMMA 4.2. *The process $\{\tilde{\eta}(t)\}_{t\geq 0}$ is time ergodic, that is, any $F \in L^2(\chi;\mu)$ such that $e^{-\widetilde{\mathscr{L}}t}F = F$, for all $t \geq 0$, is constant μ-a.s.*

PROOF. For F local and smooth we introduce the Dirichlet form

$$\widetilde{\mathscr{E}}(F,F) \equiv \langle F, \widetilde{\mathscr{L}}F\rangle = \left\langle \sum_{x\in\mathbb{Z}^d} (\partial_x F)^2 + \sum_{\alpha=1}^{d} \widetilde{V}''_\alpha(\eta)[F(\tau_\alpha\eta) - F(\eta)]^2 \right\rangle \tag{4.6}$$

and, by the first line, we can extend the definition to any $F \in \mathscr{D}(\widetilde{\mathscr{L}})$. The second equality holds as well in this generality with ∂_x interpreted in the distribution sense, by a standard approximation procedure. Observe that, by definition of domain, $e^{-\widetilde{\mathscr{L}}t}F = F$ implies that $F \in \mathscr{D}(\widetilde{\mathscr{L}})$ and that $\widetilde{\mathscr{L}}F = 0$, therefore $\widetilde{\mathscr{D}}(F,F) = 0$. But by (2.2),

$$\widetilde{\mathscr{E}}(F,F) \geq C_- \left\langle \sum_{\alpha=1}^{d} [F(\tau_\alpha\eta) - F(\eta)]^2 \right\rangle, \tag{4.7}$$

which implies that $F(\tau_\alpha\eta) - F(\eta) = 0$ for every α, μ-a.s. Therefore, F is translation invariant and, since μ is shift ergodic, F is a constant μ-a.s. □

PROOF OF PROPOSITION 4.1. We want to apply Theorem 1.8 of [15] and we have therefore to verify its conditions. First, we observe that the position of the random walk $X(t)$ is an additive functional of the process $\tilde{\eta}(t)$ and it can be written as

$$X_\alpha(t) = M_\alpha(t) + \int_0^t (D_\alpha^* V''_\alpha)(\tilde{\eta}(s))\,ds, \tag{4.8}$$

where $M_\alpha(t)$ is a $(\widetilde{\mathbb{P}}, \widetilde{\mathscr{F}})$-martingale. $\widetilde{\mathscr{F}}$ is the natural filtration associated to $\{\tilde{\eta}(s)\}_{s\geq 0}$, such that

$$\widetilde{\mathbb{E}}[M_\alpha(t)M_\beta(t)] = 2t\langle V''_\alpha\rangle \delta_{\alpha\beta}. \tag{4.9}$$

Observe that $\langle D_\alpha^* V_\alpha'' \rangle = 0$ and that there exists a constant C, depending only on C_-, such that for all $\psi \in \mathscr{D}(\widetilde{\mathscr{L}})$,

$$\big\langle \psi(\eta) D_\alpha^* V_\alpha''(\eta) \big\rangle \le \big\langle (V_\alpha'')^2 \big\rangle^{1/2} \big\langle (D_\alpha \psi)^2 \big\rangle^{1/2} \le C \big\langle \psi \widetilde{\mathscr{L}} \psi \big\rangle^{1/2}, \tag{4.10}$$

which verifies the condition (1.14) of [15]. Since the requested time ergodicity is shown by Lemma 4.2 we can apply Theorem 1.8 of [15] and we have that $\varepsilon X(\varepsilon^{-2} t)$ converges as ε tends to zero to a $\mathbb{R}^d$-Brownian motion $y(t)$, weakly in the Skorohod topology (in any finite time interval).

We now turn to computing the effective diffusion matrix q of $y(t)$. It follows by a simple time-reversal argument (cf. Theorem 2.2 in [7] and its proof) that for every $v \in \mathbb{R}^d$,

$$\begin{aligned} \widetilde{\mathbb{E}}\bigg[\bigg(\sum_{\alpha=1}^d v_\alpha M_\alpha(t)\bigg)^2\bigg] &= \widetilde{\mathbb{E}}\bigg[\bigg(\sum_{\alpha=1}^d v_\alpha X_\alpha(t)\bigg)^2\bigg] \\ &\quad + \widetilde{\mathbb{E}}\bigg[\bigg(\int_0^t \sum_{\alpha=1}^d v_\alpha (D_\alpha^* V_\alpha'')(\tilde\eta(s))\, ds\bigg)^2\bigg]. \end{aligned} \tag{4.11}$$

From (4.9) and (4.11) it follows that

$$q_{\alpha\beta} = 2\langle V_\alpha'' \rangle \delta_{\alpha\beta} - 2\int_0^\infty \widetilde{\mathbb{E}}\big[(D_\alpha^* V_\alpha'')(\tilde\eta(t))(D_\beta^* V_\beta'')(\tilde\eta(0))\big]\, dt, \tag{4.12}$$

which is equivalent to the variational formula (2.12) (cf [7], [21]).

From (2.12) we obtain the bounds

$$2\sum_{\alpha=1}^d v_\alpha^2 \big\langle (V_\alpha'')^{-1} \big\rangle^{-1} \le v \cdot qv \le 2\sum_{\alpha=1}^d v_\alpha^2 \langle V_\alpha'' \rangle. \tag{4.13}$$

The upper bound is immediate; just take ψ to be constant in (2.12) or drop the second term in (4.12). The lower bound follows since, by translation invariance and Cauchy–Schwarz, we have that

$$\begin{aligned} v_\alpha^2 = \langle v_\alpha - D_\alpha \psi \rangle^2 &= \bigg\langle \Big[(v_\alpha - D_\alpha \psi)\sqrt{V_\alpha''}\Big] \frac{1}{\sqrt{V_\alpha''}} \bigg\rangle^2 \\ &\le \Big\langle (v_\alpha - D_\alpha \psi)^2 V_\alpha'' \Big\rangle \Big\langle (V_\alpha'')^{-1} \Big\rangle. \end{aligned} \tag{4.14}$$

From (2.2) and (4.13), we derive (2.13). In particular q is strictly positive. □

5. Equilibrium fluctuations. This section is devoted to the proof of Theorem 2.1. We break the proof into four main steps:

1. Computation of the limiting covariance of the fluctuation field, which will require the heat kernel estimates of Appendix B.
2. Boltzmann–Gibbs principle, that will identify the drift of the limiting process.
3. Tightness of the fluctuation field in a suitable path space.
4. Identification of the limit.

Each one of these points will be considered in a separate subsection.

5.1. *Computation of the limit covariance.*

PROPOSITION 5.1. *For every $\alpha, \beta \in \{1, \dots, d\}$ and any $f, g \in C_0^\infty$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu[\xi_\alpha^\varepsilon(f, \varepsilon^{-2}\tau)\xi_\beta^\varepsilon(g, 0)] = \int_{\mathbb{R}^d} k_\alpha k_\beta \frac{e^{-\tau k\cdot qk}\hat{g}(k)\hat{f}(k)^*}{k\cdot qk}\, dk. \tag{5.1}$$

PROOF. By introducing the notation $f_\varepsilon(\cdot) = f(\varepsilon\cdot)$ for any $f \in C_0^\infty$ and using (3.6), we can write the covariance of the ξ-fields as

$$\begin{aligned} &\mathbb{E}_\mu\big[\xi_\alpha^\varepsilon(f, \varepsilon^{-2}\tau)\xi_\beta^\varepsilon(g, 0)\big] \\ &\quad= \int_0^\infty \varepsilon^d \sum_{x\in\mathbb{Z}^d} \nabla_\beta^* g_\varepsilon(x)\mathbb{E}_\mu^x\big[\nabla_\alpha^* f_\varepsilon(X(\varepsilon^{-2}\tau + t))\big]\, dt \\ &\quad= \int_0^\infty \varepsilon^d \sum_{x\in\mathbb{Z}^d} \varepsilon^{-1}\nabla_\beta^* g_\varepsilon(x)\mathbb{E}_\mu^x\big[\varepsilon^{-1}\nabla_\alpha^* f_\varepsilon(X(\varepsilon^{-2}(t+\tau)))\big]\, dt \\ &\quad= \int_0^\infty \varepsilon^d \sum_{x\in\mathbb{Z}^d} \nabla_\beta^{\varepsilon\,*} g(z)\mathbb{E}_\mu^{z/\varepsilon}\big[\nabla_\alpha^{\varepsilon\,*} f(\varepsilon X(\varepsilon^{-2}(t+\tau)))\big]\, dt, \end{aligned} \tag{5.2}$$

where $\mathbb{P}_\mu^x = \int \mathbf{P}_{x,\eta}\, d\mu(\eta)$ and, in the last step, we have used the notation $\nabla_\alpha^\varepsilon f(\cdot) = \varepsilon^{-1}[f(\cdot + \varepsilon e_\alpha) - f(\cdot)]$ and the analogous notation for the adjoint.

For any fixed $T > 0$ we can apply the invariance principle of the previous section (Proposition 4.1) and by [2], Theorem 5.5, and the bounded convergence theorem we conclude that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^T \varepsilon^d \sum_{z\in\varepsilon\mathbb{Z}^d} \nabla_\beta^{\varepsilon *} g(z)\mathbb{E}_\mu^{z/\varepsilon}\big[\nabla_\alpha^{\varepsilon *} f(\varepsilon X(\varepsilon^{-2}(t+\tau)))\big]\, dt \\ &\qquad= \int_0^T \int_{\mathbb{R}^d} \partial_\beta g(z)\mathbf{E}_z^q[\partial_\alpha f(y(t+\tau))]\, dt\, dz. \end{aligned} \tag{5.3}$$

Let us set

$$R_T^\varepsilon = \left| \int_T^\infty \varepsilon^d \sum_{z\in\varepsilon\mathbb{Z}^d} h_1^\varepsilon(z)\mathbb{E}_\mu^{z/\varepsilon}\big[h_2^\varepsilon(\varepsilon X(\varepsilon^{-2}(t+\tau)))\big]\, dt\right|, \tag{5.4}$$

where

$$h_1^\varepsilon(z) = \nabla_\beta^{\varepsilon\,*} g(z), \qquad h_T^\varepsilon(z) = \nabla_\alpha^{\varepsilon *} f(z). \tag{5.5}$$

We have the following lemma.

LEMMA 5.2. *With the above definitions, for every $f, g \in C_0^\infty(\mathbb{R}^d)$,*

$$\lim_{T\to\infty} \sup_{\varepsilon\in(0,1)} R_T^\varepsilon = 0. \tag{5.6}$$

Once Lemma 5.2 is proved, the proof of Proposition 5.1 is complete, since it is immediate to verify that

$$\int_0^\infty \int_{\mathbb{R}^d} \partial_\beta g(z) \mathbf{E}_z^q[\partial_\alpha f(y(t+\tau))]\, dt\, dz = \int_{\mathbb{R}^d} k_\alpha k_\beta \hat{g}(k) \hat{f}(k) \frac{e^{-\tau k\cdot qk}}{k\cdot qk}\, dk. \quad \square \tag{5.7}$$

In order to prove Lemma 5.2 we need some notation. Let us first introduce the scalar product of $L^2(\varepsilon\mathbb{Z}^d)$,

$$(f, g)_\varepsilon = \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} f(x)g(x), \tag{5.8}$$

together with

$$\|f\|_{\varepsilon, p}^p = \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} |f(x)|^p, \tag{5.9}$$

and the corresponding $L^p(\varepsilon\mathbb{Z}^d)$ spaces, for $1 \le p < \infty$ and the standard definition of L^∞. With some abuse of notation, f and g can either be functions from $\varepsilon\mathbb{Z}^d$ to $\mathbb{R}$, or functions from $\mathbb{R}^d$ to $\mathbb{R}$. Finally, if A is an operator from $L^p(\varepsilon\mathbb{Z}^d)$ to $L^q(\varepsilon\mathbb{Z}^d)$, its norm will be donoted by $\|A\|_{p,q}$.

DEFINITION 5.1 (The inhomogeneous random walk process). Let $\{P^\varepsilon_{s,t}\}_{0\le s\le t}$ be the time inhomogeneous semigroup on $L^2(\varepsilon\mathbb{Z}^d)$ characterized by

$$\frac{d}{dt}\big(g, P^\varepsilon_{s,t} f\big)_\varepsilon = \varepsilon^d \sum_{x\in\mathbb{Z}^d} \sum_{\alpha=1}^d a^\varepsilon_{x,\alpha}(t) \nabla^\varepsilon_\alpha g(\varepsilon x) \nabla^\varepsilon_\alpha P^\varepsilon_{s,t} f(\varepsilon x), \tag{5.10}$$

for all t, s such that $0 \le s \le t < \infty$, all $g, f \in L^2(\varepsilon\mathbb{Z}^d)$ and $P^\varepsilon_{s,s}$ is the identity operator. In (5.10) $a^\varepsilon_{x,\alpha}(\cdot)$ is (for convenience) assumed to be C^0 and for all x and all $\varepsilon > 0$,

$$0 < C_- \le a^\varepsilon_{x,\alpha}(\cdot) \le C_+ < \infty. \tag{5.11}$$

PROOF OF LEMMA 5.2. The link between the semigroup $P^\varepsilon_{s,t}$ and the random walk X^ε is made once we fix a trajectory $\eta \in C^0(\mathbb{R}^+; \chi)$ and define $a^\varepsilon_{x,\alpha}(t) = V''(\eta_{x,\alpha}(\varepsilon^{-2}t))$. Below we will always assume this choice. We distinguish the case $d \ge 3$ and the case $d = 2$, but the starting point is in common,

$$R^\varepsilon_T \le \sup_{\eta\in C^0(\mathbb{R}^+;\chi)} \int_T^\infty \big|\big(h^\varepsilon_1, P^\varepsilon_{0,t} h^\varepsilon_2\big)_\varepsilon\big|\, dt. \tag{5.12}$$

Therefore it is sufficient to show that there exists $\delta > 0$ and a constant $C = C(f, g) < \infty$ that

$$\big|\big(h^\varepsilon_1, P^\varepsilon_{0,t} h^\varepsilon_2\big)_\varepsilon\big| \le \frac{C}{t^{1+\delta}}, \tag{5.13}$$

for every $\varepsilon > 0$.

The case $d \geq 3$. In this case it is sufficient to use the bound $\|P^{\varepsilon}_{0,t}\|_{1,\infty} \leq C_1/t^{d/2}$, uniform in η (which is a direct consequence of Proposition B.2). In fact this implies

$$\left|\left(h^{\varepsilon}_1, P^{\varepsilon}_{0,t} h^{\varepsilon}_2\right)_{\varepsilon}\right| \leq \left\|h^{\varepsilon}_1\right\|_{\varepsilon,1} \left\|h^{\varepsilon}_2\right\|_{\varepsilon,1} \frac{C_1}{t^{d/2}}, \tag{5.14}$$

and, since f and g are in C_0^{∞}, (5.13) is proved provided $d \geq 3$.

The case $d = 2$. In this case we have to exploit the fact that

$$\varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} h^{\varepsilon}_1(x) = 0, \tag{5.15}$$

for every $\varepsilon > 0$. We keep the notation of a general d, since the estimates work in all dimensions. Let us define $h^{\varepsilon}_{2,t} = P^{\varepsilon}_{t/2,t} h^{\varepsilon}_2$ and observe that, by (5.15) and the semigroup property,

$$\begin{aligned}\left(h^{\varepsilon}_1, P^{\varepsilon}_{0,t} h^{\varepsilon}_2\right)_{\varepsilon} &= \left(h^{\varepsilon}_1, P^{\varepsilon}_{0,t} h^{\varepsilon}_{2,t}\right)_{\varepsilon} \\ &= \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} h^{\varepsilon}_1(x)\left[P^{\varepsilon}_{0,t/2} h^{\varepsilon}_{2,t}(x) - P^{\varepsilon}_{0,t/2} h^{\varepsilon}_{2,t}(0)\right].\end{aligned} \tag{5.16}$$

By Proposition B.6,

$$\begin{aligned}&\varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} h^{\varepsilon}_1(x)\left[P^{\varepsilon}_{0,t/2} h^{\varepsilon}_{2,t}(x) - P^{\varepsilon}_{0,t/2} h^{\varepsilon}_{2,t}(0)\right] \\ &\qquad \leq C_2\left[\varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}} \left|h^{\varepsilon}_1(x)\right| \left(\frac{|x|}{\sqrt{t/2}}\right)^{\varrho}\right] \left\|h^{\varepsilon}_{2,t}\right\|_{\infty}\end{aligned}$$

and applying again Proposition B.2, using $f, g \in C_0^{\infty}$, we obtain that, if $d = 2$, (5.13) is verified with $\delta = \varrho/2$. $\square$

5.2. *Boltzmann–Gibbs principle.* By (2.8), or (3.2), (2.10) and the scaling properties of Brownian motion, we have that

$$\xi^{\varepsilon}_{\alpha}(f, \varepsilon^{-2} t) = \xi^{\varepsilon}_{\alpha}(f, 0) + \int_0^t \gamma^{\varepsilon}_{\alpha}\left(\eta(\varepsilon^{-2} s)\right) ds + M^{\varepsilon}_{\alpha}(f, t), \tag{5.17}$$

where

$$\gamma^{\varepsilon}_{\alpha}(\eta) = -\varepsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \sum_{\beta=1}^{d} V'_{\beta}(\eta) \varepsilon^{-2} \nabla_{\beta} \nabla^*_{\alpha} f(\varepsilon x) \tag{5.18}$$

and $M^{\varepsilon}_{\alpha}(f, t)$ is a martingale that has the same law of

$$\varepsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \varepsilon^{-1} \nabla^*_{\alpha} f_{\varepsilon}(x) \sqrt{2} B_x(t). \tag{5.19}$$

For ease of notation we set $\gamma^{\varepsilon}_{\alpha}(s) = \gamma^{\varepsilon}_{\alpha}(\eta(\varepsilon^{-2} s))$. The following lemma will identify the limit drift.

LEMMA 5.3. *For any $f \in C_0^\infty(\mathbb{R}^d)$, any $\alpha \in 1, \dots, d$ and any $T > 0$,*

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu\left[\left(\int_0^t \left(\gamma_\alpha^\varepsilon(s) - \xi_\alpha^\varepsilon(Af, s)\right) ds\right)^2\right] = 0, \tag{5.20}$$

where

$$A = -\sum_{\beta, \theta=1}^{d} q_{\theta\beta} \frac{\partial}{\partial x_\theta} \frac{\partial}{\partial x_\beta}. \tag{5.21}$$

PROOF. Lemma 5.3 is a direct consequence of (5.1) and the reversibility of the process. In fact for every $g \in C_0^\infty$ by time reversal we have that

$$\mathbb{E}_\mu\left[(\xi_\alpha^\varepsilon(g, t) - \xi_\alpha^\varepsilon(g, 0)) \int_0^t \gamma_\alpha^\varepsilon(s)\, ds\right] = 0. \tag{5.22}$$

Then we can rewrite the argument of the limit on the left-hand side of (5.20) as

$$\begin{aligned}
&\mathbb{E}_\mu\left[\left(\left(\xi_\alpha^\varepsilon(f, t) - \xi_\alpha^\varepsilon(f, 0)\right) - M_\alpha^\varepsilon(f, t) - \int_0^t \xi_\alpha^\varepsilon(Af, s)\, ds\right)^2\right] \\
&\qquad = -\mathbb{E}_\mu\left[\left(\xi_\alpha^\varepsilon(f, t) - \xi_\alpha^\varepsilon(f, 0)\right)^2\right] + \mathbb{E}_\mu\left[M_\alpha^\varepsilon(f, t)^2\right] \\
&\qquad\quad + \mathbb{E}_\mu\left[\left(\int_0^t \xi_\alpha^\varepsilon(Af, s)\, ds\right)^2\right] + 2\mathbb{E}_\mu\left[M_\alpha^\varepsilon(f, t) \int_0^t \xi_\alpha^\varepsilon(Af, s)\, ds\right].
\end{aligned} \tag{5.23}$$

Using the martingale property and again (5.22) we can rewrite the last term as

$$\begin{aligned}
&2\mathbb{E}_\mu\left[M_\alpha^\varepsilon(f, t) \int_0^t \xi_\alpha^\varepsilon(Af, s)\, ds\right] \\
&\qquad = 2 \int_0^t \mathbb{E}_\mu[M_\alpha^\varepsilon(f, s) \xi_\alpha^\varepsilon(Af, s)]\, ds \\
&\qquad = 2 \int_0^t \mathbb{E}_\mu\left[M_\alpha^\varepsilon(f, s)\left(\xi_\alpha^\varepsilon(Af, s) - \xi_\alpha^\varepsilon(Af, 0)\right)\right] ds \\
&\qquad = 2 \int_0^t \mathbb{E}_\mu\left[\left(\xi_\alpha^\varepsilon(f, s) - \xi_\alpha^\varepsilon(f, 0)\right)\left(\xi_\alpha^\varepsilon(Af, s) - \xi_\alpha^\varepsilon(Af, 0)\right)\right] ds \\
&\qquad = 4 \int_0^t \left(\mathbb{E}_\mu\left[\xi_\alpha^\varepsilon(f, 0) \xi_\alpha^\varepsilon(Af, 0)\right] - \mathbb{E}_\mu\left[\xi_\alpha^\varepsilon(f, s) \xi_\alpha^\varepsilon(Af, 0)\right]\right) ds,
\end{aligned} \tag{5.24}$$

and we can compute now the limit of each term of the right-hand side,

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu\left[\left(\xi_\alpha^\varepsilon(f, t) - \xi_\alpha^\varepsilon(f, 0)\right)^2\right] = 2 \int k_\alpha^2 \hat{f}(k)^2 \frac{1 - e^{-(k, qk)t}}{k \cdot qk}\, dk, \tag{5.25}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu\left[M_\alpha^\varepsilon(f, t)^2\right] = 2t \int k_\alpha^2 \hat{f}(k)^2\, dk, \tag{5.26}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu\left[\left(\int_0^t \xi^\varepsilon_\alpha(Af,s)\,ds\right)^2\right] \tag{5.27}$$
$$= 2\int k_\alpha^2(k\cdot qk)\hat f(k)^2\frac{\int_0^t ds_2\int_0^{s_2} ds_1 e^{-(k,\,qk)(s_2-s_1)}}{k\cdot qk}\,dk$$
$$= 2\int k_\alpha^2\hat f(k)^2\left(t-\frac{1-e^{-(k,\,qk)t}}{k\cdot qk}\right)dk,$$

$$\lim_{\varepsilon\to 0}4\int_0^t\left(\mathbb{E}_\mu\left[\xi^\varepsilon_\alpha(f,0)\xi^\varepsilon_\alpha(Af,0)\right]-\mathbb{E}_\mu\left[\xi^\varepsilon_\alpha(f,s)\xi^\varepsilon_\alpha(Af,0)\right]\right)ds \tag{5.28}$$
$$= -4\int_0^t ds\int k_\alpha^2\,(k\cdot qk)\,\hat f(k)^2\frac{1-e^{-(k,qk)s}}{k\cdot qk}\,dk$$
$$= -4\int k_\alpha^2\,\hat f(k)^2\left(t-\frac{1-e^{-(k,\,qk)t}}{k\cdot qk}\right)dk$$

Putting everything together, the sum in (5.23) vanishes. □

5.3. *Tightness.* We will look at the process ξ^ε_α on a fixed time interval $[0,T]$, as a continuos process on the Hilbert space $\mathscr{H}_{-\kappa}$, of which we recall the definition. Let $\mathscr{H}_\kappa$ be the Sobolev space associated to the scalar product

$$(\phi,\psi)_\kappa=\int_{\mathbb{R}^d}\phi(r)(|r|^2-\Delta)^\kappa\psi(r)\,dr \tag{5.29}$$

and let $\mathscr{H}_{-\kappa}$ be its dual space (with respect to the L^2 scalar product).

LEMMA 5.4. *For every $\kappa>1+d$ and every $T\ge 0$, the family of measures $\{\mathbb{P}^\varepsilon\}_{\varepsilon>0}$ is relatively compact on $C([0,T];\mathscr{H}_{-\kappa})$.*

Lemma 5.4 is, by standard arguments, an immediate consequence of the following lemma.

LEMMA 5.5. *For $\kappa>1+d$ and for any $T>0$ we have that*:

(i) *The sequence of random $\mathscr{H}_{-\kappa}$-valued processes $\{\xi^\varepsilon_\alpha\}_{\varepsilon\in(0,1)}$ is equibounded in $L^2(\mathbb{P}_\mu)$,*

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}_\mu\left(\sup_{t\in[0,T]}\left\|\xi^\varepsilon_\alpha(t)\right\|^2_{-\kappa}\right)<\infty. \tag{5.30}$$

(ii) *The sequence $\{\xi^\varepsilon_\alpha\}_{\varepsilon\in(0,1)}$ is equicontinuous in $L^2(\mathbb{P}_\mu)$;*

$$\lim_{\delta\searrow 0}\sup_{\varepsilon\in(0,1)}\mathbb{E}_\mu\left(\sup_{t,\,s\in[0,T]:|t-s|\le\delta}\left\|\xi^\varepsilon_\alpha(t)-\xi^\varepsilon_\alpha(s)\right\|^2_{-\kappa}\right)=0. \tag{5.31}$$

We will use the following equilibrium estimate on the drift of ξ^ε, defined in (5.18).

LEMMA 5.6. *There exists* $c > 0$ *such that*

$$\sup_{\varepsilon\in(0,1)} \langle (\gamma_\alpha^\varepsilon)^2\rangle \le c\int_{\mathbb{R}^d} [\Delta f(x)]^2\,dx. \tag{5.32}$$

PROOF. By Proposition (3.1), explicit computations and summation by parts we obtain

$$\begin{aligned}\langle (\gamma_\alpha^\varepsilon)^2\rangle &= \sum_{x\in\mathbb{Z}^d} \langle \partial\gamma_\alpha^\varepsilon(x,\eta)[\mathscr{L}^{-1}\partial\gamma_\alpha^\varepsilon](x,\eta)\rangle\\ &= \varepsilon^d \sum_{x\in\mathbb{Z}^d} \langle p_{\varepsilon,\alpha}(x,\eta)(\nabla_\alpha \mathscr{L}^{-1}\nabla_\alpha^* p_{\varepsilon,\alpha})(x,\eta)\rangle,\end{aligned} \tag{5.33}$$

where $p_{\varepsilon,\alpha}$: $\mathbb{Z}^d \times \chi \to \mathbb{R}$ is defined as

$$p_{\varepsilon,\alpha}(x,\eta) = V''(\eta_{x,\alpha})g_\varepsilon(x) \tag{5.34}$$

with

$$g_\varepsilon(x) = \sum_{\beta=1}^d \nabla_\alpha^{\varepsilon *}\nabla_\beta^\varepsilon f_\varepsilon(x), \qquad x\in\mathbb{Z}^d. \tag{5.35}$$

Let us set $\widehat{C} = \{g$: $\mathbb{Z}^d\times\chi\to\mathbb{R}$: there exists $R>1$ such that $g(x,\cdot)\equiv 0$ for $|x|\ge R$ and $g(x,\cdot)\in C_b^\infty(\chi)\}$. By using the variational expression for the last term in (5.33), together with the lower bound in (2.2) we obtain the following chain of inequalities:

$$\begin{aligned}\langle (\gamma_\alpha^\varepsilon)^2\rangle &= \sup_{g\in\widehat{C}}\Big[2\sum_{x\in\mathbb{Z}^d}\langle g(x,\eta)\nabla_\alpha^* p_{\varepsilon,\alpha}(x,\eta)\rangle - \sum_{x\in\mathbb{Z}^d}\langle g(x,\eta)(\mathscr{L}g)(x,\eta)\rangle\Big]\\ &\le \sup_{g\in\widehat{C}}\Big[2\sum_{x\in\mathbb{Z}^d}\langle g(x,\eta)\nabla_\alpha^* p_{\varepsilon,\alpha}(x,\eta)\rangle - C_-\sum_{x\in\mathbb{Z}^d}\sum_{\beta=1}^d\langle[(\nabla_\beta g)(x,\eta)]^2\rangle\Big]\\ &\le \Big\langle \sup_{g\in\widehat{C}}\Big[2\sum_{x\in\mathbb{Z}^d} g(x,\eta)\nabla_\alpha^* p_{\varepsilon,\alpha}(x,\eta) - C_-\sum_{x\in\mathbb{Z}^d}\sum_{\beta=1}^d[(\nabla_\beta g)(x,\eta)]^2\Big]\Big\rangle\\ &\le \frac{1}{C_-}\varepsilon^d\sum_{x\in\mathbb{Z}^d}\langle p_{\varepsilon,\alpha}(x,\eta)(\nabla_\alpha(-\Delta_1)^{-1}\nabla_\alpha^* p_{\varepsilon,\alpha})(x,\eta)\rangle\\ &\le \frac{1}{C_-}\varepsilon^d\sum_{x\in\mathbb{Z}^d}\langle (V_\alpha''(\eta_{x,\alpha}))^2\rangle\, g_\varepsilon(x)^2 \le \frac{C_+{}^2}{C_-}\varepsilon^d\sum_{x\in\mathbb{Z}^d} g_\varepsilon(x)^2.\end{aligned} \tag{5.36}$$

In the last inequality above we have used the fact that $-\nabla_\alpha^*\Delta_1^{-1}\nabla_\alpha$ in Fourier space is multiplication by

$$2(1-\cos k_\alpha)\Big(2\sum_{\beta=1}^d(1-\cos k_\beta)\Big)^{-1} \le 1. \tag{5.37}$$

By recalling the definition of g_ε, again by Fourier decomposition, we verify that there exists c_1, independent of ε, such that

$$\varepsilon^d \sum_{x\in\mathbb{Z}^d} [g_\varepsilon(x)]^2 \le c_1 \int_{\mathbb{R}^d} (\Delta f(x))^2\, dx \tag{5.38}$$

and the proof of Lemma 5.6 is complete. □

PROOF OF LEMMA 5.5. First of all we give a more convenient representation of the $\|\cdot\|_\kappa$. Let us introduce h_n, the (normalized) Hermite polynomial of order $n \in \mathbb{Z}^+$ and let us set $\lambda_n = 2n+1$. For $\mathbf{n} \in (\mathbb{Z}^+)^d$ let us set also $h_{\mathbf{n}}(q) = \prod_{i=1}^d h_{n_i}(q_i)$ and $\lambda_{\mathbf{n}} = \sum_{i=1}^d \lambda_{n_i}$. Since the normalized Hermite polynomials form a base of $L^2(\mathbb{R}^d)$ which diagonalizes the operator $q^2 - \Delta$ we have that

$$\|\xi_\alpha^\varepsilon(t)\|_{-\kappa}^2 = \sum_{\mathbf{n}\in\mathbb{N}^d} \lambda_{\mathbf{n}}^{-\kappa} \xi_\alpha^\varepsilon(h_{\mathbf{n}}, t)^2 \tag{5.39}$$

and an analogous expression for $\xi_\alpha^\varepsilon(t) - \xi_\alpha^\varepsilon(s)$. We now use the following general result, valid for reversible Markov processes. For any $T > 0$ and any $G \in \mathscr{D}(L)$,

$$\mathbb{E}_\mu\left(\sup_{0\le t\le T} |G(\eta(t))|^2\right) \le 3\langle |G(\eta)|^2\rangle + 72T\mathscr{E}(G,G). \tag{5.40}$$

This inequality (inspired by [19]) can be proved observing that (using Doob's inequality)

$$\begin{aligned}\mathbb{E}_\mu&\left[\sup_{0\le t\le T} |G(\eta(t))|^2\right]\\ &\le 3\langle |G(\eta)|^2\rangle + 3\mathbb{E}_\mu\left(\sup_{0\le t\le T}\left|\int_0^t LG(\eta(s))\,ds\right|^2\right) + 24T\mathscr{E}(G,G)\end{aligned} \tag{5.41}$$

then using Proposition 3.4 of [17],

$$\mathbb{E}_\mu\left(\sup_{0\le t\le T}\left|\int_0^t LG(\eta(s))\,ds\right|^2\right) \le 16T\langle LG\, L^{-1}LG\rangle = 16T\mathscr{E}(G,G) \tag{5.42}$$

(which is valid for any Markov process).

Now we claim that

$$\mathbb{E}_\mu\left(\sup_{0\le t\le T} |\xi_\alpha^\varepsilon(f,t)|^2\right) \le c_1\|f\|_2^2 + c_2 T\|\underline{\partial} f\|_2^2, \tag{5.43}$$

where the constants c_1 and c_2 are independent of f, ε, and T. This is because

$$\langle \xi_\alpha^\varepsilon(f)\varepsilon^{-2}L\xi_\alpha^\varepsilon(f)\rangle = \varepsilon^d \sum_{x\in\mathbb{Z}d}\left[\varepsilon^{-1}\nabla_\alpha^* f_\varepsilon(x)\right]^2 \le c\|\nabla f\|_2^2, \tag{5.44}$$

where c is a constant independent of f and ε. Moreover, by the Brascamp–Lieb inequality [cf. [3]; it can also be obtained directly from (5.2)],

$$\begin{aligned}\langle|\xi_\alpha^\varepsilon(f)|^2\rangle &\le \frac{1}{C_-}\varepsilon^d \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \nabla_\alpha^* f_\varepsilon(x)\varepsilon^{-2}(-\Delta_1)^{-1}(x,y)\nabla_\alpha^* f_\varepsilon(y)\\ &= \frac{1}{C_-}\varepsilon^{2d} \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \nabla_\alpha^* f_\varepsilon(x)(-\Delta_1)^{-1}(\varepsilon x,\varepsilon y)\nabla_\alpha^* f_\varepsilon(y)\\ &\le \frac{2}{C_-}\int_{\mathbb{R}^d} f^2(r)\,dr,\end{aligned}\tag{5.45}$$

where $(-\Delta_1)^{-1}(x,y)$ is the matrix element of Δ_1^{-1} and we have used again (5.37). Therefore, by (5.44), (5.45) and (5.40), the claim (5.43) is proved. The proof of the first part of Lemma 5.5 is then a consequence of (5.43) and (5.39),

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}\|\xi_\alpha^\varepsilon(t)\|_{-\kappa}^2\right) &\le \mathbb{E}\left(\sup_{0\le t\le T}\sum_{\mathbf{n}\in\mathbb{Z}^{+d}}\lambda_{\mathbf{n}}^{-\kappa}\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)^2\right)\\ &\le C\sum_{\mathbf{n}\in\mathbb{N}^d}\lambda_{\mathbf{n}}^{-\kappa}(1+\lambda_{\mathbf{n}}),\end{aligned}\tag{5.46}$$

which is finite if $\kappa > d+1$ and therefore (5.30) is proved.

We are left with the proof of (5.31). We start by observing that

$$\begin{aligned}&\mathbb{E}_\mu\left(\sup_{t,\,s\in[0,T]:\ |t-s|\le\delta}\left\|\xi_\alpha^\varepsilon(t)-\xi_\alpha^\varepsilon(s)\right\|_{-\kappa}^2\right)\\ &\qquad\le \sum_{\mathbf{n}\in\mathbb{N}^d}\lambda_{\mathbf{n}}^{-\kappa}\mathbb{E}_\mu\left\{\sup_{t,\,s\in[0,T]:\ |t-s|\le\delta}\left[\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)-\xi_\alpha^\varepsilon(h_{\mathbf{n}},s)\right]^2\right\}\end{aligned}\tag{5.47}$$

and that for every $R\ge 1$,

$$\begin{aligned}&\sum_{\mathbf{n}\in\mathbb{N}^d:\ |\mathbf{n}|\ge R}\lambda_{\mathbf{n}}^{-\kappa}\mathbb{E}_\mu\left\{\sup_{t,\,s\in[0,T]:\ |t-s|\le\delta}\left[\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)-\xi_\alpha^\varepsilon(h_{\mathbf{n}},s)\right]^2\right\}\\ &\qquad\le 4\sum_{\mathbf{n}\in\mathbb{N}^d:\ |\mathbf{n}|\ge R}\lambda_{\mathbf{n}}^{-\kappa}\mathbb{E}_\mu\left\{\sup_{t\in[0,T]}\left[\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)\right]^2\right\}.\end{aligned}\tag{5.48}$$

Therefore, by the same computation as in (5.46), we obtain that for every $\delta>0$,

$$\lim_{R\to\infty}\sum_{\mathbf{n}\in\mathbb{N}^d:\ |\mathbf{n}|\ge R}\lambda_{\mathbf{n}}^{-\kappa}\mathbb{E}_\mu\left\{\sup_{t,\,s\in[0,T]:\ |t-s|\le\delta}\left[\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)-\xi_\alpha^\varepsilon(h_{\mathbf{n}},s)\right]^2\right\}=0.\tag{5.49}$$

Thanks to (5.49), we have reduced the proof of (5.47) to showing that for every $\mathbf{n}$,

$$\lim_{\delta\to 0}\mathbb{E}_\mu\left\{\sup_{t,\,s\in[0,T]:\ |t-s|\le\delta}\left[\xi_\alpha^\varepsilon(h_{\mathbf{n}},t)-\xi_\alpha^\varepsilon(h_{\mathbf{n}},s)\right]^2\right\}=0,\tag{5.50}$$

which is a consequence of the following argument. Going back to the definition of ξ_α^ε we write

$$\begin{aligned}\xi_\alpha^\varepsilon(h_{\mathbf{n}}, t) - \xi_\alpha^\varepsilon(h_{\mathbf{n}}, s) &= \int_s^t \gamma_\alpha^\varepsilon(\eta(\varepsilon^{-2}s'))\, ds' + \big[M_\alpha^\varepsilon(h_{\mathbf{n}}, t) - M_\alpha^\varepsilon(h_{\mathbf{n}}, s)\big] \\ &= I_1^\varepsilon(t, s) + I_2^\varepsilon(t, s),\end{aligned} \tag{5.51}$$

where the dependence on $h_{\mathbf{n}}$ in the right-hand side is kept implicit. By Cauchy–Schwarz, stationarity and Lemma 5.6,

$$\mathbb{E}_\mu\bigg\{\sup_{t,\, s\in[0,\, T]:\ |t-s|\le\delta} \big[I_1^\varepsilon(t, s)\big]^2\bigg\} \le \delta \int_0^T \mathbb{E}_\mu\Big[\big(\gamma_\alpha^\varepsilon(\eta(\varepsilon^{-2}s))\big)^2\Big]\, ds \le c\delta T, \tag{5.52}$$

for some $c = c(\mathbf{n})$ independent of ε and δ. For what concerns $I_2^\varepsilon(t, s)$ we simply observe that $M^\varepsilon(t)$ is equal in law to $c_\varepsilon B_t(0)$, $c_\varepsilon = \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}[(\nabla_\alpha^\varepsilon h_{\mathbf{n}\varepsilon})(x)]^2$. Therefore, by standard results on the Brownian motion, there exists $c = c(\mathbf{n})$ such that for every ε and δ,

$$\mathbb{E}_\mu\bigg\{\sup_{t,\, s\in[0,\, T]:\ |t-s|\le\delta} \big[I_2^\varepsilon(t, s)\big]^2\bigg\} \le c\delta T, \tag{5.53}$$

and also (5.31) is proved. □

5.4. *Identification of the limit.*

COMPLETION OF THE PROOF OF THEOREM 2.1. The first part of this proof follows the standard line of [23] Section 2.10, but instead of identifying the initial condition, we use directly the fact that the process is stationary. By using Lemma 5.3 one shows that any limit process ξ solves a martingale problem which has a unique solution and coincides (in law) with a process, still denoted by ξ, which solves, for every $f \in \mathscr{H}_\kappa$, the linear stochastic equation

$$d\xi_\alpha(f, t) = -\xi_\alpha(Af, t)\, dt + \sqrt{2}\, d(W(\cdot, t), \partial_\alpha f), \tag{5.54}$$

where $A = -\underline{\partial}\cdot q\underline{\partial}$ and W is Gaussian space–time white noise, that is, standard cylindrical Wiener process. The solution of (5.54) can be rewritten in integral form as

$$\xi_\alpha(f, t) = \xi_\alpha(e^{-At}f, 0) + \sqrt{2}\int_0^t d(W, e^{-(t-s)A}\partial_\alpha f). \tag{5.55}$$

By (5.1) and the lower bound on the matrix q [cf. (2.13)],

$$\mathbb{E}^0\big[\big(\xi_\alpha\big(e^{-At}f, 0\big)\big)^2\big] \le \frac{1}{2C_-}\int_{\mathbb{R}^d} e^{-2tC_-|k|^2}\big|\hat f(k)\big|^2\, dk, \tag{5.56}$$

and therefore there exists a sequence $\{t_i\}_{i\in\mathbb{Z}^+}$, $\lim_{i\to\infty} t_i = \infty$, such that

$$\lim_{i\to\infty} \xi_\alpha(e^{-At_i}f, 0) = 0 \quad \mathbb{P}^0\text{-a.s.} \tag{5.57}$$

Hence it follows that any stationary distribution for (5.54) has to be Gaussian and, again from (5.56), we can read off that the only stationary field for (5.54) is the centered Gaussian field defined with covariance specified by (2.15) and we are done. This ends the proof of Theorem 2.1. □

APPENDIX A

The diffusion matrix q and the surface tension. We recall that the limit static fluctuations have the covariance operator $(-\underline{\partial} \cdot q\underline{\partial})^{-1}$ (Corollary 2.2). We would like to equate q with the surface tension (1.5). Here we establish only a corresponding finite volume identity.

From the DLR equations (2.4) we observe that the measure μ_u for the potential V is identical to the measure μ_0 with potential $V(\cdot + u_\alpha)$ for bonds directed along e_α. This latter choice we adopt for a torus Λ of sidelength N. The finite volume surface tension is then defined by

$$\sigma_\Lambda(u) = -\frac{1}{|\Lambda|} \log \int \exp\left[-\sum_{x\in\Lambda}\sum_{\alpha=1}^{d} V(\eta_{x,\alpha} + u_\alpha)\right] d\eta_\Lambda, \tag{A.1}$$

where $d\eta_\Lambda$ is the uniform measure on χ_Λ, that is, the Lebesgue measure constrained to η configurations of zero curl (cf. [13]). Then

$$\begin{aligned}\frac{\partial}{\partial u_\alpha}\frac{\partial}{\partial u_\beta}\sigma_\Lambda(u) &= \delta_{\alpha\beta}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle V''(\eta_{x,\alpha}+u_\alpha)\rangle \\ &\quad - \frac{1}{|\Lambda|}\left[\left\langle\left(\sum_{x\in\Lambda} V'(\eta_{x,\alpha}+u_\alpha)\right)\left(\sum_{y\in\Lambda} V'(\eta_{y,\beta}+u_\beta)\right)\right\rangle\right. \\ &\quad \left. - \left\langle\sum_{x\in\Lambda} V'(\eta_{x,\alpha}+u_\alpha)\right\rangle\left\langle\sum_{y\in\Lambda} V'(\eta_{y,\beta}+u_\beta)\right\rangle\right],\end{aligned} \tag{A.2}$$

where $\langle\cdot\rangle$ is the expectation with respect to μ_Λ, the Gibbs measure on χ_Λ. Let $\eta_b(t), b \in \Lambda^*$, be the stationary diffusion process governed by (2.8) with invariant measure μ_Λ and let $x(t)$ be the random walk on Λ with symmetric jump rates $V''(\eta_{x,\alpha}(t) + u_\alpha)$ through the bond $(x, x + e_\alpha)$. The joint process $(x(t), \eta(t))$ has the invariant measure $(|\Lambda|^{-1}\sum_{x\in\Lambda}\delta_x) \otimes \mu_\Lambda$ on $\Lambda \times \chi_\Lambda$. We lift the stationary $x(t)$ on Λ to X_t on $\mathbb{Z}^d$ by periodic extension. Then $\varepsilon X_{\varepsilon^{-2}t}$ converges to a $\mathbb{R}^d$-Brownian motion as $\varepsilon \to 0$ (see, e.g., [7]) with diffusion matrix

$$\begin{aligned} q^\Lambda_{\alpha\beta} &= 2\delta_{\alpha\beta}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle V''(\eta_{x,\alpha}+u_\alpha)\rangle \\ &\quad - 2\int_0^\infty dt\, \mathbf{E}\big[j_\alpha(x(0),\eta(0))\; j_\beta(x(t),\eta(t))\big],\end{aligned} \tag{A.3}$$

where

$$j_\alpha(x,\eta) = \nabla^*_\alpha V''(\eta_{x,\alpha}+u_\alpha). \tag{A.4}$$

Here $\mathbf{E}$ denotes the expectation with respect to the stationary $(x(t), \eta(t))$-process. We note that

$$\partial_x \sum_{y\in\Lambda} V'(\eta_{y,\alpha} + u_\alpha) = j_\alpha(x, \eta), \tag{A.5}$$

with the definition of ∂_x as in (3.1). Therefore by the same argument leading to (3.6), simplified by the fact that we are in finite volume, we have

$$\begin{aligned} &|\Lambda| \int_0^\infty \mathbf{E}[\,j_\alpha(x(0), \eta(0))\; j_\beta(x(t), \eta(t))]\,dt \\ &\quad = \left\langle \left(\sum_{x\in\Lambda} V'(\eta_{x,\alpha} + u_\alpha) \right) \left(\sum_{y\in\Lambda} V'(\eta_{y,\beta} + u_\beta) \right) \right\rangle \\ &\qquad - \left\langle \sum_{x\in\Lambda} V'(\eta_{x,\alpha} + u_\alpha) \right\rangle \left\langle \sum_{y\in\Lambda} V'(\eta_{y,\beta} + u_\beta) \right\rangle \end{aligned} \tag{A.6}$$

and we conclude that

$$q^\Lambda_{\alpha\beta} = \frac{\partial}{\partial u_\alpha} \frac{\partial}{\partial u_\beta} \sigma_\Lambda(u). \tag{A.7}$$

We did not succeed in proving that $q^\Lambda \to q$ nor that the Hessian of σ_Λ tends to the Hessian of σ as $\Lambda \nearrow \mathbb{Z}^d$.

There is a however a pay-off. Since the bounds in (4.13) can be easily reproduced here (in particular the lower bound), we obtain that $q^\Lambda \geq 2C_- \mathbb{I}$; that is, σ_Λ is strictly convex, uniformly in Λ. Therefore the surface tension σ is strictly convex (see [8], Section 3.4, for an alternative proof).

APPENDIX B

Nash–Aronson estimates. Let $X \equiv \{X(t)\}_{t\geq 0}$ be the random walk on $\mathbb{Z}^d, d \in \{1, 2, \ldots\}$, which, starting from $X(0) = y \in \mathbb{Z}^d$, performs nearest neighbor jumps with time dependent Poisson rates at site x, time t, in the direction e_i, the unit vector in the i direction, $i \in \{\pm 1, \ldots, \pm d\}$, given by $a_{x,i}(t)$. We assume that $a_{x,i}(\cdot) \in C^0(\mathbb{R}^+)$ for every x and i and that the jump rates are symmetric:

$$a_{x,i}(\cdot) = a_{x+e_i,-i}(\cdot) \quad \text{for all } x \in \mathbb{Z}^d, i \in \{\pm 1, \ldots, \pm d\}. \tag{B.1}$$

Moreover, the jump rates are nondegenerate and bounded:

$$0 < c_a^- \leq a_{x,i}(t) \leq c_a^+ < \infty \qquad \text{for all } x, i \text{ and } t \geq 0. \tag{B.2}$$

The Markov semigroup on $L^2(\mathbb{Z}^d)$ associated to the stochastic process X will be denoted by $P_{s,t}, 0 \leq s \leq t < \infty$. One can easily show that, if we set

$f_t = P_{s,t}f$ for $f \in L^2(\mathbb{Z}^d)$ and $t \geq s$, $f_t(x)$ is differentiable in $t \in (s, \infty)$ for every $x \in \mathbb{Z}^d$, and

$$\frac{d}{dt}(g, f_t) = -\mathscr{E}_t(g, f_t), \qquad g \in L^2(\mathbb{Z}^d), \tag{B.3}$$

in which we have introduced

$$\mathscr{E}_t(g, f) = \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{d} a_{x,i}(t) \nabla_i f(x) \nabla_i g(x) \tag{B.4}$$

and ∇ is the discrete gradient as before:

$$\nabla f(x) = (\nabla_1 f(x), \ldots, \nabla_d f(x)), \qquad \nabla_i f(x) = f(x + e_i) - f(x).$$

The adjoint of ∇ with respect to the L^2 scalar product will be denoted by ∇^*.

The purpose of this Appendix is to establish some estimates on the transition kernel p: $\mathbb{R}^+ \times \mathbb{Z}^d \times \mathbb{R}^+ \times \mathbb{Z}^d \to [0, 1]$, defined as

$$p(s, y; t, x) = P_{s,t} \mathbf{1}_x(y).$$

When needed, we will stress the dependence of p on the rates a by writing p_a. These estimates go under the name of Aronson bounds and Nash continuity estimate (see Proposition B.3, Proposition B.4 and Proposition B.6 below) and they are already available in the literature for diffusion processes on $\mathbb{R}^d$ with time dependent coefficients [1, 10]. In the discrete set up, that is, for random walks on $\mathbb{Z}^d$, the results in [5, 24] cover the case of time independent jump rates. We will now extend these results to cover our (time dependent) case. We will rely very heavily on [10], [5] and [24].

Diagonal estimates and Nash inequality. We start off by giving the Nash inequality for our semigroup. This is a step which is crucial for the whole Appendix. We will then derive an integral bound on the semigroup which follows rather directly from the Nash inequality.

LEMMA B.1 (Nash inequality). *There exists c_1, depending only on d, such that for all $f \in L^1(\mathbb{Z}^d)$,*

$$\|f\|_2^{2+4/d} \leq c_1 \|\nabla f\|_2^2 \|f\|_1^{4/d}. \tag{B.5}$$

Moreover,

$$\inf_{t \geq 0} \mathscr{E}_t(f, f) \geq \frac{c_a^-}{c_1} \frac{\|f\|_2^{2+4/d}}{\|f\|_1^{4/d}}. \tag{B.6}$$

PROOF (See [24], Lemma 1.7). Note that (B.6) follows immediately from (B.5) and (B.2). The left-hand side of (B.5) is a priori well defined since $\|f\|_2 \leq \|f\|_1$ (this is also the key observation to extend the original proof in $\mathbb{R}^d$ by Nash to $\mathbb{Z}^d$). $\square$

Let us now set $f_t = P_{0,t} f$, for $f \in L^1(\mathbb{Z}^d)$. By (B.4) and (B.2) we obtain

$$\frac{d}{dt}\|f_t\|_2^2 \leq -2c_a^- \|\nabla f_t\|_2^2.$$

Let us now assume that f is nonnegative. Therefore, $\|f_t\|_1 = \|f\|_1$ and, if we set $u(t) = \|f_t\|_2^2$, by (B.5) we obtain

$$u'(t) \leq -Ku(t)^{1+2/d}, \qquad K = \frac{2c_a^-}{C\|f\|_1^{4/d}},$$

which directly implies

$$\|f_t\|_2^2 \leq c_2 \|f\|_1^2 t^{-d/2}, \tag{B.7}$$

with c_2 depending only on d and c_a^-. Since $P_{0,t}$ is positivity preserving, we extend immediately (B.7) to every $f \in L^1(\mathbb{Z}^d)$ and the result can be restated as

$$\|P_{0,t}\|_{1\to 2} \leq \sqrt{c_2}\, t^{-d/4}. \tag{B.8}$$

Moreover, by duality, we have that

$$\|P_{0,t} f\|_\infty = \sup_{g:\, \|g\|_1 = 1} \Bigl(P_{0,t}^* g, f\Bigr) = \|P_{0,t}^*\|_{1\to 2} \|f\|_2,$$

where $P_{0,t}^*$ is the adjoint of $P_{0,t}$. It is, however, immediate to verify that $P_{0,t}^*$ is the semigroup associated to the random walk process in which $a_{x,i}(\cdot)$ is replaced by $a_{x,i}(t-\cdot)$. Therefore (B.8) holds also for $P_{0,t}^*$, since the bound depends only on (B.2). Putting everything together and observing that (B.8) holds obviously also for $P_{T,T+t}$, for any $T \geq 0$, we obtain that

$$\|P_{0,t}\|_{1\to\infty} = \|P_{t/2,t} \circ P_{0,t/2}\|_{1\to\infty} \leq \|P_{t/2,t}\|_{2\to\infty} \|P_{0,t/2}\|_{1\to 2} \leq c_3(d, c_a^-) t^{-d/2},$$

and from this we arrive at the following result.

PROPOSITION B.2 (Estimate on the diagonal). *There exists c, depending only on d and c_a^-, such that*

$$(f, P_{s,t} f) = \sum_{x,\, y \in \mathbb{Z}^d} p(s, y; t, x) f(x) f(y) \leq c \frac{\|f\|_1^2}{(t-s)^{d/2}}, \tag{B.9}$$

for every $f \in L^1(\mathbb{Z}^d)$ and every $t, s \in \mathbb{R}^+$, with $t \geq s$.

Off-diagonal estimates: upper bound. By taking advantage of some of the results proved in [5] and [24], which deal with the time independent setting, we will now establish the following Aronson bound.

PROPOSITION B.3 (Upper bound on the kernel). *There exists $C \in (1, \infty)$, depending only on d, c_a^- and c_a^+, such that*

$$p(s, y; t, x) \le \frac{C}{1 \vee (t-s)^{d/2}} \exp\left\{-\frac{|x-y|}{1 \vee (t-s)^{1/2}}\right\}, \tag{B.10}$$

for every $x, y \in \mathbb{Z}^d$ and every $t \ge s$.

PROOF. We follow the method by Davis and look for integral estimates for the evolution semigroup $e^{-\psi} P_{s,t} e^{\psi}$, where ψ is a bounded function from $\mathbb{Z}^d$ to $\mathbb{R}$. It is sufficient to prove the result for $s = 0$, since $p_a(s, y; t, x) = p_{\tilde{a}}(0, y; t-s, x)$, where $\tilde{a}(\cdot) = a(s + \cdot)$. We therefore set for $f \in L^2(\mathbb{Z}^d)$, $f \ge 0$,

$$f_t(x) = \exp\{-\psi(x)\} \sum_{y \in \mathbb{Z}^d} p(0, y; t, x) f(y) \exp\{\psi(x)\} \tag{B.11}$$

and we have

$$\partial_t \|f_t\|_{2p}^{2p} = -2p\mathscr{E}_t\left(e^{\psi} f_t, e^{-\psi} f_t^{2p-1}\right). \tag{B.12}$$

We apply Theorem 3.9 of [5] to obtain that for every $f \in L^\infty(\mathbb{Z}^d) \cap L^1(\mathbb{Z}^d)$, $f \ge 0$, and every $t \ge 0$,

$$\mathscr{E}_t\left(e^{\psi} f, e^{-\psi} f\right) \ge \mathscr{E}_t(f, f) - \Gamma(\psi)^2 \|f\|_2^2, \tag{B.13}$$

as well as

$$\mathscr{E}_t\left(e^{\psi} f, e^{-\psi} f^{2p-1}\right) \ge p^{-1}\mathscr{E}_t(f^p, f^p) - 9p\Gamma(\psi)^2 \|f\|_{2p}^{2p}, \tag{B.14}$$

for all $p \in [1, \infty)$, where

$$\Gamma(\psi)^2 = c_a^+ \left\| \sum_{i=1}^d \left[\exp\{|\nabla_i \psi|\} - 1\right]^2 \right\|_\infty. \tag{B.15}$$

We keep the power 2 in $\Gamma(\psi)^2$ for uniformity with [5]. By (B.12), (B.13) and the fact that the semigroup is positivity preserving, we easily obtain that

$$\|f_t\|_2 \le \exp\left\{\Gamma(\psi)^2 t\right\} \|f\|_2, \tag{B.16}$$

for every $f \in L^2(\mathbb{Z}^d)$. From (B.12), (B.14) and the Nash inequality (B.6) we obtain that for $p \in [2, \infty)$ and $f \in L^1(\mathbb{Z}^d) \cap L^\infty(\mathbb{Z}^d)$, $f \ge 0$,

$$\frac{d}{dt}\|f_t\|_{2p} = -\frac{c_a^-}{pc_1} \frac{\|f_t\|_{2p}^{1+4p/d}}{\|f_t\|_p^{4p/d}} + 9p\Gamma(\psi)^2 \|f_t\|_{2p}. \tag{B.17}$$

By [5], Theorem 3.25, (B.16) and (B.17) imply that there exists $c = c(d, c_a^-, c_a^+)$ such that for every $t \geq 0$ and every $x, y \in \mathbb{Z}^d$,

$$p_a(0, y; t, x) \leq \frac{c}{t^{d/2}} \exp\{-D(2t; x, y)\}, \tag{B.18}$$

where

$$D(s; x, y) \equiv \sup_{\psi \in L^\infty(\mathbb{Z}^d)} \big[|\psi(y) - \psi(x)| - s\Gamma(\psi)^2\big], \qquad s \geq 0. \tag{B.19}$$

Note that for $x = y$ the statement follows directly from (B.18) (or from Lemma B.2). Let us then assume $x \neq y$. Choose $\psi(x) = (\xi \cdot x)\mathbf{1}_{\{|\xi \cdot x| \leq R\}} + R\mathbf{1}_{\{|\xi \cdot x| > R\}}$, with $\xi = (y - x)/|x - y|\sqrt{t}$ and $R > |x| \wedge |y|$, to obtain

$$D(2t; x, y) \geq |\xi \cdot (y - x)| - 2tc_a^+ \sum_{i=1}^d \big(e^{|\xi_i|} - 1\big)^2,$$

and if $t \geq 1$, $\sup_i |x_i| \leq 1$, therefore $e^{|\xi_i|} - 1 \leq 2\xi_i$, and we conclude that

$$D(2t; x, y) \geq \frac{|x - y|}{\sqrt{t}} - 8dc_a^+.$$

We are therefore left with the case $t \in (0, 1)$. In this case the L^2-estimate (B.16) suffices, since, observing that $\|g\|_2 \geq \|g\|_\infty$, it implies that

$$e^{-\psi(y)} P_{0,t} e^{\psi(y)} \mathbf{1}_x(y) \leq e^{\Gamma(\psi)^2 t} \quad \text{for every } x, y \in \mathbb{Z}^d.$$

By choosing ψ as above, but with $\xi = (y - x)/|y - x|$ we conclude the proof. □

Off-diagonal estimates: lower bound. We will content ourselves with the so-called estimates near the diagonal, that is, in the case in which $|x - y|/(1 \vee \sqrt{t}) \leq$ const. We do this by adapting the proof of (1.16) in [24] to our set-up. We therefore try to stick as close as possible to the notation in [24].

We now consider a random walk on $\mathbb{Z}_\alpha^d \equiv \alpha\mathbb{Z}^d$, $\alpha \in (0, 1]$. We use the notations

$$\|f\|_{\alpha, p} \equiv \left(\alpha^d \sum_{x \in \mathbb{Z}^d} |f(\alpha x)|^p\right)^{1/p}, \qquad p \geq 1$$

and

$$(f, g)_\alpha = \alpha^d \sum_{x \in \mathbb{Z}^d} f(\alpha x) g(\alpha x),$$

for f and g from $\mathbb{Z}_\alpha^d$ to $\mathbb{R}$. The semigroup, on $L^2(\mathbb{Z}_\alpha^d)$, will be denoted by $P_{s,t}^\alpha$ and it is specified by

$$\frac{d}{dt}\big(g, P_{s,t}^\alpha f\big)_\alpha = -\mathscr{E}_t^\alpha\big(g, P_{s,t}^\alpha f\big), \tag{B.20}$$

for $f, g \in L^2(\mathbb{Z}_\alpha^d)$, with

$$\mathscr{E}_t^\alpha(f, g) = \alpha^d \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{d} a_{x,i}(\alpha^{-2}t) \nabla_i^\alpha f(\alpha x) \nabla_i^\alpha g(\alpha x), \tag{B.21}$$

and $\nabla_i^\alpha f(\alpha x) = (f(\alpha(x+e_i)) - f(\alpha x))/\alpha$. Finally the kernel $p^\alpha\colon \mathbb{R}^+ \times \mathbb{Z}_\alpha^d \times \mathbb{R}^+ \times \mathbb{Z}_\alpha^d \to [0, \alpha^{-d}]$ is defined by

$$p^\alpha(s, \alpha y; t, \alpha x) = \alpha^{-d} P_{s,t}^\alpha \mathbf{1}_x(y). \tag{B.22}$$

PROPOSITION B.4 [Lower bound near the diagonal (partial Aronson lower bound)]. *There exists $\delta > 0$, depending only on d, c_a^- and c_a^+ such that*

$$p^\alpha(s, \alpha y; t, \alpha x) \geq \frac{\delta}{(\alpha \vee (t-s)^{d/2})}, \tag{B.23}$$

for every $\alpha \in (0, 1]$, every $t > s \geq 0$ and every $x, y \in \mathbb{Z}^d$ such that $|x - y| \leq \sqrt{t/\alpha^2}$.

PROOF. We start by observing that it is sufficient to take $s = 0$, as well as $y = 0$, and that the kernel p^α enjoys a scaling property, namely that for every $\alpha, \beta > 0$,

$$\beta^d p^{\alpha\beta}(0, \alpha\beta y; \beta^2 t, \alpha\beta x) = p^\alpha(0, \alpha y; t, \alpha x), \tag{B.24}$$

for every $x, y \in \mathbb{Z}^d$ and every $t \geq 0$. The proof of (B.24) goes as follows: associate to $f \in L^2(\mathbb{Z}_\alpha^d)$ the function $f_\beta \in L^2(\mathbb{Z}_{\alpha\beta}^d)$ defined as $f_\beta(\alpha\beta x) = f(\alpha x)$, and set $u_t(\alpha x) = P_{0,\beta^2 t}^{\alpha\beta} f_\beta(\alpha\beta x)$. By (B.20) and (B.21),

$$\begin{aligned} \frac{d}{dt}(g, u_t(\alpha x))_\alpha &= -\beta^{-d} \frac{d}{dt}\Big(g_\beta, P_{0,\beta^2 t}^{\alpha\beta} f_\beta\Big)_{\alpha\beta} \\ &= -\beta^{2-d} \mathscr{E}_{\beta^2 t}^{\alpha\beta}\Big(g_\beta, P_{0,\beta^2 t}^{\alpha\beta} f_\beta\Big) = -\mathscr{E}_t^\alpha(g, u_t), \end{aligned} \tag{B.25}$$

for every $t \geq 0$. By uniqueness of the semigroup we therefore obtain that $u_t = P_{0,t}^\alpha f$ and (B.24) is proved.

By using the scaling we can reduce the proof of (B.23) to an estimate at $t = 1$. In fact, (B.23) is immediate if $t < \alpha^2$: since we are assuming that $y = 0$, we have that $x = 0$ and $p^\alpha(0, 0; t, 0) \geq \alpha^{-d} \exp(-c_a^+)$, by definition of p^α and elementary properties of Poisson processes. By (B.24) we obtain that

$$p^\alpha(0, 0; t, \alpha x) = t^{-d/2} p^{\alpha/\sqrt{t}}(0, 0; 1, \alpha x/\sqrt{t}),$$

and we have reduced the problem to the existence of a $\delta = \delta(d, c_a^-, c_a^+) > 0$ such that

$$p^{\alpha/\sqrt{t}}(0, 0; 1, \alpha x/\sqrt{t}) \geq \delta, \tag{B.26}$$

for every α and t such that $\alpha/\sqrt{t} \in (0,1]$ and every $x \in \mathbb{Z}^d$ such that $|\alpha x/\sqrt{t}| \le 1$. By substituting $\alpha/\sqrt{t}$ with α we see that we are left with finding δ such that

$$p^{\alpha}(0,0;1,\alpha x) \ge \delta \qquad \text{for } \alpha \in (0,1] \text{ and } |\alpha x| \le 1. \tag{B.27}$$

The following result is taken from [24]. Choose $U \in C^{\infty}(\mathbb{R};\mathbb{R}^+)$ such that

$$\int_{\mathbb{R}} e^{-2U(r)}\,dr = 1 \quad \text{and} \quad U(r) = |r| \text{ for } |r| \ge 1. \tag{B.28}$$

Define the *probability density* g_α: $\mathbb{Z}^d \to [0,1]$ as

$$g_\alpha(x) = \alpha^{-d} \prod_{i=1}^{d} \int_{\alpha x_i}^{\alpha(x_i+1)} e^{-2U(r_i)}\,dr_i, \tag{B.29}$$

and notice that $\alpha^d \sum_{x\in\mathbb{Z}^d} g_\alpha(x) = 1$. We define also the average with respect to g_α as

$$\langle f \rangle_{g_\alpha} = \alpha^d \sum_{x\in\mathbb{Z}^d} f(\alpha x) g_\alpha(x), \qquad f \in L^{\infty}(\mathbb{Z}^d_\alpha). \tag{B.30}$$

LEMMA B.5 ([24], Lemma 1.19: Poincaré inequality for $\langle\cdot\rangle_{g_\alpha}$). *There exists* $c > 0$ *such that*

$$c\|f - \langle f\rangle_{g_\alpha}\|^2_{\alpha,2} \le \alpha^d \sum_{x\in\mathbb{Z}^d} g_\alpha(x) \sum_{i=1}^{d} [\nabla_i^\alpha f(\alpha x)]^2.$$

The reason we introduced g_α is the following: the Chapman–Kolmogorov equation, the fact that $g_\alpha \le 1$ and Jensen's inequality imply that

$$\begin{aligned} \log p^{\alpha}(0,0;1,\alpha x) \ge{}& \alpha^d \sum_{y\in\mathbb{Z}^d} g_\alpha(y) \log p^{\alpha}(0,0;1/2,\alpha y) \\ &+ \alpha^d \sum_{y\in\mathbb{Z}^d} g_\alpha(y) \log p^{\alpha}(1/2,\alpha y;1,\alpha x). \end{aligned} \tag{B.31}$$

Since by the symmetry of the transition rates one can easily show that

$$p^{\alpha}_{a}(1/2,\alpha y;1,\alpha x) = p^{\alpha}_{\tilde a}(0,\alpha x;1/2,\alpha y), \tag{B.32}$$

where $\tilde a_{x,i}(\cdot) = a_{x,i}(1-\cdot)$ and recalling once again that the only restriction on the jump rates is (B.2), the proof of (B.27) is reduced to finding M, depending only on d, c_a^- and c_a^+ such that

$$\begin{aligned} &\alpha^d \sum_{x\in\mathbb{Z}^d} g_\alpha(x) \log\ p^{\alpha}(0,\alpha y;1/2,\alpha x) \ge -M \\ &\qquad\qquad \text{for } \alpha \in (0,1] \text{ and } y \text{ s.t. } |\alpha y| \le 1. \end{aligned} \tag{B.33}$$

The proof of (B.33) is also very close to the proof of the analogous statement, Lemma 1.24, in [24]. We sketch it briefly: define for $t \in (0,1/2]$,

$$G(t) = \alpha^d \sum_{x\in\mathbb{Z}^d} g_\alpha(x) \log u_t(x), \tag{B.34}$$

where $u_t(x) = \log p^\alpha(0, \alpha y; t, \alpha x)$. By taking the time derivative and repeating the elementary steps at the beginning of the proof of Lemma 1.24 in [24] we obtain that there exists $L = L(d, c_a^-, c_a^+) \in (1, \infty)$ such that for every $\alpha \in (0, 1]$,

$$G'(t) \geq \frac{1}{L}\alpha^d \sum_{x\in\mathbb{Z}^d}\sum_{i=1}^{d} g_\alpha(x)\big(\nabla_i^\alpha \log u_t(x)\big)^2 - L.$$

Now we apply Lemma B.5 to get

$$G'(t) \geq \frac{c}{L}\alpha^d \sum_{x\in\mathbb{Z}^d} g_\alpha(x)\big(\log u_t(x) - G(t)\big)^2 - L. \tag{B.35}$$

Observe that from the upper bound on the transition kernel we can extract a weak lower bound, which is useful here. We can in fact prove that there exists $r > 0$ such that

$$\alpha^d \sum_{|\alpha x|\leq r} p^\alpha(0, \alpha y; t, \alpha x) \geq 3/4, \tag{B.36}$$

for every $\alpha \in (0, 1]$, every $t \in (0, 1]$ and $|\alpha x| \leq 1$. This follows directly (Chebyshev inequality) from the following: there exists $c = c(d, c_a^-, c_a^+)$ such that

$$\alpha^d \sum_{x\in\mathbb{Z}^d} |\alpha x - \alpha y|^2 p^\alpha(0, \alpha y; t, \alpha x) \leq ct, \tag{B.37}$$

for $\alpha \in (0, 1]$ and $t \geq 0$. By (B.24), it is sufficient to consider the case $\alpha = 1$. If $t \geq 1$, (B.37) follows from the upper bound on the kernel (B.10). If $t \in (0, 1)$, it is a direct consequence of elementary properties of Poisson processes.

By the second part of the proof of Lemma 1.24 in [24], (B.35), combined with (B.36), implies (B.33) and the proof of Proposition B.4 is complete. □

The Nash continuity estimate. In [10], in the $\mathbb{R}^d$ context, it is shown how the Harnack inequality and its consequences can be extracted from the Aronson estimates. In [24] it is shown how Propositions B.3 and B.4 imply the parabolic Harnack inequality [24], Lemma 1.30. The parabolic Harnack inequality implies directly the Nash continuity estimate ([24], Theorem 1.31). This latter result is of interest for us: we state it here and we refer to [24] for a proof.

PROPOSITION B.6 (Nash continuity estimate). *There exists $\varrho > 0$ and $c > 0$, depending only on d, c_a^- and c_a^+, such that for every $\alpha \in (0, 1]$ and $f \in L^\infty(\alpha\mathbb{Z}^d)$,*

$$\big|P_{0,t}^\alpha f(\alpha x) - P_{0,s}^\alpha f(\alpha y)\big| \leq c\|f\|_{\alpha,\infty}\bigg(\frac{|t-s|^{1/2} \vee |\alpha x - \alpha y|}{(t\wedge s)^{1/2}}\bigg)^\varrho, \tag{B.38}$$

for $t, s \geq 0$ and $x, y \in \mathbb{Z}^d$.

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