SEQUENTIAL DECISION PROBLEMS FOR PROCESSES WITH CONTINUOUS TIME PARAMETER. PROBLEMS OF ESTIMATION[1]

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Summary. In a recent paper [1] the authors began the study of the theory of sequential decision functions for stochastic processes with a continuous time parameter. This paper treated the standard problem of testing hypotheses, and the advantage of being able to stop at an arbitrary time point (not necessarily a multiple of some unit given in advance) was demonstrated in several cases, notably in that of deciding between two Poisson processes. The optimal tests were Wald probability ratio tests and thus truly sequential. In the present paper we treat the problem of estimation, and study in detail the Poisson, Gamma, Normal and Negative Binomial processes. It turns out for these processes that, with a proper weight function, the minimax (sequential) rule reduces to a fixed-time rule. Though we confine ourselves to point-estimation it is clear that similar methods apply to interval estimation. It may also be remarked that the case when the time-parameter is discrete need not be treated separately. For example, as described in Section 6.1, the results of Sections 2 and 3 imply analogous results in the case of discrete time, which in turn imply certain results proved in [3] and (in the nonsequential case) in [2] by other methods. The treatment of some other problems in estimation is discussed in Section 6. This paper may be read independently of [1].

1. Preliminaries. Let $X(\mathbf{t} \mid \omega)$, $t \geqq 0$, $\omega \, \varepsilon \, \Omega$, be a family of stochastic processes in time t which depend on a parameter ω. Let $c(t)$, $t \geqq 0$, be a given cost function which represents the cost to the statistician of observing the process up to time t. For every ω in Ω and $\bar{\omega}$ in the terminal decision space D^2 let $W(\omega, \bar{\omega})$ be the weight function, that is, the loss involved in giving the estimate $\bar{\omega}$ when ω is the correct value of the parameter. Let (T, δ) be a pair of functionals of the sample function $x(t)$ into $(0 \leqq T \leqq \infty, D)$, where δ depends on $x(t)$ only through its values for $0 \leqq t \leqq T$ if $T < \infty$ (if $T = \infty$, δ is undefined, but in accordance with our assumptions on $c(t)$ below we define the quantity of (1) to be ∞ if this event occurs with positive probability under ω). The decision rule corresponding to these functionals is: observe up to time T and then (in case T is finite) adopt the esti-

Received 11/28/52.

[1] This work was sponsored by the Office of Naval Research under a contract with Columbia University and under a National Bureau of Standards contract with the University of California at Los Angeles.

[2] In Sections 3 and 5, we take $\Omega = D$. In Section 2, $D = \Omega + \{\lambda = 0\}$. In Section 4, $D = \Omega + \{\omega = 0\} = \Omega + \{p = 1\}$.

mate δ. If T is a constant independent of the sample function $x(t)$ then the procedure is not truly sequential. It is called a fixed-size or fixed-time estimation procedure. Throughout this paper we shall by $x(t)$ mean $x(t+)$; that is, the sample functions are to be considered as continuous from the right.

For a given ω the risk associated with such a procedure is defined by

$$R_\omega(T, \delta) = E_\omega\{c(T) + W(\omega, \delta)\} \tag{1}$$

where E_ω denotes the expected value under the assumption that ω is the true value of the parameter, provided the expected value exists. Assuming the expected value to exist for every $\omega \varepsilon \Omega$ we define the maximum risk associated with T and δ by

$$R(T, \delta) = \sup R_\omega(T, \delta), \tag{2}$$

the supremum taken over all $\omega \varepsilon \Omega$.

An estimation procedure $(\hat{T}, \hat{\delta})$ is called *minimax* if

$$R(\hat{T}, \hat{\delta}) \leqq R(T, \delta) \tag{3}$$

for any functionals T and δ for which (2) is defined. If no minimax estimation rule exists, it is still possible to define a minimax sequence of decision rules $\hat{T}_n$, $\hat{\delta}_n (n = 1, 2, \cdots)$, that is, a sequence for which

$$\lim_{n=\infty} R(\hat{T}_n, \hat{\delta}_n) = \inf R(T, \delta). \tag{4}$$

In the cases we treat it will be shown that a minimax rule does exist. However, a slight relaxation of the assumptions (e.g., dropping the continuity assumption about the cost function $c(t)$) may affect the existence of minimax rules, and in such cases one has to modify the argument only slightly in order to find a minimax sequence (which, in the cases treated below, may be taken to consist of fixed-time rules).

Let ζ be a Borel field of subsets of Ω and $R_\omega(\delta, T)$ be a measurable function of ω with respect to ζ. Let $F(\omega)$ be a probability distribution on Ω. Then, assuming the integral to exist, we define

$$R_F(T, \delta) = \int_\Omega R_\omega(T, \delta) dF(\omega). \tag{5}$$

The estimation rule (T_F, δ_F) is called a Bayes rule for F if

$$R_F(T_F, \delta_F) = \inf R(T, \delta). \tag{6}$$

We shall denote by δ^T fixed time estimation rules with constant observation time T, and in this case we shall write δ^T instead of the pair (T, δ). We define

$$r_\omega(\delta^T) = E_\omega W(\omega, \delta^T) \tag{7}$$

and

$$r_F(\delta^T) = \int_\Omega r_\omega(\delta^T)\, dF(\omega). \tag{8}$$

δ_F^T is called a *T-Bayes estimation rule* for F if (8) assumes its minimum for $\delta^T = \delta_F^T$.

Let A be any set of sample curves $x(t)$ for which the probability $P\{x(t) \varepsilon A\}$ is defined and is a measurable function of ω. Let $F(\omega)$ be any distribution function over Ω. Then for every A for which $P(A) = \int_\Omega P\{x(t) \varepsilon A\} dF(\omega) > 0$ we define the a posteriori probability distribution $F(\omega \mid A)$ by assigning to every Borel set $S \varepsilon \zeta$ the probability $P(A)^{-1} \int_S P\{x(t) \varepsilon A\} dF(\omega)$. The *a posteriori T-risk* corresponding to F and δ^T is defined by

$$r_F(\delta^T \mid A) = \int_\Omega r_\omega(\delta^T)\, dF(\omega \mid A). \tag{9}$$

If $r_F(\delta^T \mid A)$ is independent of A we say that *the a posteriori risk is independent of the sample* $x(t)$. (It is assumed in the sequel that "many" sets A with the above property exist. This is of course the case for the processes usually encountered in mathematical statistics and in particular with families of processes, like those treated in this paper, with which are associated sufficient statistics of a simple nature. Since our primary interest is in statistical applications there seems to be no point in inserting a lengthy technical discussion of the precise measurability properties required in order to insure that the class of sets A will be sufficiently rich.)

We shall make frequent use of the following obvious remark, which is a familiar tool in decision theory (see, e.g., [4]).

Suppose that, for every $T \geqq 0$, there exists a sequence of probability distributions $F_n (n = 1, 2, \cdots)$ for which there are corresponding T-Bayes solutions δ_n^T with the property that the a posteriori risk associated with F_n and δ_n^T is independent of the sample $x(t)$, and suppose that there exists $\hat{\delta}^T$ for which

$$r(T) = \sup_\omega r_\omega(\hat{\delta}^T) = \lim_{n=\infty} r_{F_n}(\delta_n^T). \tag{10}$$

If there exists a T_0 $(0 \leqq T_0 < \infty)$ for which

$$c(T_0) + r(T_0) = \min_{T \geqq 0} [c(T) + r(T)] \tag{11}$$

holds, then the fixed time rule $\hat{\delta}^{T_0}$ is a minimax estimation rule.

The proof of this assertion is evident. Indeed, the conclusion remains valid under weaker assumptions. As this is not needed for the sequel we just point out that we could have dropped the assumption of risk independent of the sample and replaced (10) by

$$\sup_\omega r_\omega(\hat{\delta}^T) = \lim_{n=\infty} \inf_A r_{F_n}(\delta_n^T \mid A).$$

It may also be worth while to remark that if no T_0 satisfying (11) exists, we still have a minimax sequence of estimation rules all of which are fixed-time rules. (These results clearly remain valid also if randomized rules are considered.)

In the examples treated in the sequel $r(T)$ is a nonnegative continuous function.

We assume that the cost function $c(T)$ is nonnegative, lower semicontinuous, and tends to infinity as $T \to \infty$. These assumptions guarantee the existence of a T_0 which satisfies (11).

We remark, finally, that in the examples of Sections 2 and 3, the minimum of (11) will always be achieved for $T > 0$, since $R(0, \delta) = \infty$ for all δ. In the examples of Sections 4 and 5, this need not be the case. Analogous remarks apply to the discussion of Section 6.

2. The Poisson process. This is defined for every $\lambda > 0$ as a process $X_\lambda(t)$ with independent stationary increments which satisfies

$$P\{X_\lambda(t) = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \qquad (x = 0, 1, 2, \cdots) \tag{12}$$

for all $t \geqq 0$.

We let Ω be the half-line $0 < \lambda < \infty$ and ζ consist of the usual Borel sets.[2] Our problem is to estimate the mean λ. It is well known that $x(T)$ is a sufficient statistic for λ when the sample curve $x(t)$ is observed for $0 \leqq t \leqq T$.

As weight function we take, following Hodges and Lehmann [3] and Girshick and Savage [2],

$$W(\lambda, \delta) = \frac{1}{\lambda} (\delta - \lambda)^2. \tag{13}$$

This is the squared error measured in terms of the variance. As these authors point out, the classical squared error $(\delta - \lambda)^2$ gives, for every finite time, infinite minimax risk, and is thus of no interest unless some additional information about λ is known.

Let $F_n(\lambda)$, $n = 1, 2, \cdots$ be the probability distribution on the half-line $\lambda > 0$ with density

$$f_n(\lambda) = \frac{1}{n} e^{-\lambda/n} \qquad (0 < \lambda < \infty). \tag{14}$$

Let the process be observed during the time $0 \leqq t \leqq T$. The a posteriori probability distribution when $x(T) = x$ is well defined and its density is given by

$$f_n(\lambda \mid x) = \frac{\lambda^x}{x!} \left(T + \frac{1}{n}\right)^{x+1} e^{-\lambda(T+1/n)} \qquad (0 < \lambda < \infty).$$

The a posteriori T-risk (see (9)) is given by

$$r_n(\delta^T \mid x) = \int_0^\infty \frac{1}{\lambda} (\delta^T - \lambda)^2 f_n(\lambda \mid x)\, d\lambda.$$

It is easily seen that this is minimized by taking

$$\delta^T(x(T) = x) = 1 \Big/ \int_0^\infty \frac{1}{\lambda} f_n(\lambda \mid x)\, d\lambda = \frac{x}{T + 1/n}.$$

Therefore the T-Bayes solution corresponding to $F_n(\lambda)$ is given by

$$\hat{\delta}_n^T = \frac{x(T)}{T + 1/n}. \tag{15}$$

The corresponding a posteriori risk is

$$r_n(\hat{\delta}_n^T \mid x) = \frac{1}{T + 1/n}, \tag{16}$$

which is independent of $x(T)$ (hence, $x(T)$ being a sufficient statistic, of the sample).

On the other hand, taking

$$\hat{\delta}^T = \frac{x(T)}{T} \tag{17}$$

we see that

$$r_\lambda(\hat{\delta}^T) = e^{-\lambda T} \sum_{x=0}^{\infty} x \frac{(\lambda T)^x}{x!} = \frac{1}{T}$$

for all $\lambda > 0$. Thus we have the following.

For the Poisson process (12) *with* $0 < \lambda < \infty$ *and weight function* (13) *the fixed-time estimate* (17) *with* $T = T_0$ *given by*

$$c(T_0) + \frac{1}{T_0} = \min_{T>0} \left[c(T) + \frac{1}{T} \right] \tag{18}$$

is minimax.

3. The Gamma process. This is defined for every pair of positive numbers r and θ as a process $X_{r,\theta}(t)$ with independent stationary increments such that $X_{r,\theta}(0) = 0$ and for every $t > 0$ and $x > 0$

$$P\{X_{r,\theta}(t) < x\} = \int_0^x \frac{x^{rt-1}}{\Gamma(rt)\theta^{rt}} e^{-x/\theta} \, dx. \tag{19}$$

The parameter r will be assumed known, and the space Ω will consist of the half-line $0 < \theta < \infty$, the Borel sets being the ordinary ones.[2] Here again it is well known that if the sample curve $x(t)$ is given only for $t \leqq T$ then $x(T)$ is a sufficient statistic for θ.

As weight function we take

$$W(\theta, \delta) = \left[\left(\frac{\delta}{\theta} \right)^\gamma - 1 \right]^2, \tag{20}$$

γ being an arbitrary positive number. For $\gamma = 1$ this weight function, like (13), is proportional to the square error of the mean measured in terms of the variance and occurs in Hodges and Lehmann [3] and Girshick and Savage [2].

Let $F_n(\theta)$, $n = 1, 2, \cdots$ be the probability distribution over $\theta > 0$ with density

$$f_n(\theta) = \frac{\theta^{-1-1/n}e^{-1/\theta}}{\Gamma(1/n)} \qquad (0 < \theta < \infty). \tag{21}$$

The a posteriori probability distribution when $x(T) = x$ has the density

$$f_n(\theta \mid x) = \frac{(x+1)^{rT+1/n}e^{-(x+1)/\theta}}{\Gamma(rT + 1/n)\theta^{rT+1+1/n}} \qquad (0 < \theta < \infty). \tag{22}$$

The a posteriori T-risk is given by

$$r_n(\delta^T \mid x) = \int_0^\infty \left[\left(\frac{\delta}{\theta}\right)^\gamma - 1\right]^2 f_n(\theta \mid x)\, d\theta. \tag{23}$$

It is minimized by taking

$$\begin{aligned} \hat{\delta}_n^T(x(T) = x) &= \left[\int_0^\infty \theta^{-\gamma} f_n(\theta \mid x)\, d\theta \Big/ \int_0^\infty \theta^{-2\gamma} f_n(\theta \mid x)\, d\theta\right]^{1/\gamma} \\ &= (x+1)\left[\frac{\Gamma(rT + \gamma + 1/n)}{\Gamma(rT + 2\gamma + 1/n)}\right]^{1/\gamma} \end{aligned} \tag{24}$$

Substituting this value in (23) we obtain

$$r_n(\hat{\delta}_n^T \mid x) = 1 - \frac{\Gamma^2(rT + \gamma + 1/n)}{\Gamma(rT + 1/n)\Gamma(rT + 2\gamma + 1/n)}, \tag{25}$$

which is independent of $x(T)$.

On the other hand the estimator

$$\hat{\delta}^T = \left[\frac{\Gamma(rT + \gamma)}{\Gamma(rT + 2\gamma)}\right]^{1/\gamma} x(T) \tag{26}$$

gives

$$r_\theta(\hat{\delta}^T) = 1 - \frac{\Gamma^2(rT + \gamma)}{\Gamma(rT)\Gamma(rT + 2\gamma)} \tag{27}$$

for all $0 < \theta < \infty$. Since (27) is independent of θ and is the limit of (25) as $n \to \infty$ we have:

For the Gamma process (14) *with fixed r, unknown θ $(0 < \theta < \infty)$ and weight function* (20), *the fixed time estimate* (26) *with $T = T_0$ given by*

$$c(T_0) - \frac{\Gamma^2(rT_0 + \gamma)}{\Gamma(rT_0)\Gamma(rT_0 + 2\gamma)} = \min_{T>0}\left[c(T) - \frac{\Gamma^2(rT + \gamma)}{\Gamma(rT)\Gamma(rT + 2\gamma)}\right] \tag{28}$$

is minimax.

If instead of using the weight function (20) we use, following Girshick and Savage [2],

$$W_1(\theta, \delta) = \log^2 \frac{\delta}{\theta}, \tag{29}$$

that is, the squared error when $\log \theta$ is considered as the parameter, then we find that for the distributions (21) the a posteriori T-risk is minimized by taking

$$\hat{\delta}_n^T(x(T) = x) = \exp \int_0^\infty \log \theta f_n(\theta \mid x)\, d\theta = (x+1) \exp\left\{-\frac{\Gamma'(rT + 1/n)}{\Gamma(rT + 1/n)}\right\}$$

and that the corresponding value of the a posteriori risk is

$$\frac{\Gamma''(rT + 1/n)}{\Gamma(rT + 1/n)} - \left[\frac{\Gamma'(rT + 1/n)}{\Gamma(rT + 1/n)}\right]^2,$$

which again is independent of x. Since

$$\delta^T = x(T)e^{-\Gamma'(rT)/\Gamma(rT)} \tag{30}$$

has the constant risk function

$$\frac{\Gamma''(rT)}{\Gamma(rT)} - \left[\frac{\Gamma'(rT)}{\Gamma(rT)}\right]^2,$$

we have as before:

If, instead of (20), *the weight function* (29) *is used, then the fixed time estimate* (30) *with* $T = T_0$ *given by*

$$c(T_0) + \frac{\Gamma''(rT_0)}{\Gamma(rT_0)} - \left[\frac{\Gamma'(rT_0)}{\Gamma(rT_0)}\right]^2 = \min_{T>0}\left\{c(T) + \frac{\Gamma''(rT)}{\Gamma(rT)} - \left[\frac{\Gamma'(rT)}{\Gamma(rT)}\right]^2\right\} \tag{31}$$

is minimax.

4. The Negative Binomial process.

This is defined for every $\omega > 0$ as a process $X_\omega(t)$ with independent stationary increments satisfying $X_\omega(0) = 0$ and

$$P\{X_\omega(t) = x\} = \frac{\Gamma(t+x)}{\Gamma(t)\Gamma(x+1)} \frac{\omega^x}{(1+\omega)^{x+t}} \quad (x = 0, 1, 2, \cdots) \tag{32}$$

for every $t > 0$. It is customary to put

$$p = \frac{1}{1+\omega}, \qquad q = \frac{\omega}{1+\omega}; \tag{33}$$

then (32) becomes

$$P\{X_\omega(t) = x\} = \frac{\Gamma(t+x)}{\Gamma(t)\Gamma(x+1)} p^t q^x \qquad (x = 0, 1, 2, \cdots).$$

As Ω we take the half line $0 < \omega < \infty$, the Borel sets being the usual ones.[2] It is easy to see that $x(T)$ is a sufficient statistic for ω when $x(t)$ is observed for $0 \leqq t \leqq T$.

$X_\omega(t)$ has mean ωt and variance $\omega(1+\omega)t$. It is easily seen that the square error $(\delta - \omega)^2$ would give an infinite minimax risk. We therefore use as weight function

$$W(\delta, \omega) = \frac{(\delta - \omega)^2}{\omega(1 + \omega)} = \frac{p^2}{q}\left(\delta - \frac{q}{p}\right)^2 \tag{34}$$

which is proportional to the square error measured in units equal to the variance.

Let $F_n(p)$, $n = 1, 2, \cdots$ be the probability distribution over $0 < p < 1$ with density

$$f_n(p) = \frac{\Gamma(2 + 1/n)}{\Gamma(2)\Gamma(1/n)} p^{-1+1/n} q \qquad (0 < p < 1). \tag{35}$$

If the process is observed for the period $0 \leqq t \leqq T$ and $x(T) = x$ then the a posteriori probability distribution has density

$$f_n(p \mid x) = \frac{\Gamma(x + 2 + T + 1/n)}{\Gamma(x + 2)\Gamma(T + 1/n)} p^{T-1+1/n} q^{x+1} \qquad (0 < p < 1).$$

The a posteriori T-risk is

$$r_n(\delta^T \mid x) = \int_0^1 \frac{p^2}{q}\left(\delta^T - \frac{q}{p}\right)^2 f_n(p \mid x) dp,$$

and is minimized by taking

$$\delta^T(x(T) = x) = \frac{\int_0^1 p f_n(p \mid x) dp}{\int_0^1 \frac{p^2}{q} f_n(p \mid x) dp} = \frac{x + 1}{T + 1 + 1/n}.$$

Therefore the T-Bayes estimate corresponding to the a priori distribution $F_n(p)$ with density (35) is given by $\hat{\delta}_n^T = (x + 1)/(T + 1 + 1/n)$. The corresponding a posteriori risk is

$$r_n(\hat{\delta}_n^T \mid x) = \frac{1}{T + 1 + 1/n}, \tag{36}$$

which is independent of $x(T)$ and, therefore, of the sample.

For every given ω, $0 < \omega < \infty$, the estimator

$$\hat{\delta}^T = \frac{x(T)}{T + 1} \tag{37}$$

gives the risk

$$\frac{p^2}{q} \sum_{x=0}^{\infty} \left(\frac{x}{T + 1} - \frac{q}{p}\right)^2 \frac{\Gamma(T + x)}{\Gamma(T)\Gamma(x + 1)} p^T q^x = \frac{T + q}{(T + 1)^2}.$$

The supremum of this expression for $0 \leqq \omega < \infty$ is $1/(T + 1)$ by (33); that is, equal to the limit of (36) as $n \to \infty$. Hence (10) holds and the remark of Section 2 may be applied to give the following.

For the Negative Binomial process (32) *with* $0 < \omega < \infty$ *and weight function* (34) *the fixed time estimate* (37) *with* $T = T_0$ *given by*

$$c(T_0) + \frac{1}{T_0 + 1} = \min_{T \geqq 0}\left[c(T) + \frac{1}{T+1}\right] \tag{38}$$

is a minimax estimate.

It may be worth while to remark that (37) is a biased estimate. Indeed the expected value of it for given ω is $T\omega/(T + 1)$. The unbiased estimate x/T gives a constant risk $1/T$.

5. The Normal (Wiener) process. This is defined for every real μ and positive σ as a process with stationary independent increments such that $X_{\mu,\sigma}(0) = 0$ and

$$P\{X_{\mu,\sigma}(t) < x\} = \frac{1}{\sqrt{2\pi t}\sigma}\int_{-\infty}^{x} e^{-(x-\mu t)^2/2\sigma^2 t}\,dx$$

for every real x and $t > 0$.

The parameter σ will be assumed known and the space Ω will consist of the real line $-\infty < \mu < \infty$, the Borel sets being the ordinary ones.[2]

As weight function we take any function of the form

$$W(\mu, \delta) = w(\,|\,\mu - \delta\,|\,) \tag{39}$$

with $w(x)$ nonnegative and nondecreasing for $x \geqq 0$.

In the present case it is not necessary to perform computations similar to those of the preceding sections, since it is easily seen that the arguments of Wolfowitz [4], where a discrete time parameter was considered, carry over to the present case.

The fixed time estimator

$$\hat{\delta}^T = \frac{x(T)}{T} \tag{40}$$

gives for $T > 0$ and any real μ the T-risk

$$r_\mu(\hat{\delta}^T) = \frac{\sqrt{2T}}{\sqrt{\pi}\sigma}\int_0^\infty w(x)e^{-x^2 T/2\sigma^2}\,dx\,. \tag{41}$$

Moreover it is easily seen that $r_\mu(\hat{\delta}^T)$ is, as a function of T, for $T > 0$, nonincreasing, continuous from the right and that

$$\lim_{T \downarrow 0} r_\mu(\hat{\delta}^T) = \lim_{x\to\infty} w(x).$$

Disregarding the trivial cases 1) when

$$\int_0^\infty w(x)e^{-x^2 h}\,dx \tag{42}$$

is divergent for all h, that is, when the risk is always infinite, and 2) $w(x) \equiv 0$ when the value of the estimator is of no consequence, we have:

For the Normal process with fixed variance σ^2, *unknown mean* μ *and weight function* (39) *with* $w(x)$, $x \geqq 0$, *not identically zero, nondecreasing, and such that* (42) *converges for at least one value of* h, *the fixed time estimate* (40) *with* $T = T_0$ *given by*

$$(43)\quad c(T_0) + \frac{\sqrt{2T_0}}{\sqrt{\pi}\sigma}\int_0^\infty w(x)e^{-x^2T_0/2\sigma^2}\,dx = \min_{T\geqq 0}\left[c(T) + \frac{\sqrt{2T}}{\sqrt{\pi}\sigma}\int_0^\infty w(x)e^{-x^2T/2\sigma^2}\,dx\right]$$

is minimax, where the term following $c(T)$ *and* $c(T_0)$ *in* (43) *is replaced by* $\sup_x w(x)$ *for* $T = 0$ *or* $T_0 = 0$, *and where* (40) *may be replaced by any estimator if the minimum of* (43) *is at* $T = 0$ (*which can only occur if* w *is bounded*).

For further remarks on the normal process see the next section, especially 6.4 and 6.5.

6. Generalizations and other remarks.

6.1. It is by no means impossible to have practically continuous observation of a stochastic process. However, our results apply without any modification to stochastic processes with a discrete time parameter or, more generally, to the case when the observations can be made only at times belonging to some set given in advance (there is, of course, no loss of generality in assuming the time-parameter continuous). Indeed, if I is any closed subset of the reals and an observation at time T may be made only if $T \varepsilon I$, all our results remain valid provided T_0 and T in (11) [respectively (18), (28), (31), (38) and (43)] are restricted by the condition that they belong to I. (The same end could be achieved by having $c(t) = \infty$ for $t \not\varepsilon I$ and dropping or suitably modifying the assumption that $c(t)$ is continuous.)

For the special case $I = \{0, 1, 2, \cdots\}$ we have the usual discrete time case; if, furthermore, $c(t)$ is a linear function of t we have the classical sequential case. In this classical case, Hodges and Lehmann [3] obtained by a different method the fact that the fixed-sample estimator of Section 2 on the Poisson process as well as the first estimator of Section 3 on the Gamma process (for the weight function (20) with $\gamma = 1$), both with $T = n$, minimize $\sup_\omega E_\omega W(\omega, \delta)$ subject to $\sup_\omega E_\omega N \leqq n$, where N is the number of observations required (a chance variable) and n is a given positive integer. These results are implied by, but do not imply, ours (see [5], Lemma 5); as remarked in [3], their method does not seem applicable to our problem. The Gamma process with the weight function (29) was considered nonsequentially by Girshick and Savage [2] who, using a different method, established that (30) is a minimax estimator for the fixed sample size problem. As far as we know the Negative Binomial process has never been treated before even nonsequentially.

6.2. The impossibility, in practice, of observing a process for a continuous range of time may be taken care of in the following manner. We replace $c(t)$ by $c(n; t_1, t_2, \cdots, t_n)$ which represents the cost of taking n observations at times $t_1 < t_2 < \cdots < t_n$. The function $c(n; t_1, \cdots, t_n)$ is assumed to satisfy appropriate conditions such as being nonnegative and satisfying $c(n + 1; t_1, \cdots, t_{n+1}) \geqq c(n; t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$ for $n = 0, 1, \cdots$ and $i = 1, 2, \cdots, n + 1$. Our results easily carry over to this case. Thus, for example, for the Poisson process with the weight function considered in Section 2 a minimax estimation procedure is to take a single observation at time $T = T_0$ for which $c(1; T) + 1/T$ becomes a minimum, and to estimate λ by $x(T_0)/T_0$.

This modification of the problem may be combined with that of Section 6.1 by considering only times belonging to a given set I.

6.3. Another modification of the sequential estimation problem is the following: The statistician is required to estimate ω continuously by a function $\delta(t)$ which is a functional of the observed process up to time t, and the loss function is $\int_0^\infty W(\delta(t), \omega) dG(t)$ where $G(t)$ is a monotone nondecreasing function. Our methods apply also to this modified problem. For example in the case of the Poisson process with the weight function used in Section 2 a minimax procedure is obtained by taking $\delta(t) = x(t)/t$.

This formulation may be combined with that of Section 6.2 by having a cost function $c(n; t_1, \cdots, t_n; v, \tau_1, \cdots, \tau_v)$ which expresses the cost of observing the process at times $t_1 < \cdots < t_n$ and changing the estimator $\delta(t)$ at times $\tau_1 < \cdots < \tau_v$. Here again if for every T we can find a sequence of probability distributions with T-Bayes solutions for which the a posteriori risk is independent of the sample and an estimator $\hat{\delta}_T$ satisfying (10) we deduce that a minimax procedure is obtained as follows: choose $n, t_1, \cdots, t_n, v, \tau_1, \cdots, \tau_v$ so as to minimize

$$c(n; t_1, \cdots, t_n; v, \tau_1, \cdots, \tau_n) + \int_0^\infty r(\bar{t})\, dG(t)$$

where $\bar{t} = \max_{t_i \leqq \tau_j} t_i$ for $\tau_j \leqq t < \tau_{j+1}$ (with $\tau_{v+1} = \infty$), and estimate by $\delta(t) = \hat{\delta}^{\bar{t}}$. It is easily seen that if c reduces to a function of n and v only which is monotone in both arguments, then one can choose the τ_v from among the t_i.

This modification may also be considered together with that of Section 6.1. We may further combine it with a weight function which is dependent on the time, etc.

6.4. Throughout the paper we dealt with the problem of point estimation, but it is possible to treat similarly the problem of sequential interval estimation (including that of one-sided estimation). In particular, for the case of the Normal process the results of Wolfowitz [4] carry over to the case of a continuous time parameter.

6.5. We would like now to make some remarks about a class of estimation problems best exemplified by the problem of estimating the variance of a Normal Process.

Let T_1 and $T_2 > T_1$ be any two nonnegative numbers and put $t_{m,n} = T_1 + (n/2^m)(T_2 - T_1)$ for $m = 1, 2, \cdots; n = 0, 1, \cdots, 2^m$. It is well known (see [1]) that if $x(t)$ is a sample function of the Normal process then

$$\lim_{m=\infty} \sum_{n=1}^{2^m} [x(t_{m,n}) - x(t_{m,n-1})]^2 = \sigma^2(T_2 - T_1)$$

with probability 1. Therefore if one could observe a Normal process *without error* for an arbitrarily short period of time one would know the correct value of σ with probability 1. To make the problem practical it is necessary to modify the problem somewhat, for example, in the manner suggested in Section 6.2. We observe that if $X(t)$ is the Normal process with mean μ and variance σ^2 then, for every positive T and t, the random variable $(1/2t)[X(T + t) - X(T) - \mu t]^2$ has the Gamma distribution given by (19) with $r = \frac{1}{2}$, $\theta = \sigma^2$ and $t = 1$. Therefore, if the mean is known and the problem is to estimate the variance we could apply the results of Section 3 (with the modifications suggested in Section 6.2). If both the mean and the variance are unknown again only a slight change, corresponding to the loss of one degree of freedom in the chi square distribution, is necessary.

The situation encountered in this last subsection does not occur if the process is observed continuously but not exactly. This may be done in various ways, a suggestive one for estimating the variance being the following. The process is observed continuously but only deviations exceeding a prescribed size Δ are recorded; that is, we are given a sequence of real numbers $0 = t_0 < t_1 < t_2 < \cdots$ having the property that $| x(t_{n+1}) - x(t_n) | \geqq \Delta$ while $| x(t) - x(t_n) | < \Delta$ for $t_n < t < t_{n+1}$, $(n = 0, 1, 2, \cdots)$.

6.6. One may also consider for continuous (in time) processes such problems as those of sequential unbiased estimation (see [6]) and of unbiased estimation at the conclusion of sequential hypothesis testing (see [7], [8]). For example, for the first of these, one can prove an analogue of the extension of the Cramér-Rao inequality proved in [6], where for our setup En is replaced by ET and $f(x, \theta)$ is replaced by the probability function or density function of $x(1)$ in equation (4.5) of [6]. Under regularity conditions analogous to those of [6], and which are satisfied for the four processes considered herein, the proof (also valid for biased estimators) may be carried out by dividing the time axis into intervals of equal length and allowing the length to approach zero. In particular, the fixed duration estimator of duration T and with estimator $x(T)/T$ is an unbiased estimator of λ, $r\theta$, $(1/p) - 1$, and μ (for the cases considered in Sections 2, 3, 4, and 5, respectively) for which equality holds in this extended Cramér-Rao inequality. This inequality could also be used to apply the technique of [3] to our problems. The two problems described above will both be considered in detail in a future paper.

Finally, we remark that many of Wald's general results on decision functions (complete class theorems, etc.) carry over to the present case of continuous time processes under suitable assumptions. As in [1], the main difficulties in the general theory are ones of measurability, and we shall not bother with them here. We shall return to these problems in a future publication.

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