THE USE OF GROUP DIVISIBLE DESIGNS FOR CONFOUNDED ASYMMETRICAL FACTORIAL ARRANGEMENTS[1]

By Marvin Zelen

National Bureau of Standards

1. Introduction and summary. A factorial experiment involving m factors such that the ith factor has m_i levels is termed an asymmetrical factorial design. If the number of levels is equal to one another the experiment is termed a symmetric factorial experiment. When the block size of the experiment permits only a sub-set of the factorial combinations to be assigned to the experimental units within a block, resort is made to the theory of confounding. With respect to symmetric factorial designs, the theory of confounding has been highly developed by Bose [1], Bose and Kishen [4], and Fisher [11], [12]. An excellent summary of the results of this research appears in Kempthorne [13]. However, these researches are closely related to Galois field theory resulting in (i) only symmetric factorial designs being incorporated into the current theory of confounding; (ii) the common level must be a prime (or power of a prime) number; and (iii) the block size must be a multiple of this prime number.

The theory of confounding for asymmetric designs has not been developed to any great degree. Examples of asymmetric designs can be found in Yates [19], Cochran and Cox [9], Li [15], and Kempthorne [13]. Nair and Rao [16] have given the statistical analysis of a class of asymmetrical two-factor designs in considerable detail.

Kramer and Bradley [14] discuss the application of group divisible designs to asymmetrical factorial experiments, however their paper is mainly confined to the two-factor case and its intra-block analysis.[2] It is the purpose of this paper, which was done independently of their work, to outline the general theory for using the group divisible incomplete block designs for asymmetrical factorial experiments.

The use of incomplete block designs for asymmetric factorial experiments results in (i) no restriction that the levels must be a prime (or power of a prime) number, (ii) no restriction with respect to the dependence of the block size on the type of level, and (iii) unlike the previous referenced works on asymmetric factorial designs, the resulting analysis is simple, does not increase in difficulty with an increasing number of factors, and "automatically adjusts" for the effects of partial confounding.

Received January 10, 1957; revised June 18, 1957; revised November 1, 1957.

[1] This paper is an extension of results presented at the Annual Meeting of the American Statistical Society, September, 1954 (cf. [22]).

[2] *Note added in proof:* The Editor has pointed out that the paper by K. R. Nair, "A note on group divisible incomplete block designs", *Calcutta Statistical Association Bulletin,* Vol. 5, No. 17, (1953), pp. 30–35, together with Nair and Rao [16] essentially contains the results for the intra-block analysis of the two-factor asymmetrical designs.

Section 2 states three useful lemmas, Section 3 contains the main results of this paper, and Section 4 outlines the recovery of inter-block information.

2. Some useful lemmas.

We state here three lemmas which will be referred to in later sections. Since the proofs are trivial they are omitted.

Let $X' = (X_1, X_2, \cdots, X_n)$ have a multivariate normal distribution such that

$$E(X') = m' = (m_1, m_2, \cdots, m_n),$$
$$E[(X - m)(X - m)'] = M\sigma^2.$$

LEMMA 2.1. *The expected value of the quadratic form $X'AX$ is*

$$E(X'AX) = m'Am + \sigma^2 \text{ trace } (AM).$$

LEMMA 2.2. *If $M^2 = \lambda M$ (λ a scalar), then the quadratic form*

$$\frac{(X - m)'(X - m)}{\lambda}$$

follows a $\sigma^2\chi^2$ distribution with r degrees of freedom where $r \leqq n$ is the rank of M.

LEMMA 2.3. *Define the direct-product of two square matrices A and B of dimensions m and n respectively by*

$$(A{*}B) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}.$$

If $A^2 = \alpha A$ and $B^2 = \beta B$ (α and β are scalars), then $(A * B)^2 = \alpha\beta(A * B)$. In general, given p matrices $A, B, C, \cdots$ such that $A^2 = \alpha A$, $B^2 = \beta A$, $C^2 = \gamma C$, $\cdots$ we have $(A * B * C * \cdots)^2 = (\alpha\beta\gamma \cdots)(A * B * C * \cdots)$.

3. Analysis of group divisible designs used as asymmetrical factorials.

3.1. Estimation. The group divisible designs are partially balanced incomplete block designs with two associate classes. These were first discussed extensively by Bose and Connor [3] and Bose and Shimamato [5]. A large catalogue of such experiment plans giving full details of the analysis can be found in Bose, Clatworthy, and Shrikhande [2]. Designs with block size $k = 2$ have been enumerated by Clatworthy [7]. Bose, Shrikhande, and Battacharya [6], and Clatworthy [8] give methods for constructing group divisible designs.

Briefly group divisible designs can be characterized by having b blocks with

k experimental units such that each of the $v = mn$ treatments is replicated r times. The $v = mn$ treatments can be divided into m groups of n treatments each, where any two treatments in the same group are 1st associates and two treatments in different groups are 2nd associates. With respect to any treatment, there will be $(n - 1)$ 1st associates and $n(m - 1)$ 2nd associates.

Consider a factorial experiment with $(g + h)$ factors $A_1, A_2, \cdots, A_g, B_1, \cdots, B_h$ such that the number of levels associated with A_s is m_s for $s = 1, \cdots, g$ and the number of levels associated with B_r is n_r $r = 1, 2, \cdots, h$. Furthermore, let these levels be such that $m = \prod_{s=1}^{g} m_s$ and $n = \prod_{r=1}^{h} n_r$. Then one can use the group divisible designs for a $\prod_{s=1}^{g} m_s \times \prod_{r=1}^{h} n_r$ factorial design by arranging the $v = mn$ treatments in an $n \times m$ array and assigning the m factorial combinations among the A factors to the columns (groups) and the n factorial combinations among the B factors to the rows.

Let the measurement of the uth treatment combination $(u = 1, 2, \cdots, v)$ measured in the zth block be denoted by y_{uz} and let the underlying mathematical model be

$$y_{uz} = m + t_u + b_z + \epsilon_{uz}, \tag{3.1}$$

where m is a constant common to all measurements, t_u is the effect of the uth treatment combination, b_z is the constant associated with the zth block

$$(z = 1, 2, \cdots, b),$$

and $\{\epsilon_{uz}\}$ is a sequence of uncorrelated random variables having a zero mean and (unknown) variance σ^2. For making all tests of significance, we shall further assume that the $\{\epsilon_{uz}\}$ follow a normal distribution.

Due to the factorial nature of the experiment, a treatment combination t_z can be written as

$$\begin{aligned} t_z = & \sum_{s=1}^{g} (a_s)_{i_s} + \sum_{q=1}^{h} (b_q)_{j_q} \\ & + \sum_{t=2}^{g} \sum_{s=1}^{t} (a_{st})_{i_s i_t} + \sum_{r=2}^{h} \sum_{q=1}^{r} (b_{qr})_{j_q j_r} \\ & + \sum_{q=1}^{h} \sum_{s=1}^{g} (a_s b_q)_{i_s j_q} + \cdots + (a_{12\cdots g} b_{12\cdots h})_{i_{12}\cdots g j_{12}\cdots h}. \end{aligned} \tag{3.2}$$

The $(a_s)_{i_s}$ are constants associated with the main effect of A_s at level i_s; the $(a_{st})_{i_s i_t}$ are constants associated with the two factor interaction between A_s and A_t at levels i_s and i_t, etc. Similar interpretations hold for the constants associated with the main effects and interactions of the B factors, and also for the constants associated with the interactions composed of both A and B factors. It is well known that these parameters are not all linearly independent and satisfy the following relations:

$$
(3.3)\quad \begin{cases}
\sum_{i_s=1}^{m_s} (a_s)_{i_s} = 0, & s = 1, 2, \cdots, g, \\
\sum_{j_q=1}^{n_q} (b_q)_{j_q} = 0, & q = 1, 2, \cdots, h, \\
\sum_{i_\alpha=1}^{m_\alpha} (a_{st})_{i_{st}} = 0, & \alpha = s, t;\ \ s < t = 1, 2, \cdots, g, \\
\sum_{j_\beta=1}^{n_\beta} (b_{qr})_{j_{qr}} = 0, & \beta = q, r;\ \ q < r = 1, 2, \cdots, h, \\
\vdots & \vdots \\
\sum_{i_\alpha=1}^{m_\alpha} (a_{12\cdots g} b_{12\cdots h})_{i_{12\cdots g} j_{12\cdots h}} & \\
= \sum_{j_\beta=1}^{n_\beta} (a_{12\cdots g} b_{12\cdots h})_{i_{12\cdots g} j_{12\cdots h}} = 0, & \alpha = 1, 2, \cdots g;\ \ \beta = 1, 2, \cdots h.
\end{cases}
$$

If the adjusted treatment total for the uth treatment is defined by

$$Q_u = (u\text{th treatment total}) - \begin{pmatrix}\text{sum of the block averages in} \\ \text{which the } u\text{th treatment occurs}\end{pmatrix},$$

then the treatment estimates can conveniently be written as

$$(3.4)\qquad \hat{t}_u = \frac{1}{r(k-1)}\,[kQ_u + c_1 S_1(Q_u) + c_2 S_2(Q_u)].$$

Here $S_1(Q_u)$ and $S_2(Q_u)$ are the sum of the adjusted treatment totals for the 1st and 2nd associates with respect to treatment u, and c_1, c_2 are constants calculated from the design parameters. (All catalogues of group divisible designs [2], [5], [7], [8], give numerical values of c_1 and c_2).

Since these estimates satisfy the restraint $\sum_{u=1}^{v} \hat{t}_u = 0$, the variance of a treatment estimate can be written as

$$(3.5)\qquad \operatorname{Var} \hat{t}_u = \left[\frac{vk - [k + (n-1)c_1 + n(m-1)c_2]}{r(k-1)v}\right]\sigma^2,$$

and the covariance between treatments which are (say) sth associates ($s = 1, 2,$) is

$$(3.6)\qquad \operatorname{Cov}(\hat{t}_i, \hat{t}_j) = \left[\frac{vc_s - [k + (n-1)c_1 + n(m-1)c_2]}{r(k-1)v}\right]\sigma^2$$

for $s = 1, 2$.

Let (say) $A_1, A_2, \cdots, A_p$ ($p \leqq g$) and $B_1, B_2, \cdots, B_q$ ($q \leqq h$) be a selection of the A and B factors and let them be associated with the particular levels $i = (i_1, i_2, \cdots, i_p)$ and $j = (j_1, j_2, \cdots, j_q)$ respectively. We define an S-function associated with these particular factors and levels by

$$(3.7)\qquad S[1, 2, \cdots, p;\ 1, 2, \cdots, q \mid i, j] = \frac{\prod_{i=1}^{p} m_i \prod_{j=1}^{q} n_j}{v} \sum_u{}' \hat{t}_u,$$

where the summation $\sum_u'$ refers to the sum over all treatment estimates which have the same levels $i_1, i_2, \cdots, i_p; j_1, j_2, \cdots, j_q$ with respect to the factors $A_1, A_2, \cdots, A_p; B_1, B_2, \cdots, B_q$. If an S-function contains no A factors, we shall denote this by $S[0; 1, 2, \cdots, q \mid j]$ with a similar notation for the absence of B factors. (Note that these S-functions are simply the cell averages in any $(p + q)$ way table associated with these factors). Then the expected value of (3.7) is

$$(3.8)\qquad \begin{aligned} E\{S[1, 2, \cdots, p; 1, 2, \cdots, q \mid i, j]\} &= \sum_{s=1}^{p} (a_s)_{i_s} + \sum_{r=1}^{q} (b_r)_{j_r} \\ &+ \sum_{t=2}^{p} \sum_{s=1}^{t} (a_{st})_{i_{st}} + \sum_{s=2}^{q} \sum_{r=1}^{s} (b_{rs})_{j_{rs}} + \cdots + (a_{12\cdots p} b_{12\cdots q})_{i_{12\cdots p} j_{12\cdots q}}, \end{aligned}$$

where the summations refer to the particular factors A_i $(i = 1, 2, \cdots, p)$, B_j $(j = 1, 2, \cdots, q)$ and the levels $i_{st\cdots} j_{rs\cdots}$ refer only to $i = (i_1, i_2, \cdots, i_p)$ and $j = (j_1, j_2, \cdots, j_q)$. There will be only $(v - 1)$ linearly independent treatment estimates and since the relations (3.3) imply that there exist $(v - 1)$ linearly independent factorial constants, the condition of unbiasedness is sufficient to insure unique estimates of the factorial constants. Therefore the estimates of the main effects and interaction parameters are given by

$$(3.9)\qquad \begin{cases} (\hat{a}_s)_{i_s} = S[s; 0 \mid i_s], \\ (\hat{b}_q)_{j_q} = S[0; q \mid j_q], \\ (\hat{a}_{st})_{i_{st}} = S[s, t; 0 \mid i_s, i_t] - \{S[s; 0 \mid i_s] + S[t; 0 \mid i_t]\}, \\ (\hat{b}_{qu})_{j_{qu}} = S[0; q, u \mid j_q, j_u] - \{S[0; q \mid j_q] + S[0; u \mid j_u]\}, \\ (a_s\hat{b}_q)_{i_s j_q} = S[s; q \mid i_s, j_q] - \{S[s; 0 \mid i_s] + S[0; q \mid j_q]\}, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ (a_{12\cdots g}\hat{b}_{12\cdots h})_{i_{12\cdots g} j_{12\cdots h}} \\ \quad = S[1, 2, \cdots, g; 1, 2, \cdots, h \mid i_1, \cdots, i_g, j_1, \cdots, j_h] \\ \quad - \{S[1, 2, \cdots, g - 1; 1, 2, \cdots, h \mid i_1, \cdots, i_{g-1}, j_1, \cdots, j_h] + \cdots\}. \\ \quad + \cdots + (-1)^{g+h-1}\{S[1; 0 \mid i_1] + \cdots\}. \end{cases}$$

The estimate for a $(p + q)$th interaction involving the factors (say) $\{A_s\}$, $\{B_r\}$ associated with the respective levels i_s, j_r $(s = 1, 2, \cdots, p; r = 1, 2, \cdots, q)$ can conveniently be written as

$$(3.10)\qquad (a_{12\cdots p}\hat{b}_{12\cdots q})_{i_{12\cdots p} j_{12\cdots q}} = (-1)^{p+q} \sum_{w=1}^{p+q} (-1)^{w}\{w\},$$

where $\{w\}$ denotes the sum of all S-functions involving exactly $w \leqq p + q$ factors from the above set.

3.2. Variances, covariances, and tests of significance. In this section we shall obtain the variances and covariances of the main effects and interaction terms. It will be shown that these can be written as direct products of matrices and this leads directly to the appropriate sums of squares for the analysis of variance. Four lemmas pertaining to the S-functions are derived and are used for proving three basic theorems pertaining to the analysis.

LEMMA 3.1. *The variance of* $S = S[1, 2, \cdots, p; 1, 2, \cdots, q \mid i, j]$ *is*

$$\text{Var } S = \frac{\sigma^2}{r(k-1)v}[(MN-1)(k-c_1) + n(M-1)(c_1-c_2)], \tag{3.11}$$

where $M = \prod_{s=1}^{p} m_s$, $N = \prod_{r=1}^{q} n_r$.

PROOF. The number of treatments summed in S is $v/MN = mn/MN$ which can be regarded as m/M groups of n/N treatments each, such that treatments within the same group are first associates and treatments in different groups are second associates. Then there are $\binom{mn/MN}{2}$ different pairs of treatments among the mn/MN treatments in S, of which

$$\frac{m}{M}\binom{n/N}{2} = \frac{v(n-N)}{2N^2M}$$

are 1st associates and

$$\frac{1}{2}\left(\frac{m}{M} - 1\right)\left(\frac{n}{N}\right)\frac{v}{NM}$$

are 2nd associates. Therefore the variance of S is

$$\left\{\begin{aligned} \text{Var } S = \frac{\sigma^2 M^2 N^2}{v^2}\Bigg\{ & \frac{v}{MN}\left[\frac{vk - [k + c_1(n-1) + c_2 n(m-1)]}{r(k-1)v}\right] \\ & + \frac{2v(n-N)}{2N^2M}\left[\frac{c_1 v - [k + c_1(n-1) + c_2 n(m-1)]}{r(k-1)v}\right] \\ & + \frac{2v(m-M)n}{2(MN)^2}\left[\frac{c_2 v - [k + c_1(n-1) + c_2 n(m-1)]}{r(k-1)v}\right]\Bigg\}, \end{aligned}\right. \tag{3.12}$$

which on simplifying gives the desired result.

LEMMA 3.2. *Let* $S = S[1, 2, \cdots, p; 1, 2, \cdots, q \mid i, j]$ *and*

$$S' = S[1', 2', \cdots, p'; 1', 2', \cdots, q' \mid i', j']$$

be two S-functions having a A factors and b B factors in common, such that for a_1 and b_1 of these factors, the levels are identical and for a_2 and b_2 of these (common) factors, the levels are different ($a = a_1 + a_2$, $b = b_1 + b_2$). *Then*

$$\text{Cov}(S, S') = \frac{\sigma^2}{r(k-1)v}\left\{\begin{aligned}&(M_1N_1 - 1)(k - c_1)\\ &+ n(M_1 - 1)(c_1 - c_2)\end{aligned}\right\}, \tag{3.13}$$

where $M_1 = \prod_{i=1}^{a_1} m_i$ *(product of the levels of the* a_1 *factors having common levels) and* $N_1 = \prod_{j=1}^{b_1} n_j$ *(product of the levels of the* b_1 *factors having common levels).*

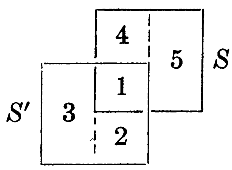
PROOF. The number of treatments summed in S and S' are v/MN and $v/M'N'$ respectively. These treatments can be regarded as consisting of two rectangular treatment arrays of dimensions $(n/N) \times (m/M)$ and

$$(n/N') \times (m/M')$$

respectively. The two arrays will overlap if they have common treatments and the number of such common treatments is

$$\frac{vM_1N_1}{(MN)(M'N')} = \left(\frac{mM_1}{MM'}\right)\left(\frac{nN_1}{NN'}\right).$$

It is convenient to depict the intersection of the rectangular arrays by the five regions as shown below,



where region (1) is an array representing the common treatments having (nN_1/NN') rows and (mM_1/MM') columns. If $\sum (i)$ represent the sum of the treatments in the ith region ($i = 1, 2, 3, 4, 5$), then

$$\text{(3.14)} \qquad \begin{cases} S = \sum(1) + \sum(4) + \sum(5), \\ S' = \sum(1) + \sum(2) + \sum(3). \end{cases}$$

Hence, in order to find the covariance between S and S', it is necessary to find the number of pairs of 1st and 2nd associates formed from the multiplication of S and S'. These will give pairs formed from $\sum(1)^2$, $\sum(1)\sum(2)$, $\sum(1)\sum(3)$, $\sum(1)\sum(4)$, $\sum(2)\sum(4)$, $\sum(3)\sum(4)$, $\sum(1)\sum(5)$, $\sum(2)\sum(5)$, and $\sum(3)\sum(5)$.

Define

$$\text{(3.15)} \qquad \begin{cases} m_1 = \dfrac{mM_1}{MM'}, & n_1 = \dfrac{nN_1}{NN'}, \\[2ex] m_2 = \dfrac{m}{MM'}(M - M_1), & n_2 = \dfrac{n}{NN'}(N - N_1), \\[2ex] m_3 = \dfrac{m}{MM'}(M' - M_1), & n_3 = \dfrac{n}{NN'}(N' - N_1). \end{cases}$$

Then the dimensions of the five regions are:

region (1): $n_1 \times m_1$,

region (2): $n_2 \times m_1$,

$$\text{region (3):}\quad (n_1 + n_2) \times m_2,$$
$$\text{region (4):}\quad n_3 \times m_1,$$
$$\text{region (5):}\quad (n_1 + n_3) \times m_3.$$

Since the treatments in the same row are 1st associates of each other and treatments in different rows 2nd associates, it is an easy matter to count the number of 1st and 2nd associates arising from pairs formed from $\sum (i) \sum (j)$. Performing the necessary algebra, we find that the total number of 1st associate pairs is

$$\frac{vM_1(n - N_1)}{(NN')(MM')}$$

and the total number of 2nd associate pairs is

$$\frac{vn(m - M_1)}{(NN')(MM')}.$$

Therefore,

$$(3.16)\quad \begin{cases} \operatorname{Cov}(S, S') = \dfrac{\sigma^2(MM')(NN')}{v^2}\left\{\dfrac{vM_1N_1}{(MM'NN')}\ [\operatorname{Var} t]\right. \\ \qquad + \dfrac{vM_1(n - N_1)}{(NN'MM')}\ \operatorname{Cov}\ (\text{1st associates}) \\ \qquad \left. + \dfrac{vn(m - M_1)}{(NN'MM')}\ \operatorname{Cov}\ (\text{2nd associates})\right\}. \end{cases}$$

On simplifying we get the desired result.

LEMMA 3.3. *Let* $(\hat{ab}) = (a_{12\cdots p}b_{12\cdots q})_{i_{12}\cdots_p j_{12}\cdots_q}$ *be the estimate of the* $(p + q)$th *factor interaction associated with the factors* $\{A_s\}(s = 1, 2, \cdots, p)$ *and*

$$\{B_r\}(r = 1, 2, \cdots, q).$$

Let $S' = S[1; 2', \cdots, p'; 1', 2', \cdots, q' \mid i', j']$ *be an S-function which is not associated with all factors (regardless of level) of* $(\hat{ab})$. *Then*

$$(3.17)\qquad \operatorname{Cov}[(\hat{ab}), S'] = 0.$$

PROOF. Let α be the number of common A factors between (ab) and S', and α_1 and α_2 $(\alpha = \alpha_1 + \alpha_2)$ be the number of these common factors having the same levels and different levels, respectively. Define β, β_1, and β_2 in the same manner with respect to the B factors. Since the interaction $(\hat{ab})$ can be written in the form

$$(\hat{ab}) = (-1)^{p+q} \sum_{w=1}^{p+q} (-1)^w \{w\},$$

consider a fixed $\{w\}$ and a particular S-function in $\{w\}$ having the characteristics a, a_1, a_2, b, b_1, b_2 as defined in Lemma 3.2.

Define

$$
(3.18)\quad \begin{cases} C(0, 0) = -1, \\ C(a_1) = \sum \cdots \sum (m_{s_1} m_{s_2} \cdots m_{s_{a_1}} - 1), & a_1 \leqq \alpha_1, \\ C(b_1) = \sum \cdots \sum (n_{r_1} n_{r_2} \cdots n_{r_b} - 1), & b_1 \leqq \beta_1, \\ C(a_1, b_1) = \sum \cdots \sum (m_{s_1} m_{s_2} \cdots m_{s_{a_1}} n_{r_1} n_{r_2} \cdots n_{r_{b_1}} - 1), \\ \qquad\qquad a_1 \leqq \alpha_1, \quad b_1 \leqq \beta_1, \end{cases}
$$

where the summations are only over combinations of A and B factors taken a_1 and b_1 at a time respectively, such that these factors are those in which $(\hat{ab})$ and S have in common at the same level.

Then the covariance between S' and $\{w\}$ can be written

$$
(3.19)\quad \begin{cases} \operatorname{Cov}[S', \{w\}] = \dfrac{\sigma^2}{r(k-1)v} \left\{ \sum_{a_1+b_1=w} \dbinom{p+q-\alpha-\beta}{w-a_1-b_1} \right. \\ \qquad \cdot \left[(k-c_1)C(a_1, b_1) + n \dbinom{\beta_1}{b_1} (c_1-c_2)C(a_1) \right] \\ \quad - \sum_{\substack{a_1+b=w \\ b_2 \neq 0}} \dbinom{p+q-\alpha-\beta}{w-a_1-b} \\ \qquad \cdot \left[\dbinom{\alpha_1}{a_1} \dbinom{\beta}{b} (k-c_1) - n \dbinom{\beta}{b} (c_1-c_2)C(a_1) \right] \\ \quad \left. + \sum_{a_2=1}^{w} \dbinom{p+q-\alpha-\beta}{w-a_2} \dbinom{\alpha_2}{a_2} [(k-c_1) + n(c_1-c_2)] \right\}. \end{cases}
$$

Note that the first summation is for those S-functions in $\{w\}$ for which

$$a_2 = b_2 = 0;$$

the second summation refers to $a_2 = 0$, $b_2 \neq 0$; and the third summation is when $a_2 \neq 0$. Since the covariance between S' and $(\hat{ab})$ is

$$
(3.20)\qquad \operatorname{Cov}[S', (\hat{\ })] = (-1)^{p+q} \sum_{w=1}^{p+q} (-1)^w \operatorname{Cov}[S', \{w\}],
$$

we can substitute (3.19) in (3.20) to obtain an explicit expression for (3.20). Now with respect to fixed values of a_1, a_2, b_1, and b_2 the only terms contributing to the first summation in (3.19) is when

$$w = a_1 + b_1, \cdots, p + q + a_1 + b_1 - \alpha - \beta;$$

the value of w contributing to the second summation in (3.19) is for

$$w = a_1 + b, \cdots, p + q + a_1 + b - \alpha - \beta$$

and the contributing value of w for the last summation in (3.19) is when

$$w = a_2, \cdots, p + q + a_2 - \alpha - \beta.$$

Therefore collecting coefficients of

$$\left[(k - c_1)C(a_1, b_1) + n\binom{\beta_1}{b_1}(c_1 - c_2)C(a_1)\right]$$

in (3.20) gives

$$(3.21) \qquad (-1)^{a_1+b_1} \sum_{w=0}^{p+q-\alpha-\beta} \binom{p+q-\alpha-\beta}{w} (-1)^w = 0$$

for all a_1 and b_1. Collecting coefficients of

$$\binom{\beta}{b}\left[\binom{\alpha_1}{a_1}(k - c_1) - n(c_1 - c_2)C(a_1)\right]$$

results in

$$(-1)^{a_1+b+1} \sum_{w=0}^{p+q-\alpha-\beta} \binom{p+q-\alpha-\beta}{w} (-1)^w = 0$$

for all a_1, b_1, and b_2. Finally, with respect to the coefficient of

$$\binom{\alpha_2}{a_2}[(k - c_1) + n(c_1 - c_2)]$$

in (3.20) we have

$$(-1)^{a_1} \sum_{w=0}^{p+q-\alpha-\beta} \binom{p+q-\alpha-\beta}{w} (-1)^w = 0.$$

LEMMA 3.4. *Let* $(\hat{ab}) = (a_{12\cdots p}b_{12\cdots q})_{i_{12}\cdots{}_p j_{12}\cdots{}_q}$ *be an estimate of the* $(p + q)$ *factor interaction associated with the factors* $\{A_i\}(i = 1, 2, \cdots, p)$ *and*

$$\{B_j\}(j = 1, 2, \cdots, q).$$

Let $S' = S[1, 2, \cdots, p; 1, 2, \cdots, q \mid i, j]$ *be an S-function associated with the same factors as* $(\hat{ab})$ *such that* α_1 *and* β_1 *of the A and B factors have common levels. Then,*

$$(3.22) \quad \operatorname{Cov}[(\hat{ab}), S'] = \begin{cases} (-1)^{p+q+\alpha_1+\beta_1}\theta(1, 2, \cdots, \alpha_1)\phi(1, 2, \cdots, \beta_1)\dfrac{\sigma^2}{E_b rv}, & \text{if } q \neq 0, \\ (-1)^{p+\alpha_1}\theta(1, 2, \cdots, \alpha_1)\dfrac{\sigma^2}{E_a rv}, & \text{if } q = 0, \end{cases}$$

where

$$(3.23) \quad \begin{aligned} \theta(1, 2, \cdots, \alpha_1) &= (m_1 - 1)(m_2 - 1)\cdots(m_{\alpha_1} - 1), \\ \phi(1, 2, \cdots, \beta_1) &= (n_1 - 1)(n_2 - 1)\cdots(n_{\beta_1} - 1), \end{aligned}$$

and

$$(3.24)\qquad \begin{cases} E_a = \dfrac{(k-1)}{(k-c_1)+n(c_1-c_2)}, \\ E_b = \dfrac{k-1}{k-c_1}. \end{cases}$$

PROOF. If we expand $(a^\wedge b)$ in terms of S-functions (Eq. 3.10), we can write the covariance between S' and a fixed $\{w\}$ for $q \neq 0$ as

$$(3.25)\qquad \begin{aligned} \operatorname{Cov}[S', \{w\}] = \frac{\sigma^2}{r(k-1)v}\Bigg\{\Bigg[&(k-c_1)\sum_{a_1+b_1=w} C(a_1, b_1) \\ &+ n(c_1-c_2)\sum_{a_1+b_1=w}\binom{\beta_1}{b_1}C(a_1)\Bigg] \\ &- \Bigg[\sum_{\substack{a_2+s=w \\ a_2\neq 0}}\binom{\alpha_2}{a_2}\binom{q+\alpha_1}{s}[(k-c_1)+n(c_1-c_2)]\Bigg] \\ &- \Bigg[\sum_{\substack{a_1+b_1+b_2=w \\ b_2\neq 0}}\binom{\alpha_1}{a_1}\binom{\beta_1}{b_1}\binom{\beta_2}{b_2}[(k-c_1)] \\ &- n(c_1-c_2)\sum_{\substack{a_1+b_1+b_2=w \\ b_2\neq 0}}\binom{\beta_1}{b_1}\binom{\beta_2}{b_2}C(a_1)\Bigg]\Bigg\}, \end{aligned}$$

where the first bracket is when $a_2 = b_2 = 0$; the second bracket is the case $a_2 \neq 0$; and the third bracket refers to $a_2 = 0$, $b_2 \neq 0$. Substituting (3.25) in

$$(3.26)\qquad \operatorname{Cov}[S', (a^\wedge b)] = (-1)^{p+q}\sum_{w=1}^{p+q}(-1)^w \operatorname{Cov}[S', \{w\}]$$

results in the first bracket being written as (neglecting the constant term)

$$(3.27)\qquad \begin{aligned} (-1)^{p+q+\alpha_1+\beta_1}\Bigg[&\sum_{w=1}^{\alpha_1+\beta_1}(-1)^w\sum_{a_1+b_1=w}C(a_1, b_1)(k-c_1) \\ &+ n(c_1-c_2)\sum_{w=1}^{\alpha_1+\beta_1}(-1)^w\sum_{a_1+b_1=w}\binom{\beta_1}{b_1}C(a_1)\Bigg] \\ = (-1)&^{p+q+\alpha_1+\beta_1}\theta(1, 2, \cdots, \alpha_1)\phi(1, 2, \cdots, \beta_1)(k-c_1) \\ &+ (-1)^{p+q}n(c_1-c_2)\sum_{a_1=1}^{\alpha_1}C(a_1)\sum_{w=a_1}^{a_1+q}(-1)^w\binom{\beta_1}{w-a_1}. \end{aligned}$$

With respect to the bracket when $a_2 \neq 0$, we can write these terms after substituting in (3.26) as

$$(3.28)\qquad [k-c_1+n(c_1-c_2)]\left[\sum_{r=1}^{p+q}(-1)^r\binom{p+q}{r} - \sum_{r=1}^{\alpha_1+q}(-1)^r\binom{\alpha_1+q}{r}\right] = 0.$$

Finally for the terms where $a_2 = 0$, $b_2 \neq 0$, after substituting in (3.26), we can write

$$(3.29) \quad \begin{cases} (-1)^{p+q+1}\left[\sum_{w=1}^{\alpha_1+q}(-1)^w\binom{\alpha_1+q}{w} - \sum_{w=1}^{\alpha_1+\beta_1}(-1)^w\binom{\alpha_1+\beta_1}{w}(k-c_1)\right. \\ +\ (-1)^{p+q}n(c_1-c_2)\left[\sum_{a_1=1}^{\alpha_1}C(a_1)\sum_{w=a_1}^{a_1+q}(-1)^w\left[\binom{\beta_1+\beta_2}{w-a_1} - \binom{\beta_1}{w-a_1}\right]\right]. \end{cases}$$

The first term in (3.29) is identically zero and combining the second term of the right hand side of (3.27) with the second term of (3.29) gives

$$(-1)^{p+q}n(c_1-c_2)\left[\sum_{a_1=1}^{\alpha_1}C(a_1)\sum_{r=a_1}^{a_1+q}(-1)^r\binom{q}{r-a_1}\right] = 0.$$

Thus the Lemma is true for $q \neq 0$. For $q = 0$, the covariance between S' and $\{w\}$ will be

$$(3.30) \quad \begin{cases} \operatorname{Cov}\,[S', \{w\}] = \dfrac{\sigma^2}{r(k-1)v}\left\{[(k-c_1)+n(c_1-c_2)]C(a_1)\right. \\ \qquad\left. - \displaystyle\sum_{a_1+a_2=w}\binom{\alpha_1}{a_1}\binom{\alpha_2}{a_2}[(k-c_1)+n(c_1-c_2)]\right\} \end{cases}$$

and following the same reasoning as for $q \neq 0$, we can prove the Lemma for $q = 0$.

THEOREM 3.1. *Let* $(\hat{ab}) = (a_{12\cdots p}b_{12\cdots q})_{i_{12\cdots p}j_{12\cdots q}}$ *be an estimate of the*

$$(p+q)\text{th}$$

factor interaction associated with the factors $\{A_i\}$ $(i = 1, 2, \cdots, p)$ *and*

$$\{B_j\}\ (j = 1, 2, \cdots, q);$$

let $(\hat{ab})' = (a'_{12\cdots r}\hat{b}'_{12\cdots s})_{i'_{12\cdots r}j'_{12\cdots s}}$ *be a* $(r+s)$ *factor interaction associated with the factors* $\{A'_i\}$ $(i = 1, 2, \cdots, r)$ *and* $\{B'_j\}$ $(j = 1, 2, \cdots, s)$, *such that all factors are not identical between* $(\hat{ab})$ *and* $(\hat{ab})'$. *Then the two different interactions are uncorrelated.*

PROOF. $(\hat{ab})$ can be expanded in terms of S-functions, such that no S-function contains all the factors of $(\hat{ab})'$. Hence by Lemma 3.3, the covariance between all such S-functions and $(\hat{ab})'$ are zero which proves the theorem.

THEOREM 3.2. *The variance of the* $(p+q)$ *factor interaction*

$$(\hat{ab}) = (a_{12\cdots p}\hat{b}_{12\cdots q})_{i_{12\cdots p}j_{12\cdots q}}$$

associated with the factors $\{A_i\}$ $(i = 1, 2, \cdots, p)$ *and* $\{B_j\}$ $(j = 1, 2, \cdots, q)$ *is*

$$(3.31) \qquad \operatorname{Var}(\hat{ab}) = \begin{cases} \theta(1, 2, \cdots, p)\dfrac{\sigma^2}{E_a rv}, & \text{if } q = 0, \\ \theta(1, 2, \cdots, p)\phi(1, 2, \cdots, q)\dfrac{\sigma^2}{E_b rv}, & \text{if } q \neq 0. \end{cases}$$

PROOF. Using Lemma 3.3, we can show that

$$\operatorname{Var}(\widehat{ab}) = \operatorname{Cov}[(\widehat{ab}), S],$$

where S denotes that S-function coinciding in all factors and levels with the interaction $(\widehat{ab})$. Hence, by Lemma 3.4 the theorem is proved.

THEOREM 3.3. *Let $(\widehat{ab})_{ij}$ and $(\widehat{ab})_{i'j'}$ be two estimates of a $(p + q)$ factor interaction associated with the factors $\{A_i\}$ $(i = 1, 2, \cdots, p)$ and*

$$\{B_j\} \ (j = 1, 2, \cdots, q)$$

such that for α_1 of the A factors and β_1 of the B factors, the levels are identical. Then

$$\operatorname{Cov}[(\widehat{ab})_{ij}, (\widehat{ab})_{i'j'}] = \begin{cases} (-1)^{p+\alpha_1}\theta(1, 2, \cdots, \alpha_1) \dfrac{\sigma^2}{E_a rv}, & \text{if } q = 0, \\ (-1)^{p+q+\alpha_1+\beta_1}\theta(1, 2, \cdots, \alpha_1)\phi(1, 2, \cdots, \beta_1) \dfrac{\sigma^2}{E_b rv}, & \text{if } q \neq 0. \end{cases} \tag{3.32}$$

PROOF. Expanding $(\widehat{ab})_{i'j'}$ in terms of S-functions, taking the covariance of $(\widehat{ab})_{ij}$ with each of the S-functions associated with $(\widehat{ab})_{i'j'}$ and using Lemma 3.3, results in

$$\operatorname{Cov}[(\widehat{ab})_{ij}, (\widehat{ab})_{i'j'}] = \operatorname{Cov}[(\widehat{ab})_{ij}, S'],$$

where S' is that S-function associated with the factors $\{A_i\}$ and

$$\{B_j\} \ (i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$$

and levels $i'_{12\cdots p}j'_{12\cdots q}$.

Hence, by Lemma 3.4 the theorem is proved.

Theorems 3.1 through 3.3 give the variances and covariances of any general interaction. Define the square matrices $M(i)$ and $N(j)$ of dimension m_i and n_j, respectively, by

$$\begin{cases} M(i) = m_i I - J, & i = 1, 2, \cdots, g, \\ N(j) = n_j I - J, & j = 1, 2, \cdots, h, \end{cases} \tag{3.33}$$

where J is a matrix of appropriate dimension having all elements unity. Then the variance-covariance matrix of the estimates of the $(p + q)$ factor interaction $(a_{12\cdots p}\hat{b}_{12\cdots q})_{i_{12\cdots p}j_{12\cdots q}}$ ranging over all the $\prod_{i=1}^{p} m_i \prod_{j=1}^{q} n_j$ combinations is given by the direct matrix product

$$\frac{\sigma^2}{E_a rv}[M(1)*M(2)*\cdots *M(p)], \qquad \text{if } q = 0, \tag{3.34}$$

or

$$\frac{\sigma^2}{E_b rv}[M(1)*\cdots *M(p)*N(1)*\cdots *N(q)], \qquad \text{if } q \neq 0. \tag{3.35}$$

Therefore, since $M(i)^2 = m_i M(i)$, $N(j)^2 = n_j N(j)$, and using Lemmas (2.2) and (2.3) the sums of squares

$$\frac{E_a rv}{\prod_{s=1}^{p} m_s} \sum_{i_{12\cdots p}} (\hat{a}_{12\cdots p})^2_{i_{12\cdots p}},$$

(3.36)

$$\frac{E_b rv}{\prod_{s=1}^{p} m_s \prod_{r=1}^{q} n_r} \sum_{i_{12\cdots p} j_{12\cdots q}} (a_{12\cdots p} \hat{b}_{12\cdots q})^2_{i_{12\cdots p} j_{12\cdots q}}, \qquad \text{if } q \neq 0,$$

follow $\chi^2\sigma^2$ distributions with $\prod_{s=1}^{p} (m_s - 1)$ and $\prod_{s=1}^{p} \prod_{r=1}^{q} (m_s - 1)(n_r - 1)$ degrees of freedom respectively if the null hypothesis of no interaction effect is true.

Using Lemma 2.1 these sums of squares have the expected values

$$\frac{E_a rv}{\prod_{s=1}^{p} m_s} \sum_{i_{12\cdots p}} (a_{12\cdots p})^2_{i_{12\cdots p}} + \prod_{i=1}^{p} (m_i - 1)\sigma^2$$

and

$$\frac{E_b rv}{\prod_{s=1}^{p} m_s \prod_{r=1}^{q} n_r} \sum_{i_{12\cdots p} j_{12\cdots q}} (a_{12\cdots p} b_{12\cdots q})^2 + \prod_{i=1}^{p} (m_i - 1) \prod_{j=1}^{q} (n_j - 1)\sigma^2.$$

The entire intra-block analysis of variance can be summarized in Table 1 where B represents the $b \times 1$ vector of block totals, Q is the $v \times 1$ vector of adjusted treatment totals, $\hat{t}$ is the $v \times 1$ vector of treatment estimates,

$$g = \frac{(\text{grand total})^2}{bk},$$

and terms such as

$$\sum_{i_{12\cdots p}} (\hat{a}_{12\cdots p})^2_{i_{12\cdots p}}$$

are written as $(\hat{a}_{12\cdots p})^2$, etc.

The computations for the analysis of variance are straightforward and actually amount to treating the $\hat{t}_u$'s as observations on a one replicate factorial experiment, where all sums of squares are multiplied by $E_a r$ or $E_b r$. It is also clear from the analysis of variance that the various interactions are estimated with one of a possible two types of efficiencies. If the interaction is composed only of A factors the efficiency is E_a, otherwise the efficiency will be E_b.

Extension to the balanced incomplete block designs. The balanced incomplete block designs can also be used for asymmetric factorial arrangements by assigning the v factorial combinations to the v treatments of the balanced incomplete block design. All results immediately follow by regarding the balanced incomplete block designs as a "degenerate" partially balanced design. Then

TABLE 1
Summary of intra-block analysis of variance

Source	Degrees of freedom	Sum of squares
Blocks (unadjusted)	$b - 1$	$\frac{B'B}{k} - g$
Treatments (adjusted)	$v - 1$	$\hat{t}'Q$
A_1	$(m_1 - 1)$	$\frac{v}{m_1} E_a r(\hat{a}_1)^2$
A_2	$(m_2 - 1)$	$\frac{v}{m_2} E_a r(\hat{a}_2)^2$
$\vdots$	$\vdots$	$\vdots$
B_1	$(n_1 - 1)$	$\frac{v}{n_1} E_b r(\hat{b}_1)^2$
$\vdots$	$\vdots$	$\vdots$
A_1B_1	$(m_1 - 1)(n_1 - 1)$	$\frac{v}{m_1 n_1} E_b r(\widehat{a_1 b_1})^2$
$\vdots$	$\vdots$	$\vdots$
$A_1A_2 \cdots A_gB_1 \cdots B_h$	$\prod_{s=1}^{g} (m_s - 1) \prod_{r=1}^{h} (n_r - 1)$	$\frac{v}{\prod_1^g m_s \prod_1^h n_r} E_b r(a_{12\ldots g} \hat{b}_{12\ldots h})^2$
Error	$n_e = bk - b - v + 1$	$S_e = \sum_{i,j} y_{ij}^2 - t'Q - \frac{B'B}{k}$
Total	$bk - 1$	$\sum_{i,j} y_{ij}^2 - g$

$c_1 = c_2 = k/v$ in (3.4), $E_a = E_b = E = v(k - 1)/k(v - 1)$, and all main effects and interactions are estimated with an efficiency factor E.

4. The recovery of inter-block information. If the block effects b_j in (3.4) can be regarded as a sequence of random variables such that

$$\begin{cases} E(b_j) = 0, \quad \operatorname{Var} b_j = \sigma_b^2, \\ \operatorname{Cov}(b_j, b_{j'}) = 0, \\ \operatorname{Cov}(\epsilon_{ij}, b_{j'}) = 0, \end{cases} \qquad \text{for } j \neq j', \tag{4.1}$$

it will be possible to extract additional information from the block totals. This additional analysis is sometimes termed the recovery of inter-block information or the interblock analysis. With respect to the balanced incomplete block designs, the inter-block analysis was first developed by Yates [20] and appears in the books by Cochran and Cox [9], Federer [10], and Kempthorne [13]. The inter-block analysis with respect to the partially balanced designs is discussed in a particularly simple form by Bose and Shimamoto [5] and by Bose, Clatworthy, and Shrikhande [2]. Generally it will be possible to use this inter-block information in two ways. The preceding references discuss how one may combine the inter-block information with the intra-block information in order to obtain the most precise treatment estimates. The paper by Zelen [21] discusses how one can use this inter-block information for obtaining additional independent tests of significance for every hypothesis pertaining to the treatments.

Define $Q_i^* = T_i - Q_i - r\bar{y}$, where Q_i is the ith adjusted treatment total, T_i is the total for the ith treatment and $\bar{y}$ is the grand average of all measurements. Then the best estimates of the treatments using both the intra- and inter-block information can be written as

$$\bar{t}_i = \frac{1}{R(k-1)} \{kP_i + d_1 S_1(P_i) + d_2 S_2(P_i)\}, \tag{4.2}$$

where

$$\begin{cases} P_i = WQ_i + W^*Q_i^*, \\ R = r\left[W + \dfrac{W^*}{k-1}\right], \\ W = \dfrac{1}{\sigma^2}, \qquad W^* = \dfrac{1}{\sigma^2 + k\sigma_b^2}. \end{cases} \tag{4.3}$$

The constants d_1, d_2 are usually tabulated with all the designs. Note that (4.2) is the same form as (3.4) except that P_i replaces Q_i, R replaces r, and d_1, d_2 replace c_1, c_2. Thus all results in Section 3 carry over directly by substituting the above changes in the formulas of Section 3 and replacing σ^2 by unity.

On the other hand, under certain conditions which are elaborated in [4], one can also obtain additional independent tests of significance using only the inter-block information. Three cases have to be considered depending on whether the group divisible design is a regular, singular, or semi-regular design. These are the three exhaustive classes of group divisible designs introduced by Bose and Connor [3].

For the regular group divisible designs the inter-block treatment estimates can be written as

$$t_i^* = \frac{kQ_i^* + c_1^* S_1(Q_i^*) + c_2^* S_2(Q_i^*)}{r} \tag{4.4}$$

and will have a variance of

$$\operatorname{Var} t_i^* = \left[\frac{vk - [k + (n-1)c_1^* + n(m-1)c_2^*]}{rv}\right](\sigma^2 + k\sigma_b^2).$$

Also if t_i^* and t_j^* are sth associates $(s = 1, 2)$,

$$\operatorname{Cov}(t_i^*, t_j^*) = \left[\frac{vc_s^* - [k + (n-1)c_1^* + n(m-1)c_2^*]}{rv}\right](\sigma^2 + k\sigma_b^2).$$

The quantities c_1^* and c_2^* are defined by

$$c_s^* = \frac{c_s \Delta - r\lambda_s}{\Delta - rH + r^2} \qquad (s = 1, 2),$$

where the parameters c_s, Δ, λ_s, and H are the usual parameters for partially balanced designs (cf. [2], [5]). Therefore all results for Section 3 apply directly by replacing σ^2 by $(\sigma^2 + k\sigma_b^2)$, c_s by c_s^*, and $r(k - 1)$ by r. This results in the two efficiencies being

$$E_a^* = \frac{1}{k - c_1^* + n(c_1^* - c_2^*)},$$

$$E_b^* = \frac{1}{k - c_1^*},$$

and the breakdown of the $v - 1$ treatment sum of squares, using only the inter-block information, is similar to Table 1. If $b > v$, there will be an independent estimate of $\sigma^2 + k\sigma_b^2$, thus permitting independent inter-block tests of significance for the main effects and interactions.

With respect to the singular designs, the intra-block efficiencies are

$$E_a < 1, \qquad E_b = 1.$$

Hence it is only possible to obtain inter-block estimates for those main effects and interactions associated only with the A factors. Since treatments in the same group are first associates, $1/n[t_i + S_1(t_i)]$ represents the average of the group to which treatment i belongs. This average is estimated by

$$\frac{1}{n}[t_i^* + S_1(t_i^*)] = \frac{Q_i^*}{E_a^* r}, \qquad E_a^* = \frac{mn - k}{k(m - 1)}. \tag{4.5}$$

There will be m such estimates, thus making it possible to have S-functions of the form $S[1, 2, \cdots, p; 0 \mid i]$ for $p \leqq g$. Then all results of Section 3 follow by replacing E_a by E_a^* and σ^2 by $\sigma^2 + k\sigma_b^2$. If $b > m$, this will permit an estimate of $\sigma^2 + k\sigma_b^2$ and thus we can have independent inter-block tests of significance for the A factor.

The semi-regular group divisible designs have the intra-block efficiencies $E_a = 1$, $E_b < 1$. Therefore it is possible to obtain inter-block estimates of those main effects and interactions having the intra-block efficiency E_b. These $(v - m)$ contrasts can be found by using the normal equations

$$\sum_{s=1}^{v} \frac{\lambda_{is}}{k} t_s^* = Q_i^*, \qquad i = 1, 2, \cdots, v, \tag{4.6}$$

where $\lambda_{ii} = r$, and λ_{is} = number of blocks in which treatments i and s appear together. The rank of Eqs. (4.6) is exactly $(v - m)$. If $b > (v - m)$, then it will be possible to have an independent estimate of $(\sigma^2 + k\sigma_b^2)$, thus allowing independent inter-block tests of significance for testing these contrasts. An open problem is to simplify this analysis.

Extension to balanced incomplete block designs. Similar results apply to the recovery of inter-block information for the balanced incomplete block designs. The best combined estimate can be written as

$$\bar{t}_i = \frac{P_i}{r[WE + W^*(1 - E)]} = \frac{P_i}{\bar{E}R}, \qquad \bar{E} = \frac{R(k - 1) + \lambda(W - W^*)}{R}.$$

Therefore all results of Section 3 can also be carried over by substituting unity for σ^2, R for r, etc. This produces an efficiency of

$$\bar{E} = \frac{R(k-1) + \lambda(W - W^*)}{R}.$$

In addition if one wished to obtain additional independent significance tests using the inter-block information only, the treatment estimates can be written

$$t_i^* = \frac{Q_i^*}{(1-E)r}$$

and all results of Section 3 follow by replacing $\sigma^2 + k\sigma_b^2$ for σ^2, and

$$E_a = E_b = 1 - E.$$

Again we will have two independent tests of significance for testing every null hypothesis pertaining to the main effects and interactions.

Acknowledgement. The author is indebted to Professor R. C. Bose and Dr. Max Halperin for drawing the author's attention to Lemmas 2.1 and 2.2, respectively. Also thanks go to Dr. Alan Goldman for reading a preliminary version of this paper.

REFERENCES

[1] R. C. Bose, "Mathematical theory of the symmetric factorial design," *Sankhyā*, Vol. 8 (1947), pp. 107–166.
[2] R. C. Bose, W. H. Clatworthy, and S. S. Shrikhande, *Tables of Partially balanced designs with two associate classes*, Institute of Statistics of Univ. of North Carolina, Reprint series No. 50 (1954).
[3] R. S. Bose and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.
[4] R. C. Bose and K. Kishen, "On the problem of confounding in general symmetrical factorial designs," *Sankhyā*, Vol. 5 (1940), pp. 21–26.
[5] R. C. Bose and T. Shimamoto, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.
[6] R. C. Bose, S. S. Shrikhande, and K. N. Bhattacharya, "On the construction of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 167–195.
[7] W. H. Clatworthy, "Partially balanced incomplete block designs with two associate classes and two treatments per block," *Journal of Research of the National Bureau of Standards*, Vol. 54 (1955), pp. 177–190.
[8] W. H. Clatworthy, *Contributions on partially balanced incomplete block designs with two associate classes*, National Bureau of Standards Applied Mathematics Series 47, 1956.
[9] W. G. Cochran and G. M. Cox, *Experimental Designs*, New York, John Wiley and Sons, 1950.
[10] W. T. Federer, *Experimental Designs*, New York, Macmillan, 1955.
[11] R. A. Fisher, *The Design of Experiments*, Oliver and Boyd, Edinburgh.
[12] R. A. Fisher, "The theory of confounding in factorial experiments in relation to the theory of groups," *Ann. Eugenics*, Vol. 11 (1942), pp. 341–353.
[13] O. Kempthorne, *The Design and Analysis of Experiments*, New York, John Wiley and Sons, 1952.

[14] C. Y. KRAMER AND R. A. BRADLEY, "Intra-block analysis for factorials in two-associate class group divisible designs," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 349–361.

[15] J. LI, "Design and statistical analysis of some confounded factorial experiments," *Iowa Agricultural Experimental Station Research Bulletin*, *333*, 1934.

[16] K. R. NAIR AND C. R. RAO, "Confounding in asymmetrical factorial experiments," *J. Roy. Stat. Soc.*, Series B, Vol. 10 (1948), pp. 109–131.

[17] C. R. RAO, "General methods of analysis for incomplete block designs," *J. Amer. Stat. Assn.*, Vol. 42 (1947), pp. 541–561.

[18] C. R. RAO, "On the recovery of inter-block information in varietal trials," *Sankhyā*, Vol. 17 (1956), pp. 105–114.

[19] F. YATES, *The design and analysis of factorial experiments*, Imperial Bureau Soil Science, Harpenden, England, 1937.

[20] F. YATES, "The recovery of inter-block information in balanced incomplete block designs," *Ann. Eugenics*, Vol. 10 (1940), pp. 317–325.

[21] M. ZELEN, "The analysis of incomplete block designs," *J. Amer. Stat. Assn.*, Vol. 52 (1957), pp. 204–217.

[22] M. ZELEN, "The use of incomplete block designs for factorial experiments" (abstract), *J. Amer. Stat. Assn.*, Vol. 50 (1955), p. 599.