ABSTRACTS OF PAPERS

(Abstract of a paper presented at the Cambridge, Massachusetts Meeting of the Institute, August 25–28, 1958.)

31. **Markov Renewal Processes.** RONALD PYKE, Columbia University. (Invited Paper presented under the title, "On Multi-event Renewal Processes.")

Let $\mathfrak{Q} = \| Q_{ij} \|$, $1 \leqq i, j \leqq m$, $m < \infty$ be a matrix of transition distributions, i.e. each Q_{ij} is a mass function satisfying $Q_{ij}(t) = 0$ for $t \leqq 0$ and $\sum_{j=1}^{m} Q_{ij}(+\infty) = 1$. For discrete probabilities $a_1, \cdots, a_m$, let $\{(J_n, X_n); n \geqq 0\}$ be a stochastic process satisfying

$$P[J_0 = k] = a_k, \qquad X_0 = 0,$$

and $P[J_n = k, X_n \leqq x \mid J_0, J_1, X_1, \cdots, J_{n-1}, X_{n-1}] = Q_{J_{n-1},k}(x)$ a.s. For $t \geqq 0$, $1 \leqq j \leqq m$, define $N_j(t)$ as the number of times $J_n = j$ and $S_n \leqq t$ for $n > 0$, where

$$S_n = X_1 + \cdots + X_n .$$

The vector process $\{N_1(t), \cdots, N_m(t); t \geqq 0\}$ is called a Markov Renewal Process (M.R.P.). Alternatively, it is possible to define an M.R.P. as an equivalent 1-dimensional process. Set $N(t) = N_1(t) + \cdots + N_m(t)$, and define $Z_t = J_{N(t)}$. The process

$$\{Z_t : t \geqq 0\}$$

is called a Semi-Markov Process (S.-M.P.). An M.R.P. is an S.-M.P. (with a finite state space) if and only if $Q_{ii} \equiv 0$ for all i. Let $P_{ij}(t) = P[Z_t = j \mid Z_0 = i]$ and

$$G_{ij}(t) = P[N_j(t) > 0 \mid Z_0 = i],$$

the latter being the first passage-time distribution from state i to state j. Relationships between the Q_{ij}, P_{ij} and G_{ij} are derived. These can be solved to obtain expressions for the P_{ij} and G_{ij} in terms of the Q_{ij}. For example, one may show $\| P_{ij} \| = (I - \mathfrak{Q})^{(-1)}(I - \mathfrak{H})$ where the elements of $\mathfrak{H}$ are given by $H_{ij} = \delta_{ij} \sum_{k=1}^{m} Q_{ik}$. M.R.P.'s are generalizations both of discrete and continuous parameter Markov Chains. They have many applications, the one which motivated the author's definition and study of these processes being to the theory of multiple channel electronic counters.

(Abstracts of papers presented at the Washington, D. C., Annual Meeting of the Institute, December 27–30, 1959.)

24. **Main-Effect Designs for Asymmetrical Factorial Experiments.** SIDNEY ADDELMAN, Iowa State University.

A method of constructing orthogonal designs which allow the estimation of main effects for a general class of asymmetrical factorial experiments is presented. By the use of the suggested method of construction, it is possible to obtain a design in which all main effects are preserved, for the $s_1^{t_1} \times s_2^{t_2} \times \cdots \times s_k^{t_k}$ experiment in s_1^n observations, where s_1 is a prime or a power of a prime, $s_1 > s_2 > \cdots > s_k$, and $\sum_{i=1}^{k} t_i = (s_1^n - 1)/(s_1 - 1)$. As an interesting consequence of the above method of construction, one is able to obtain main-effect designs for symmetrical factorial experiments in which the number of levels of each factor is not a prime or a power of a prime.

25. A Probability Model for Theory of Organization of Groups with Multi-Valued Relations Between Persons. JOHN L. BAGG, Florida State University.

We are concerned with relations between ordered pairs of distinct individuals in a finite group of n individuals. Let there be $(k + 1)$ distinct types of relations where the $(k + 1)$st relation usually denotes the null relation. The relations between ordered pairs $(i, j), i, j = 1, 2, \cdots, n, i \neq j$ of individuals are represented by an $n \times n \times k$ matrix C with elements $c_{iju} = 0$ or 1, $u = 1, 2, \cdots, k$. We let $r_{iu} = \sum_{j=1}^{n} c_{iju}$ represent the total number of choices made by individual i and $s_{ju} = \sum_{i=1}^{n} c_{iju}$ represent total number of choices received by individual j with respect to the uth relation where $c_{iju} = 1$ if individual i chooses individual j, $i \neq j$, otherwise $c_{iju} = 0$. $c_{iiu} \equiv 0$ for all i and u. We insist that one and only one of the $(k + 1)$ relations exist between each ordered pair (i, j), i.e., $\sum_{u=1}^{k} c_{iju} \leqq 1$. We let $\mathbf{r}^u = (r_{1u}, \cdots, r_{nu})$ and $\mathbf{s}^u = (s_{1u}, \cdots, s_{nu})$ be the marginal row and column total vectors for the $n \times n$ submatrix of c for the uth relation. We let $(\text{R}, \text{S}) = (r', s', \cdots, r^k, s^k)$. The main theorem gives a procedure for counting the exact number of matrices C for any given fixed $2nk$ dimensional vector (R, S) subject to the previous restrictions on the elements c_{iju}. This is an extension of a result obtained by Katz and Powell (*Proc. Amer. Math. Soc.*, Vol. 5, 1954).

26. Multiple-Decision Ranking Problems Arising from Factorial Experiments on Variances of Normal Populations (Preliminary report). ROBERT E. BECHHOFER, Cornell University.

A multiplicative model is considered as a basis for analyzing multifactor experiments which are conducted to study the effect of changes in the levels of the factors on the variance of a normally distributed chance variable. A single-sample multiple-decision procedure for ranking the treatment "effects" on the variance when the experiment is conducted in blocks (and the block "effects" are thus removed) is proposed. The procedure is a generalization of the one described in these *Annals*, Vol. 25, pp. 273–289. Similar procedures can be used in multifactor experiments for ranking simultaneously the "effects" of two or more factors. Tables of the type given in the above reference are being prepared. Some of these tables can also be used for testing hypotheses concerning the "effects" or for forming interval estimates of the "effects."

27. A "Renewal" Limit Theorem for General Stochastic Processes. V. E. BENEŠ, Bell Telephone Laboratories and Dartmouth College.

Let x_t be a stochastic process on a space X, and let $\{x_t \varepsilon A\}$ be a measurable set, $A \subset X$. Let t_n, $n = 0, \pm 1, \pm 2, \cdots$ be a real discrete-parameter process on the same measure space, with $t_{n+1} > t_n$ a.s. and $H(x) - H(y) =$ expected number of $t_n \varepsilon (y, x) < \infty$. $\Pr\{x_t \varepsilon A\}$ can always be written as $\int_{-\infty}^{t} K_A(t, u) \, dH(u)$. The event $\{x_t \varepsilon A\}$ is called weakly stationary w.r. to $\{t_n\}$ if its representative kernel K_A is a difference kernel, $K_A(t, u) = K_A(t - u)$. *Theorem*: Let y_t be the time from t to the next t_n, i.e., $y_t = \min\{t_n - t \mid t_n > t\}$. If $\{y_t < \infty\}$ and $\{x_t \varepsilon A\}$ are both weakly stationary w.r. to $\{t_n\}$ with respective L_1 kernels Y and K_A, if the $\{t_n\}$ are "aperiodic" in the sense that the Fourier transform of Y does not vanish, and if $H(\cdot + 1) - H(\cdot)$ is bounded, then $\lim \Pr\{x_t \varepsilon A\} = \| K_A \| / \| Y \|$ (L_1 norm) as $t \to \infty$.

28. Use of Prior Knowledge in Finding the Maximum Response. R. J. Buehler, Iowa State University.

In seeking the value of a vector of control variables x which maximize an expected yield, $Ey = f(x)$, the choices of x for the initial observations must of necessity depend on a subjective judgement based on prior knowledge. The following problems are considered: (1) Under what prior assumptions does the "path of steepest ascent" have optimal properties? (2) What are the properties of some other paths, for example those determined by choosing the nth vector x_n to maximize the conditional expectation of the nth yield y_n given the first $n - 1$ observations.

29. A Subfield Containing a Sufficient Subfield is Not Necessarily Sufficient. D. L. Burkholder, University of Illinois.

Let X be Euclidean two-space, S be the sigma-field of Borel sets of X, and P be the class of all probability measures p on S of the form $p = q \times q$ where q is a probability measure on the sigma-field of Borel sets of the real line. Let

$$t_0(x) = (\min\{x_1, x_2\}, \max\{x_1, x_2\})$$

if $x = (x_1, x_2)$ is in X. Let $t_1(x) = x$ if x is in B, $= t_0(x)$ if x is in $X - B$, where B is a subset of X not in S such that $x_1 > x_2$ for each x in B. Let S_i be the subfield of S induced by the statistic t_i for $i = 0, 1$. Then t_0 is a sufficient statistic and S_0 is a sufficient subfield for the measures P on S. However, t_1 and S_1 are not sufficient. This is in spite of the fact that $t_0 = F(t_1)$ for some function F and $S_0 \subset S_1$. This example provides a negative answer to a question posed by Bahadur on page 441 of his paper "Sufficiency and statistical decision functions," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 423–462.

30. Optimum Properties and Admissibility of Sequential Tests. D. L. Burkholder and R. A. Wijsman, University of Illinois.

Suppose $X_1, X_2, \cdots$ are independent and identically distributed, with common density p_0 or p_1, and it is desired to test one possibility against the other. In the following, $i = 0, 1$. If S is a sequential test, let $\alpha_i(S)$ denote the error probabilities, $\nu_i(S) = E_i c_i(N)$, where N is the sample size, $0 = c_i(0) \leqq c_i(1) \leqq \cdots < c_i(\infty) = \infty$, and $c_i(n) \to \infty$ as $n \to \infty$. S will be called inadmissible if there is an S^* such that $\alpha_i(S^*) \leqq \alpha_i(S)$, $\nu_i(S^*) \leqq \nu_i(S)$, with strict inequality in at least one of the four. S^* is said to have optimum property $I(OP_I)$ if $\nu_i(S^*) < \infty$, and $\nu_i(S) \geqq \nu_i(S^*)$ for each S satisfying $\nu_i(S) < \infty$ and

$$\alpha_i(S) \leqq \alpha_i(S^*).$$

S^* has OP_{II} if $\nu_i(S) \geqq \nu_i(S^*)$ for each S satisfying $\alpha_i(S) \leqq \alpha_i(S^*)$. *Theorem* 1. If S^* has OP_I then it has OP_{II}. For $c_i(n) = n$, Wald and Wolfowitz have shown that the Wald SPRT with barriers $0 < B < 1 < A < \infty$ satisfies OP_I. Hence, by Theorem 1, it must satisfy OP_{II}. In the next theorem it is assumed that $c_i(n) = n$. *Theorem* 2. If S is a SPRT with either $B < A < 1$ or $1 < B < A$, then S is inadmissible. S can be improved upon by a mixture of at most 3 tests, one of which does not take any observations, such that the mixture is not only admissible but possesses OP_I and therefore also OP_{II}.

31. Conditional Expectations of Banach-Valued Random Variables. S. D. Chatterji, Michigan State University. (Introduced by K. J. Arnold.)

The notion of conditional expectation of Banach-valued random variables has been introduced and a study of martingales of such random variables has been made. Three different cases arise, depending upon the topology used: the strong, weak and weak star if the Banach space of values is the dual space of another Banach space. The corresponding notions of integration used are Bochner, Pettis and Gelfand respectively. Owing to the non-existence of theorems of Radon-Nikodym type for Banach-valued measures, separate proofs for the existence of conditional expectations had to be given. The theory simplifies in the strong topology and the usual properties of conditional expectations are valid in this case. Convergence of martingales of the type $X_n = E(Z \mid \mathfrak{F}_n) n \geqq 1$ and $X_{-n} = E(Z \mid \mathfrak{F}_{-n})$, both almost everywhere and in L_p are proved independently of the classical theory. Convergence in L_p has also been proved for above martingales when $\mathfrak{X}$ is reflexive by extending a method due to Jerison (*Proc. Amer. Math. Soc.* Vol. 10, 1959) using mean ergodic theorems. For doing this, weak completeness and compactness properties of spaces $L_p(\Omega, \mathcal{B}, P, \mathfrak{X}) = \{X(\omega): X(\omega) \text{ strongly measurable}, \int \| X(\omega) \|^p \, dP < \infty\}$ have been studied. The results are used to prove the strong law of large numbers for Banach-valued random variables and the theory of derivatives in Banach spaces.

32. **Certain Extensions of a Theorem of Marcinkiewicz** (Preliminary Report). INGE CHRISTENSEN, Catholic University.

This paper considers the function $f(t) = K_n f_1(t) e_n[P_m(t)]$, where K_n is a constant and where $P_m(t)$ is a polynomial of degree in and with complex coefficients $\alpha_v + i\beta_v$, and where the iterated exponentials $e_n(z)$ are defined as follows: $e_1(z) = \exp(z)$, $e_2(z) = \exp[e_1(z)]$, $\cdots$, $e_k(z) = \exp[e_{k-1}(z)]$. Using the analytic approach of E. Lukacs (*Pacific J. Math.*, Vol. 8 (1958), pp. 487–501), it has been shown that if $m > 2$, then $f(t)$ cannot be a characteristic function in the following cases: (i) $f_1(t) = \exp[\gamma_1(e^{it} - 1) + \gamma_2(e^{-it} - 1)]$; (ii) $f_1(t) = \exp[g(t) - 1]$ where $g(t)$ is an entire characteristic function belonging to a lattice distribution with the origin as a lattice point; (iii) $f_1(t)$ is the characteristic function of a bionomial distribution. In the case where $n = 1$ and $f_1(t)$ is the characteristic function of a gamma distribution, it has been shown that $f(t)$ cannot be characteristic function if $m > 3$ or if $m = 3$ and β_3 is zero or negative.

33. **Minimax Sequential Tests of Some Composite Hypotheses.** MORRIS H. DEGROOT, Carnegie Institute of Technology.

Let $\{X(t); t \geqq 0\}$ be a Wiener process with unknown mean μ per unit time and known variance per unit time. The problem is to test the hypotheses $H_0 : \mu \leqq \mu_0$ and $H_1 : \mu > \mu_0$, where μ_0 is a given constant. Let the cost of accepting an incorrect hypothesis when μ is the true mean be of the form $c \mid \mu - \mu_0 \mid^r$, where $c > 0$ and $0 < r \leqq 2$. Let the cost of observing the process for a time T be bT, where $b > 0$. Under these conditions it is shown that the minimax test is a specific sequential probability ratio test; i.e., a test under which the process is observed as long as $h_1 + st < X(t) < h_2 + st$ for appropriate constants h_1, h_2, and s. The analogous problem of testing composite hypotheses about the mean of a normal distribution is considered and it is shown that if the cost per observation is large, the minimax test is to take exactly one observation and then accept one of the hypotheses.

34. **Small Sample Behavior of Estimators of Parameters in a Linear Functional Relationship.** MARTIN DORFF AND JOHN GURLAND, Iowa State University.

Housner and Brennan (1948) and Durbin (1954) have proposed a very simple consistent estimator of the slope in a linear functional relationship between two variables subject to error: $b = (\Sigma w_i y_i)/(\Sigma w_i x_i)(\Sigma w_i = 0)$, where the weights w_i are very simply related to the serial order of the observations; that is, $w_i = i - \bar{\imath}$. If one knew that the true values X_i corresponding to the observed values x_i were uniformly spaced, this would clearly be a desirable estimator; in fact, it is precisely the usual least-squares estimator. The question arises, how does this estimator behave when the X_i are not uniformly spaced. It is possible to obtain the bias and mean square error of this estimator for various error distributions without undue difficulty if it is assumed that ordering the points according to the x_i is the same as ordering the points according to the X_i. It is shown that the bias and mean square error are surprisingly insensitive to even wide-spread departures from uniform spacing, and in particular, that the bias is much less than that obtained when using the ordinary least-squares estimator.

35. On the Distribution of a Noncircular Serial Correlation Coefficient with Lag 1 When the Mean of the Observations is Unknown. FRIEDHELM EICKER, University of North Carolina.

In the theory of time series several serial correlation coefficients have been used for testing the independence between the observations x_i. In this paper the x_i are considered to be distributed like $N(m, 1)$ where m is unknown. As a suitable noncircular serial correlation coefficient with lag 1 for the test of independence is considered $r = q/p$ with $q = \sum_{i=1}^{n-1} (x_i - \bar{x})(x_{i+1} - \bar{x})$, $p = \sum_{i=1}^{n} (x_i - \bar{x})^2$, where $\bar{x} = 1/n \sum_{i=1}^{n} x_i$ and n is the sample size. So far not much seems to be known about the distribution of r. In this paper its first cumulants are derived. This is done by starting from a divisor of the characteristic polynomial of the matrix of the quadratic form in the numerator. Thereby use is made of a symmetry in its characteristic vectors and of the relations between power sums and complete symmetric functions. Some results of Siddiquis work on noncircular coefficients for known mean m are utilized here. As is to be expected our results do not differ very much from his. Besides these exact results, bounds are found for all cumulants. The method used here is related to perturbation theory and another theory developed mainly by Schaefke for characteristic value problems with two parameters.

36. Partnership Games with Secret Conventions Prohibited. MARTIN FOX AND HERMAN RUBIN, Michigan State University.

The ethics of bridge prohibit the use of secret signals by any partnership. This is explicitly stated in Law 5 of "The Laws of Duplicate Contract Bridge" (Ely Culbertson, *Bidding and Play in Duplicate Contract Bridge*, John C. Winston, Philadelphia, 1946, pp. 223–224.) Two game-theoretical formulizations of this rule are: 1. Whenever an agent of either player is required to make a bid or to play a card as defender he must announce his behavioral strategy as well as the bid or play which results from the randomization required by the behavioral strategy. 2. Instead of announcing the behavioral strategy to all other agents, the agent who is moving announces it to a referee. The referee announces to each of the other agents their a posteriori probabilities of each distribution of the cards unseen by them given the previous sequence of bids. It is shown that with rule 1. bridge has a value. Furthermore, each player has a good strategy in which the behavioral strategies at each move depend only on the a posteriori probabilities.

37. A Simplified Method for Finding Confidence Limits on the Relative Risk in 2 × 2 Tables. JOHN J. GART, Johns Hopkins University. (By title)

Consider a 2×2 table with a total of m positives of which x are from the first sample of size n_1 and $m - x$ are from the second sample of size n_2; the designation of the samples and the positives being defined by the relations; $m \leqq n_1 + n_2 - m$ and $x/n_1 \leqq (m - x)/n_2$. Cornfield (*Proc. of Third Berk. Symp.*, IV, pp. 135–148) and Cox (*J. R. S. S. (B)*, Vol. 20 (1958), pp. 215–238) have proposed methods for finding approximate confidence limits on the relative risk, namely: $\psi = p_1q_2/p_2q_1$, where p_1 and p_2 are the population proportions. The method proposed here involves an approximation to the sum of the hypergeometric probabilities similar to the first term approximation of Wise (*Biometrika,* Vol. 41 (1954), pp. 317–329). It yields the lower limit,

$$\psi_1 = (2n_2 - (m - x))/(2n_1 - x + 1)x/(m - x + 1)1/(F_{1-\alpha/2}[2(m - x + 1), 2x]),$$

and the upper limit,

$$\psi_2 = (2n_2 - (m - x) + 1)/(2n_1 - x)(x + 1)/(m - x)F_{1-\alpha/2}[2(x + 1), 2(m - x)],$$

where the approximate confidence coefficient is $1 - \alpha$. Several examples have shown that this method yields confidence coefficients which are comparable to those found using the previously proposed methods.

38. A Single Sample Decision Procedure for Selecting a Subset Containing the Best of Several Normal Populations and Some Extensions. S. S. Gupta, Bell Telephone Laboratories.

Let $\bar{x}_1$, denote the sample mean and sample variance based on n_i observations from a normal population Π_i with mean μ_i and a common variance σ^2 (μ_i and σ^2 unknown). A single sample decision procedure for selecting a non-empty, small subset of the k populations such that the probability that the population with the largest mean is included in the selected subset is at least equal to a pre-assigned value P^* (regardless of the true unknown values of the parameters) is given. The procedure is "Select the population Π_i if and only if $\bar{x}_i \geqq \max (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{i-1}, \bar{x}_{i+1}, \cdots, \bar{x}_k) - cs/(n_i)$" where s^2 is the usual pooled estimate of σ^2 and c is determined to satisfy the required probability condition. Expressions for the probability of a correct selection are derived and in the case of common number of observations, the constants c's are shown to be the percentage points of a certain statistic. The case of unequal but known variances σ_i^2 is also treated. Formulae are obtained for the expected number of populations retained in the selected subset and, for selected cases, tables are given for the expected proportion of populations retained. The latter tables can be used to determine the common number of observations required to control the expected size (or proportion) of the retained subset when the best population has a certain "distance" from the others. Some extensions of the procedure to other parametric cases are given.

39. On the Distribution of the Ratio of the Smallest of Several Chi-Squares to an Independent Chi-Square. S. S. Gupta and M. Sobel, Bell Telephone Laboratories. (By title)

This paper deals with the problem of finding lower percentage points of the distribution of $y = \chi^2_{\min}/\chi^2_0$ where $\chi^2_{\min}$ is the smallest of p independent chi-squares and χ^2_0 is a chi-square independent of the p others. The case of a common even number of degrees of freedom for all $p + 1$ chi-squares is the principal case considered and the only case for which computation was carried out. Tables give the 25%, 10%, 5%, and 1% points for common $\nu = 2(2)50$ and $p = 1(1)10$; the case $p = 1$ which reduces to an F-distribution was used as a check. Relationships to the distribution of $\chi^2_{\max}/\chi^2_0$ are considered. The tables computed have immediate application to the problem of selecting a subset of $k (= p + 1)$ normal populations, based on a common number of observations from each, which contains the population with the smallest variance with any pre-assigned probability.

40. On a Single Sample Procedure for Selecting from Several Normal Populations a Subset Containing the Population with the Smallest Variance. S. S. Gupta and M. Sobel.

A procedure is studied for selecting a subset of several given normal populations which includes the population with the smallest variance. For given numbers of observations from each of k normal populations, the procedure R selects a subset which is small (the exact size depends on the observed results), never empty and yet large enough to guarantee with preassigned probability that, regardless of the true unknown values of the variances, it will include the population with the smallest variance. If s_i^2 based on ν_i degrees of freedom denotes the sample variance from the population Π_i then the procedure is "Select Π_i if and only if $cs_i^2 < \min (s_1^2, s_2^2, \cdots, s_{i-1}^2, s_{i+1}^2, \cdots, s_k^2)$" where the constant $c(0 < c < 1)$ is determined so as to satisfy the required probability condition. Expressions are derived for the probability of a correct selection, and for the expected number of populations included in the selected subset. The relationship to the problem of selecting the population with the largest variance is discussed.

41. Almost Linearly-Optimum Combination of Correlated Unbiased Estimates by Regression Methods. Max Halperin, Knolls Atomic Power Laboratory.

Suppose one has available a multi-normal sample $(y_{1i}, y_{2i}, \cdots, y_{ki})$, $i = 1, 2, \cdots, n$, with mean vector $\mu\mathbf{j}$ (μ a scalar, $\mathbf{j}$ a unit row vector) and arbitrary covariance matrix Σ. The coefficients of the minimum variance linear unbiased estimate (MVLUE) of μ will, of course, involve the (unknown) elements of the covariance matrix. We can estimate these coefficients and still have an unbiased estimate of μ with, however, an unknown distribution almost certainly involving nuisance parameters. Transform the sample into $(y_{1i}, d_{2i}, \cdots, d_{ki})$, $i = 1, 2, \cdots, n$, where $d_{ji} = y_{1i} - y_{ji}$ and consider the distribution of $y_{11}, \cdots, y_{1n}$ given $(d_{2i}, \cdots, d_{ki})$, $i = 1, 2, \cdots, n$. Defining σ_{opt}^2 as the variance of the MVLUE based on a single observation, $(y_1, \cdots, y_k)$, one finds that, conditionally, y_{1i} is normal with variance σ_{opt}^2 and expected value, $\mu + \sum_2^k \beta_j d_{ji}$. It follows immediately that the regression estimate of μ is given by $\hat{\mu} = \bar{y}_1 - \sum_2^k \hat{\beta}_j \bar{d}_j$, where the carets denote maximum likelihood estimates, and that Var $\hat{\mu} = (\sigma_{\text{opt}}^2/n)\{1 + T^2/(n-1)\}$ where T^2 is Hotelling's T^2 statistic with $(k-1)$ and $(n-k+1)$ degrees of freedom for the vectors $(d_{2i}, \cdots, d_{ki})$, $i = 1, 2, \cdots, n$. This variance is identical to the variance of the MVLUE except for the factor $T^2/(n-1)$, which is trivial for n at all large. An estimate of σ_{opt}^2 with $(n-k)$ d.f. is available from the sum of squares of deviation from regression, so that exact confidence intervals for μ are available. Note that these results apply also to k independent samples of equal size, with no intrinsic pairing from sample to sample, by the introduction of randomization.

42. Certain Uncorrelated Statistics. Robert V. Hogg, University of Iowa.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution symmetric about θ. Let $T = T(X_1, X_2, \cdots, X_n)$ be a statistic such that $E(T) = \theta$, $T(X_1 + h, \cdots, X_n + h) = T(X_1, \cdots, X_n) + h$, and $T(-X_1, \cdots, -X_n) = -T(X_1, \cdots, X_n)$. Let $S = S(X_1, \cdots, X_n)$ be a statistic such that $S(X_1 + h, \cdots, X_n + h) = S(X_1, \cdots, X_n)$ and $S(-X_1, \cdots, -X_n) = S(X_1, \cdots, X_n)$. If the correlation coefficient of T and S exists, it is equal to zero.

43. Further Results on Hypothesis of No Interaction in Multidimensional Table (Preliminary report). P. R. KRISHNAIAH, University of Minnesota AND V. K. MURTHY, University of North Carolina.

In a contingency table, any dimension is defined as a factor or response according as its marginal totals are fixed or random. Roy and Kastenbaum (these *Annals*, 1956) discussed the hypothesis of no interaction in a three way table when all dimensions are responses. In this paper, the extension of the above results to multiway table are discussed when some dimensions are factors and the rest are responses.

44. Remarks on "Standard Coefficients" in Normal Regression Analysis. P. R. KRISHNAIAH AND M. M. RAO, University of Minnesota. (By title)

In some applications of the normal regression analysis, for computational reasons, the so-called "standard partial regression (or beta) coefficients" are in use. For estimation of the usual multiple regression coefficients the use of either this procedure or the direct calculation is immaterial. But the fact that the "beta" coefficients are *not* normally distributed and generally the usual test procedures (the "Student's" t for testing the regression coefficients to have specified values not necessarily zero, and the confidence bounds obtained therefrom) are not valid for these standard coefficients, is overlooked. The only valid one in the "new procedure" is the over-all test for the hypothesis of *no* regression. The correct procedure and the distribution of the beta's (in series form) are indicated in this note.

45. On Characterization Problems Connected with Quadratic Regression. R. G. LAHA AND E. LUKACS, Catholic University.

Let X and Y be two random variables. Then Y is said to have polynomial regression of order p on X, if the conditional expectation of Y given X is a polynomial of degree p in X. In particular, if $p = 2$, Y is said to have quadratic regression on X. Let $X_1, X_2, \cdots, X_n$ be n independently and identically distributed random variables with a common distribution function having a finite variance. Let $\Lambda = X_1 + X_2 + \cdots + X_n$ be the sum and $Q = Q(X_1, X_2, \cdots, X_n)$ be a quadratic polynomial statistic. In the present paper all the distribution functions which have the property that Q has quadratic regression on Λ are investigated in detail. It is also proved that in each case the distribution function is uniquely determined by this property. These results contain as special cases the earlier investigation of M. C. K. Tweedie [cf. *London Math. Soc.*, Vol. 21, (1946) pp. 22–28] on the regression of the sample variance on the sample mean.

46. Distribution of Sample Size in Sequential Sampling. L. L. LASMAN, Florida State University AND E. J. WILLIAMS, North Carolina State College.

Suppose it is desired to sample sequentially from a mixture of s populations and to cease sampling when some criteria have been attained. Suppose further that these criteria can be specified in terms of a function of the numbers of observations obtained in the sampling process and that an observation can be identified by population only after it has been drawn. Then it might be desired to estimate the average size sample that would be needed to satisfy the criteria. Under certain assumptions on the functions involved, the asymptotic distribution of the sample size for such a procedure is obtained through the use of Wald's fundamental identity. The mean and variance turn out as relatively simple functions of the criteria specified and of the mixture probabilities. Sampling until the standard error of the difference between two means reaches a given value, is given as an example for the case $s = 2$.

47. Generalizations of Thompson's Distribution, II. ANDRE G. LAURENT, Wayne State University.

Let **X** be a $\underline{n} \times \underline{p}$ random matrix with probability density $f(\mathbf{X}) = h(\mathbf{X}'\mathbf{X})$, ξ be a $\underline{k} \times \underline{p}$ submatrix of **X**, $\mathbf{X}'\mathbf{X} = \mathbf{TT}'$, **T** lower triangular, $\mathbf{Y} = \mathbf{XT}'^{-1}$, $\mathbf{n} = \xi T'^{-1}$, (Schmidt's orthogonalization process), the marginal and conditional (given $\mathbf{X}'\mathbf{X}$) distribution of **X**, ξ, **Y**, **n**, $\xi'\xi$, $\mathbf{Y}'\mathbf{Y}$, $\mathbf{n}'\mathbf{n}$ are derived. For example, $f(\xi \mid \mathbf{X}'\mathbf{X}) = K \mid I - (\mathbf{X}'\mathbf{X})^{-1}\xi'\xi \mid^{(n-k-p-1)/2} \mid \mathbf{X}'\mathbf{X} \mid^{-k/2}$, $f(\mathbf{n}) = C \mid I - \mathbf{n}'\mathbf{n} \mid^{(n-k-p-1)/2}$, $p \leqq n - k$, in the proper domain. Applications are described (U.M.V. unbiased estimates, roots of determinantal equations, bombing problems, etc.). In case $n = p$, Cayley parametric representation **U** of **Y**, $\mathbf{U} = \mathbf{U}^*$ (**Y** exceptional), $\mathbf{U} = \mathbf{U}^{**}$ (**Y** non exceptional) is $A \mid I + \mathbf{U} \mid^{-(p-1)} d\mathbf{U}$ distributed (Haar invariant measure). The problem of constructing a "random" basis in the euclidean space is considered. Other lines of generalization of Thompson's distribution are studied. The results generalize (and sometimes specialize) results previously given (*Ann. Math. Stat.*, 1956, p. 1184; *Journ. Soc. Stat. Paris*, 1955, pp. 262–296; *Journ. Oper. Res. Soc.*, 1957, pp. 75–89).

48. Optimum Decision Procedures for a Poisson-Process Parameter. JAMES A. LECHNER, University of Maryland.

Rules are discussed for deciding whether λ, the parameter of a continuous-time Poisson process, is less than or greater than a given constant k. If the cost of observation is proportional to the length of time the process is observed, and the cost of a wrong decision is proportional to the magnitude of the error, that is, to $\mid \lambda - k \mid$, then an optimum non-randomized sequential decision procedure is proved to exist and is found, where by an optimum procedure is meant a procedure which minimizes the total expected cost with respect to any given prior distribution for λ of the incomplete Gamma form with mean y/t and variance y/t^2, $t > 0$, y a positive integer. Some of the results hold time for other cost functions, prior distributions, and/or random processes; indications are made of some of these.

49. Reduction of Multiple Regression System by Use of Direct Products of Matrices. JULIUS LIEBLEIN, U. S. Navy Department.

Let R represent a high order multi-variable polynomial regression, such as might occur in a large factorial experiment. Let the independent variables of R be arbitrarily separated into two groups, S and S', and suppose R is arranged according to ascending powers of the variables in S. For every fixed set of values of the variables in S', this gives a new regression $R(S \mid S')$, with coefficients $C(S')$, depending on S'. These coefficients $C(S')$ may then themselves each be taken as a polynomial regression $R(S')$ over the variables in S'. Let the matrices of the normal equations for the three systems of regressions given by R, $R(S \mid S')$, $R(S')$ be, respectively, M, M_S, $M_{S'}$. Then it was essentially shown by E. A. Cornish (*Biometrics*, March 1957, pp. 19–27) that (*) $M = M_S \otimes M_{S'}$ (Kronecker or direct product), in a certain class of cases. The present paper generalizes this relationship to any number of sets $S, S', S'', \cdots$, and to other regressions than polynomials and finds the conditions for it to hold or not. Considerable savings in computation, and a mathematical check on regression calculations, were shown by Cornish (*ibid.*) to be possible when (*) holds. In general, however, R will have missing terms corresponding to non-significant interactions and (*) will not hold. The present paper shows also how to obtain computational savings and a mathematical check even in such cases, especially in conjunction with high-speed digital computers.

50. On the Characterization of a Family of Populations which includes the Poisson Population. Eugene Lukacs, Catholic University.

A random variable Y which has finite expectation is said to have constant regression on a random variable X if the relation $E(Y \mid X) = E(Y)$ holds almost everywhere. The k-statistic of order j is the symmetric, homogeneous polynomial statistic whose expectation is the j-th cumulant; it is denoted by k_j. The following theorem is proved: Let $X_1, X_2, \cdots, X_n$ be a sample of size n taken from a population with distribution function $F(x)$. Let $p \geqq 1$ and $r \geqq 1$ be two positive integers and assume that the moment of order $p + r$ of $F(x)$ exists. The distribution $F(x)$ is the convolution of a Poisson Distribution, the conjugate to a Poisson Distribution and a normal distribution if, and only if, $k_{p+r} - k_p$ has constant regression on k_1. (One or two of the components of $F(x)$ may be absent).

51. On Queues in Tandem. Gregory E. Masterson, Burroughs Research Center and Seymour Sherman, University of Pennsylvania.

A queueing system is considered which consists of an infinite number of identical servers in tandem. The service times for all customers and all servers are independent random variables with identical probability distributions. The distribution is arbitrary, except that it has a finite mean. The interarrival times of customers at the input to the system i.e., at the first server, are also independent random variables with identical probability distributions. Again, the distribution is arbitrary, except that it has a finite mean. When a customer has been served, he immediately proceeds to the next server, where he may have to join a queue if that server has not yet finished serving the previous customers. Customers may, of course, have to queue at the input to the system. The service discipline is "first come, first served." It is shown that the chance that the interdeparture time between the jth and the $j + 1$th customer, from the nth server, is less than x, tends to zero as n tends to infinity for each positive x, except in the unique case of constant service times.

52. Power Characteristics of the Control Chart for Number of Defects, No Standard Given. Edmund M. McCue, Ohio University.

A standard procedure for testing an industrial process for control with respect to defects-per-unit is to compare the numbers of defects observed in each of k samples with upper and lower control limits based on the total number of defects in the k samples. Methods are given for obtaining the probability of a Type I error and the power to detect single slippages. It is found that there is considerable variation in the probability of a Type I error and power for various values of k. Utilizing this information, procedures are developed to increase the effectiveness of the control chart for number of defects. Some asymptotic properties of the power are obtained, and it is shown how approximations based on these properties can be used in practice.

53. On the Distribution of the Sum of Circular Serial Correlation Coefficients and the Effect of Non-Normality on its Distribution. V. K. Murthy, University of North Carolina.

Let $x_1, x_2, \cdots, x_{2m+1}$ be a random sample of size $(2m + 1)$ from a normal distribution with zero mean and unit variance. Let r_L denote the circular serial correlation coefficient of lag L defined by $r_L = \sum_{j=1}^{2m+1} x_j x_{j+L} / \sum_{j=1}^{2m+1} / x_j^2$, where $x_j = x_{2m+1+j}$ for all j. Define $\bar{r} = \sum_{j=0}^{2m} r_j / (2m + 1)$. It is then shown that the distribution of $\bar{r}$ is a beta-distribution. The effect of non-normality on the distribution of $\sum_{j=1}^{2m} r_j$ is studied by the method of David and Johnson (*Ann. Math. Stat.*, 1951).

54. Generalized Power Series Distribution and Certain Characterization Theorems. G. P. Patil, University of Michigan.

Let T be an arbitrary countable non-null subset of non-negative numbers and define the generating function $f(\theta) = \sum_{x \varepsilon T} a_x \theta^x$ with $a_x \geqq 0$; $\theta \geqq 0$ so that $f(\theta) > 0$, is finite and differentiable. Then we can define a random variable X taking values in T with probabilities Prob $\{X = x\} = (a_x \theta^x)/(f(\theta))$, $x \varepsilon T$ and call this distribution a Generalized Power Series Distribution (gpsd). The Binomial, Poisson, Negative Binomial and the Logarithmic Series distributions and their truncated forms can be obtained as special cases of the gpsd by proper choice of T, a_x and hence of $f(\theta)$. Recurrence relations are obtained for central, raw and factorial moments, cumulants etc. which are generalizations of corresponding results obtained by Romanovsky, Frisch, Haldane, etc. An explicit functional relationship between the variance and the mean of a gpsd is obtained and based on this relationship, some characterization theorems are presented. To mention one, the gpsd with equal variance and mean for all admissible parameter values is characterized to be Poisson distribution. Some problems of estimation and others have been studied for the gpsd and will be presented elsewhere.

55. Stationary Probabilities for a Semi-Markov Process with Finitely Many States. Ronald Pyke, Columbia University.

A process $\{Z_t : t \geqq 0\}$ is called a Semi-Markov process (S.-M.P) if, roughly speaking, it moves from one to another of $m(\leqq \infty)$ states in accordance with a transition matrix as does a Markov Chain, but where the time between two successive transitions may depend on the states between which the transition is being made. These processes are then generalizations of both discrete and continuous parameter Markov Chains. Let J_n denote the state entered at the n-th transition and let X_n denote the time taken between the $(n - 1)$-th and n-th transitions. An S.-M.P. (or alternatively, a Markov Renewal process) is said to be regular if for all choices of initial probabilities, $N(t) = \sup \{n \geqq 0; X_1 + X_2 + \cdots + X_n \leqq t\} < \infty$ a.s. for every $t \geqq 0$. Almost all sample functions of a regular process are step functions. A characterization of, as well as several sufficient conditions for regularity are derived. A classification of states analogous to that for Markov Chains is presented and studied. Limit theorems are proven under weak restrictions for random variables $W_f(t) = \sum_{n=1}^{n(t)} f(J_{n-1}, J_n, X_n)$, for arbitrary real functions f defined on R_3. Furthermore, the a.s. convergence of ratios (Doeblin Ratios) of the form $W_f(t)/W_g(t)$ is studied.

56. On the Decomposition of Certain Characteristic Functions (Preliminary report). B. Ramachandran, Catholic University. (Introduced by Eugene Lukacs.)

A family of characteristic functions is said to be factor-closed if the factors of every element of the family belong to the family. It is known that the Normal, the Poisson and the Binomial families are factor-closed. Recently Yu. V. Linnik (*Teor. Veroyat.* 2, 1957) proved that the characteristic functions of the compositions of a Normal and a Poisson distribution form a factor-closed family. In the present paper it is shown that the characteristic functions of compositions of a (standard) Poisson and a (standard) Binomial distribution constitute a factor-closed family. An example is given to demonstrate that the characteristic functions of the compositions of a Normal and a Binomial distribution do not form a factor-closed family.

57. Generalization of a Theorem of Polya, and Applications. R. Ranga Rao, Indian Statistical Institute. (Introduced by R. R. Bahadur.) (By title)

Let $\mu_n (n = 1, 2, \cdots)$ and μ be measures on the Borel sets of a separable and complete metric space X. Let F be a given family of continuous mappings from X into the k-dimensional Euclidean space E_k. Let $\mathcal{R}$ be the class of all sets of the form $f^{-1}(R)$, with $f \varepsilon F$ and R a k-dimensional rectangle. *Theorem* 1. Suppose that F is compact under the topology corresponding to uniform convergence on compacta, and that μf^{-1} has continuous marginal distributions for each $f \varepsilon F$. Then $\mu_n \rightarrow \mu$ (weak convergence) implies that sup $\{| \mu_n(A) - \mu(A) |, A \varepsilon \mathcal{R}\} \rightarrow 0$. (When X is the real line and F consists of the single function $f(x) = x$, this theorem reduces to a well known theorem of Polya). *Theorem* 2. If $X = E_k$, and $\mu \ll$ Lebesgue measure, then $\mu_n \rightarrow \mu$ if and only if sup $\{| \mu_n(C) - \mu(C) |, C$ measurable and convex$\} \rightarrow 0$. As an application, we have the following generalization of previous results of Wolfowitz, and of Fortet and Mourier. Let $\xi_n \varepsilon E_k$ $(n = 1, 2, \cdots)$ be independent random vectors with common distribution μ. Let μ_n be the sample distribution function based on the first n observations $\xi_1, \cdots, \xi_n$. For each fixed positive integer m, let $\mathcal{H}_m$ be the class of all sets which are intersections of m half-spaces. Then sup $\{| \mu_n(A) - \mu(A) |, A \varepsilon \mathcal{H}_m\} \rightarrow 0$ with probability one. If $\mu \ll$ Lebesgue measure, then sup $\{| \mu_n(C) - \mu(C) |, C$ measurable and convex$\} \rightarrow 0$, with probability one.

58. The Method of Moments Applied to a Mixture of Two Exponential Distributions. Paul R. Rider, Aeronautical Research Laboratory.

The method of moments is used to estimate the parameters of a mixed exponential distribution. Variances of the estimators are derived.

59. When to Stop. Herbert Robbins, Columbia University. (By title)

Let $\{x_n\}$ be independent random variables with a common distribution function F. We observe the x_n sequentially and can stop at any time; if we stop with x_n we receive the payoff $f_n(x_1, \cdots, x_n)$. *Problem*: what stopping rule maximizes the expected payoff? It is shown that for $f_n(x_1, \cdots, x_n) = \max(x_1, \cdots, x_n) - cn$, $c > 0$, the optimum stopping rule when the first moment of the x_n exists is: stop with the first $x_n > \alpha$ where α is the root of the equation $\int (x - \alpha)^+ dF(x) = c$; the expected payoff is then α.

60. On Estimating the Mean of a Finite Population. J. Roy, Indian Statistical Institute, and I. M. Chakravarti, University of North Carolina. (Introduced by R. C. Bose.)

Consider a population consisting of a finite number N of distinguishable elementary units u_i with associated real numbers (variate-values) y_i $i = 1, 2, \cdots, N$. Let the mean and the variance of the population be respectively $\mu = 1/N \sum_{i=1}^{n} y_i$ and $\sigma^2 = 1/N \sum_{i=1}^{n} (y_i - \mu)^2$. Let $\{U\}$ denote a countable collection of derived units $U(x)$ $x = 1, 2, \cdots \cdots$ formed by combining the elementary units. Only one of the derived units is to be selected, the probability of selecting $U(x)$ being $p(x)$ and the variate-values for all the elementary units in the selected derived unit are to be determined. The estimate of μ is the random variable $T = t(X) = \sum_{i=1}^{n} y_i a_i(x)$ where Prob $(X = x) = p(x)$ $x = 1, 2, \cdots$ and the set of coefficients $a_i(x)$ associated with a derived unit $U(x)$ are chosen so that T is an unbiased estimate of μ and has finite variance. In this paper an admissible estimate and a complete class of estimates of μ have been obtained. If the sampling scheme is "balanced", a best estimate of μ in the class of linear unbiased estimates T which have variances proportional to σ^2 is shown to exist.

61. A Solution of the Classification Problem. S. N. Roy, University of North Carolina. (By title)

For one-way classification the problem is the following. Given k observed random samples of experimental units (on each of which p kinds of observations have been made) drawn from k populations with known distribution forms but unknown parameters, and given another experimental unit carrying p kinds of observations, how to assign the experimental unit to one of the k populations? A heuristic solution of this problem is offered when the k-populations are p-variate normal ones (with unknown parameters), and this is then extended to the case of two-way or multi-way classification under the models of multivariate analysis of variance. The method offered is then formally extended to one- or multi-way classification problems under distribution forms, not necessarily normal, with $p = 1$ one would have the univariate case. A comparison with the solution (not yet available) in terms of the general decision function approach is desirable, for it is felt by the author that the latter solution, while much more difficult, would be better and more rational than the easier but heuristic one offered here.

62. On the Determinants and Characteristic Equations of a Class of Patterned Matrices. S. N. Roy, B. G. Greenberg, and A. E. Sarhan, University of North Carolina.

In three previous papers by the authors, inverses were given of a class of patterned matrices that occur in a wide variety of problems, including those in univariate and multivariate analysis of variance, the exploration and study of response surfaces and the handling of censored data. For some aspects of these problems one may need to obtain, in addition to the inverses, (i) the determinants, (ii) the characteristic equations and (iii) the characteristic roots of these patterned matrices. This paper obtains (i) and (ii), and in forms that turn out to be nearly as simple and patterned as the inverses obtained earlier. For special values of some of the parameters involved, (iii) comes out in a simple form, but for the more general cases one has to resort to the numerical solution of the characteristic equations (using any of the various methods in vogue), in order to obtain the characteristic roots. These roots, both for the general and the special cases, again happen to be patterned in the same sense as the inverses and the characteristic equations.

63. On the Efficiency of Experimental Designs. S. N. Roy, S. S. Shrikhande, and P. R. Krishnaiah, University of North Carolina.

With the randomized block design furnishing the yardstick, the efficiency of two dimensional designs has been studied from the standpoint of point estimation in the case of a single response type and under certain further well-known restrictions. It is the purpose of this paper to start a study of efficiency, for a single response type, from the viewpoint of (i) the power function, (ii) the confidence bounds, both total and partial, on parametric functions measuring departures from the total and partial hypotheses, and also (iii) point estimation under assumptions broader than usual, and furthermore to generalize this study to the case of experiments with multiple response types. Most of the requisite basic concepts are discussed here, and detailed formulae are given for some classes of BIB and PBIB designs. Further work along the same lines is underway.

64. The Estimation of the Location of a Discontinuity in Density. Herman Rubin, Michigan State University.

Let $x_1, \cdots, x_n$ be independent multivariate random variables with common density of the form $\phi_i(x \mid \theta)$ for $x \varepsilon R_i(\theta)$. Then under suitable regularity conditions, hyperefficient estimates, including maximum likelihood estimates, exist for the parameters of the R_i. The results are similar to those obtained by Chernoff and Rubin in "The estimation of the location of a discontinuity in density," *Proc. Third Berkeley Symposium on Math. Stat. and Prob.*, Vol. 1. The limiting distribution of the estimates involves a study of stochastic processes with multidimensional "time" which have some Markov properties.

65. **A Modified Procedure for Group Testing.** MILTON SOBEL, New York University and Bell Telephone Laboratories.

In group-testing a binomial sample of size N is given and any number $x(1 \leqq x \leqq N)$ of units can be tested simultaneously. Each test determines either that all x units are good or that at least one defective is present (it is not known how many or which ones are bad). It has been shown in [1] that for *known* a priori probability q of a unit being good a procedure R_1 based on recursion formulae is optimal under a certain restriction, namely that in selecting a group to be tested one should not *mix* "binomial" units with units from a set known to contain at least one defective. A modified procedure R_0 is now developed which allows a certain "small" amount of mixing and it furnishes an improvement over R_1. The extent of the improvement is numerically investigated for selected values of N and q. It is not yet known whether (or to what extent) the procedure R_0 is optimal in the unrestricted case.

66. **A Problem in Restrictive Group-Testing.** MILTON SOBEL AND PHYLLIS A. GROLL, Bell Telephone Laboratories.

In group-testing a binomial sample of size N is given and any number $x(1 \leqq x \leqq N)$ of units can be tested simultaneously. Each test determines either that all x units are good or that at least one defective is present (it is not known how many or which ones are bad). In some applications of group testing it is desirable to apply the restriction that any one unit not be included in more than k group tests. Based on recursion formulae, a group testing procedure is developed, for known a priori probability q of a unit being good, which satisfies the above restriction. In the special case $k = 2$, it is clear that for any set containing at least one defective, each unit in this set must be tested separately; this type of solution was proposed by Dorfman for the unrestricted problem. In the special case $k = 3$, tables and explicit rules are prepared for all values of q up to $n = 8$. One particular application is the field of pooled blood testing, where the restriction insures that it will not be necessary to take more than one blood sample from each patient.

67. **On the Probability of Detection of Noise-Like Signals.** W. M. STONE, Boeing Airplane Co. and Oregon State College, AND K. J. HAMMERLE, Boeing Airplane Co. (Introduced by J. Bryce Tysver.)

The classical paper of Kac and Siegert (*J. Appl. Phys.*, 18: 383–397) dealt with the detection of nonrandom signals by a receiver system. In the present paper the signal is assumed to be a random process with a prescribed spectrum. A properly chosen adjustment on the transfer function of the bandpass filter has the effect of modifying the distribution of the output of the system in terms of the signal bandwidth. Third order cumulants are obtained, also suitable approximations to the probability of detection.

68. **Identifiability of Mixtures.** HENRY TEICHER, Purdue University.

If $\mathfrak{F} = \{F(x; \alpha), \alpha \varepsilon R^m\}$ is a family of distribution functions (c.d.f.'s) and $\mathcal{G} = \{G(\alpha)\}$, a class of non-degenerate m-dimensional c.d.f.'s then a class $\mathfrak{H} = \{H\}$ of G-mixtures of $\mathfrak{F}$ (i.e. (*) $H(x) = \int F(x; \alpha)\, dG(\alpha)$) is called identifiable if (*) effects a 1–1 correspondence between $\mathcal{H} \cup \mathfrak{F}$ and $\mathcal{G} \cup \mathcal{I}$ where $\mathcal{I}$ is the class of degenerate distributions assigning mass one to a single point of R^m, ("On the Mixture of Distributions," *Ann. Math. Stat.*, March, 1960). It is shown that if $m = 1$ and $\mathfrak{F}$ is an additively closed family with α varying over the non-negative integers, rationals or reals then the induced class $\mathcal{H}$ of mixtures is identifiable. Some scale parameter mixtures not therein encompassed may be handled by a method indicated. Applications are made to the classical families of Gamma, Uniform and Binomial distributions.

69. On the Problem of Negative Estimates of Variance Components. William A. Thompson, Jr., University of Delaware.

The usefulness of variance component techniques is frequently limited by the occurrence of negative estimates of essentially positive parameters. This paper demonstrates that the principle of maximum likelihood, properly applied, will remove this objectionable characteristic in certain cases. From a conceptual viewpoint, the solution of the problem of negative estimates of variance components, at least in so far as maximum likelihood is concerned, is that the likelihood function should be maximized *subject to the constraints* that all variances should be non-negative. The results of Kuhn and Tucker on nonlinear programming greatly facilitates carrying out the mechanics of this objective. The technique has been successfully applied for the following random models: one and two factor experiments with multiple observations in each cell, two factor experiment with a single observation per cell, and the n-fold hierarchal classification. The problem of determining the precision of instruments in the two instrument case [Grubb, *J.A.S.A.*, 1948] is dealt with, and a surprising though not unreasonable answer is obtained.

70. An Infinite Packing Theorem for Spheres: A New Application of the Borel-Cantelli Lemma. Oscar Wesler, University of Michigan.

A classic example (Wolff, 1921) in the theory of functions of a complex variable involves removing a sequence of disjoint circles from a given circle in such a way that only a set of measure zero remains behind. Borel observed indirectly that the areas of such circles necessarily form a convergent series whose rate of convergence is less rapid than that of the series with general term $e^{-e^{k^2}}$, and wondered what the order of magnitude of these circles might be, and whether one could determine it directly. It turned out that Borel's bound was incredibly weak: the convergence is actually so much slower that the radii of the circles form a divergent series! Various proofs have been given using the relatively heavy machinery of complex function theory. In this paper a direct and simple proof is given using the easier half of the Borel-Cantelli lemma. In fact, our method is such that it yields at once a result of much greater generality: we show that the infinite packing theorem just mentioned holds not only for circles in the plane, but that analogous results hold for spheres in n-space, as well as for more general figures.

71. On Time Series Analysis and Reproducing Kernel Spaces. N. Donald Ylvisaker, Columbia University.

Let $X(\cdot)$ be a real valued, weakly stationary of second order, continuous parameter process with $\mathcal{E}[X(s)X(t)] = K(s, t) = k(s - t)$. The reproducing kernel space $H(K)$ of functions, which is associated with the kernel K, is a representation of the process $X(\cdot)$. The realization of the group of unitary operators in $H(K)$ is given. The properties of functions in $H(K)$ are related to the properties of the kernel K and, in particular, a sufficient condition is given that $H(K)$ consist of quasi analytic functions. The notion, due to Kolmogorov, of processes subordinate to $X(\cdot)$ is treated, and the reproducing kernel space corresponding to a subordinate process is characterized relative to $H(K)$. The linear extrapolation problem is viewed in $H(K)$. Specifically, necessary and sufficient conditions are given that $H(K)$ correspond to a deterministic, non-deterministic, or regular non-deterministic process. Sufficient conditions are given in this context, that a process subordinate to a deterministic (regular non-deterministic) process be itself deterministic (regular non-deterministic). A subordinating operation is given for which the subordinate process is deterministic and mutually subordinate with the original process $X(\cdot)$.

72. Some Randomization Consequences in Balanced Incomplete Blocks. GEORGE ZYSKIND, University of North Carolina and Iowa State University.

The analysis of balanced incomplete blocks is developed directly from the randomization consequences of the experimental procedure and under the general case of no additivity assumptions. It is shown that expected values of squares of partial observational means, as well as the expected values of products of individual observations, admit simple and easily specifiable expressions in terms of Σ's—linear functions of the population variances, uniquely determined by the structure of the experiment. The expected values of mean squares in the analysis of variance tables and the expression for the average variance of estimated treatment differences are then derived as a simple consequence. Extension to the case where the intended amounts of treatment amounts are subject to error are indicated. The correlational structure of the observations under the simplifying additivity and/or homogeneity assumptions is examined. The relationship of this structure with the ones generally given in connection with assumed models is exhibited. Some estimation problems are discussed.

73. Optimum Experimental Designs. J. KIEFER, Cornell University, (Invited paper).

Let $f_1, \cdots, f_k$ be functions on a space $\mathfrak{X}$. We consider the regression problem where an experiment at x yields an observation with expectation $\Sigma\theta_i f_i(x)$. A design is (approximately) a probability measure ξ on $\mathfrak{X}$ which describes the proportion of observations to be taken at each value x. Let $M(\xi)$ be the matrix of elements $\int f_i f_j \, d\xi$, and write M_3 for the lower right-hand $(k - s) \times (k - s)$ submatrix of M. Write $f^{(2)}$ for the vector of the last $k - s$ functions of the vector f of f_i's. The results of Kiefer and Wolfowitz (*Canadian J.*, 1960) are generalized to the case where we are interested in $s < k$ parameters: *Theorem.* The following are equivalent if $M(\xi^*)$ is nonsingular (with an analogous result in the singular case): (1) ξ^* minimizes the generalized variance of the best linear estimators of $\theta_1, \cdots, \theta_s$; (2) ξ^* minimizes $\max_x d(x, \xi)$, where

$$d(x, \xi) = f(x)'M^{-1}(\xi)f(x) - f^{(2)}(x)'M_3^{-1}(\xi)f^{(2)}(x);$$

(3) $\max_x d(x, \xi^*) = s$. A characterization of the set of all such ξ^* is also given. The results complement those of Kiefer and Wolfowitz (*Ann. Math. Stat.*, 1959), and yield improved computational techniques in many cases. Numerous applications are given, e.g., to problems of polynomial regression on a q-dimensional cube or simplex; in particular, it is shown which of the designs considered by Scheffe (*J.R.S.S.* (*Ser. B*), 1958) are optimum.

74. Semi-Markov Processes: Countable State Space. RONALD PYKE, Columbia University, (Invited paper).

Let $A = (a_1, a_2, \cdots, a_m)$ be a vector of $m < \infty$ probabilities and let $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ be two matrices of transition distributions. Let $\{(J_n; X_n): n \geqq 0\}$ be a process satisfying $X_0 = 0$, $P[J_0 = k] = a_k$, $P[J_1 = j, X_1 \leqq x \mid J_0 = i] = \tilde{\mathfrak{Q}}_{ij}(x)$ and for $n > 1$, $P[J_n = j, X_n \leqq$

$x \mid J_0, J_1, \cdots J_{n-1}, X_1, \cdots X_{n-1}] = Q_{J_{n-1}j}(x)$. Define $N(t) = \sup\{k \geqq 0: X_0 + X_1 + \cdots + X_k \leqq t\}$ and $Z_t = J_{N(t)}$. The process $\{Z_t: t \geqq 0\}$ thus defined is called a general Semi-Markov process (G.S.-M.P.) determined by $(m, A, \tilde{\mathfrak{Q}}, \mathfrak{Q})$. Essentially, therefore a G.S.-M.P. is an S.-M.P. with random starting conditions. For the S.-M.P. determined by $(m, A, \mathfrak{Q})$, define $R^i_{jk}(x; t) = P[Z_t = j, J_{N(t)+1} = k, S_{N(t)+1} \leqq t + x \mid Z_0 = i]$. The complete limiting behavior of this function as $t \to \infty$ is obtained. In particular, if state j is recurrent and G_{jj}, the recurrence time distribution of state j, is non-lattice, then $\lim_{t\to\infty} R^i_{jk}(x; t) = c_{ij}\mu^{-1}_{jj} \int_0^x [Q_{jk}(+\infty) - Q_{jk}(y)]\, dy$ where c_{ij} is the probability of reaching state j from state i, and where μ_{jj} is the mean recurrence time of state j. From this result, it is possible to obtain specific quantities $\tilde{A}$ and $\tilde{\mathfrak{Q}}$ such that the G.S.-M.P. determined by $(m, \tilde{A}, \tilde{\mathfrak{Q}}, \mathfrak{Q})$ has the property that the three dimensional "age" process $\{(J_{N(t)}, J_{N(t)+1}, S_{N(t)+1} - t): t \geqq 0\}$ is a wide sense stationary process.

75. Generalized Bayes Solutions in Estimation Problems. JEROME SACKS, Columbia University, (Invited paper).

For simplicity consider the estimation on the basis of a sample of size one of the mean ω of a normal distribution with variance one with the loss function being squared error. If ξ is an a priori distribution then the Bayes estimate with respect to ξ is $E^x_\xi \omega$ (the a posteriori expected value of ω). Let F be a distribution function whose total variation over the space $\Omega = \{\omega\}$ is infinite but having the property that $E^x_F\omega$ is finite for all x. Call $E^x_F\omega$ a Generalized Bayes Solution (G.B.S.). These G.B.S. arise as limits of ordinary Bayes solutions. In case Ω is a half-infinite interval it can be proved that the class of G.B.S. together with the class of B.S. form a complete class. A sidelight of these considerations is this: Take $\Omega = [0, \infty)$ and F to be Lebesgue measure on Ω, then an admissible minimax estimate of ω is $E^x_F\omega$. These notions are extended to other loss functions. For some classes of distributions other than the normal class a complete class theorem of the type mentioned above is proved.