# THE NON-ABSOLUTE CONVERGENCE OF GIL-PELAEZ' INVERSION INTEGRAL

By J. G. Wendel

*University of Michigan*

Let $\varphi(t)$ be the characteristic function corresponding to a distribution function $F(x) = \{F(x - 0) + F(x + 0)\}/2$,

$$\varphi(t) = \int_{-\infty}^{+\infty} \exp(itx)\, dF(x). \tag{1}$$

Gil-Pelaez [1] has given an attractive expression for the inverse correspondence, which we may write in the form

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_{\to 0}^{\to\infty} \operatorname{Im} e^{-itx} \varphi(t)/t\, dt; \tag{2}$$

the arrows signify that the integral might be improper at either or both limits, as is implicit in Gil-Pelaez' proof.

Specializing to the case $x = 0$ we reduce (2) to the expression

$$F(0) = \frac{1}{2} - \frac{1}{\pi} \int_{\to 0}^{\to\infty} \operatorname{Im} \varphi(t)/t\, dt, \tag{3}$$

from which (3) may be recovered by a translation of the random variable.

Trivial instances where the integral in (3) is improper at the upper limit abound, e.g., $\varphi(t) = \exp(iat)$, $a \neq 0$. The lower limit is, however, a more delicate matter; although an isolated example of nonabsolute convergence at $t = 0$ may be drawn from ([4], Section 6.11), the "standard" distributions do not exhibit the phenomenon. Some misunderstanding on this point may have crept into the literature ([3], pp. 402, 411), and it is therefore thought that the following result may be of interest.

Let $\mathfrak{X}$ be the space of distribution functions $F$, metrized by

$$\rho(F, G) = \|F - G\| = \text{total variation of } F(x) - G(x).$$

Let $\mathfrak{A}$ be the subset of $\mathfrak{X}$ consisting of those $F$ for which (3) is proper at the lower limit.

Theorem. *$\mathfrak{A}$ is a set of the first category in $\mathfrak{X}$.*

As $\mathfrak{X}$ is a complete metric space, hence of second category, the theorem shows not only that $\mathfrak{X} - \mathfrak{A}$ is nonempty, but even that $\mathfrak{A}$ is a very "sparse" subset of $\mathfrak{X}$. (Category-theoretic existence proofs are well known in analysis; see, for example, ([2], p. 327), where the method is elegantly used to verify the existence of nowhere differentiable continuous functions.)

In order to prove the result we must show that $\mathfrak{A}$ is contained in the union of

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countably many closed sets having empty interiors. To this end let

$$\mathfrak{F}_n = \{F \,\Big|\, \int_0^1 |\operatorname{Im} \varphi(t)|/t\, dt \leqq n\}.$$

$\mathfrak{F}_n$ is closed, by an easy application of Fatou's lemma; clearly $\mathfrak{A}$ is the union of the $\mathfrak{F}_n$.

Suppose now that some $\mathfrak{F}_n$ has nonempty interior. Then there exists $F \in \mathfrak{F}_n$ and $\epsilon > 0$ such that $\rho(F, G) < 3\epsilon$ implies $G \in \mathfrak{F}_n$. In particular, let $E_c$ be the distribution function of a unit mass at $c$, and put $G_c = (1 - \epsilon)F + \epsilon E_c$; then $\rho(F, G_c) = \|F - G_c\| \leqq \epsilon\{\|F\| + \|E_c\|\} = 2\epsilon$, so that $G_c \in \mathfrak{F}_n$. For the corresponding characteristic functions $\psi_c$ we have

$$\psi_c(t) = (1 - \epsilon)\varphi(t) + \epsilon \exp(ict),$$

whence

$$\operatorname{Im} \psi_c(t) = (1 - \epsilon) \operatorname{Im} \varphi(t) + \epsilon \sin(ct).$$

Therefore

$$|\sin(ct)| \leqq \epsilon^{-1}\{|\operatorname{Im} \psi_c(t)| + |\operatorname{Im} \varphi(t)|\}.$$

Dividing through by $t$ and integrating from 0 to 1 yields

$$\int_0^1 |\sin(ct)|/t\, dt \leqq \epsilon^{-1}\{n + n\} = 2n\epsilon^{-1}.$$

But the left member is unbounded as $c \to \infty$. This contradiction completes the proof.

## REFERENCES

[1] J. Gil-Pelaez, "Note on the inversion theorem," *Biometrika*, Vol. 38 (1951), pp. 481–482.

[2] Casimir Kuratowski, *Topologie I*, Monografie Matematyczne, Warsaw, 1948.

[3] Emanuel Parzen, *Modern Probability Theory and its Applications*, John Wiley and Sons, New York, 1960.

[4] E. C. Titchmarsh, *Theory of Fourier Integrals*, Oxford University Press, Oxford, 1937.