CONFIDENCE INTERVALS FROM CENSORED SAMPLES

By Max Halperin[1]

Knolls Atomic Power Laboratory

1. Summary. Suppose a random sample of size n is drawn from a normal population with mean μ and standard deviation σ and that the sample has been censored either to the right or the left. Suppose the censoring is at a fixed point of the distribution or at a pre-specified sample percentage point, or is a combination of these two types of censoring. In this paper we present small sample bounded confidence intervals for μ and σ, based on a joint bounded confidence region at a confidence level with fixed bound. The limits for μ and σ so obtained converge in probability, as $n \to \infty$, to the parameter values. The procedure of the paper allows similar results for some other scale-translation families of distributions. One such case, which is briefly discussed, is that of the exponential distribution with unknown initial point. The somewhat general applicability of the procedure mitigates the fact that it is not based on sufficient statistics.

2. Derivation of results: normal distribution.

a. *Fixed point censoring.* In this discussion we assume censoring is to the right; the changes if censoring is to the left will be obvious.

Thus, we consider a random sample of n on $N(\mu, \sigma)$ censored to the right at a known number T. For m the number of noncensored observations greater than zero, denote the ordered non-censored observations by

$$x_1 < x_2 < \cdots < x_m < T.$$

The number m is a random variable having a binomial distribution with parameters $\Phi[(T - \mu)/\sigma]$ and n, where Φ is the unit-normal cumulative distribution function. The density of $x_1, x_2, \cdots, x_m$ given m is

$$(2.1)\qquad \begin{cases} m!\,[\Phi(T-\mu)/\sigma]^{-m} \prod_{i=1}^{m} \varphi[(x_i - \mu)/\sigma], & x_1 < x_2 < \cdots < x_m < T, \\ 0, & \text{elsewhere,} \end{cases}$$

where φ is the unit-normal density.

On the basis of (2.1) and the distribution of m, we would like to obtain a reasonable confidence set on (μ, σ) at a bounded confidence level β, where β is specified in advance. By "reasonable" we mean that the set should be bounded (providing this is possible), and that as $n \to \infty$ any confidence limit should converge to the appropriate parameter. We consider in detail the case where two sided limits on μ and σ are desired.

First we observe that, from the binomial distribution of m, we can get confidence limits on $\Phi[(T - \mu)/\sigma]$ or Φ_T for short, at say the β_1 confidence level, as

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[1] Now at the General Engineering Laboratory, General Electric Co., Schenectady.

$[\Phi_L \leqq \Phi_T \leqq \Phi_U]$. Now suppose that from (2.1) we could make a confidence statement, conditional on m, concerning Φ_T and some other function of μ and σ, or at least the latter, at say the β_2 level or greater for each m. Denote this conditional confidence statement by C_m. The total probability that both C_m and the statement about Φ_T are true is clearly

$$\sum \Pr\{C_m \text{ is true} \mid m\} \Pr\{m\}, \tag{2.2}$$

where the sum runs over all values of m such that the associated statement about Φ_T is true. It then follows that the probability (2.2) is $\geqq \beta_2\beta_1$. Since μ and σ are functions of Φ_T and whatever other parameter is involved we should then be able to get confidence limits on μ and σ, as well as other functions, at least at the $\beta_1\beta_2$ level; this presumes Φ_T and the "other parameter" are a one-to-one transformation of μ, σ. The question of optimum choice of β_1 and β_2 such that $\beta_1\beta_2 = \beta$ is not discussed; for simplicity we take $\beta_1 = \beta_2 = \beta^{\frac{1}{2}}$.

Turn to the definition of C_m. Suppose we let

$$z_i = \Phi[x_i - \mu)/\sigma]/\Phi_T. \tag{2.3}$$

Then the density of the z_i's given m is

$$\begin{aligned} &m!, \qquad 0 \leqq z_1 < z_2 < \cdots < z_m \leqq 1, \\ &0, \qquad \text{elsewhere,} \end{aligned} \tag{2.4}$$

providing $m > 0$. If $m = 0$, we can obviously only make a useful statement about Φ_T; any statement made about any other parameter must include all possibilities in order to be correct and hence will be trivial. Thus if $m = 0$, the confidence region cannot be bounded. However, $m = 0$ is clearly a degenerate case for either point or interval estimation of μ and σ no matter how one goes about solving these problems. Hence, we take care of this case in a purely formal way to insure the bounded confidence level property. This will be made explicit shortly.

Now, suppose $m \geqq 1$ and that we can find numbers r and s (integral or zero) and a number δ such that

$$0 \leqq s < r \leqq m + 1, 0 < \delta < 1, \qquad \Pr\{z_s \leqq \delta \leqq z_r \mid m\} \geqq \beta^{\frac{1}{2}}, \tag{2.5}$$

where $z_0 = 0$ and $z_{m+1} = 1$. Of course, s, r and δ will depend on m. As will be apparent shortly, we must require $\delta \leqq \frac{1}{2}$. Before indicating in detail how to determine r, s and δ, we turn to the inequalities on μ and σ separately which are implied by (2.5) and a $\beta^{\frac{1}{2}}$ confidence interval on Φ_T. From (2.3), the inequality on δ specified by (2.5) can be written (for $s > 0$, $r < m + 1$) as

$$(x_s - \mu)/\sigma \leqq \Phi^{-1}(\delta\Phi_T) \leqq (x_r - \mu)/\sigma, \tag{2.6}$$

where $\Phi^{-1}(\alpha)$ is the standardized normal deviate exceeded with probability $1 - \alpha$. Since we can write, for example,

$$(x_s - \mu)/\sigma = [(x_s - T)/\sigma] + [(T - \mu)/\sigma] = [(x_s - T)/\sigma] + \Phi^{-1}(\Phi_T), \tag{2.7}$$

and $\Phi^{-1}(\delta\Phi_T) - \Phi^{-1}(\Phi_T) < 0$, for $0 < \delta < 1$, it follows from (2.6) that

$$\sigma \leqq (T - x_s)/[\Phi^{-1}(\Phi_T) - \Phi^{-1}(\delta\Phi_T)]. \tag{2.8}$$

A lower inequality for σ based on x_r and of the same form follows similarly. Differentiation of the right hand side of (2.8) with respect to Φ_T shows that it will be monotone decreasing in Φ_T providing one has

$$\exp\{-\tfrac{1}{2}[\Phi^{-1}(\delta\Phi_T)]^2\} - \delta \exp\{-\tfrac{1}{2}[\Phi^{-1}(\Phi_T)]^2\} > 0. \tag{2.9}$$

The left hand side of (2.9) is zero at $\Phi_T = 0$ and differentiation with respect to Φ_T shows that it is monotone increasing. Thus one has as limits for σ, if $s > 0$, $r < m + 1$, since $T - x_s > T - x_r$,

$$\begin{aligned}(T - x_r)/[\Phi^{-1}(\Phi_U) - \Phi^{-1}(\delta\Phi_U)] &\leqq \sigma \\ &\leqq (T - x_s)/[\Phi^{-1}(\Phi_L) - \Phi^{-1}(\delta\Phi_L)].\end{aligned} \tag{2.10}$$

Clearly if $r = m + 1$, $s > 0$, the lower bound in (2.10) becomes zero, while if $r < m + 1$, $s = 0$, the upper bound is infinite.

Now we turn to the determination of limits for μ. First we observe that $\mu = T - \sigma\Phi^{-1}(\Phi_T)$ so that

$$\begin{aligned}\frac{\partial\mu}{\partial\sigma} &= -\Phi^{-1}(\Phi_T) \leqq 0, \text{ if } \Phi_T \geqq \tfrac{1}{2}, \\ &> 0, \text{ if } \Phi_T < \tfrac{1}{2}, \\ \frac{\partial\mu}{\partial\Phi_T} &= -\sigma\,(2\pi)^{\frac{1}{2}} \exp \tfrac{1}{2}[\Phi^{-1}(\Phi_T)]^2 < 0.\end{aligned}$$

Thus we need to consider three cases,

$$\text{(A) } \Phi_L \geqq \tfrac{1}{2}, \qquad \text{(B) } \Phi_U < \tfrac{1}{2}, \qquad \text{(C) } \Phi_U \geqq \tfrac{1}{2}, \quad \Phi_L < \tfrac{1}{2}.$$

In case (A), $(\partial\mu/\partial\sigma) < 0$, $(\partial\mu/\partial\Phi_T) \leqq 0$. Consequently the upper limit to μ must lie on $\sigma = (T - x_r)/[\Phi^{-1}(\Phi_T) - \Phi^{-1}(\delta\Phi_T)]$ so that

$$\mu \leqq T - (T - x_r)\Phi^{-1}(\Phi_T)/[\Phi^{-1}(\Phi_T) - \Phi^{-1}(\delta\Phi_T)]. \tag{2.11}$$

Differentiating the right hand side of (2.11) with respect to Φ_T it is found that the derivative is negative providing

$$\delta\Phi^{-1}(\Phi_T)\exp\{-\tfrac{1}{2}[\Phi^{-1}(\Phi_T)]^2\} - \Phi^{-1}(\delta\Phi_T)\exp\{-\tfrac{1}{2}[\Phi^{-1}(\delta\Phi_T)]^2\} > 0. \tag{2.11a}$$

If $\Phi_T \geqq \frac{1}{2}$ and $\delta \leqq \frac{1}{2}$ (which has been assumed) (2.11a) is clearly satisfied. If $\Phi_T < \frac{1}{2}$, the derivative of the left hand side of (2.11a) is found to be positive. Since the left hand side of (2.11a) is zero at $\Phi_T = 0$, (2.11a) is always satisfied and it follows that

$$\mu \leqq T - (T - x_r)\Phi^{-1}(\Phi_L)/[\Phi^{-1}(\Phi_L) - \Phi^{-1}(\delta\Phi_L)]. \tag{2.11b}$$

Similarly, the lower limit for μ must lie on $\sigma = (T - x_s)/[\Phi^{-1}(\Phi_T) - \Phi^{-1}(\delta\Phi_T)]$ and we conclude that

$$\mu \geqq T - (T - x_s)\Phi^{-1}(\Phi_U)/[\Phi^{-1}(\Phi_U) - \Phi^{-1}(\delta\Phi_U)]. \tag{2.11c}$$

Note that if $r = m + 1, s > 0$, (2.11b) is replaced by $\mu \leqq T$, while if $r < m + 1$, $s = 0$, (2.11c) is replaced by $\mu \geqq -\infty$.

In case (B) $(\partial\mu/\partial\sigma) > 0$, $(\partial\mu/\partial\Phi_T) \leqq 0$. Arguments analogous to these above lead to the interval for μ

$$\begin{aligned} (2.12) \qquad &\{T - (T - x_r)\Phi^{-1}(\Phi_U)/[\Phi^{-1}(\Phi_U) - \Phi^{-1}(\delta\Phi_U)], \\ &T - (T - x_s)\Phi^{-1}(\Phi_L)/[\Phi^{-1}(\Phi_L) - \Phi^{-1}(\delta\Phi_L)]\}, \end{aligned}$$

if $r < m + 1$, $s > 0$. If $r = m + 1$, $s > 0$, the lower bound for μ becomes T; if $r < m + 1$, $s = 0$, the upper limit becomes $+\infty$.

In case (C) we break the interval (Φ_L, Φ_U) into $(\Phi_L, \frac{1}{2})$ and $(\frac{1}{2}, \Phi_U)$. In the interval $(\Phi_L, \frac{1}{2})$ one has from case (B), if $r < m + 1$, $s > 0$,

$$(2.13a) \qquad T \leqq \mu \leqq T - (T - x_s)\Phi^{-1}(\Phi_L)/[\Phi^{-1}(\Phi_L) - \Phi^{-1}(\delta\Phi_L)].$$

In the interval $(\frac{1}{2}, \Phi_U)$ we have by case (A),

$$(2.13b) \qquad T - (T - x_s)\Phi^{-1}(\Phi_U)/[\Phi^{-1}(\Phi_U) - \Phi^{-1}(\delta\Phi_U)] \leqq \mu \leqq T,$$

and we combine (2.13a) and (2.13b) in the obvious way. If $r = m + 1$, $s > 0$ (2.13a) and (2.13b) remain unchanged. If $r < m + 1$, $s = 0$, the upper limit to (2.13a) becomes $+\infty$ and the lower limit to (2.13b) becomes $-\infty$ leading to a trivial interval on μ. Note that in all cases considered the confidence region is closed providing we never take z_0 as a lower limit in (2.5); i.e., providing we exclude the choice $s = 0$.

The above discussion can be modified in an obvious way to yield one sided intervals on μ or σ but we omit that analysis.

Now we need to discuss the choice of r, s and δ to achieve the desired $\beta^{\frac{1}{2}}$ confidence. First we note that the confidence levels achievable will be somewhat limited by the confidence levels which can be obtained for Φ_T. Aside from this we note that the smallest value of m, given δ, for which we can take $s > 0$, $r < m + 1$ and still achieve conditional protection of at least $\beta^{\frac{1}{2}}$ is determined by

$$(2.14) \qquad 1 - (1 - \delta)^m - \delta^m \geqq \beta^{\frac{1}{2}}.$$

We observe the left hand side of (2.14) has a maximum, for given m, at $\delta = \frac{1}{2}$. Hence taking $\delta = \frac{1}{2}$ will allow us to get two sided limits on μ and σ with $r < m + 1$, $s > 0$ for a lower value of m than will be permitted for any other value of δ. For values of m so low that (2.14) cannot be satisfied even for $\delta = \frac{1}{2}$, our earlier discussion suggests taking $s = 1$, $r = m + 1$ and then choosing δ so that

$$(2.15) \qquad 1 - (1 - \delta)^m \geqq \beta^{\frac{1}{2}}.$$

Unfortunately for values of β used in practice (2.15) may violate our assumption $\delta \leqq \frac{1}{2}$ for m small enough. Thus, for very small m we would take $s = 0$, $r = m$ and choose δ to insure

$$(2.16) \qquad 1 - \delta^m \geqq \beta^{\frac{1}{2}},$$

which can always be done. Thus for some small values of m one will get unbounded intervals in the effort to insure bounded confidence. For values of m for which it is feasible to take $\delta = \frac{1}{2}$ it is natural to take r and s as the (symmetric) extremes from the maximum and minimum order statistics, allowing, (2.5) to be satisfied. Since for small samples this may be somewhat conservative, an alternative is to take δ sufficiently less than $\frac{1}{2}$ for the given r and s, so that a conditional coverage of exactly $\beta^{\frac{1}{2}}$ is obtained. Other alternatives may occur to the reader but the lack of exactness is a trivial question in any case. The case $m = 0$, as indicated earlier, is degenerate and can be included in the discussion above by associating with $m = 0$ the statement $0 < \sigma < \infty$ as well as the appropriate interval on Φ_T (which implies $-\infty < \mu < \infty$).

Now we turn to the question of the asymptotic behavior of the intervals we have defined. To this end we note that (2.5) can be written as

$$\sum_{t=s}^{r-1} \binom{m}{t} \delta^t (1-\delta)^{m-t} \geqq \beta^{\frac{1}{2}}. \tag{2.17}$$

It is clear that for large m (2.17) can be written approximately as

$$(2\pi)^{-\frac{1}{2}} \int_{\Phi_1^{-1}}^{\Phi_2^{-1}} \exp\left(-\tfrac{1}{2}u^2\right) du = \beta^{\frac{1}{2}} \tag{2.18}$$

where

$$r - 1 = \delta m + \Phi_2^{-1} m^{\frac{1}{2}} \delta^{\frac{1}{2}} (1-\delta)^{\frac{1}{2}},$$
$$s = \delta m + \Phi_1^{-1} m^{\frac{1}{2}} \delta^{\frac{1}{2}} (1-\delta)^{\frac{1}{2}},$$

and Φ_2^{-1} and Φ_1^{-1} are both finite. For m large our discussion has taken $\delta = \frac{1}{2}$ and $\Phi_1^{-1} = -\Phi_2^{-1} = -\Phi^{-1}[\frac{1}{2}(1 - \beta^{\frac{1}{2}})]$, although we could also take any δ, $0 < \delta \leqq \frac{1}{2}$ and any finite pair Φ_1^{-1}, Φ_2^{-1} satisfying (2.18). In any cases it is easy to show that as $n \to \infty$, with r and s defined by (2.18),

$$\begin{aligned} \omega_r &= \left(\frac{m}{\delta(1-\delta)}\right)^{\frac{1}{2}} \left[z_r - \delta - \Phi_2^{-1} \left(\frac{\delta(1-\delta)}{m}\right)^{\frac{1}{2}} \right], \\ \omega_s &= \left(\frac{m}{\delta(1-\delta)}\right)^{\frac{1}{2}} \left[z_s - \delta - \Phi_1^{-1} \left(\frac{\delta(1-\delta)}{m}\right)^{\frac{1}{2}} \right], \end{aligned} \tag{2.19}$$

are each $N(0, 1)$. This, of course, requires taking into account the asymptotic distribution of m, which, for simplicity, is not displayed explicitly in (2.19). It follows that for large n and any $\epsilon > 0$

$$\begin{aligned} \Pr\left\{ \left| z_r - \delta - \frac{\Phi_2^{-1}[\delta(1-\delta)]^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right| > \epsilon \right\} &\leqq O\left(\frac{1}{n\epsilon}\right), \\ \Pr\left\{ \left| z_s - \delta - \frac{\Phi_1^{-1}[\delta(1-\delta)]^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right| > \epsilon \right\} &\leqq O\left(\frac{1}{n\epsilon}\right), \end{aligned} \tag{2.20}$$

so that z_r and z_s both converge in probability to δ. This fact, in conjunction with the convergence of Φ_U and Φ_L to Φ_T allows us to conclude that the confidence intervals on μ and σ separately will indeed converge as $n \to \infty$ for any $\delta \leqq \frac{1}{2}$ and for r and s chosen in any of the ways suggested above.

b. *Fixed sample percent point censoring.* We again assume censoring is to the right. This case can be treated in a manner identical to that of case (a) by proper identification. Thus, here we suppose that the $n - m - 1$ largest observations are censored. Denoting the first $m + 1$ order statistics by $x_1, x_2, \cdots, x_{m+1}$, we identify $\Phi[(x_{m+1} - \mu)/\sigma]$ with $\Phi[(T - \mu)/\sigma]$ and consider the conditional distribution of $x_1, x_2, \cdots, x_m$ given x_{m+1}. The same reasoning as used in case (a) leads to formally identical results with x_{m+1} replacing T. It is clear that the confidence intervals obtained for this case will converge to points for large samples for $\delta \leqq \frac{1}{2}$ and r and s chosen as indicated in case (a). The only real difference from case (a) is the random nature of x_m rather than m and this causes no difficulty because of the stochastic convergence of x_m to the appropriate percent point of the normal distribution.

It should be remarked that for this case one can derive many other confidence regions for μ and σ without resorting to the conditional procedure outlined above. For example, if we obtained two sided confidence limits on two percentiles, this would generally give a closed region on μ and σ. The approach of this paper does have the virtue of providing a uniform treatment of both types of censoring. A more meaningful criterion to use in choosing between our approach and the unconditional approach would be the shortness of the confidence intervals obtained. However, we do not pursue this question.

c. *Mixed censoring.* As a practical matter many life-testing experiments are a mixture of cases (a) and (b) above. Thus one frequently finds that a life test is terminated if either r out of n items have failed or if some fixed length of time, T, has elapsed. In such cases r will generally be a large fraction of n, reflecting the thought that the further results give only a modest amount of information; the choice of T, on the other hand, may reflect the extreme life expected or of concern, or perhaps the urgency of the need for information. Procedures analagous to those of case (a) or (b) may also be developed for this situation and are sufficiently different to warrant separate consideration.

In this case the sample likelihood has two forms depending on whether m, the observed number of uncensored observations is equal to r or less than r. If $m < r$, we have exactly the situation described in case (a) and can follow the procedures described for that case. If $m = r$, we are in a situation like case (b) but not identical. The distinguishing feature is that we must have $x_r < T$, i.e., the distribution of m is truncated at $m = r$. The procedure of case (*b*) still applies to the boundaries of the confidence region defined by the conditional distribution of $x_1, x_2, \cdots, x_{r-1}$ given x_r. But because of the truncated distribution of m, at $m = r$ one only has the trivial upper confidence limit of unity for Φ_T. However, one can obtain on the basis of usual binomial theory a non-trivial lower confidence limit for Φ_T at the $\beta^{\frac{1}{2}}$ level, say Φ_L. It is also appropriate to assert that $\Phi[(x_r - \mu)/\sigma] \leqq \Phi_L$ because of the manner in which the statement on Φ_T is obtained. From $\Phi_T \geqq \Phi_L$ we have

$$\Phi^{-1}(\Phi_T) \geqq \Phi^{-1}(\Phi_L)$$

or writing

$$\Phi^{-1}(\Phi_T) = [(T - x_r)/\sigma] + [(x_r - \mu)/\sigma],$$

we are led to conclude

$$\Phi[\Phi^{-1}(\Phi_L) - (T - x_r)/\sigma] \leqq \Phi_r \leqq \Phi_L, \tag{2.21}$$

where $\Phi_r = \Phi[(x_r - \mu)/\sigma]$.

The related conditional statement at level $\beta^{\frac{1}{2}}$, for appropriately chosen δ, s and t $(r > t > s > 0)$ is just

$$(T - x_t)/[\Phi^{-1}(\Phi_r) - \Phi^{-1}(\delta\Phi_r)] \leqq \sigma \leqq (T - x_s)/[\Phi^{-1}(\Phi_r) - \Phi^{-1}(\delta\Phi_r)]. \tag{2.22}$$

Of course, if $(r = t, s > 0)$ or $(r > t, s = 0)$ (2.22) must be modified as indicated earlier. We observe that the left hand side of (2.21) is monotone increasing in σ and approaches Φ_L as $\sigma \to \infty$. Coupling this with previous discussion it is clear that at least for $(r > t, s > 0)$ (2.21) and (2.22) provide a closed region in the (σ, Φ_r) plane.

It is clear, from what has been said, that for $r > t$, $s > 0$ an upper limit to σ is given from (2.21) by

$$\sigma = (T - x_r)/[\Phi^{-1}(\Phi_L) - \Phi^{-1}(\Phi_r)], \tag{2.23a}$$

and from (2.22) by

$$\sigma = (T - x_s)/[\Phi^{-1}(\Phi_r) - \Phi^{-1}(\delta\Phi_r)]. \tag{2.23b}$$

Since in (2.23a) $d\sigma/d\Phi_r > 0$ while in (2.23b) $d\sigma/d\Phi_r < 0$ it follows that (2.23a) and (2.23b) have a single point of intersection and that the value of σ at this point, σ_U say, is an fact the upper limit to σ implied by (2.21) and (2.22). The value of σ_U is probably most easily found by trial and error using the monotonicity properties just described. A lower limit for σ follows from previous discussion by substituting Φ_L for Φ_r in the left hand side of (2.22). Again the results must be slightly modified if $(r = t, s > 0)$ or $(r > t, s = 0)$. The details are omitted.

To obtain limits for μ we must go through an argument similar to that of case (a). There are three cases to consider. We consider only the situation for which $(r > t, s > 0)$. The necessary changes if $(r = t, s > 0)$ or $(r > t, s = 0)$ can easily be ascertained as indicated earlier. The three cases are, as in (a),

(A) Φ_{rL}, the lower limit to Φ_r, is $\geqq \frac{1}{2}$, (B) $\Phi_L \leqq \frac{1}{2}$,

(C) $\Phi_{rL} \leqq \frac{1}{2}, \Phi_L > \frac{1}{2}$.

To determine Φ_{rL} one must solve (2.23a) with

$$\sigma = (T - x_t)/[\Phi^{-1}(\Phi_r) - \Phi^{-1}(\delta\Phi_r)].$$

This again is most easily done by trial and error. The analysis parallels that of case (a) and one is led to, for case (A), using $D(\delta, p) = \Phi^{-1}(p) - \Phi^{-1}(\delta p)$,

$$x_r - (T - x_s)\Phi^{-1}(\Phi_L)/D(\delta, \Phi_L) \leqq \mu \leqq x_r - (T - x_t)\Phi^{-1}(\Phi_{rL})/D(\delta, \Phi_{rL}), \tag{2.24a}$$

and for case (B),

$$x_r - (T - x_t)\Phi^{-1}(\Phi_L)/D(\delta, \Phi_L) \leqq \mu \leqq x_r - (T - x_t)\, \Phi^{-1}(\Phi_{rL})/D(\delta, \Phi_{rL}), \tag{2.24b}$$

and for case (C),

$$x_r - (T - x_s)\Phi^{-1}(\Phi_L)/D(\delta, \Phi_L) \leqq \mu \leqq x_r - (T - x_t)\Phi^{-1}(\Phi_{rL})/D(\delta, \Phi_{rL}). \tag{2.24c}$$

To meaningfully discuss asymptotic behavior of the confidence intervals for this case, it is clear that one must require r/n to approach a constant, say p, as $n \to \infty$. It is obvious that if $p > \Phi_T$, we will in the limit be essentially in case (a) so that for suitably restricted δ, t and s the argument of case (a) applies with almost no change. It is only if $p \leqq \Phi_T$ that a new asymptotic argument is needed. It is clear that such an argument will follow the lines indicated for case (b) and it is omitted.

3. Extension of results. It is clear that the results of Section 2 do not depend crucially on the assumption of normality except in the implications of the regions on σ and Φ_T (or Φ_r) for limits on μ and σ. Thus the procedure should be applicable to other distributions depending only on scale and location parameters. We illustrate this by sketching the extension of the results of Section 2(a) to the case of sampling from

$$p(x) = (1/\sigma) \exp[-(x - \mu)/\sigma], \qquad 0 \leqq \mu \leqq x < \infty, \qquad \sigma > 0. \tag{3.1}$$

The entire argument of Section 2 up to and including (2.8) then is applicable to (3.1) by the trivial re-definition of Φ as the cumulative distribution function of $(x - \mu)/\sigma$, where x obeys (3.1), and $\Phi^{-1}(\alpha)$ as the value of $(x - \mu)/\sigma$ exceeded with probability $1 - \alpha$. The same type of argument as that following (2.8) leads to limits for σ which are identical with those given by (2.10) upon proper definition of Φ^{-1} and Φ. The discussion concerning limits for μ follows identical lines being rendered even simpler and not requiring $\delta \leqq \frac{1}{2}$ because of the non-negative character of $\Phi^{-1}(\alpha)$; the results for μ are in fact identical with those given in Section 2, upon proper identification and, of course, taking into account that μ is essentially non-negative (although, formally, this is not required). Obviously there is no difficulty in the consistency argument for properly restricted r and s.

It is evident that the results may hold for many classes of distributions under appropriate conditions on the parent densities. However, we do not pursue this question further.

4. An example. As an example, we consider some data falling in case (a); the maximum likelihood estimates of μ and σ are included as a matter of interest. Thus consider a random sample of 30, drawn from a normal distribution with zero mean and unit variance, with $T = 1$. The data, obtained from a table of random normal deviates, are summarized below in order of size.

i	x_i	i	x_i	i	x_i
1	−1.805	11	−0.482	21	0.658
2	−1.787	12	−0.439	22	0.906
3	−1.501	13	−0.105	23	>1
4	−1.399	14	−0.005	24	"
5	−1.376	15	0.041	25	"
6	−1.339	16	0.060	26	"
7	−1.186	17	0.159	27	"
8	−1.132	18	0.199	28	"
9	−1.010	19	0.279	29	"
10	−0.690	20	0.464	30	"

A 95% confidence interval on Φ_T based on 22 out of 30 sample units uncensored is given by $.53 \leqq \Phi_T \leqq .88$. For a sample of 22 one finds that the shortest symmetric confidence interval for the median at least at the 95% level is given by x_6 and x_{17}. The exact confidence level is in fact .9831. One easily finds using binomial tables that $\delta = .42$ makes x_6 and x_{17} almost exactly 95% confidence limits on the 42% point of the distribution. The formulae derived above immediately give

$$-.83 \leqq \mu \leqq .916,$$

$$.62 \leqq \sigma \leqq 2.79,$$

as confidence limits each at least at the 90% level. Computation of the maximum likelihood estimates of μ and σ for these data gives estimates,

$$\hat{\mu} = .12, \qquad \hat{\sigma} = 1.28$$

Note that $\hat{\mu}$ is near zero and well within our confidence limits on μ while $\hat{\sigma}$ is near the lower limit on σ. This, of course, is a matter over which we have no control, since the confidence interval estimates and point estimates are based on different principles. We would, however, generally expect our point estimates to lie within the confidence intervals, especially with increase in sample size.

5. Some final remarks. Some consideration was given to the problem of obtaining confidence intervals for μ and σ when one has double censoring. A procedure along the lines of Section 2 was studied but it appeared that the confidence region based on order statistics of the conditional truncated distribution of uncensored items and the frequency split of the data into the three possible categories might not define a useful region on μ and σ. In particular, there was some indication that the confidence statement on μ derivable from the region might, in some instances, involve disjoint intervals which would be unacceptable. Attention was also given to the use of order statistics for obtaining confidence limits on μ and σ in the singly truncated (to the right) normal distribution. Using four order statistics and two percent points to obtain a region along the lines of Section 2, it turned out that although σ could be bounded as a function of Φ_T and various order statistics, one could not bound Φ_T other than trivially. It is possible but not, in the writer's opinion, likely, that schemes more complex than that of Section 2 for using the order statistics might allow a solution.

Although the point has not been emphasized, it is clear that the results of this paper are conservative in two senses. On the one hand we are using a bounded confidence region. On the other hand we are using selected order statistics of the uncensored portion of our sample rather than all the uncensored data. It is felt that the conservatism is considerably mitigated by the fairly general nature of the results and the bounded confidence property.

Some readers of an earlier version of this paper have suggested that confidence intervals based on asymptotic parametric theory can hardly be so bad that the procedure of this paper should be preferred. This may very well be true but unfortunately appears almost impossible to verify analytically and has not yet been investigated numerically. Thus, a choice between the procedures of this paper and asymptotic parametric theory is presently a choice between somewhat long intervals at known confidence levels and relatively short intervals with conjectural confidence properties.

Various papers treating the asymptotic parametric theory are given in the bibliography compiled by Mendenhall [1].

REFERENCE

[1] Mendenhall, William, "A bibliography on life testing and related topics," *Biometrika*, Vol. 45 (1958), 521–543.