REFERENCES

[1] Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain 'goodness-of-fit' criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

[2] Birnbaum, Z. W. (1953). On the power of a one sided test of fit for continuous probability functions. *Ann. Math. Statist.* **24** 484–489.

[3] Rosenblatt, Judah (1959). Goodness-of-fit tests for approximate hypotheses, microfilmed Ph.D. thesis, Columbia Univ.

[4] Kiefer, J. and Wolfowitz, J. (1958). On the deviations of the empiric distribution function of vector chance variables. *Trans. Amer. Math. Soc.*, **87** 173–186.

[5] Rosenblatt, Judah (1962). Some modified Kolmogorov-Smirnov tests of approximate hypotheses and their properties. *Ann. Math. Statist.* **33** 513–524.

[6] Simpson, Paul B. (1951). Note on the estimation of a bivariate distribution function. *Ann. Math. Statist.* **23** 476–478.

[7] Halmos, Paul R. (1958). *Measure Theory.* Van Nostrand, New York.

THE UNIQUENESS OF THE SPACING OF OBSERVATIONS IN POLYNOMIAL REGRESSION FOR MINIMAX VARIANCE OF THE FITTED VALUES

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1. Introduction. A spacing of $n(p + 1)$ observations in the interval $[-1, 1]$ in order to minimize the maximum variance of $\hat{u}(x)$, a pth degree polynomial fitted by least squares, in the interval $[-1, 1]$ has been given by P. G. Guest [1]. The spacing places n observations at -1, 1, and each of the $p - 1$ zeros of $P'_p(x)$, where $P_p(x)$ is the Legendre polynomial of degree p. The purpose of this note is to establish the uniqueness of P. G. Guest's solution when the observations are made at $p + 1$ distinct points.

2. Statement of problem. Guest defines

$$F(x) = \prod_{j=0}^{p} (x - x_j) \tag{1}$$

where the x_j are the distinct points in the interval $[-1, 1]$ at which the observations are to be made. The minimax variance condition requires

$$x_0 = -1, \qquad x_p = 1 \tag{2}$$

and

$$F''(x_j) = 0, \qquad \text{for } j = 1, 2, \cdots, p - 1. \tag{3}$$

After defining $\phi(x)$ by the equation

$$F(x) = \alpha(x^2 - 1)\phi(x), \tag{4}$$

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the problem of finding the x_j to satisfy (1) and (3) becomes that of selecting $\phi(x)$ such that

$$F''(x_j) = \phi(x_j) = 0 \tag{5}$$

for $p - 1$ distinct (but unspecified) values of x_j in the interval $(-1, 1)$. We choose α so that the leading coefficient of $\alpha\phi(x)$ is unity. The question of the uniqueness of $\alpha\phi(x)$ (and thus the points of observation) is now raised.

3. The solution. P. G. Guest [1] has shown that

$$\alpha\phi(x) = \frac{2^p(p!)^2}{p(2p)!} P'_p(x) \tag{6}$$

where $P_p(x)$ is the pth degree Legendre polynomial, does satisfy (5) as required. We shall now show that if condition 5 is satisfied, $\alpha\phi(x)$ is necessarily as given above.

Let

$$\alpha\phi(x) = \sum_{i=0}^{p-1} a_i x^i \tag{7}$$

where $a_{p-1} = 1$.

Substituting (7) into (4) and differentiating, we obtain

$$\begin{aligned} F''(x) = p(p + 1)x^{p-1} &+ a_{(p-2)}p(p - 1)x^{p-2} \\ &+ \sum_{i=0}^{p-3} (i + 2)(i + 1)(a_i - a_{(i+2)})x^i. \end{aligned} \tag{8}$$

From (5) it follows that $F''(x_j) - \alpha p(p + 1)\phi(x_j) = 0$, for $j = 1, 2, \cdots, p - 1$, so that

$$\begin{aligned} &a_{p-2}[p(p - 1) - p(p + 1)]x_j^{p-2} \\ &+ \sum_{i=0}^{p-3} \{[(i + 2)(i + 1) - p(p + 1)]a_i - (i + 2)(i + 1)a_{i+2}\}x_j^i = 0, \end{aligned} \tag{9}$$

for $j = 1, 2, \cdots, p - 1$.

Since (9) is of degree $p - 2$, requiring it to be zero for $p - 1$ distinct values of x_j requires it to be identically zero. Thus $a_{p-2} = 0$, and

$$a_i = (i + 2)(i + 1)(i - p + 1)^{-1}(i + p + 2)^{-1}a_{i+2},$$

which along with $a_{p-1} = 1$, fully determines $\alpha\phi(x)$. That $\phi(x) = P'_p(x)$ for

$$\alpha = \frac{2^p(p!)^2}{p(2p)!}$$

follows by an inspection of the recursion formula for the coefficients of $P'_p(x)$.

REFERENCES

[1] Guest, P. G. (1958). The spacing of observations in polynomial regression. *Ann. Math. Statist.* **29** 294–299.