AN INVARIANCE PRINCIPLE IN RENEWAL THEORY[1]

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1. Introduction. Let $X_1, X_2, \cdots$, be a sequence of independent, identically distributed positive random variables, and let S_n be their nth partial sum. Define a stochastic process $\{Y_t\}$ by letting

$$Y_t = t - \max\{S_n \mid S_n \leqq t\}; \tag{1.1}$$

$\{Y_t\}$ is Markovian and has stationary transition probabilities. It has been shown by Dynkin [3] (see also [7]) that the random variable Y_t/t has a non-degenerate limiting distribution as $t \to \infty$ if and only if $F(x)$, the common distribution function of the X_i, satisfies

$$1 - F(x) = x^{-\alpha}L(x), \qquad 0 < \alpha < 1, \tag{1.2}$$

where $L(x)$ is a slowly varying function. (That is, $L(cx)/L(x) \to 1$ as $x \to \infty$ for every positive c. These functions were introduced by Karamata in [5].) More recently it has been shown [8] that the same is true of M_t/t, where

$$M_t = \sup_{\tau \leqq t} Y_\tau . \tag{1.3}$$

These facts suggest that perhaps, if (1.2) holds, the *processes* $\{\xi^{-1}Y_{\xi t}\}$ converge as $\xi \to \infty$ to a limiting process in somewhat the manner of Donsker's *invariance principle* [2]. The purpose of the present paper is to investigate this question.

In Section 2, it will be shown that the transition probability function of the Markov process $\{\xi^{-1}Y_{\xi t}\}$ converges to a limit as $\xi \to \infty$; the limit is explicitly obtained. It follows easily that the finite-dimensional distributions of the process also converge to ascertainable limits. In Section 3 it is then shown that the limiting distribution of any suitably continuous path functional exists; in other words, that an invariance principle holds. The principle tool (in addition to the convergence of the finite dimensional distributions) is a general theorem of Skorohod [10] which is easily applied to the present situation.

The next two parts of the paper study the limits, and show that they can be identified with secondary processes derived from some well-known objects. For example, let $\{x_t\}$ be a one-dimensional Wiener (Brownian motion) process with $x_0 = 0$ and define

$$y_t = t - \max\{\tau \leqq t \mid x_\tau = 0\}. \tag{1.4}$$

The process $\{y_t\}$ thus obtained is the limit process to which $\{\xi^{-1}Y_{\xi t}\}$ converges in the case $\alpha = \frac{1}{2}$. This fact can be generalized to include other values of α by con-

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sidering symmetric stable processes and their "subordinators" (Section 4), or by considering a certain class of diffusion processes (Section 5). Some corollaries are pointed out in the concluding Section 6.

2. Finite-dimensional distributions. We first record the fact that

$$(2.1)\qquad \lim_{\xi\to\infty} \Pr\left(\frac{Y_{\xi t}}{\xi} \leqq x\right) = F_{1-\alpha}\left(\frac{x}{t}\right) = \frac{\sin \pi\alpha}{\pi} \int_0^{\min(1,x/t)} u^{-\alpha}(1-u)^{\alpha-1}\,du$$

provided (1.2) holds [3]. The distributions $F_\alpha(\ )$, $0 < \alpha < 1$, are called "generalized arc-sine laws" and were discovered by E. S. Andersen in connection with quite different problems in fluctuation of sums of random variables. The next step will be to study the transition probabilities of the process $\{Y_t\}$:

THEOREM 2.1. *If the distribution of X_i satisfies* (1.2) *and if $y > 0$, then*

$$(2.2)\qquad \begin{aligned} \lim_{\xi\to\infty} \Pr\left(\frac{Y_{\xi t}}{\xi} \leqq x \,\middle|\, \frac{Y_0}{\xi} = y\right) &= \left(\frac{y}{t+y}\right)^{\alpha} H(x-y-t) \\ &\quad + \frac{\alpha t}{y}\int_0^1 F_{1-\alpha}\left[\frac{x}{t(1-u)}\right]\left(\frac{y}{y+ut}\right)^{1+\alpha} du = p_t^{(\alpha)}(y, x). \end{aligned}$$

($H(u) = 0$ if $u < 0$, 1 if $u \geqq 0$, If $y = 0$, $p_t^{(\alpha)}(0, x)$ is of course defined by the right side of (2.1).)

PROOF. It is sufficient to consider the case $t = 1$, for (2.2) is obtained from that case by a change of variable. Our starting point will be the relation

$$(2.3)\qquad \begin{aligned} \Pr(Y_\xi \leqq x \mid Y_0 = y) &= \frac{1 - F(y+\xi)}{1 - F(y)} H(x - y - \xi) \\ &\quad + \int_{u=0}^{\xi} \Pr(Y_{\xi-u} \leqq x \mid Y_0 = 0)\, d\,\frac{F(y+u)}{1-F(y)}. \end{aligned}$$

This is clear since the first term represents the event that no partial sum falls between 0 and ξ, while the second term is of the familiar "renewal" type. If x and y are replaced by ξx and ξy, a change of variable is made in the integral, and (1.2) incorporated, (2.3) becomes

$$(2.4)\qquad \begin{aligned} \Pr\left(\frac{Y_\xi}{\xi} \leqq x \,\middle|\, \frac{Y_0}{\xi} = y\right) &= \left(\frac{y}{1+y}\right)^{\alpha} \frac{L[\xi(y+1)]}{L(\xi y)} H(x-y-1) \\ &\quad - \int_{u=0}^{1} \Pr\left(\frac{Y_{\xi(1-u)}}{\xi(1-u)} \leqq \frac{x}{1-u} \,\middle|\, Y_0 = 0\right) d\left(1 + \frac{u}{y}\right)^{-\alpha} \frac{L[\xi(y+u)]}{L(\xi y)}. \end{aligned}$$

Using (2.1) and the definition of a slowly varying function, we see that convergence of (2.4) to the limit in (2.2) is at least formally indicated.

To justify the convergence we use the following elementary

LEMMA. *If $g_n(x)$ converges uniformly in $[a, b]$ to a continuous limit $g(x)$, and if $F_n(x)$ are distribution functions converging to a distribution $F(x)$ at each continuity point, then*

$$(2.5)\qquad \lim_{n\to\infty} \int_a^b g_n(x)\,dF_n(x) = \int_a^b g(x)\,dF(x).$$

The integral in (2.4) satisfies all the assumptions of the lemma except possibly the uniform convergence of g_n to g; establishing this will complete the proof. Convergence of the integrand in (2.4) to $F_{1-\alpha}[x(1-u)^{-1}]$ is certainly uniform in a neighborhood of $u = 1$ for any $x > 0$, as the probabilities in question are unity when $x/(1-u) \geqq 1$. But the limit (2.1) holds uniformly in x for $t = 1$, simply because of monotonicity and the continuity of $F_{1-\alpha}$. It follows that

$$\left|\Pr\left(\frac{Y_{\xi(1-u)}}{\xi(1-u)} \leqq \frac{x}{1-u} \,\middle|\, Y_0 = 0\right) - F_{1-\alpha}\left(\frac{x}{1-u}\right)\right| \tag{2.6}$$

must be small uniformly in u if ξ is large and u is bounded from one. Hence convergence is uniform for $u \,\varepsilon\, [0, 1]$, and Theorem 2.1 is proved.

It is not surprising that the finite dimensional distributions of the process $\{\xi^{-1}Y_{\xi t}\}$ also converge as $\xi \to \infty$:

THEOREM 2.2. *If* (1.2) *holds and* $0 \leqq t_1 \leqq \cdots \leqq t_n$, *then*

$$\lim_{\xi\to\infty} \Pr\left(\frac{Y_{\xi t_1}}{\xi} \leqq x_1, \cdots, \frac{Y_{\xi t_n}}{\xi} \leqq x_n \mid Y_0 = 0\right)$$

exists and is obtained by iterating the transition function[2] $p_t^{(\alpha)}(y, x)$.

PROOF. In case $n = 2$, the statement of the theorem becomes

$$\begin{aligned} \lim_{\xi\to\infty} \int_0^{x_1} \Pr\left(\frac{Y_{\xi(t_2-t_1)}}{\xi} \leqq x_2 \,\middle|\, \frac{Y_0}{\xi} = y\right) d \Pr\left(\frac{Y_{\xi t}}{\xi} \leqq y \mid Y_0 = 0\right) \\ = \int_0^{x_1} p_{t_2-t_1}^{(\alpha)}(y, x_2)\, dp_{t_1}^{(\alpha)}(0, y). \end{aligned} \tag{2.7}$$

The justification for the interchange of limits is very similar to that in our earlier result, but we will use a slightly modified form of the previous lemma in which it is required only that $g_n \to g$ uniformly in any interval $[a + \epsilon, b]$, $\epsilon > 0$, provided that g_n are uniformly bounded and that $F(x)$ is continuous at a. We further note that even if $g(x)$ has a jump at a continuity point of $F(x)$, (2.5) remains valid. To apply the lemma in this form to (2.7), we only need to show that the limit (2.2) holds uniformly in y in any interval $0 < \epsilon \leqq y \leqq x_1$.

But one of the basic facts concerning slowly varying functions is that the convergence of $L(cx)/L(x)$ to one as $x \to \infty$ is uniform in c if c is bounded away from 0 and ∞ [5], so that the first term in (2.4) certainly converges uniformly for $y \,\varepsilon\, [\epsilon, x_1]$. The second term in (2.4) is almost as easy, for the uniform smallness of (2.6) for large ξ allows the integrand to be replaced by its limit with small error for any integrator, and an integration by parts then allows the uniform convergence property of slowly varying functions to be applied here also. This gives all that is needed for the case $n = 2$ of the theorem; the general case can be handled by an induction which will not be explicitly carried out.

3. The invariance principle. Let K be the space of all right-continuous functions on [0, 1] having left-hand limits everywhere, and left-continuous at one.

[2] The fact that this really is a transition function is a corollary of the theorem, so it is not necessary to verify it directly.

Notice that *if the distribution function* $F(x)$ *is continuous, for each* ξ *the random function* $\xi^{-1}Y_{\xi t}$ *defined by* (1.1) *belongs* a.s. to K; this is clear from the definition. If $F(x)$ is not continuous, $\xi^{-1}Y_{\xi t}$ may not be a.s. in K, since, for certain values of ξ, $S_n = \xi$ has positive probability. However, this is very improbable if ξ is large, and it is easily seen that this possibility does not at all effect the general validity of the results below. We will use Skorohod's "J_1-topology" for the space K, according to which a sequence of functions $x_n(t) \in K$ converges to $x(t) \in K$ if and only if there exists a sequence of continuous, one-to-one functions $\lambda_n(t)$ mapping [0, 1] onto itself such that

$$(3.1) \qquad \lim_{n\to\infty} |x_n(t) - x(\lambda_n(t))| = 0, \qquad \lim_{n\to\infty} |\lambda_n(t) - t| = 0$$

where both limits are uniform in t [10]. Our main result is

THEOREM 3.1. *For each* $\alpha \in (0, 1)$, *there exists a Markov process* $\{y_t^{(\alpha)}\}$ *with* $y_0^{(\alpha)} = 0$, *path functions* a.s. *in* K, *and with* $p_t^{(\alpha)}(y, x)$ *as defined in Section* 2 *for its transition probability function. If* (1.1) *and* (1.2) *hold and if* $f(\ )$ *is a real functional defined on* K *and continuous in the* J_1 *topology almost everywhere with respect to the measure of the process* $y_t^{(\alpha)}$, *then*

$$(3.2) \qquad \lim_{\xi\to\infty} \Pr [f(\xi^{-1}Y_{\xi t}) \leqq x] = \Pr [f(y_t^{(\alpha)}) \leqq x]$$

for each x *at which the right-hand side is continuous.*

PROOF. The first part of the theorem can be obtained from general theories of the path-functions of Markov processes, but in the present case a simple direct proof is possible and yields more information about the paths. We begin by considering a process $\{y_t^{(\alpha)}\}$ with time parameter running over the rationals in [0, 1] and with $p_t^{(\alpha)}(y, x)$ for its transition probability function. Observe that for $s > 0$, $a \geqq 0$,

$$\Pr (a \leqq y_t^{(\alpha)} \leqq b \mid a + s \leqq y_{t+s}^{(\alpha)} \leqq b + s) = 1;$$

this is readily verified from the formula (2.2) for $p_t(x, y)$. It is a consequence that

$$\Pr (y_{t+s}^{(\alpha)} - y_t^{(\alpha)} = s \mid y_{t+s}^{(\alpha)} \geqq s) = 1.$$

Therefore if $y_t^{(\alpha)} = y$ is positive, as it a.s. is by (2.1), with probability one $y_{t-h}^{(\alpha)} = y - h$ for all rational h such that $y - h > 0$. A path function of $\{y_t^{(\alpha)}$, t rational$\}$ thus a.s. consists of a finite or countable collection of line segments with slope 1 and extending so that their left ends approach the t-axis.

We now define the random function $y_t^{(\alpha)}$ for all t from the version for rational t by

$$(3.3) \qquad y_t^{(\alpha)} = \lim_{r\to t+0} y_r^{(\alpha)}, \qquad r \text{ rational.}$$

From the results of the previous paragraph this amounts to "filling in" a dense collection of line segments, and to defining $y_t^{(\alpha)}$ to be 0 if t does not lie in the interior of one of the segments. In particular, except for a path set of probability zero the limit in (3.3) exists for all t and defines a function in K.

It remains to show that the process $\{y_t^{(\alpha)}\}$ (as now defined for all $t \in [0, 1]$) has finite-dimensional distribution functions given by iterating $p_t^{(\alpha)}(y, x)$. (This will imply that it is a Markov process.) From the nature of the paths it is clear that if $a - h \geqq 0$, $h > 0$, and $t - h \geqq 0$,

$$\Pr(a - h \leqq y_{t-h}^{(\alpha)} \leqq b - h) \geqq \Pr(a \leqq y_t^{(\alpha)} \leqq b) \geqq \Pr(a + h \leqq y_{t+h}^{(\alpha)} \leqq b + h);$$

in fact, the events of which we are considering the probabilities satisfy the corresponding inclusion relations. But if h is given values such that $t + h$ or $t - h$ is rational, the bounding probabilities are determined by $p_t^{(\alpha)}$ and approach the desired value as a limit when $h \to 0$. Thus for each t

$$\Pr(a \leqq y_t^{(\alpha)} \leqq b) = p_t^{(\alpha)}(0, b) - p_t^{(\alpha)}(0, a).$$

The argument applies without essential change to the joint probabilities, and this completes the first part of the proof.

The second part of the theorem is a simple consequence of a general theorem of Skorohod [10], according to which the conclusion follows provided that (i) the processes $\{\xi^{-1}Y_{\xi t}\}$ and $\{y_t^{(\alpha)}\}$ have (a.s.) paths in K, (ii) the finite-dimensional distributions converge, and (iii) that

$$\lim_{c\to 0} \limsup_{\xi\to\infty} \Pr(\Delta(c, \xi^{-1}Y_{\xi t}) > \epsilon) = 0 \qquad \text{for all } \epsilon > 0, \tag{3.4}$$

where

$$\Delta(c, y(t)) = \sup_{t-c<t_1\leqq t_2<t+c} \min(|y(t_1) - y(t)|, |y(t_2) - y(t)|). \tag{3.5}$$

It is now almost evident that these conditions hold, for (ii) was proved in Section 2, and (iii) is true because the paths of $\xi^{-1}Y_{\xi t}$ are "random saw tooth" functions for which it is easily seen that

$$\Delta(c, \xi^{-1}Y_{\xi t}) \leqq 2c$$

for all ξ. The second part of (i) was proved above, and the first part is automatic if $F(x)$ is continuous. Even if it is not there is no problem, for altering the paths of $\xi^{-1}Y_{\xi t}$ at $t = 1$ to make them belong to K affects only a vanishingly small set of paths for ξ large. This completes the proof of the theorem.

From Theorem 3.1 the *existence* of many limiting distributions for functionals of $\xi^{-1}Y_{\xi t}$ (and so indirectly for certain functionals of the sequence $\{S_n\}$) can be inferred. It is, however, more difficult to actually obtain the distributions; this can be attempted either by computations with a specific distribution $F(x)$ obeying (1.2) or by direct study of the process $\{y_t^{(\alpha)}\}$. Also of interest is the inverse problem, whereby a limit theorem for $\{\xi^{-1}Y_{\xi t}\}$ gives information about $\{y_t^{(\alpha)}\}$. As an example of this we have

THEOREM 3.2. *The processes* $\{y_t^{(\alpha)}\}$ *satisfy*

$$\Pr(\sup_{t\in[0,1]} y_t^{(\alpha)} \leqq x) = 1 - H_\alpha(1/x), \tag{3.6}$$

where the distributions H_α *are determined*[3] *by*

$$\int_0^\infty e^{-\lambda t}\, dH_\alpha(t) = \left[1 + \lambda \int_0^1 \xi^{-\alpha} e^{\lambda(1-\xi)}\, d\xi\right]^{-1}. \tag{3.7}$$

[3] The inversion of these transforms is discussed in [8].

PROOF. The functional $f(x_t) = \sup x_t$, $t \varepsilon [0, 1]$, is J_1-continuous at all points of K. It was proved in [8] that

$$\lim_{\xi\to\infty} \Pr\,(f(\xi^{-1}Y_{\xi t}) \leqq x) = 1 - H_\alpha(1/x)$$

provided Y_t is defined by (1.1) and (1.2) holds. (3.6) is therefore a consequence of Theorem 3.1 (specifically, of (3.2)).

4. Relation to stable processes. We shall first derive a connection between the processes $\{y_t^{(\alpha)}\}$, $0 < \alpha \leqq \frac{1}{2}$, and the one-dimensional symmetric stable processes.

THEOREM 4.1. *If $\{x_t\}$ is a Brownian motion process with $x_0 = 0$, then*

$$y_t = t - \sup\{\tau \leqq t \mid x_\tau = 0\} \tag{4.1}$$

defines the same stochastic process as $\{y_t^{(\frac{1}{2})}\}$. If $\{x_t\}$ is a separable symmetric stable process of index $\gamma \varepsilon (1, 2)$ with right-continuous paths, and again $x_0 = 0$, (4.1) defines a process equivalent to $\{y_t^{(1-(1/\gamma))}\}$.

PROOF. Whatever the nature of the zero-set of $\{x_t\}$, provided only that $t = 1$ is almost surely not an isolated zero, the function y_t belongs to K with probability one. Also in all cases, the strong Markov property for $\{y_t\}$ follows from that for $\{x_t\}$. It is only necessary to identify the transition probabilities with $p_t^{(\alpha)}$ to complete the proof.

In the Brownian motion case, the facts necessary to accomplish this are found in [9, ch. 6]. One of these is the statement that if $\{x_t\}$ is Brownian motion and $\{y_t\}$ defined by (4.1), then

$$\Pr\,(y_{t+s} \leqq x \mid y_s = 0) = F_{\frac{1}{2}}(x/t), \tag{4.2}$$

where $F_{\frac{1}{2}}$ is given in (2.1). Another useful fact is that in case L is the length of a zero-free interval for x_t, and $x_0 = 0$, then

$$\Pr\,(L \leqq x \mid L \geqq y > 0) = 1 - (y/x)^{\frac{1}{2}}, \qquad x \geqq y. \tag{4.3}$$

From these results the transition probabilities for $\{y_t\}$ are easily computed. Indeed, (4.2) is the transition probability for $\{y_t\}$ from the state 0, and equals $p_t^{(\frac{1}{2})}(0, x)$. When $y > 0$ we have by the method of "renewal at the next zero" that

$$\Pr\,(y_{t+s} \leqq x \mid y_s = y) = \left(\frac{y}{y+t}\right)^{\frac{1}{2}} H(x - y - t) + \int_0^t F_{\frac{1}{2}}\left(\frac{x}{t-u}\right) d\left[1 - \left(\frac{y}{y+u}\right)^{\frac{1}{2}}\right].$$

This is easily seen to be the same as $p_t^{(\frac{1}{2})}(y, x)$, given in (2.2). Thus when $\{x_t\}$ is Brownian motion, the process of (4.1) is $\{y_t^{(\frac{1}{2})}\}$.

The proof in the cases $0 < \alpha < \frac{1}{2}$ is of the same kind. If the stable process of index $\gamma \varepsilon (1, 2)$ replaces Brownian motion in (4.1), the analogues of the two results of Lévy used above are the following:

$$\Pr\,(y_{t+s} \leqq x \mid y_s = 0) = F_{1/\gamma}(x/t); \tag{4.4}$$

$$\Pr\,(L \leqq x \mid L \geqq y > 0) = 1 - (y/x)^{1-(1/\gamma)}, \qquad x \geqq y. \tag{4.5}$$

(It is of some interest to compare (4.4) with Theorem 8.3 of [7], which gives the same law as a limit distribution when y_t is defined with respect to a set rather than the single state 0.)

The transition probability of $\{y_t\}$ can be computed from these formulas in the manner illustrated above, and $p_t^{(\alpha)}(y, x)$ is the result if $\gamma = (1 - \alpha)^{-1}$. Equations (4.4) and (4.5) are quite easy consequences of the fact that

$$\begin{aligned} &\Pr\,(x_t = 0 \text{ for some } t \in [a, b] \mid x_0 = 0) \\ (4.6)\qquad &\qquad = \frac{\sin\,(\pi/\gamma)}{\pi} \int_0^{(b-a)/a} u^{(1/\gamma)-1}(1 + u)^{-1}\, du. \end{aligned}$$

This result is obtained in a paper by Blumenthal and Getoor [1], where it is derived using a theorem of Kac [4].[4] A "zero" of x_t is taken in [1] and [4] to mean either $x_t = 0$ or $\lim x_{t-0} = 0$, but it will be shown in an appendix that (4.6) remains correct if the strict interpretation of $x_t = 0$ is employed.

Another identification for $\{y_t^{(\alpha)}\}$, valid for all $\alpha \in (0, 1)$, can be made as follows.[5] Let $\{T(t)\}$ be a process with independent increments whose transition probabilities are determined by

$$(4.7)\qquad E(e^{-\lambda[T(s+t)-T(s)]}) = e^{-t\lambda^\beta}, \qquad 0 < \beta < 1.$$

These increments are a.s. positive, so it is possible to find a version of $\{T(t)\}$ which has right-continuous, monotonic sample functions; let $T(0) = 0$. This process is the *stable subordinator* of index β, so called because it can be used as a "random clock" to obtain stable processes from Brownian motion.

THEOREM 4.2. *Let $\{T(t)\}$ be the stable subordinator of index β and let*

$$(4.8)\qquad y(t) = t - \sup\,\{\tau \leqq t \mid T(s) = \tau \text{ for some } s\}.$$

Then $\{y(t)\}$ is the same as the process $\{y_t^{(\beta)}\}$.

PROOF. Again we need only compare the transition probabilities for the (Markov) process defined by (4.8) with $p_t^{(\alpha)}(y, x)$. It is shown in [1] that

$$(4.9)\qquad \Pr\,(T(s) \in [a, b] \text{ for some } s) = \frac{\sin \pi\beta}{\pi} \int_0^{(b-a)/a} u^{-\beta}(1 + u)^{-1}\, du.$$

From this it is easy to see that

$$(4.10)\qquad \Pr\,(y_t \leqq x) = F_{1-\beta}(x/t),$$

and it is also not difficult to obtain

$$(4.11)\qquad \Pr\,(y_{t+s} \leqq x + s \mid y_t = x) = 1 - [x/(x + s)]^\beta.$$

(The derivations are the same as those of (4.4) and (4.5) from (4.6).) These quantities are sufficient to determine the transition function, which is just that of $\{y_t^{(\beta)}\}$ as given in (2.2).

[4] The author is much indebted to Professor Getoor for sending him a copy of [1] before publication, and for pointing out the derivations, parallel to those in the Brownian motion case, of (4.4) and (4.5) from (4.6).

[5] This identification was suggested by R. K. Getoor.

5. Relation to diffusion processes. A unique diffusion process on the positive x-axis is defined by the initial condition $x_0 = 0$, the backward differential equation

$$u_t = (\gamma/x)u_x + \tfrac{1}{2}u_{xx}, \qquad -\tfrac{1}{2} < \gamma < \tfrac{1}{2}, \tag{5.1}$$

and the reflecting-barrier boundary condition

$$\lim_{x\to 0+} x^{2\gamma} u_x(t, x) = 0. \tag{5.2}$$

We shall also need to consider symmetrized versions of these processes on the whole real axis. If the infinitesimal generator is written in the intrinsic form $Af = D_v D_u f$, then the choices

$$u(x) = [2/(1 - 4\gamma^2)]|x|^{-2\gamma}x, \qquad v(x) = |x|^{2\gamma}x \tag{5.3}$$

define a diffusion which is regular on $(-\infty, \infty)$ if $-\frac{1}{2} < \gamma < \frac{1}{2}$, which is symmetric about $x = 0$ and whose backward equation reduces to (5.1) if $x \neq 0$; the absolute values of these processes are the one-sided diffusions defined above. The purpose of this section is to prove the following:

THEOREM 5.1. *If $\{x_t^{(\gamma)}\}$ denotes the above diffusion process, either one or two-sided, then*

$$y_t = t - \sup\{\tau \leqq t \mid x_\tau^{(\frac{1}{2}-\alpha)} = 0\} \tag{5.4}$$

is the same stochastic process as $\{y_t^{(\alpha)}\}$ for each $\alpha \in (0, 1)$.

REMARK. It has been suggested that this result might be deduced from Theorem 4.2 by using the theory of "local time" for diffusions as developed especially by K. Ito and H. P. McKean. The author is insufficiently familiar with this (as yet largely unpublished) theory to carry through the suggestion, and hopes that in any case the method of proof used below may itself have some interest.

PROOF OF THEOREM 5.1. First notice that in case $\alpha = \frac{1}{2}$ the theorem reduces to Theorem 4.1, because the diffusion $\{x_t^{(0)}\}$ is just the Wiener process or its absolute value. For clarity we shall give another proof in that case, and then explain how the new proof can be extended to the other cases in question. It is only necessary to consider the finite-dimensional distributions.

Let $\{z_i, i = 0, 1, \cdots\}$ be the random variables of the simple random walk process (steps ± 1, each with probability $\frac{1}{2}$) with $z_0 = 0$. Let X_i be the times between the $i - 1$st and the ith visits to the state 0. The distribution of the X_i is well known, and it is easily found that using them as the random variables in (1.1) leads to a process which converges to $\{y_t^{(\frac{1}{2})}\}$ by Theorem 3.1. On the other hand, the $\{z_i\}$ process itself converges to Brownian motion in the sense of Donsker's theorem [2]. Now consider the functionals

$$f(y_t) = y_{t_0}, \; g(x_t) = t_0 - \sup\{\tau \leqq t_0 \mid x_\tau = 0\}. \tag{5.5}$$

From our Theorem 3.1,

$$\lim_{\xi\to\infty} \Pr[f(\xi^{-1}Y_{\xi t}) \leqq x] = \Pr[f(y_t^{(\frac{1}{2})}) \leqq x]. \tag{5.6}$$

But if $x_t^{(n)}$ is defined as $z_i/n^{\frac{1}{2}}$ if $i/n = t$, and by linear interpolation between these points for other t, then by Donsker's invariance principle

$$\lim_{n\to\infty} \Pr [g(x_t^{(n)}) \leqq x] = \Pr [g(x_t) \leqq x] \tag{5.7}$$

where $\{x_t\}$ is Brownian motion. Now examining the definitions reveals that

$$g(x_t^{(n)}) = n^{-1}Y_{nt_0} = f(n^{-1}Y_{nt}).$$

Thus the left sides of (5.6) and (5.7) are the same, and we see from the right sides that the distribution of $y_{t_0}^{(\frac{1}{2})}$ is the same as that of y_{t_0} in (4.1) or (5.4) when $\{x_t\}$ is Brownian motion; this obviously remains true if $\{x_t\}$ is the absolute value of Brownian motion instead. The joint distributions can be handled in the same way by redefining, for instance,

$$f(y_t) = \sum_{i=0}^{k} a_i y_{t_i}, \; g(x_t) = \sum_{i=0}^{k} a_i[t_i - \sup \{\tau \leqq t_i \mid x_\tau = 0\}], \tag{5.5'}$$

where $a_0, \cdots, a_k$ are arbitrary constants.

To adapt this method of proof to the cases where $\alpha \neq \frac{1}{2}$, we shall replace the simple random walk process used above by a spatially inhomogeneous birth-and-death process $\{z_t^{(\gamma)}\}$, symmetric about 0. Such a process can be defined by specifying the birth and death rates β_n and δ_n in each state n; we choose $z_0 = 0$ and let

$$\begin{aligned} &\beta_n = (1 - 4\gamma^2)^{-1}[1 + (1/n)]^{2\gamma}, \qquad \delta_n = (1 - 4_\gamma^2)^{-1} \qquad \text{if } n > 0,\\ &\beta_0 = (1 - 2\gamma)^{-1} = \delta_0,\\ &\beta_n = (1 - 4\gamma^2)^{-1}, \quad \delta_n = (1 - 4\gamma^2)^{-1}(1 - 4\gamma^2)^{-1}[1 + 1/(|n|)]^{2\gamma} \quad \text{if } n < 0. \end{aligned} \tag{5.8}$$

(If $\gamma = 0$, this is the random walk used above except that the waiting times between transitions have become exponentially distributed random variables.) Let X_i denote the time between the beginnings of the $i - 1$st and the ith visits to state 0. The distribution of the X_i has been studied by Karlin and McGregor in [6], and their results tell us that this distribution satisfies (1.2) if $\gamma = \frac{1}{2} - \alpha$. Thus if again the function $f(\ )$ is defined by (5.5) and a process $\{Y_t\}$ by (1.1) with respect to the random variables X_i, we have

$$\lim_{\xi\to\infty} \Pr [f(\xi^{-1}Y_{\xi t}) \leqq x] = \Pr [f(y_t^{(\alpha)}) \leqq x] \tag{5.9}$$

provided $\gamma = \frac{1}{2} - \alpha$.

To obtain the convergence of $\{z_t^{(\gamma)}\}$ to the two-sided diffusion $\{x_t^{(\gamma)}\}$, we use in place of Donsker's theorem a recent result of C. Stone [11, Section 6]. We have chosen the processes $\{x_t^{(\gamma)}\}$ and $\{z_t^{(\gamma)}\}$ so that this theorem is applicable, but a slight modification is convenient. Let $z_t^{(\gamma)'}$ be a continuous function obtained from the step function $z_t^{(\gamma)}$ by joining the left end-points of the "steps" with straight line segments. Then by Stone's theorem

$$\lim_{\xi\to\infty} \Pr [g(\xi^{-\frac{1}{2}}z_{\xi t}^{(\gamma)'}) \leqq x] = \Pr [g(x_t^{(\gamma)}) \leqq x], \tag{5.10}$$

where $g(\ )$ is defined in (5.5). Now, as before, we note that

$$g(\xi^{-\frac{1}{2}} z_{\xi t}^{(\gamma)\prime}) = \xi^{-1} Y_{\xi t_0} = f(\xi^{-1} Y_{\xi t}),$$

and combining this with (5.9) and (5.10) we have shown

$$\Pr(y_{t_0}^{(\alpha)} \leqq x) = \Pr(y_{t_0} \leqq x),$$

where $\{y_t\}$ is defined by (5.4). The argument can be adapted to the finite-dimensional joint distributions by using (5.5′) instead of (5.5) as before.

One technical point deserves a further comment: the functionals f and g must possess suitable continuity properties in order for the invariance principles we have cited to be valid. There is no difficulty with f; g is also continuous under uniform approximation by continuous functions at a "point" (function) which changes sign in every neighborhood of its last zero prior to t_0. Almost all paths of the *two-sided* diffusion $\{x_t^{(\gamma)}\}$ do have this property, which is essential in justifying (5.10). However, it is apparently not possible to prove Theorem 5.1 in this way for the one-sided diffusions of (5.1) and (5.2) without the aid of the two-sided processes as intermediaries.

6. Corollaries. There are, of course, any number of specific limit theorems for positive random variables contained in our work above. The corollaries we shall point out here, however, are results about the diffusions and stable processes of Sections 4 and 5;[6] the invariance principle serves only as an agent in the proof. For instance we have an extension of Lévy's formulas for Brownian motion to the diffusions defined above:

THEOREM 6.1. *Let $\{x_t^{(\gamma)}\}$ be either the one or two-sided diffusion process of Section 5. Then*

$$\begin{aligned}\Pr(x_t^{(\gamma)} = 0 \text{ for some } t \in [a, b]) &= F_{\gamma+\frac{1}{2}}[(b-a)/b] \\ &= \frac{\cos \pi\gamma}{\pi} \int_0^{(b-a)/a} u^{\gamma-\frac{1}{2}}(1+u)^{-1}\, du. \end{aligned} \tag{6.1}$$

If L is the length of a zero-free interval for $x_t^{(\gamma)}$, then

$$\Pr(L \leqq x \mid L \geqq y > 0) = 1 - (y/x)^{\frac{1}{2}-\gamma}. \tag{6.2}$$

PROOF. The first equality of (6.1) follows from (2.1) and Theorem 5.1; the second is an easy calculation. Equation (6.2) can then be derived from (6.1) by the same argument which leads from (4.2) to (4.3) or from (4.6) to (4.5).

The other result we shall mention seems to be new even in the case of Brownian motion ($\alpha = \frac{1}{2}$), although there is no doubt an easier proof in that case.

THEOREM 6.2. *Let $Z^{(\alpha)}$ represent the length of the longest subinterval of* [0, 1] *during which $\{x_t\}$ has no zeros, where $\{x_t\}$ is either a symmetric stable process of index $(1-\alpha)^{-1}$ (here $\alpha \leqq \frac{1}{2}$) or else a diffusion satisfying* (5.1) *and* (5.2) *with $\gamma = \frac{1}{2} - \alpha$. Then*

$$\Pr(Z^{(\alpha)} \leqq x) = 1 - H_\alpha(1/x), \tag{6.3}$$

[6] Without explicit mention it will be taken for granted that the previous assumptions about the initial conditions, separability, etc. of these processes are still in force.

where H_α is defined by (3.7). *If $Z^{(\alpha)}$ is the longest subinterval of* [0, 1] *not intersecting the range of a stable subordinator $T(u)$ of index α,* (6.3) *again holds.*

PROOF. This follows immediately upon combining Theorem 3.2 with the various results of Sections 4 and 5.

APPENDIX

We will prove a lemma which allows certain results from [4] and [1] to be recast into the form which was needed and used in Section 4.

PROPOSITION. *Let $\{x_t\}$ be a process with stationary independent increments, where either the increments or x_0 have a continuous distribution function. Assume that $\{x_t\}$ is separable and has right-continuous paths. Then if*

$$E_y = \{\omega: \exists t > 0 \ni \lim_{\tau\to t-0} x_\tau(\omega) = y, x_t(\omega) \neq y\},$$

we have $\Pr(E_y) = 0$ *for each real number* y.

PROOF. We will actually work with sets $E_y(h, H)$ defined as above except that t is restricted to a finite interval $0 \leqq h < t < H < \infty$; clearly it is sufficient to show that $\Pr[E_y(h, H)] = 0$ for all h, H. It is known that under our assumptions the path functions of $\{x_t\}$ have (except for an ω set of probability 0) only jumps, and for $t < H$ only finitely many jumps of length exceeding $\epsilon > 0$. We can thus speak of the largest, second, etc. jumps in (h, H), using time of occurrence to complete the ordering in case of ties. Since $E_y(h, H)$ is the union of the events that the ith jump in (h, H) occurs "from y," we can see that $E_y(h, H)$ (and so E_y) is a measurable set.

We observe next that the set

$$A(h, H) = \{y: \Pr[E_y(h, H)] > 0\}$$

is countable. Indeed, the events

$$B_i(y) = \{\omega: i\text{th largest jump is "from } y\text{"}\}$$

are disjoint as y varies (for fixed i) and so for only a countable number of y can $\Pr[B_i(y)]$ exceed 0. But then

$$A(h, H) = \bigcup_{i=1}^{\infty} \{y: \Pr[B_i(y)] > 0\}$$

must also be countable.

Consider finally the set $A^* = A(0, H - h)$, defined in the case $x_0 = 0$. Because of the temporal and spatial homogeneity of the process $\{x_t\}$, if it is given that $x_h = u$ the resulting set $A_u(h, H)$ is A^* translated by u. Now for fixed y there can be only countably many values of u such that $y \in A^* + u$, so that unless u lies in a certain countable set, $\Pr[E_y(h, H) \mid x_h = u] = 0$. But

$$\Pr[E_y(h, H)] = \int_{-\infty}^{\infty} \Pr[E_y(h, H) \mid x_h = u] \, d \Pr(x_h \leqq u),$$

and from the assumptions the integrator assigns measure 0 to any countable set. Thus $\Pr[E_y(h, H)] = 0$ for any $0 < h < H < \infty$.

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