PROPERTIES OF GENERALIZED RAYLEIGH DISTRIBUTIONS

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1. Introduction. Some years ago we defined generalized Rayleigh processes [6], [7] and considered some of their many properties. Briefly, if x_i, $1 \leqq i \leqq n$, are Gaussian variates and r^2 is the sum of their squares, then we called r a generalized Rayleigh random variable. Besides the references in [6], [7] we have noted other investigations in this direction [2], [8]. In the present paper we wish to continue our study of Rayleigh distributions.

Our results exploit the methods employed in the theory of linear vector spaces and are of two types. The next three sections deal with explicit formulas; the last two sections with symbolic representations. In Section 2 we compute the joint p-dimensional Rayleigh distribution for a certain class of covariance matrices. The result is expressed in terms of modified Bessel functions of the first kind, [cf. (2.1)]. In Section 3 we compute the distribution of the inner product of two Gaussian vectors. The result is expressed in terms of a modified Bessel function of the second kind, [cf. (3.1)]. In Section 4 we compute the distribution of the difference of squares of norms of two Gaussian vectors. The result is expressed in terms of Whittaker functions, [cf. (4.1)]. Precise definitions and assumptions are made in the theorems leading to (2.1), (3.1) and (4.1).

The problem of computing the p-dimensional Rayleigh distribution in a useful form for arbitrary covariance matrices appears intractable. However we do obtain symbolic (operator) forms for the p-dimensional distribution for both the biased and unbiased cases, [cf. (5.1)]. These results seem to be of theoretical value in related investigations. In the final section, Section 6, we obtain a symbolic form for the density function of a Rayleigh variate when the variances of the Gaussian components are not necessarily equal, [cf. (6.1)].

We are indebted to the referee for pointing out certain additional references, as well as for making the observation that the theorems of Sections 2 and 3 could be approached by starting with the Wishart distribution.

2. The p-dimensional distribution. Let $Y_1, Y_2, \cdots, Y_n$ be p-dimensional column vectors which are independent and identically normally distributed with mean zero and covariance matrix M. Let X_k be an n-dimensional vector comprised of the kth components of the Y_j, $1 \leqq j \leqq n$. Then under certain restrictions on M we shall compute the joint p-dimensional distribution of the norms of the X_k, $1 \leqq k \leqq p$.

A word on notation: Primes will always denote transposes, that is, row vectors.

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Thus $X_k'X_{k'}$ is the inner product of X_k and $X_{k'}$, and $|X_k| = (X_k'X_k)^{\frac{1}{2}}$ is the norm of X_k.

If $n \geqq p$, then $A = \sum_{j=1}^n Y_jY_j'$ has a Wishart distribution ([1], page 157) and our problem can be formulated as finding the joint distribution of $v_1, v_2, \cdots, v_p$ where $v_k = a_{kk}^{\frac{1}{2}}$ and a_{kk}, $1 \leqq k \leqq p$, are the diagonal terms of A. However, we believe our approach below is simpler.

The precise theorem we shall prove is:

THEOREM. *Let $Y_1, Y_2, \cdots, Y_n$ be p-dimensional column vectors which are independent and identically normally distributed with mean zero and positive definite covariance matrix M. Let $W = M^{-1} = (w_{kk'})_{1 \leqq k,k' \leqq p}$ have the property that $w_{kk'} = 0$ for $|k - k'| > 1$. Let r_k be the norm of the n-dimensional vector X_k composed of the kth components of the Y_j. Let $R = \{r_1, r_2, \cdots, r_p\}$ be the p-dimensional vector of norms. Then the frequency function $g(R)$ of R is*

$$\begin{aligned} g(R) &= \frac{|W|^{n/2}}{2^{(n-2)/2}\Gamma(n/2)} r_1^{(n-2)/2} r_p^{n/2} \exp(-w_{pp}r_p^2/2) \\ &\times \prod_{k=1}^{p-1} [|w_{k,k+1}|^{-(n-2)/2} r_k \exp(-w_{kk}r_k^2/2) I_{(n-2)/2}(|w_{k,k+1}|r_kr_{k+1})], \qquad (2.1) \\ &\qquad r_k \geqq 0, 1 \leqq k \leqq p \\ g(R) &= 0, \qquad \textit{otherwise}. \end{aligned}$$

Before proving this theorem we would like to make two comments:

(i) The condition $w_{kk'} = 0$ for $|k - k'| > 1$ is not unreasonable. For example, it occurs in the important practical case where M is the Toeplitz matrix $(\lambda^{|k-k'|})_{1 \leqq k,k' \leqq p}$.

(ii) If $w_{k,k+1} = 0$ for some k, say $k = \alpha$, then the term $|w_{\alpha,\alpha+1}|^{-(n-2)/2} I_{(n-2)/2} \cdot (|w_{\alpha,\alpha+1}|r_\alpha r_{\alpha+1})$ in (2.1) will be replaced by

$$(r_\alpha r_{\alpha+1})^{(n-2)/2}/2^{(n-2)/2}\Gamma(n/2). \qquad (2.2)$$

We now consider the proof of our theorem.

PROOF. Let f_q be the q-dimensional normal frequency function. Then

$$\begin{aligned} f_{np}(Y_1, Y_2, \cdots, Y_n) &= \prod_{j=1}^n f_p(Y_j) \\ &= \prod_{j=1}^n [1/(2\pi)^{p/2}|M|^{\frac{1}{2}}] \exp(-\tfrac{1}{2}Y_j'WY_j) \qquad (2.3) \\ &= [1/(2\pi)^{np/2}|M|^{n/2}] \exp[-\tfrac{1}{2}\sum_{k=1}^p w_{kk}|X_k|^2 - \sum_{k=1}^{p-1} w_{k,k+1}X_k'X_{k+1}] \end{aligned}$$

since $w_{kk'} = 0$ for $|k - k'| > 1$.

Since $r_k = |X_k|$ we may write the marginal distribution $g(R)$ as

$$\begin{aligned} g(R) &= \int_{\substack{|X_k|=r_k \\ 1 \leqq k \leqq p}} f_{np}(Y_1, Y_2, \cdots, Y_n)\, d\sigma_1\, d\sigma_2 \cdots d\sigma_p \qquad (2.4) \\ &= \frac{|W|^{n/2}}{(2\pi)^{np/2}} \exp[-\tfrac{1}{2}\sum_{k=1}^p w_{kk}r_k^2][\prod_{k=1}^{p-1} Q_k] \int_{|X_1|=r_1} d\sigma_1 \end{aligned}$$

where

$$Q_k = \int_{|X_{k+1}|=r_{k+1}} \exp\left(-w_{k,k+1}X_k'X_{k+1}\right) d\sigma_{k+1}, \qquad 1 \leqq k \leqq p-1$$

and $d\sigma_k$, $1 \leqq k \leqq p$, is the element of surface area. (In the integral Q_k choose X_k as the polar axis.) Then

$$Q_k = [|w_{k,k+1}|r_k]^{-(n-2)/2}(2\pi)^{n/2} r_{k+1}^{n/2} I_{(n-2)/2}(|w_{k,k+1}|r_k r_{k+1}), \tag{2.5}$$

while

$$\int_{|X_1|=r_1} d\sigma_1 = \frac{2\pi^{n/2} r_1^{n-1}}{\Gamma(n/2)} \tag{2.6}$$

is just the surface area of an n-dimensional sphere of radius r_1. Substituting (2.5) and (2.6) in (2.4) leads to (2.1).

3. Distribution of the inner product. Let X and Y be Gaussian random vectors. Under certain restrictions we shall compute the probability density function of the inner product $X'Y$. This slightly generalizes a result of Wishart and Bartlett [11] obtained by a more complicated argument using characteristic functions. If $n = 1$, we have simply the distribution of the product of two Gaussian random variables.

THEOREM. *Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ be n-dimensional random vectors with means zero. Let the two-dimensional vectors $\{x_j, y_j\}$, $1 \leqq j \leqq n$, be independent and normally distributed with positive definite covariance matrix $M = (m_{kk'})_{1 \leqq k,k' \leqq 2}$ (independent of j). Let $z = X'Y$ be the inner product of X and Y. Then the one-dimensional frequency function $h(z)$ of z is*

$$h(z) = \frac{|z|^{(n-1)/2} e^{-w_{12}z}}{\pi^{1/2}\Gamma(n/2)2^{(n-1)/2}|M|^{\frac{1}{2}}(m_{11}m_{22})^{(n-1)/4}} K_{(n-1)/2}(|z|(w_{11}w_{22})^{\frac{1}{2}}) \tag{3.1}$$

where $W = M^{-1} = (w_{kk'})_{1 \leqq k,k' \leqq 2}$ and K_ν is the modified Bessel function of the second kind and order ν.

PROOF. Let $r = |X|$ and $s = |Y|$. Then $z = X'Y = rs \cos \phi_1$, $0 \leqq \phi_1 \leqq \pi$, where ϕ_1 is the angle between X and Y. Following the techniques used in [6] we may show that the joint distribution $g(r, s, \phi_1)$ of r, s, ϕ_1 is

$$g(r, s, \phi_1) = [1/2^{n-2}|M|^{n/2}\Gamma(n/2)\Gamma((n-1)/2)\Gamma(\tfrac{1}{2})](rs)(rs \sin \phi_1)^{n-2}$$
$$\times \exp\left[-\tfrac{1}{2}(w_{11}r^2 + w_{22}s^2) - w_{12}rs \cos \phi_1\right].$$

Now make the change of variable $z = rs \cos \phi_1$, $\xi = rs \sin \phi_1$, $\zeta = r$ with Jacobian $J = \zeta^{-1}(z^2 + \xi^2)^{-\frac{1}{2}}$. Then the marginal distribution of z is

$$h(z) = \int_0^\infty d\xi \int_0^\infty d\zeta J g(r, s, \phi_1) = [1/2^{n-2}|M|^{n/2}\Gamma(n/2)\Gamma((n-1)/2)\Gamma(\tfrac{1}{2})]e^{-w_{12}z}$$
$$\times \int_0^\infty d\xi \int_0^\infty d\zeta \xi^{n-2}\zeta^{-1} \exp(-w_{11}\zeta^2/2) \exp\left[-\tfrac{1}{2}w_{22}(z^2 + \xi^2)\zeta^{-2}\right].$$

Using a simple transformation of the Gamma function we arrive at

$$h(z) = \frac{e^{-w_{12}z}}{2^{(n-1)/2}|M|^{n/2}\Gamma(n/2)\Gamma(\frac{1}{2})w_{22}^{(n-1)/2}} \int_0^\infty \zeta^{n-2} \exp\left[-\tfrac{1}{2}(w_{11}\zeta^2 + w_{22}z^2\zeta^{-2})\right] d\zeta.$$

From [10, page 185] we infer

$$\int_0^\infty \zeta^{2\nu-1} \exp\left(-\alpha\zeta^2 - \beta/\zeta^2\right) d\zeta = \left(\frac{\beta}{\alpha}\right)^{\nu/2} K_\nu(2(\alpha\beta)^{\frac{1}{2}}), \qquad \alpha, \beta > 0.$$

Thus

$$h(z) = \frac{e^{-w_{12}z}|z|^{(n-1)/2}}{2^{(n-1)/2}|M|^{n/2}\Gamma(n/2)\Gamma(\frac{1}{2})(w_{11}w_{22})^{(n-1)/4}} K_{(n-1)/2}(|z|(w_{11}w_{22})^{\frac{1}{2}})$$

which immediately reduces to (3.1).

One could also approach the above problem by starting with the bivariate Wishart distribution; but we believe that our solution is easier to visualize and manipulate.

4. The difference of squares of norms. Let X_n and Y_m be normally distributed random vectors. Then under certain restrictions we shall compute the probability density function of $|X_n|^2 - |Y_m|^2$. This problem has been considered by Gurland [4].

THEOREM. *Let* $X_n = \{x_1, x_2, \cdots, x_n\}$ *and* $Y_m = \{y_1, y_2, \cdots, y_m\}$ *be independent and normal random vectors with means zero. Let* Var $x_j = \Phi$, $1 \leqq j \leqq n$; Var $y_k = \Psi$, $1 \leqq k \leqq m$. *Then the frequency function of* $t = |X_n|^2 - |Y_m|^2$ *is*

$$\phi(t) = \frac{|t|^{(n+m-4)/4} \exp\left[-\frac{1}{4}t(1/\Phi - 1/\Psi)\right]\Phi^{m/4}\Psi^{n/4}}{(2\Phi)^{n/4}(2\Psi)^{m/4}(\Phi + \Psi)^{(n+m)/4}[\Gamma(n/2)\Gamma(m/2)]^{\frac{1}{2}}} \times \left[\frac{\Gamma(m/2)}{\Gamma(n/2)}\right]^{\pm\frac{1}{2}} W_{\pm\mu,\nu}(\tfrac{1}{2}|t|(1/\Phi + 1/\Psi)) \tag{4.1}$$

where $\mu = (n - m)/4$, $\nu = (n + m - 2)/4$ *and* $W_{\mu,\nu}$ *is the Whittaker function. The plus sign is taken if* $t > 0$ *and the negative sign if* $t < 0$.

PROOF. The density function of $u = |X_n|^2$ is

$$h(u) = [1/(2\Phi)^{n/2}\Gamma(n/2)]u^{(n-2)/2}e^{-u/2\Phi}, \qquad u \geqq 0 \tag{4.2}$$

and of $v = |Y_m|^2$,

$$k(v) = [1/(2\Psi)^{m/2}\Gamma(m/2)]v^{(m-2)/2}e^{-v/2\Psi}, \qquad v \geqq 0. \tag{4.3}$$

Thus the first order probability density function $\phi(t)$ of t may be written $\phi(t) = \int_0^\infty h(u)k(u - t)\, du$, $t < 0$, and $\phi(t) = \int_0^\infty h(v + t)k(v)\, dv$, $t > 0$. Substituting (4.2) and (4.3) in the above formulas and recalling the definition of the Whittaker function, ([5], page 90) leads to (4.1).

For certain special values of m and n, for example, n and m both even, (4.1) reduces to an elementary function. If $n = m$, then $\phi(t)$ is essentially the modified Bessel function of the second kind and order $(n - 1)/2$.

5. The p-dimensional biased distribution. In these last two sections we shall show that if one extends the domain and range of functions to include the space of operators on ordinary functions then certain rather intractable problems can be solved in an elegant form. Our approach is to start with some simple operation, A, (for example, a differential operator) and consider a function to be "known" if it can be expressed in the form $f(A)g$ where g is an ordinary function and $f(A)$ can be "easily" computed (for example, as a power series). We refer to $f(A)g$ as a "symbolic expression"; and the main problem is to obtain f, A and g in as simple a form as possible.

THEOREM. *Let $Y_1, Y_2, \cdots, Y_n$ be p-dimensional random vectors, independent and normally distributed with mean vector $EY_j = A_j$, $1 \leqq j \leqq n$, and common covariance matrix M. Let M be positive definite and set $W = M^{-1} = (w_{kk'})_{1 \leqq k,k' \leqq p}$. Let $r_k = |X_k|$ be the norm of X_k, $1 \leqq k \leqq p$, where X_k is the n-dimensional vector composed of the kth components of the Y_j. Let $R = \{r_1, r_2, \cdots, r_p\}$ be the p-dimensional vector of norms. Then a symbolic expression for the frequency function $g(R)$ of R is*

$$g(R) = \exp\left[-\tfrac{1}{2}\sum_{j=1}^{n} A_j'WA_j\right]\left\{\prod_{k=1}^{p} r_k^{n/2} D_k^{-(n-2)/4} I_{(n-2)/2}(2r_k D_k^{\frac{1}{2}})\right\} h(T)\,|_{T=0} \tag{5.1}$$

where

$$\begin{aligned} h(T) &= 2^p|2M + T|^{-n/2} \exp\left[\tfrac{1}{2}\sum_{k=1}^{n} A_k'(2M + T)^{-1}TWA_k\right] \\ T &= (t_k\delta_{kk'})_{1 \leqq k,k' \leqq p}, \\ D_k &= \partial/\partial t_k, \qquad\qquad 1 \leqq k \leqq p, \end{aligned} \tag{5.2}$$

and I_ν is the modified Bessel function of the first kind and order ν.

PROOF. Using the notation of Section 2,

$$\begin{aligned} g(R) &= \frac{|W|^{n/2}}{(2\pi)^{np/2}} \cdot \int_{\substack{|X_k| = r_k \\ 1 \leqq k \leqq p}} \exp\left[-\tfrac{1}{2}\sum_{j=1}^{n} (Y_j - A_j)'W(Y_j - A_j)\right] d\sigma_1 \cdots d\sigma_p \\ &= \frac{|W|^{n/2}\prod_{j=1}^{p} r_j^{n-1}}{(2\pi)^{np/2}} \cdot \int_{\substack{|X_k| = 1 \\ 1 \leqq k \leqq p}} \exp\left[-\tfrac{1}{2}\sum_{k,j=1}^{p} w_{kj}(r_kX_k - B_k)'(r_jX_j - B_j)\right] d\sigma_1 \cdots d\sigma_p \end{aligned} \tag{5.3}$$

where B_k is the mean of r_kX_k, $1 \leqq k \leqq p$, and we have normalized the X_k vectors.

We assert that the integral in (5.3) is a function only of $r_1^2, r_2^2, \cdots, r_p^2$.

Assume this for the moment. Then (5.3) has a series expansion of the form

$$(5.4)\quad g(R) = \frac{|W|^{n/2}\exp\left[-\frac{1}{2}\sum_{j=1}^{n} A_j'WA_j\right]}{(2\pi)^{np/2}} \sum_{k_1,\cdots,k_p=0}^{\infty} \frac{b_{k_1,\cdots,k_p}}{k_1!\cdots k_p!}\prod_{j=1}^{p} r_j^{2k_j+n-1}$$

where the b's do not depend on the r's. If $\operatorname{Re}[s_j] > 0$, $1 \leqq j \leqq p$, then from (5.4)

$$(5.5)\quad \begin{aligned} &\int_0^\infty dr_1 \cdots \int_0^\infty dr_p\, g(R)\exp\left[-\sum_{j=1}^{p} s_j r_j^2\right] \\ &\quad = \frac{|W|^{n/2}\exp\left[-\frac{1}{2}\sum_{j=1}^{n} A_j'WA_j\right]}{2^p(2\pi)^{np/2}|S|^{n/2}} \sum_{k_1,\cdots,k_p=0}^{\infty} b_{k_1,\cdots,k_p}\prod_{j=1}^{p}\frac{\Gamma(k_j+n/2)}{k_j!\,s_j^{k_j}} \end{aligned}$$

where $S = (s_k\delta_{kk'})_{1\leqq k,k'\leqq p}$. Also

$$(5.6)\quad \begin{aligned} \int_0^\infty dr_1 \cdots \int_0^\infty dr_p\, g(R)\exp\left[-\sum_{j=1}^{p} s_j r_j^2\right] &= E\exp\left[-\sum_{j=1}^{p} s_j r_j^2\right] \\ &= E\exp\left[-\sum_{k=1}^{n} Y_k'SY_k\right] = \prod_{k=1}^{n} E\exp\left[-Y_k'SY_k\right] \\ &= |S|^{-n/2}|2M + S^{-1}|^{-n/2} \\ &\quad \cdot\exp\left[-\frac{1}{2}\sum_{j=1}^{n} A_j'WA_j + \frac{1}{2}\sum_{k=1}^{n} A_k'(2M+S^{-1})^{-1}S^{-1}WA_k\right]. \end{aligned}$$

Equating coefficients of $s_1^{-k_1}\cdots s_p^{-k_p}$ in the expansions of (5.5) and (5.6) we obtain

$$(5.7)\quad \frac{|W|^{n/2}}{(2\pi)^{np/2}} b_{k_1,\cdots,k_p} = \left[\prod_{j=1}^{p}\Gamma\left(k_j+\frac{n}{2}\right)\right]^{-1} \frac{\partial^{k_1+\cdots+k_p}}{\partial t_1^{k_1}\cdots\partial t_p^{k_p}} h(T)\bigg|_{T=0}$$

where $h(T)$ is defined by (5.2). Substituting (5.7) into (5.4) we obtain the expanded form of the symbolic expression (5.1).

We now prove the assertion concerning the form of the integral in (5.3). Let $1 \leqq \beta \leqq p$. We shall show that the integral depends on r_β through r_β^2 only. Expanding out the sum in the exponential of the integrand we obtain

$$\begin{aligned} &\sum_{k,j=1}^{p} w_{kj}(r_kX_k - B_k)'(r_jX_j - B_j) \\ &\quad = w_{\beta\beta}r_\beta^2 + 2r_\beta X_\beta'\left[-w_{\beta\beta}B_\beta + \sum_{\substack{j=1\\ j\neq\beta}}^{p} w_{j\beta}(r_jX_j - B_j)\right] + U_\beta \end{aligned}$$

where U_β does not involve r_β or X_β. But if C is any n-dimensional vector which is not a function of r_β or X_β, then

$$\int_{|X_\beta|=1} \exp(r_\beta X_\beta'C)\,d\sigma_\beta = \frac{(2\pi r_\beta)^{n/2}|C|^{-(n-2)/2}}{r_\beta^{n-1}} I_{(n-2)/2}(r_\beta|C|)$$

which is a function of r_β^2 alone. This proves the assertion.

The unbiased case is simply obtained by letting $A_j = 0$, $1 \leqq j \leqq n$, in (5.1).

6. Weighted Rayleigh distribution. The variate r has a weighted Rayleigh distribution if $r = |X|$ where $X = \{x_1, x_2, \cdots, x_n\}$ has an n-dimensional normal distribution with mean zero and covariance matrix $M = (m_k^{-2}\delta_{kk'})_{1 \leq k,k' \leq n}$. This problem has been discussed by Gurland [4]. Techniques for the numerical evaluation of the density function and the distribution function of r are discussed in [3]. We now prove the following theorem:

THEOREM. *A symbolic expression for the frequency function $p(r)$ of r is*

$$p(r) = 2^{-(n-2)/2} r^{n/2} |W|^{\frac{1}{2}} D^{-(n-2)/4} I_{(n-2)/2}(2r\, D^{\frac{1}{2}}) h(t) \big|_{t=0} \tag{6.1}$$

where $W = M^{-1}$, $D = d/dt$, $h(t) = |E + (t/2)W|^{-\frac{1}{2}}$ and E is the $n \times n$ identity matrix.

PROOF. By definition

$$p(r) = \int_{|X|=r} \frac{|W|^{\frac{1}{2}}}{(2\pi)^{n/2}} e^{-\frac{1}{2}X'WX}\, d\sigma$$

and following the development in Section 5 we may also write

$$p(r) = \sum_{k=0}^{\infty} (a_k/k!) r^{2k+n-1}. \tag{6.2}$$

Hence for Re $[s] > 0$,

$$Ee^{-sr^2} = \int_0^\infty e^{-sr^2} p(r)\, dr = \sum_{k=0}^{\infty} \frac{a_k \Gamma(k + n/2) s^{-k-n/2}}{2k!}. \tag{6.3}$$

Also

$$\begin{aligned} Ee^{-sr^2} &= E \exp\left[-s\sum_{k=1}^{n} x_k^2\right] = \prod_{k=1}^{n} Ee^{-sx_k^2} \\ &= \prod_{k=1}^{n} \frac{m_k}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-sx^2} e^{-m_k^2 x^2/2}\, dx \\ &= (2s)^{-n/2} |W|^{\frac{1}{2}} |E + (2s)^{-1}W|^{-\frac{1}{2}}. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4) we find that

$$a_k = 2^{-n/2+1}[\Gamma(k + n/2)]^{-1} |W|^{\frac{1}{2}} (d^k/dt^k) |E + (t/2)W|^{-\frac{1}{2}} \big|_{t=0}. \tag{6.5}$$

If we substitute (6.5) in (6.2) we obtain the expanded form of (6.1).

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