THE ASYMPTOTIC NORMALITY OF TWO TEST STATISTICS ASSOCIATED WITH THE TWO-SAMPLE PROBLEM[1]

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0. Summary. In this paper we prove the asymptotic normality of two statistics which have been proposed to test the hypothesis that two samples come from the same parent population. One statistic is the number of runs of X's and Y's in the combined sample of X's and Y's; the other is the sum of squares of "S_i's" where S_i is the number of X's falling between the ith and $(i-1)$st largest Y's. Both statistics have been studied previously, both lead to consistent tests, and both were known to be asymptotically normal under the null distribution. Here we prove limiting normality under a fairly wide class of alternatives. By means of limiting power against a sequence of alternatives which approach the null hypothesis, we compare these tests with one another and with the Smirnov test based on the sample c.d.f.'s. Against a rather large class of alternatives, the Smirnov test is seen to be considerably more powerful. The method of proving limiting normality used here is based on studying conditional moments and can be used to prove limiting normality of "combinatorial" statistics other than the ones studied herein.

1. Introduction. The purpose of this paper is to demonstrate the asymptotic normality of certain statistics which have been proposed for testing the "two sample" problem. Chief among these are the Wald-Wolfowitz run statistic which has been studied extensively in [13] and [17], and a statistic studied by Dixon [6] and by Blum and Weiss [1]. Since previous proofs of normality under the null hypothesis exist (Wald and Wolfowitz [13], Blumenthal [2]), the main contribution here is the proof of normality under a fairly wide class of alternative distributions. Using this result power can be computed for the tests in question. A comparison of limiting powers for these tests is made in Section 7. It is shown there that for a large class of alternatives, these tests have a limiting efficiency of zero when compared to the Smirnov test based on the sample distribution functions. In view of similar results by Cibisov [3] for the goodness of fit problem, this is not surprising. We believe also that the remarks of Weiss [16] regarding the behaviour against different alternatives hold here also.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two sets of independent random variables, the first set with common c.d.f. $F(x)$ and the second set with common c.d.f. $G(x)$. We assume that both $F(x)$ and $G(x)$ are absolutely continuous, and have continuous differentiable density functions $f(x)$ and $g(x)$, respectively.

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We assume that $(m/n) = r + r_n$ where $n^{\frac{1}{2}} r_n \to 0$ as n increases. In the sequel, we treat m/n as a constant r without loss of generality.

Let $Z_0 = G^{-1}(0)$, $Z_{n+1} = G^{-1}(1)$, and $Z_1 < \cdots < Z_n$ be the values of the Y's arranged in increasing order. For each $i = 1, \cdots, n + 1$, let S_i be the number of X's which lie in the interval $[Z_{i-1}, Z_i]$. All the statistics to be considered can be expressed as functions of the S_i. Since the S_i are invariant under probability transformations, we shall assume hereafter that $f(x) = 1$ for $0 \leqq x \leqq 1$, that $G(x)$ assigns unit mass to $[0, 1]$, that $G^{-1}(0) = 0$, $G^{-1}(1) = 1$ and that $g(x)$ is bounded above and is positive on $(0, 1)$. This last assumption assures the uniqueness of the inverse $G^{-1}(x)$ for all x in $[0, 1]$.

We shall denote the difference, or sample spacing, $Z_i - Z_{i-1}$ by W_i, $i = 1, \cdots, n + 1$. The statistic proposed by Dixon is

$$V^2 = n^{-1} \sum_{i=1}^{n+1} S_i^2 = n^{-1} \sum_{i=1}^{n+1} S_i(S_i - 1) + m/n = n^{-1} \sum_{i=1}^{n+1} S_i(S_i - 1) + r.$$

In Section 3, we study the distributions of "combinatorial" statistics of the form

$$\frac{1}{n} \sum_{i=1}^{n+1} \binom{S_i}{k}.$$

Clearly, V^2 has the same limiting distribution as

$$r + \frac{2}{n} \sum_{i=1}^{n+1} \binom{S_i}{2}.$$

Further, it is obvious that as test statistics, V^2 and $n^{-1} \sum_{i=1}^{n+1} \binom{S_i}{2}$ will have the same properties. One could, in fact, consider the possibility of using $n^{-1} \sum_{i=1}^{n+1} \binom{S_i}{k}$ as a test statistic for k other than 2. Consistency or lack thereof can be established easily using the convergence theorem of Blum and Weiss [1], and power could be computed using the results of our Section 3. We see no point in doing this here since in [1], V^2 was shown to have better local power than these tests against a class of "linear" alternatives and in Section 7, V^2 itself is shown to have poor power against these same alternatives when compared to the Smirnov test. In Section 4, the limiting distribution of V^2 is written out explicitly.

We might note also that V^2 is not the locally most powerful rank test against these linear alternatives but only the locally most powerful "sample spacings" test. In [9a] Lehmann derived the locally best rank test for this case. The fact that this test is not a "sample spacings" test combined with the power result of Section 7 indicates that the class of "sample spacings" tests is too narrow.

The run test is studied in Section 5 and its relation to the quantities $S_1, \cdots, S_{n+1}$ is indicated there. This relation is exploited to prove the limiting normality of the run statistic by obtaining the limiting normality of a certain function of $S_1, \cdots, S_{n+1}$, namely $n^{-1}\sum_{i=1}^{n+1} \delta_0(S_i)$, where $\delta_0(x) = 1$ if $x = 0$ and 0 otherwise.

The methods of proof in Sections 3 and 5 are similar and are justified by the argument given in Section 2.

It should be mentioned that tests for the one-sample goodness of fit problem which are based on statistics analogous to the above mentioned ones were proposed and studied by David [5], Kitabatake [9] and Okamoto [10], [11]. In the one-sample case, the sample intervals are $[F_0^{-1}((i-1)/n), F_0^{-1}(i/n)]$ $(i = 1, \cdots, n)$ where $F_0(x)$ is the hypothesized distribution. The S_i are then the numbers of X's in these intervals (now fixed instead of being random). Because of the strong resemblance of the statistics, many of the computational schemes used by Kitabatake and Okamoto can be used for the two-sample case (see Sections 3 and 5).

2. Geneial approach. In both proofs of normality (Sections 3 and 5) a conditional method of moments is used to establish the asymptotic normality given the $Y_1, \cdots, Y_n$ of a function $H(S_1, \cdots, S_{n+1})$. (In Section 3, $H(S_1, \cdots, S_n)$ is $n^{-1}\sum_{i=1}^{n+1} S_i^2$, and in Section 5 it is $n^{-1}\sum_{i=1}^{n+1} \delta_0(S_i)$ where $\delta_0(x) = 1$ if $x = 0$ and 0 otherwise.) This normality will be shown to hold for almost every sample sequence $Y_1, Y_2, \cdots$. We now justify the particular method employed. Denote $H(S_1, \cdots, S_{n+1})$ by $H_n(S)$. Denote conditional expectation given $Y_1, \cdots, Y_n$ as $E_n(\cdot \mid Y)$. Our goal is to show that as n increases

$$E \exp [itn^{\frac{1}{2}}(H_n(S) - EH_n(S))] \to \exp(-t^2c/2). \tag{2.1}$$

We summarize our assumptions and result as

THEOREM 2.1. *If $H_n(S)$ and $E_n(H_n(S) \mid Y)$ are as given above, if $n^{\frac{1}{2}}[E_n(H_n(S) \mid Y) - EH_n(S)]$ considered as a function of $(Y_1, \cdots, Y_n)$ has a limiting non-degenerate Normal distribution, $N(0, c_1)$ and if*

$$\begin{aligned} n^{p/2}E_n\{[H_n(S) - E_n(H_n(S) \mid Y)]^p \mid Y\} & \\ \to (p-1)(p-3)\cdots 3.1c_2^{p/2} & \quad \textit{if } p \textit{ is even} \\ \to 0 & \quad \textit{if } p \textit{ is odd} \end{aligned} \tag{2.2}$$

with probability one, (where c_2 is some constant) then (2.1) is true with $c = c_1 + c_2$.

PROOF. We can rewrite the expectation in (2.1) as

$$\begin{aligned} E \exp \{itn^{\frac{1}{2}}[E_n(H_n(S) \mid Y) - EH_n(S)]\} & \\ \cdot E_n [\exp [itn^{\frac{1}{2}}(H_n(S) - E_n(H_n(S) \mid Y))] \mid Y]. & \end{aligned} \tag{2.3}$$

The normality proof then consists of showing that the random variable $E_n [\exp [itn^{\frac{1}{2}}(H_n(S) - E_n(H_n(S) \mid Y))] \mid Y]$ approaches $\exp(-t^2c_2/2)$ with probability one as n increases, where c_2 is the constant mentioned in the hypothesis, and of showing that $E \exp \{itn^{\frac{1}{2}}[E_n(H_n(S) \mid Y) - EH_n(S)]\}$ approaches $\exp(-t^2c_1/2)$ as n increases. This latter convergence follows easily from the Lévy uniqueness theorem for characteristic functions and from our normality assumption. Thus if we can show the above convergence with probability one, because of the boundedness in absolute value of the exponentials in the expectations in (2.3), it must be that the limit as n increases of (2.3) is

$$\text{(2.4)} \qquad \exp\,(-t^2c_2/2)\lim_{n\to\infty} E \exp\,\{itn^{\frac{1}{2}}[E_n(H_n(S)\mid Y) - EH_n(S)]\},$$

which is in turn (by the result noted above) $\exp\,[-(t^2/2)(c_1 + c_2)]$, and this is the desired result.

A series expansion (with error term) of the expression $E_n[\exp\,[itn^{\frac{1}{2}}(H_n(S) - E_n(H_n(S)\mid Y))]\mid Y]$ shows that the Condition (2.2) is sufficient to imply the desired convergence with probability one. This proves the theorem.

In the cases which we are studying, previous work (Weiss [15], Proschan and Pyke [12]) has established the limiting normality of $E_n(H_n(S)\mid Y)$. Thus we shall direct our efforts in Sections 3 and 5 to establishing the validity of (2.2).

3. Asymptotic normality of combinatorial statistics. In this section, we shall consider the limiting distributions of statistics of the form

$$\text{(3.1)} \qquad H_n^k(S) = \frac{1}{n}\sum_{i=1}^{n+1}\frac{S_i(S_i-1)\cdots(S_i-k+1)}{k!} = \frac{1}{n}\sum_{i=1}^{n+1}\binom{S_i}{k}.$$

$H_n^k(S)$ has the following interpretation: Consider all $\binom{m}{k}$ k-tuples $(X_{i_1}, \cdots, X_{i_k})$ $1 \leqq i_1 < \cdots < i_k \leqq m$ of the X's, $nH_n^k(S)$ is the number of these such that all of $X_{i_1}, \cdots, X_{i_k}$ fall in the same sample interval $[Z_{j-1}, Z_j]$, $j = 1, \cdots, n+1$. Although we shall carry out the details only for $k = 2$, it will be seen that the method will suffice for any k, and in fact will suffice to show the limiting joint normality of any finite set $(H_n^{k_1}(S), \cdots, H_n^{k_p}(S))$ of p of these quantities. Noting that $H_n^1(S) = n^{-1}\sum_{i=1}^{n+1} S = m/n = r$, it can then be seen that the result obtained for $H_n^k(S)$ implies the limiting normality of $n^{-1}\sum_{i=1}^{n+1} S_i^k$ since the latter is a linear combination of $H_n^p(S)$, $p \leqq k$. The same argument shows that finite collections of the form, $(n^{-1}\sum_{i=1}^{n+1} S_i^{k_1}, \cdots, n^{-1}\sum_{i=1}^{n+1} S_i^{k_p})$ have a limiting joint normal distribution.

For real numbers $x_1, \cdots, x_k$ such that $0 < x_i < 1$, $(i = 1, \cdots, k)$, we define

$$\text{(3.2)} \qquad \begin{aligned} t_k(x_1, \cdots, x_k) &= 1 \quad \text{if } x_1, \cdots, x_k \text{ fall in the same sample interval} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Note that implicitly $t_k(x_1, \cdots, x_k)$ is a function of $Y_1, \cdots, Y_n$ as well as of $x_1, \cdots, x_k$. Since the X's are independent, we have that $P[t_k(X_{i_1}, \cdots, X_{i_k}) = 1 \mid Y] = \sum_{i=1}^{n+1} W_i^k$ where W_i is the length of the ith sample interval (based on $Y_1, \cdots, Y_n$). Note that we can write

$$\text{(3.3)} \qquad H_n^k(S) = n^{-1}\sum t_k(X_{i_1}, \cdots, X_{i_k}).$$

The sum $\sum$ extends over all k-tuples $(i_1, \cdots, i_k)$ $1 \leqq i_1 < \cdots < i_k \leqq m$ unless otherwise stated. In the form (3.3), $H_n^k(S)$ looks deceptively like a "U statistic", which it is not in the strictest sense. Thus we cannot use the theorems for "U statistics" but must treat this separately. Note that

$$E_n(t_k(X_{i_1}, \cdots, X_{i_k}) \mid Y) = \sum_{i=1}^{n+1} W_i^k. \tag{3.4}$$

Thus we have that

$$\begin{aligned} E_n(H_n^k(S) \mid Y) &= n^{-1} \sum E_n(t(X_{i_1}, \cdots, X_{i_k}) \mid Y) = n^{-1} \sum \left(\sum_{i=1}^{n+1} W_i^k\right) \\ &= \binom{m}{k} n^{-1} \left(\sum_{i=1}^{n+1} W_i^k\right) = n^{k-1} \frac{r^k}{k!} \left(\sum_{i=1}^{n+1} W_i^k\right) + \delta_n, \end{aligned} \tag{3.5}$$

where $n^{\frac{1}{2}}\delta_n$ approaches 0 stochastically as n increases (see (3.21)). The limiting standard normality of

$$\frac{n^{\frac{1}{2}}\left(n^{k-1} \sum_{i=1}^{n+1} W_i^k - k! \int_0^1 g^{1-k}(x)\, dx\right)}{\left\{[(2k)! - 2k(k!)^2] \int_0^1 g^{1-2k}(x)\, dx - \left[(k-1)k! \int_0^1 g^{1-k}(x)\, dx\right]^2\right\}^{\frac{1}{2}}} \tag{3.6}$$

has been demonstrated by Weiss [15], and again by Proschan and Pyke [12].

In view of Theorem 2.1, it remains to study the conditional moments of

$$n^{\frac{1}{2}}\left[n^{-1} \sum \left(t_k(X_{i_1}, \cdots, X_{i_k}) - \sum_{i=1}^{n+1} W_i^k\right)\right] \tag{3.7}$$

in order to verify (2.2).

Counting the various terms involved in the moments becomes very complicated, and to avoid excessive notational troubles, we shall study in detail only the case $k = 2$.

We shall prove

THEOREM 3.1. *Let $t_2(X_i, X_j)$ be defined by* (3.2), *and $g(x)$ the density of the Y's be differentiable on* [0, 1], *then*

$$\begin{aligned} \lim_{n\to\infty} E_n &\left[\left\{\frac{1}{n^{\frac{1}{2}}} \sum \left(t_2(X_i, X_j) - \sum_{i=1}^{n+1} W_i^2\right)\right\}^p \Bigg| Y\right] \\ &= 0 \qquad\qquad p = 1, 3, 5 \cdots \\ &= [(p-1)(p-3) \cdots 3.1]c^{p/2} \qquad p = 2, 4 \cdots \end{aligned} \tag{3.8}$$

with probability one, where $\sum$ extends over all pairs $(i < j)$. The constant c is given by $c = r^2[\int_0^1 g^{-1}(x)\, dx + 6r \int_0^1 g^{-2}(x)\, dx - 4r(\int_0^1 g^{-1}(x)\, dx)^2]$.

PROOF. Our methods of counting in the proof of (3.8) are based on those used by Daniels [4], Hoeffding [8], and Okamoto [11], chiefly the last.

Let

$$\psi_2(X_i, X_j) = t_2(X_i, X_j) - \sum_{i=1}^{n+1} W_i^2. \tag{3.9}$$

We are studying

$$\mu_p = E_n[n^{-p/2}(\sum \psi_2(X_i, X_j))^p \mid Y]. \tag{3.10}$$

The right side of (3.10) can be partitioned into sums of products of conditional expectations because X_i's with different subscripts are independent. The details are exactly as in [11]. The conditional expectations are polynomials in $\sum_{i=1}^{n+1} W_i^k$. From a result due to Weiss [14], it follows that with probability one

$$\lim_{n\to\infty} n^{k-1} \sum_{i=1}^{n+1} W_i^k = \Gamma(k+1) \int_0^1 g^{1-k}(x)\, dx. \tag{3.11}$$

Using (3.11) we replace terms of the form $\sum_{i=1}^{n+1} W_i^k$ by the appropriate constants, getting new expressions which do not involve random variables and which are valid with probability one. Again following the format in [11], it is easy to see that for p odd, μ_p is $o(1)$ as n increases, and that for p even, the only terms making a contribution to μ_p are (w.p. 1) products with factors of the form $E_n[\psi_2(X_i, X_j)\psi_2(X_k, X_l)]$. In [11], further simplification resulted because it was necessary to have $i = k$ and $j = l$. Here both these factors, and factors where only one of the four equalities $i = k$, $i = l$, $j = k$, $j = l$ is satisfied contribute to μ_p. Use of (3.11) and straightforward counting techniques yield the conclusion of the theorem.

4. Asymptotic distribution of V^2 statistic. We can combine the results (3.5), (3.6), and Theorem 3.1 to infer the following

THEOREM 4.1. *Under the assumptions of Theorem* 3.1, *the distribution of*

$$\frac{n^{\frac{1}{2}}\left(n^{-1}\sum_{i=1}^{n+1}\binom{S_i}{2} - r^2\int_0^1 g^{-1}(x)\,dx\right)}{r\left[\int_0^1 g^{-1}(x)\,dx + 6r\int_0^1 g^{-2}(x)\,dx + 2r^2\int_0^1 g^{-3}(x)\,dx - r(r+4)\cdot\left(\int_0^1 g^{-1}(x)\,dx\right)^2\right]^{\frac{1}{2}}} \tag{4.1}$$

approaches the standard normal distribution as n increases.

Since

$$V^2 = r + 2\left(n^{-1}\sum_{i=1}^{n+1}\binom{S_i}{2}\right),$$

we can compute the power of tests based on V^2 using the theorem above. Note that we obtain a nontrivial approximation for this test only for alternatives of the form studied in Section 7. A test of the hypothesis that $G(x) = F(x)$ (the uniform distribution) $(0 \leqq x \leqq 1)$ based on V^2 would reject this hypothesis whenever V^2 exceeds $C_n(\alpha)$ where α is the desired level of significance. We shall use the following standard notation:

$$\Phi(y) = (2\pi)^{-\frac{1}{2}}\int_y^\infty e^{-(t^2/2)}\,dt \tag{4.2}$$

and $K(\alpha)$ is the number such that $\Phi(K(\alpha)) = \alpha$.

Then the above theorem shows that for large n, $C_n(\alpha)$ is approximately equal to

$$(r/n^{\frac{1}{2}})[n^{\frac{1}{2}}(1 + 2r) + 2(r + 1)K(\alpha)]. \tag{4.3}$$

5. Limiting conditional normality of the run statistic. In this section, we consider the limiting conditional distribution of

$$\begin{aligned} H_0(S_1, \cdots, S_{n+1}) \\ = (n+1)^{-1} \text{ times (the number of } S_1, \cdots, S_{n+1} \text{ which equal zero).} \end{aligned} \tag{5.1}$$

We shall abbreviate $H_0(S_1, \cdots, S_{n+1})$ by H_0. Denote the number of runs of X's and Y's in the combined ordered sample by U_n. It is easily seen that the number of runs of X's is the same as the number of cells containing at least one X, which is $(n + 1)(1 - H_0)$, and from the definition of U_n, we see that U_n differs from twice the number of runs of X's by at most one. Formally, we have

$$|(U_n/n) - 2((n+1)/n)(1 - H_0)| \leqq n^{-1}. \tag{5.2}$$

From (5.2), we see that if H_0 is asymptotically normal with mean μ and variance σ^2 ($n\sigma^2$ becoming infinite with increasing n), U_n/n will be asymptotically normal with mean $2(1 - \mu)$ and variance $4\sigma^2$. We shall now examine the distribution of H_0.

Since we have

$$P\{S_i = 0 \mid Y\} = (1 - W_i)^m \tag{5.3}$$

where W_i is the length of the ith spacing, it follows that

$$E_n(H_0 \mid Y) = (1/(n+1)) \sum_{i=1}^{n+1} (1 - W_i)^m. \tag{5.4}$$

Using the results of Proschan and Pyke [12], it can be shown that the asymptotic distribution of

$$\frac{n^{\frac{1}{2}}\left(\frac{1}{n}\sum_{i=1}^{n+1}(1 - W_i)^m - \int_0^1 \frac{g^2(x)}{r + g(x)}\,dx\right)}{\left[\int_0^1 \frac{g^2(x)}{2r + g(x)}\,dx + \int_0^1 \frac{rg(x)}{r + g(x)}\,dx - \int_0^1 \frac{g^2(x)}{r + g(x)}\,dx - 2r^2 \int_0^1 \frac{g^2(x)}{(r + g(x))^3} - r^4\left(\int_0^1 \frac{g(x)}{(r + g(x))^2}\right)^2\right]^{\frac{1}{2}}} \tag{5.5}$$

is the standard normal.

From Theorem 2.1 of Section 2, it follows that we need only consider the limiting behavior of the conditional moments of $n^{\frac{1}{2}}(H_0 - E_n(H_0 \mid Y))$. Using the computational scheme which Kitabatake [9] employed to solve the one-sample analogue of this problem, we shall prove the following

THEOREM 5.1. *Let H_0 be defined by* (5.1) *and let $g(x)$ be bounded on* [0, 1], *then*

$$\lim_{n\to\infty} E_n\{n^{l/2}(H_0 - E_n(H_0 \mid Y))^l \mid Y\} \tag{5.6}$$
$$= 0 \qquad \text{if } l = 1, 3, 5, \cdots$$
$$= (l-1)(l-3)\cdots 1C^{l/2} \qquad \text{if } l = 2, 4, 6, \cdots$$

with probability one. The constant C is given by

$$C = \int_0^1 \frac{g^2(x)}{r+g(x)}\,dx - \int_0^1 \frac{g^2(x)}{2r+g(x)}\,dx - r\left(\int_0^1 \frac{g^2(x)}{(r+g(x))^2}\,dx\right)^2. \tag{5.7}$$

PROOF. Let $J_0 = (n+1)H_0$. Let $V_i = 1$ if $S_i = 0$, and 0 otherwise. Then,

$$J_0 = \sum_{i=1}^{n+1} V_i, \tag{5.8}$$

and

$$E_n\{V_i \mid Y\} = (1 - W_i)^m. \tag{5.9}$$

Also,

$$E_n\{J_0^{(s)} \mid Y\} = \sum_{nP_s}(1 - W_{i_1} - \cdots - W_{i_s})^m \tag{5.10}$$

where

$$J_0^{(s)} = J_0(J_0-1)\cdots(J_0-s+1) \quad \text{if} \quad s > 0$$
$$J_0^{(0)} = 1$$

and $\sum_{nP_s}$ stands for summation over all permutations $(i_1, \cdots, i_s)$ of $(n+1)$ integers such that $1 \leqq i_j \leqq n+1$, $i_j \neq i_k$ if $j \neq k$ $(j, k = 1, 2, \cdots, n+1)$.

We note also that we can write (w.p. 1)

$$(1 - W_{i_1} - \cdots - W_{i_s})^m = \exp\left[-r\sum_{j=1}^{s}(nW_{ij})\right] \cdot\left[1 - \frac{r}{2n}\left(\sum_{j=1}^{s}(nW_{ij})\right)^2 + O(1/n^2)\right]. \tag{5.11}$$

Using these definitions and relations and applying the analytical method used by Kitabatake [9] to study $E_n\{n^{l+2/2}(J_0/(n+1) - E_n(H_0 \mid Y))^{l+2} \mid Y\}$, we can establish the following equations

$$E_n\{n^l(H_0 - E_n(H_0 \mid Y))^{2l} \mid Y\} = (2l-1)(2l-3)\cdots 5.3.1\cdot\left[\sum_{i=1}^{n+1} \cdot\frac{\exp(-rnW_i)}{n+1} - \sum_{i=1}^{n+1}\frac{\exp(-2rnW_i)}{n+1} - r\left(\sum_{i=1}^{n+1}\frac{(nW_i)\exp(-rnW_i)}{n+1}\right)^2\right]^l + O(n^{-1}) \tag{5.12}$$

and

$$E_n\{n^{l+\frac{1}{2}}(H_0 - E_n(H_0 \mid Y))^{2l+1} \mid Y\} = O(1/n^{\frac{3}{2}}).$$

Applying the Weiss convergence result, (3.11), we obtain (5.5).

6. Asymptotic distribution of the run statistic. Now we can put together the remark following (5.2) with the results of Theorem 2.1, Theorem 5.1 and Equation (5.5), to obtain explicitly the limiting distribution of U_n, the number of runs in the combined sample of X's and Y's.

THEOREM 6.1. *Under the assumptions of Theorem* 5.1 *the distribution of*

$$(6.1)\quad \frac{n^{\frac{1}{2}}\left(\frac{1}{n}U_n - 2\int_0^1 \frac{rg(x)}{r+g(x)}\,dx\right)}{2\left[\int_0^1 \frac{rg(x)}{r+g(x)}\,dx - 2r^2\int_0^1 \frac{g^2(x)}{(r+g(x))^3}\,dx - r\left(\int_0^1 \frac{g^2(x)}{(r+g(x))^2}\,dx\right)^2 - r^4\left(\int_0^1 \frac{g(x)}{(r+g(x))^2}\,dx\right)^2\right]^{\frac{1}{2}}}$$

approaches the standard normal as n increases.

Using the theorem, we can set up a test of the hypothesis that $G(x) = F(x)$ (the uniform distribution) based on U_n and having size of approximately α for large n. Letting $\Phi(v)$ and $K(\alpha)$ be defined by (4.2), the test based on U_n will reject the hypothesis of equality whenever U_n/n is less than

$$(6.2)\qquad (2r/(1+r))[1 - (K(\alpha)(1+r)^{\frac{1}{2}}/(1+r)n^{\frac{1}{2}})].$$

When the alternatives are of the type studied in Section 7, we can obtain nontrivial approximations to the power of these tests.

7. A comparison of limiting power. As an application of the results of Sections 4 and 6, we shall compute the limiting power of the V^2 test and the run test against sequences of alternatives approaching the uniform distribution, and we shall then compare these tests to the Smirnov test of the same hypothesis. We consider a sequence of densities $g_n(x)$ given by

$$(7.1)\qquad g_n(x) = 1 + (c/n^{\frac{1}{4}})h(x)$$

where $c > 0$ and we have

$$(7.2)\qquad \int_0^1 h(x)\,dx = 0; \qquad |h(x)| < B < \infty.$$

We define $K(\alpha)$ and $\Phi(v)$ as in (4.2).

We shall use the results of Noether (see Fraser [7], pp. 272–273) to compute limiting power and efficiency with respect to the sequence $g_n(x)$. Using Theorem 4.1 and the following remarks, it is easy to verify the conditions of Noether's Theorem for the V^2 test. It then follows that the limiting power of this test against the sequence (7.1) is given by

$$\Phi\left(K(\alpha) - (c^2r/(r+1)) \int_0^1 h^2(x)\, dx\right). \tag{7.3}$$

From Theorem 6.1 we can again verify the conditions of Noether's Theorem for the test based on U_n, and verify that the limiting power against the sequence of alternatives (7.1) is

$$\Phi\left(K(\alpha) - (c^2r/(r+1)^{3/2}) \int_0^1 h^2(x)\, dx\right). \tag{7.4}$$

It is then easily verified that the efficiency of the run test relative to the V^2 test is $1/(r+1)$. Thus as r, the ratio m/n, increases the relative efficiency of the run test decreases to zero.

It is more informative to use the above results to compare these tests to the Smirnov test instead of to one another. It is easily seen that if instead of the sequence (7.1), we had considered

$$g_n(x) = 1 + (c/n^{\frac{1}{2}})h(x) \tag{7.5}$$

the limiting power of either of these tests would equal the size of the test. It is simple to compute a lower bound for the power of the Smirnov test which rejects H_0 when $\sup_{0 \leqq x \leqq 1} |F_n(x) - G_m(x)|$ exceeds an appropriate constant, where F_n and G_m are the sample distribution functions. From the lower bound, it is easily shown that if c is sufficiently large, the Smirnov test will have limiting power exceeding its size against the sequence (7.5). Thus relative to the Smirnov test, the two tests studied here have efficiency of zero against (7.5). (This relation was pointed out by the referee.) This result is foreshadowed by the work of Cibisov [3] for the "one sample" or goodness of fit problem. It should also be remarked that this relation between the relative efficiencies could be reversed by considering alternative sequences similar to those mentioned by Weiss in his review [16] of the work in [3]. Thus the comparison of efficiency depends strongly on the sequence of alternatives used. Most investigators seem to consider the sequence (7.5) a reasonable one.

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