POISSON COUNTS FOR RANDOM SEQUENCES OF EVENTS

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1. Introduction. We shall be concerned in this paper with the properties of random sequences of events, such as the arrivals of customers at a queue. If T_n denotes the instant of the nth event, then any such sequence, occurring in the time interval $(0, \infty)$, can be identified with the sequence of random variables

(1) $$\mathfrak{J} = (T_1, T_2, \cdots)$$

satisfying

(2) $$0 < T_1 \leqq T_2 \leqq \cdots \leqq T_n \leqq \cdots.$$

We make without further comment the assumption that only finitely many events occur in any finite time, so that

(3) $$T_n \to \infty \qquad (n \to \infty).$$

There are a number of different ways of specifying the distributions of $\mathfrak{J}$, which are convenient in different contexts. In the theory of queues, for instance, it is usual to specify the distributions of the process $(t_1, t_2, \cdots)$, where

(4) $$t_n = T_n - T_{n-1}.$$

(Here and elsewhere, we make the notational convention that $T_0 = 0$.) The process $\mathfrak{J}$ is called a *renewal sequence* if the t_n are independent and identically distributed; if in addition the t_n have a negative exponential distribution, $\mathfrak{J}$ is a *Poisson sequence*.

Another way of describing $\mathfrak{J}$ is in terms of the *counts* of the sequence in successive intervals. Thus we consider intervals $(0, a]$, $(a, 2a]$, $\cdots$, and denote by $C_n = C_n(a)$ the number of events in the nth interval:

(5) $$C_n(a) = \text{number of } r \text{ with } (n-1)a < T_r \leqq na.$$

It is, however, clear that a knowledge of the distributions of the process $\{C_n(a)\}$ for any one value of a does not suffice to determine the distributions of $\mathfrak{J}$. Again, if $\mathfrak{J}$ is a renewal sequence, the structure of $\{C_n\}$ is, in general, exceedingly complex. For these and other reasons it seems that the count process $\{C_n\}$ is not well adapted to describe the sequence of events $\mathfrak{J}$.

Some of the disadvantages of the count process can be avoided by considering the counts of $\mathfrak{J}$ in intervals of unequal length, and this suggests considering the counts in intervals of random length. It will be shown in Section 7 that this apparently arbitrary procedure arises very naturally in the theory of queues, and it is suggested that the idea of a "randomized count process" may prove useful

Received March 26, 1963.

in the theoretical analysis of systems involving random sequences of events. We shall not, however, be concerned in this paper with the use of randomized count processes in problems of *inference*.

Suppose, then, that $\mathfrak{I}$ and $\mathfrak{I}'$ are two independent random sequences of events. We define the *count process* $\mathfrak{N} = \{N_n ; n = 1, 2, \cdots\}$ of $\mathfrak{I}$ *in* $\mathfrak{I}'$ by

$$N_n = \text{number of } r \text{ with } T'_{n-1} < T_r \leqq T'_n . \tag{6}$$

The *dual count process* $\mathfrak{N}^* = \{N_n^*\}$ is the count process of $\mathfrak{I}'$ in $\mathfrak{I}$, and so $\mathfrak{N}^{**} = \mathfrak{N}^*$.

Notice that the processes $\mathfrak{N}$ and $\mathfrak{N}^*$ are different ways of describing the relative configuration of the sequences of points T_n and T'_m on the line. Thus suppose that, for each m, n, $T_n \neq T'_m$, and let the points $T_1, T_2, \cdots, T'_1, T'_2, \cdots$ be written in ascending order as $\tau_1, \tau_2, \cdots$. Now define a sequence $\{\xi_n\}$ by letting $\xi_n = 0$ or 1 according as τ_n belongs to $\mathfrak{I}$ or to $\mathfrak{I}'$. Then the sequence $\{\xi_n\}$, or equivalently the binary decimal $\xi = 0.\xi_1\xi_2 \cdots$, determines uniquely the relative configuration of $\mathfrak{I}$ and $\mathfrak{I}'$, and the processes $\mathfrak{N}$ and $\mathfrak{N}^*$ can be read off from it. Moreover, the sequence $\{\xi_n\}$ can be constructed from a knowledge of $\mathfrak{N}$ (or of $\mathfrak{N}^*$). We shall not make explicit use of this device, but simply remark that, so long as, for each m, n, we have $T_n \neq T'_m$, the sequence $\mathfrak{N}$ determines the sequence $\mathfrak{N}^*$, and vice versa.

2. Poisson counts. The most interesting case of a randomized count process of the type considered in the previous section is certainly that in which $\mathfrak{I}'$ is a Poisson sequence. If $\mathfrak{I}'$ is a Poisson sequence of rate λ independent of $\mathfrak{I}$, then we call the count process of $\mathfrak{I}$ in $\mathfrak{I}'$ the *Poisson count process* (of rate λ) of $\mathfrak{I}$, and denote it by $\mathfrak{N}(\lambda) = \{N_n(\lambda)\}$, and its dual process by $\mathfrak{N}^*(\lambda) = \{N_n^*(\lambda)\}$. Then $\mathfrak{N}(\lambda)$ and $\mathfrak{N}^*(\lambda)$ can be regarded as randomized statistics of $\mathfrak{I}$.

For most of the sequel we shall be concerned with just one value λ_0 of λ, and we shall then fix the time scale so that $\lambda_0 = 1$. When this is done, $\mathfrak{N}(1)$ and $\mathfrak{N}^*(1)$ will be written simply as $\mathfrak{N}$ and $\mathfrak{N}^*$.

Because of the well-known fact that the numbers of events of a Poisson sequence in disjoint intervals are independent Poisson variables, we can readily calculate the distributions of the dual Poisson count process $\mathfrak{N}^*$. In fact, if $t_n = T_n - T_{n-1}$, then we have at once

$$\mathbf{P}\{N_j^* = k_j ; j = 1, 2, \cdots, n\} = \mathbf{E}\left\{\prod_{j=1}^{n} e^{-t_j} t_j^{k_j}/k_j\,!\right\}. \tag{7}$$

Since the T'_n have absolutely continuous distributions, there is zero probability that, for any m, n, $T_n = T'_m$. It follows that the sequence $\mathfrak{N}^*$ determines the sequence $\mathfrak{N}$ with probability one, so that (7) enables us, in principle, to compute the distributions of the Poisson count process $\mathfrak{N}$.

We illustrate this procedure by calculating the Poisson count process for the very important case of a renewal sequence. The result will be expressed in terms of "delayed recurrent events", for which we use the notation and terminology of Feller ([5], Section XIII, 5).

THEOREM 1. *Let $\mathfrak{I}$ be a renewal sequence with lifetime distribution function F, and write*

$$a_k = \int_0^\infty \frac{e^{-t}t^k}{k!}\, dF(t). \tag{8}$$

Then the Poisson count process $\mathfrak{N}$ (of rate 1) of $\mathfrak{I}$ is given by

$$N_n = \epsilon_n \nu_n, \tag{9}$$

where

(i) $\{\epsilon_n\}$ *and* $\{\nu_n\}$ *are independent,*

(ii) ϵ_n *takes the values* 0 *and* 1 *and is such that* $\{\epsilon_n = 1\}$ *is a delayed recurrent event with first occurrence probabilities* a_{n-1} $(n = 1, 2, \cdots)$ *and recurrence time distribution*

$$f_n = a_n/(1 - a_0) \qquad (n = 1, 2, \cdots), \tag{10}$$

and

(iii) *the* ν_n *are independent and identically distributed, with*

$$\mathbf{P}\{\nu_n = \nu\} = (1 - a_0)a_0^{\nu-1} \qquad (\nu = 1, 2, \cdots). \tag{11}$$

PROOF. Equation (7) for the dual count process $\mathfrak{N}^*$ shows that the N_n^* are independent, with common distribution $\mathbf{P}\{N_n^* = k\} = a_k$, $(k = 0, 1, 2, \cdots)$. Now let $q, r_1, \cdots, r_q, k_1, \cdots, k_q$ be any strictly positive integers, write $R(v) = r_1 + r_2 + \cdots + r_v$, $K(v) = k_1 + k_2 + \cdots + k_v$, and consider the probability $p(r_1, \cdots, r_q; k_1, \cdots, k_q)$ that $N_{R(v)} = k_v$, $(v = 1, 2, \cdots, q)$, and that all other N_n, for $n < R(q)$, are zero. Then it is clear from the definitions of N_n and N_n^* that this event differs by an event of zero probability from the event that $N_1^* = r_1 - 1$, $N_{K(v)+1}^* = r_{v+1}$, $(v = 1, 2, \cdots, q-1)$, $N_{K(q)+1}^* \geqq 1$, and that all other N_n^*, for $n \leqq K(q)$ are zero. Hence

$$p(r_1, \cdots, r_q; k_1, \cdots, k_q) = a_{r_1-1}a_{r_2} \cdots a_{r_q}(1 - a_0)a_0^{K(q)-q}. \tag{12}$$

Summing over the k_v, we see that the probability that $N_{R(v)} \neq 0$ for $v = 1, 2, \cdots, q$ and that $N_n = 0$ for all other $n < R(q)$ is

$$p(r_1, \cdots, r_q) = a_{r_1-1}f_{r_2} \cdots f_{r_q},$$

where the f_n are given by (10). It follows that the event $\{N_n \neq 0\}$ is a delayed recurrent event with first occurrence probabilities a_{n-1} and recurrence time distribution f_n. Thus, if $\epsilon_n = 0$ or 1 according as $N_n = 0$ or $N_n \neq 0$, then $\{\epsilon_n\}$ has the properties (ii). Furthermore, conditional on the occurrence of $\{\epsilon_n = 1\}$ at $n = R(v)$ $(v = 1, 2, \cdots, q)$, and at no other values of n in $n < R(q)$, Equation (12) shows that the distribution of the $N_{R(v)}$ is given by

$$\mathbf{P}\{N_{R(v)} = k_v; v = 1, 2, \cdots, q\} = (1 - a_0)^q a_0^{K(q)-q} = \prod_{v=1}^{q} [(1 - a_0)a_0^{k_v-1}].$$

This shows that the $N_{R(v)}$ are conditionally independent and geometrically distributed, and it follows at once that N_n can be written in the form (9), where the ν_n satisfy (i) and (iii). This completes the proof.

COROLLARY. *If* $\mathfrak{I}$ *is a Poisson process of rate* μ, *then the* N_n *are independent, with*

$$\mathbf{P}\{N_n = k\} = (1 - \beta)\beta^k, \qquad (k = 0, 1, 2, \cdots), \tag{13}$$

where $\beta = \mu(1 + \mu)^{-1}$.

It follows from (9) and (10) by the usual formulas of recurrent event theory that the probabilities $u_n = \mathbf{P}\{\epsilon_n = 1\}$ are given by

$$U(x) = \sum_{n=1}^{\infty} u_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n / 1 - \sum_{n=1}^{\infty} f_n x^n$$
$$= xA(x)(1 - a_0)/[1 - A(x)], \qquad (|x| < 1),$$

where

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \int_0^{\infty} e^{-(1-x)t}\, dF(t) = F^*(1 - x), \qquad \text{(say)}.$$

Hence, for $|x| < 1$, we have

$$U(x) = x\{1 - F^*(1)\}F^*(1 - x)/[1 - F^*(1 - x)], \tag{14}$$

which determines, in principle, the probabilities u_n. The distribution of N_n is then given by

$$p_{n,k} = \mathbf{P}\{N_n = k\} = 1 - u_n \qquad (k = 0) \tag{15}$$
$$= u_n(1 - a_0)a_0^{k-1} \qquad (k \geqq 1).$$

This may easily be thrown into the alternative form

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} p_{n,k}\, x^n y^k = \frac{x}{1 - x} - \frac{x(1 - y)\{1 - F^*(1)\}F^*(1 - x)}{\{1 - yF^*(1)\}\{1 - F^*(1 - x)\}}. \tag{16}$$

More calculations on these lines may be carried out without difficulty. However, the important point is that it is possible to specify, in fairly simple and explicit terms, the Poisson count process of any renewal sequence. This is to be contrasted with the much more complex problem of finding the joint distributions of the counts of a renewal sequence in successive intervals of equal lengths.

If, instead of being a renewal sequence, $\mathfrak{I}$ is a "modified renewal sequence" in which the distribution of t_1 is different from the common distribution of $t_2, t_3, \cdots$, then the analysis of Theorem 1 goes through, the only change being that the first occurrence probabilities become

$$\mathbf{P}\{\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{n-1} = 0, \epsilon_n = 1\} = \mathbf{E}\{t_1^{n-1}e^{-t_1}/(n - 1)!\}. \tag{17}$$

In particular, if $\mu = \mathbf{E}(t_2) < \infty$, and if the distribution of t_1 is chosen to make $\mathfrak{I}$ an "equilibrium renewal sequence" (i.e. if $\mathbf{P}\{x < t_1 \leqq x + dx\} = \mathbf{P}\{x < t\}\, dx/\mu$, c.f. Cox [2]), then (17) becomes, after a little calculation,

$$\mathbf{P}\{\epsilon_1 = \cdots = \epsilon_{n-1} = 0, \epsilon_n = 1\} = \sum_{r=n}^{\infty} f_r \Big/ \sum_{r=1}^{\infty} r f_r.$$

Hence, in this particular case, the values of n for which $\epsilon_n = 1$ form an equilibrium renewal process in discrete time, and so $\{\epsilon_n\}$ is stationary. It follows at once that the Poisson count process of an equilibrium renewal sequence is stationary.

3. Bulk renewal sequences. In our definition of a random sequence of events we have not required that the sequence $\{T_n\}$ be strictly increasing, so that the definition admits the possibility that several events may occur at the same instant of time. In the terminology of Khintchine [8], our sequence of events need not be "orderly", and for many purposes it is both necessary and convenient to consider such processes.

Let $\mathfrak{I}$ be any sequence of events, and let $\mathbf{m} = (m_1, m_2, \cdots)$ be any sequence of strictly positive integers. Then we define a new sequence $\mathfrak{I} \times \mathbf{m}$ by replacing the nth event of $\mathfrak{I}$ by a group of m_n simultaneous events. More formally, we define $\mathfrak{I} \times \mathbf{m} = \{\tilde{T}_n\}$, where

$$\tilde{T}_1 = \tilde{T}_2 = \cdots = \tilde{T}_{m_1} = T_1, \tilde{T}_{m_1+1} = \cdots = \tilde{T}_{m_1+m_2} = T_2, \cdots. \tag{18}$$

In general, both $\mathfrak{I}$ and $\mathbf{m}$ will be random, and then $\mathfrak{I} \times \mathbf{m}$ will be a random sequence of events.

In particular, suppose that $\mathfrak{I}$ is a renewal sequence, and that the m_n are independent of $\mathfrak{I}$ and of each other, and have the common distribution

$$b_k = \mathbf{P}\{m_n = k\}. \tag{19}$$

Then we say that $\mathfrak{I} \times \mathbf{m}$ is a *bulk renewal sequence*. If $\mathfrak{I}$ is a Poisson sequence, then $\mathfrak{I} \times \mathbf{m}$ is called a *bulk Poisson sequence*. The distribution $\{b_k\}$ will be called the *batch distribution*.

It will be clear that the Poisson count process of $\mathfrak{I} \times \mathbf{m}$ bears a simple relation to that of $\mathfrak{I}$. Thus the Poisson count process of a bulk renewal sequence can readily be computed, and we state the result below, omitting the proof.

THEOREM 2. *Let $\mathfrak{I}$ be a bulk renewal sequence with batch distribution $\{b_k\}$, and let a_k be defined by* (8) *from the lifetime distribution function F of the renewal sequence from which $\mathfrak{I}$ is derived. Then the Poisson count process $\mathfrak{N}$ of $\mathfrak{I}$ is given by*

$$N_n = \epsilon_n \nu_n, \tag{20}$$

where

(i) *$\{\epsilon_n\}$ and $\{\nu_n\}$ are independent,*

(ii) *ϵ_n takes the values 0 and 1 and is such that $\{\epsilon_n = 1\}$ is a delayed recurrent event with first occurrence probabilities a_{n-1} $(n = 1, 2, \cdots)$ and recurrence time distribution $f_n = a_n/(1 - a_0)$ $(n = 1, 2, \cdots)$, and*

(iii) *the ν_n are independent, with common distribution*

$$\mathbf{P}\{\nu_n = \nu\} = \sum_{k=1}^{\nu} (1 - a_0) a_0^{k-1} b_\nu^{(k)}, \tag{21}$$

where $\{b_n^{(k)}\}$ is the k-fold convolution of $\{b_n\}$ with itself.

COROLLARY. *If $\mathfrak{I}$ is a bulk Poisson sequence derived from a Poisson sequence of rate μ, then the N_n are independent, with common distribution*

$$\mathbf{P}\{N_n = \nu\} = (1-\beta)\sum_{k=0}^{\nu} \beta^k b_\nu^{(k)}, \tag{22}$$

where $\beta = \mu(1+\mu)^{-1}$, *and* $b_\nu^{(0)} = \delta_{\nu 0}$.

Thus the Poisson count process of a bulk renewal sequence has the same structure as that of a renewal sequence, except that the geometric distribution of ν_n is replaced by a more general distribution.

In view of the simple character of the Poisson count process of a bulk renewal sequence, it is natural to ask whether there are other sequences of events with similar properties. More particularly, are there sequences of events, other than bulk Poisson sequences, whose Poisson counts are independent and identically distributed? These questions lead naturally to a study of the extent to which the Poisson count process determines the stochastic structure of the sequence, and to such a study we turn in the next section.

4. The information contained in the Poisson count process. We recall that a knowledge of the joint distributions of the counts C_n of a sequence $\mathfrak{I}$ in successive intervals $(0, a]$, $(a, 2a]$, $\cdots$ is not sufficient to determine the distributions of $\mathfrak{I}$. The situation, however, is quite different for the Poisson count process, as the following theorem shows.

THEOREM 3. *Let* ${}^1\mathfrak{I}$ *and* ${}^2\mathfrak{I}$ *be random sequences of events, whose Poisson count processes* ${}^1\mathfrak{N}$ *and* ${}^2\mathfrak{N}$ *(for rate* $\lambda = 1$*) have the same joint distributions. Then the joint distributions of* ${}^1\mathfrak{I}$ *and* ${}^2\mathfrak{I}$ *coincide.*

PROOF. For $\alpha = 1, 2$, consider an event of the form

$${}^\alpha N_n^* = k_n, \quad (n = 1, 2, \cdots, \nu),$$

where $k_n \geqq 0$, $k_\nu \geqq 1$. As in the proof of Theorem 1, this differs by an event of zero probability from an event of the form

$${}^\alpha N_n = l_n \ (n = 1, 2, \cdots, \mu), \qquad {}^\alpha N_{\mu+1} \geqq 1,$$

where the numbers μ, l_n are determined by the numbers ν, k_n. Hence

$$\mathbf{P}\{{}^\alpha N_n^* = k_n ; n = 1, 2, \cdots, \nu\}$$

is independent of α, and hence so is

$$\mathbf{E}\left\{\prod_{n=1}^{\nu} e^{-{}^\alpha t_n}\, {}^\alpha t_n^{k_n}/k_n\,!\right\}.$$

This holds for all $\nu \geqq 1$, $k_1, \cdots, k_{\nu-1} \geqq 0$, $k_\nu \geqq 1$, and hence by summation over k_ν for all $k_1, \cdots, k_\nu \geqq 0$. Multiplying by $z_1^{k_1} \cdots z_\nu^{k_\nu}$ and summing over the k_n, we see that $\mathbf{E}\{\prod_{n=1}^{\nu} \exp[{}^\alpha t_n(z_n - 1)]\}$ is independent of α whenever $|z_n| < 1$, and hence by analytic continuation

$$\mathbf{E}\left\{\exp - \sum_{n=1}^{\nu} \theta_n {}^1t_n\right\} = \mathbf{E}\left\{\exp - \sum_{n=1}^{\nu} \theta_n {}^2t_n\right\}$$

whenever $\operatorname{Re}\theta_n \geqq 0$. Hence the random vectors $({}^1t_1, \cdots, {}^1t_\nu)$ *and* $({}^2t_1, \cdots, {}^2t_\nu)$

and so also $({}^1T_1, \cdots, {}^1T_\nu)$ *and* $({}^2T_1, \cdots, {}^2T_\nu)$ have the same distribution, and the proof is complete.

Thus the stochastic structure of a sequence $\mathfrak{J}$ is completely determined by that of its Poisson count process $\mathfrak{N}(\lambda)$ for any one value of λ. This perhaps somewhat surprising result will be examined from a different point of view in the next section, but before doing this, we apply Theorem 3 to give converses to some of the results already established.

THEOREM 4. *Let $\mathfrak{J}$ be a random sequence of events with the property that its Poisson counts N_n (at rate 1) are independent and identically distributed. Then $\mathfrak{J}$ is a bulk Poisson sequence.*

PROOF. Let $p_k = \mathbf{P}\{N_n = k\}$, and write $Z_n = T_n - T_1$. Then

$$\begin{aligned}\mathbf{E}\{Z_n^k e^{-Z_n}/k!\} &= \mathbf{P}\{k \text{ events of } \mathfrak{J}' \text{ occur in } (T_1, T_n)\}\\ &= \sum_{r=0}^{\infty} \mathbf{P}\{T_r' \leqq T_1 < T_{r+1}', T_{r+k}' \leqq T_n < T_{r+k+1}'\}\\ &= \sum_{r=0}^{\infty} \mathbf{P}\{N_1 = \cdots N_r = 0, N_{r+1} \geqq 1, N_{r+1} + \cdots + N_{r+k} < n\\ &\qquad \leqq N_{r+1} + \cdots + N_{r+k+1}\}\\ &= \sum_{r=0}^{\infty} p_0^r \mathbf{P}\{N_1 \geqq 1, N_1 + \cdots N_k < n \leqq N_1 + \cdots + N_{k+1}\}\\ &= (1-p_0)^{-1}\mathbf{P}\{N_1 \geqq 1, N_1 + \cdots + N_k < n \leqq N_1 + \cdots + N_{k+1}\}\\ &= \mathbf{P}\{N_1 + \cdots + N_k < n \leqq N_1 + \cdots + N_{k+1} \mid N_1 \geqq 1\},\end{aligned}$$

where sums of the type $N_{r+1} + \cdots + N_{r+k}$ are taken as zero when $k = 0$. It follows that

$$\phi_n(x) = \mathbf{E}\{e^{-(1-x)Z_n}\} = 1 - (1-x)\sum_{k=1}^{\infty} x^{k-1}\mathbf{P}\{N_1 + \cdots + N_k < n \mid N_1 \geqq 1\},$$

and so, by the standard manipulations of renewal theory, we have

$$\sum_{n=1}^{\infty} \phi_n(x) y^n = \frac{y}{1-y}\left[1 - \frac{p(y) - p(0)}{1 - p(0)} \frac{1-x}{1 - xp(y)}\right], \tag{23}$$

where $p(y) = \sum_{n=0}^{\infty} p_n y^n$. This equation holds, in the first instance, whenever $|x|, |y| < 1$. Now fix y so that $|y| < 1$. Then $\phi_n(x)$ is regular and bounded in $\operatorname{Re} x < 1$, and the right hand side is regular in $x \neq 1/p(y)$. Hence, for fixed y, (23) holds in a domain which includes some half-plane $\operatorname{Re} x < c$. Letting $x \to -\infty$, so that $\phi_n(x) \to \mathbf{P}\{Z_n = 0\}$, we get

$$\sum_{n=1}^{\infty} \mathbf{P}\{Z_n = 0\} y^n = \frac{y}{1-y}\left[1 - \frac{p(y) - p(0)}{p(y)\{1 - p(0)\}}\right].$$

Now $\mathbf{P}\{Z_n = 0\}$ is a decreasing function of n, taking the value 1 at $n = 1$, and tending to zero as $n \to \infty$. Hence the numbers $b_n = \mathbf{P}\{Z_n = 0\} - \mathbf{P}\{Z_{n+1} = 0\}$,

($n = 1, 2, \cdots$), form a probability distribution, with

$$B(y) = \sum_{n=1}^{\infty} b_n y^n = [p(y) - p_0]/(1 - p_0)p(y).$$

Comparison with (22) shows that the Poisson counts of $\mathfrak{I}$ have the same joint distributions as those of a bulk Poisson sequence with batch distribution $\{b_k\}$, and so, by Theorem 3, $\mathfrak{I}$ is a bulk Poisson sequence. Thus the proof is complete.

In exactly the same way, the simple structure of $\mathfrak{N}$ given in Theorem 2 is characteristic of the bulk renewal sequence. These results are disappointing from a practical point of view, for it might have been hoped that a comparatively wide class of sequences would have simple Poisson count processes.

5. Poisson sampling. Let $\mathfrak{I}$ be a random sequence of events. Then one way of describing $\mathfrak{I}$ which has been widely used (see for instance [8]) is by means of its *cumulative count process* $C(t)$ $(t \geqq 0)$ defined as the number of events of $\mathfrak{I}$ occurring in the interval $(0, t]$;

$$C(t) = \max\{n \geqq 0;\, T_n \leqq t\}. \tag{24}$$

It is clear that the various count processes of $\mathfrak{I}$ can be defined in terms of $C(t)$; for instance $C_n(a) = C\{na\} - C\{(n-1)a\}$.

In particular, the Poisson count process $\mathfrak{N}$ of $\mathfrak{I}$ is given by

$$N_n = C(\tau_n) - C(\tau_{n-1}), \tag{25}$$

where $(\tau_0 = 0, \tau_1, \tau_2, \cdots)$ is a Poisson sequence of unit rate independent of $\mathfrak{I}$. (Thus τ_n is what was denoted earlier by T'_n.) Theorem 3 can then be re-stated as follows: *A knowledge of the joint distributions of $C(\tau_n)$ $(n = 1, 2, \cdots)$, together with the knowledge that $C(t)$ is the cumulative count process of some sequence of events, suffices to determine the distributions of $C(t)$.*

The process $C(\tau_n)$ is said to be obtained by *Poisson sampling* (at rate 1) from the process $C(t)$. Such sampling is not, of course, confined to cumulative count processes, and we shall see that the above "determining property" of Poisson sampling holds for a much wider class of processes.

We must, however, digress for a moment to discuss a peculiar difficulty which arises when we try to consider sampling of general stochastic processes. Let $X(t)$ $(t \geqq 0)$ be a real-valued stochastic process, and let τ be a random variable, independent of $X(t)$, with continuous distribution function G. Intuitively we would expect that $X(\tau)$ is a random variable with distribution function

$$\mathbf{P}\{X(\tau) \leqq x\} = \int_0^\infty \mathbf{P}\{X(t) \leqq x\}\, dG(t). \tag{26}$$

However, there is no reason why $X(\tau)$ should be a random variable at all, and even if it is, we can alter its value quite arbitrarily without changing the finite-dimensional distributions of X, or interfering with the independence of X and τ, (since, for any t, there is zero probability that $t = \tau$). The difficulty lies in the fact that independence of uncountably infinite families of random variables is a rather weak concept, and this difficulty largely disappears if we make stronger independence assumptions.

The simplest such assumption seems to be the following. Suppose that $X(t, \omega_1)$ is a stochastic process on the probability space $(\Omega_1, \mathcal{A}_1, \mathbf{P}_1)$, and that $\tau(\omega_2)$ is a random variable on a different space $(\Omega_2, \mathcal{A}_2, \mathbf{P}_2)$. Then X and τ are also defined on the product space $(\Omega, \mathcal{A}, \mathbf{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \mathbf{P}_1 \times \mathbf{P}_2)$. Then, if $X(t, \omega_1)$ is a measurable process (in the sense of Doob [4]), the random variable $Y = X(\tau)$ is defined on $(\Omega, \mathcal{A}, \mathbf{P})$ by

$$Y(\omega) = Y(\omega_1, \omega_2) = X\{\tau(\omega_2), \omega_1\}. \tag{27}$$

Equation (26) is then an immediate consequence of Fubini's theorem.

When we talk of a process $\{X(\tau_n)\}$ obtained by Poisson sampling from a process $X(t)$, it will be implicit that the $X(\tau_n)$ are defined on a product space in this way, the τ_n forming a Poisson sequence on $(\Omega_2, \mathcal{A}_2, \mathbf{P}_2)$. Thus Poisson sampling is only meaningful when applied to measurable processes.

We are now in a position to state and prove the basic theorem on Poisson sampling, which is the counterpart to Theorem 3. This will be stated for processes which are continuous in probability, i.e. which satisfy

$$\text{plim}_{s \to t} X(s) \doteq X(t); \tag{28}$$

as is well-known [4], such processes always possess measurable versions.

THEOREM 5. *Let $X_1(t)$, $X_2(t)$ be two measurable, real-valued stochastic processes, both continuous in probability, and suppose that the processes obtained from X_1 and X_2 by Poisson sampling (at rate 1) have the same joint distributions. Then X_1 and X_2 have the same finite-dimensional distributions.*

PROOF. Let n be any positive integer, $x_1, \cdots, x_n$ any real numbers, and write, for $\alpha = 1, 2$,

$$F_\alpha(t_1, \cdots, t_n) = \mathbf{P}\{X_\alpha(t_j) \leqq x_j; j = 1, 2, \cdots, n\}.$$

Now let $k_1, \cdots, k_n$ be non-negative integers, and set

$$K(q) = k_1 + \cdots + k_q, \; \xi_q = \tau_{K(q)}, \; \eta_q = \xi_q - \xi_{q-1}.$$

Then, as in (26),

$$\begin{aligned} p_\alpha &= \mathbf{P}\{X_\alpha(\xi_j) \leqq x_j; \quad j = 1, 2, \cdots, n\} \\ &= \mathbf{E}\{F_\alpha(\xi_1, \cdots, \xi_n)\} \\ &= \int_0^\infty \cdots \int_0^\infty F_\alpha(\eta_1, \eta_1 + \eta_2, \cdots, \eta_1 + \cdots + \eta_n) \prod_{j=1}^n \frac{\eta_j^{k_j - 1} e^{-\eta_j}}{(k_j - 1)!} d\eta_1 \cdots d\eta_n. \end{aligned}$$

But p_α is independent of α, and hence so is the final expression. Multiplying this by $z_1^{k_1} \cdots z_n^{k_n}$, summing over the k_j, and writing $\theta_j = 1 - z_j$, we see that

$$\int_0^\infty \cdots \int_0^\infty F_\alpha(\eta_1, \eta_1 + \eta_2, \cdots, \eta_1 + \cdots + \eta_n) \exp\left\{-\sum_{j=1}^n \theta_j \eta_j\right\} d\eta_1 \cdots d\eta_n$$

is independent of α whenever $|\theta_j - 1| < 1$, and hence by analytic continuation whenever Re $\theta_j > 0$. But, since X_α is continuous in probability, the function F_α is continuous (its arguments being time variables). It follows that F_α is independent of α, and the theorem is proved.

The assumption of continuity in probability (which is in any case fairly weak) was only used for the final step. However, it seems appropriate to make this assumption, since it enables us to guarantee the existence of a measurable version. The cumulative count process of a random sequence of events $\mathfrak{I}$ is continuous in probability if and only if each of the random variables T_n has a continuous distribution function. Thus Theorem 5 does not quite contain Theorem 3.

The assumption that the processes are real-valued is expendable. The theorem extends at once to processes taking values in R^n, and without difficulty to more general state spaces.

The results on Poisson sampling obtained here suggest that it might in some cases be profitable from a practical point of view to sample stochastic processes at random instants of time. Such a procedure has been suggested by more than one author (see, for example, [6], p. 58), but we shall not pursue the matter further in the present paper.

6. Further properties of Poisson sampling. In this section we refer briefly to properties of Poisson sampling, and in particular to characteristics of stochastic processes which are preserved under Poisson sampling.

Consider first the Markov property. Let $X(t)$ be a Markov process whose state space S we take, for simplicity, to be countable, and suppose that the transition probabilities

$$p_{ij}(t) = \mathbf{P}\{X(s+t) = j \mid X(s) = i\} \tag{29}$$

are independent of s. Suppose moreover that $X(t)$ is *standard*, i.e. that

$$p_{ij}(t) \to \delta_{ij}, \qquad (t \to 0). \tag{30}$$

Then a trivial calculation shows that the process $X(\tau_n)$ obtained from $X(t)$ by Poisson sampling is a discrete time Markov process on S with stationary transition probabilities

$$\bar{p}_{ij} = \mathbf{P}\{X(\tau_{n+1}) = j \mid X(\tau_n) = i\} = \int_0^\infty e^{-t} p_{ij}(t)\, dt. \tag{31}$$

In the special case when the original process is q-bounded, i.e. where

$$\sup_{i \varepsilon S} [-p'_{ii}(0)] < \infty, \tag{32}$$

and in particular when S is finite, the transition matrix $P(t) = (p_{ij}(t))$ can be written in the form

$$P(t) = \exp(Qt), \tag{33}$$

where $Q = P'(0)$, and then the transition matrix $\bar{P}$ of $X(\tau_n)$ is just

$$\bar{P} = \int_0^\infty e^{-t} \exp(Qt)\, dt = (I - Q)^{-1}. \tag{34}$$

This may be inverted to give

$$Q = I - \bar{P}^{-1}, \tag{35}$$

so determining $P(t)$ in terms of $\bar{P}$.

Equation (31) shows the close connection, for Markov processes, between Poisson sampling and the "method of collective marks" of van Dantzig [3]. If $X(t)$ is a Markov process, then $X(\tau_n)$ is a "derived Markov process" in the sense of Cohen [1]. Notice also that $\bar{P}$ is a value of the "resolvent" of the semi-group $\{P(t)\}$, and for processes which are not q-bounded Q must be replaced by the "infinitesimal generator" of the semigroup.

The main point, however, is that a process obtained from a Markov process by Poisson sampling is itself Markovian. This result holds for more general state spaces, but we omit the precise formulation and proof. Theorem 5 leads us to expect a converse result, that if $X(\tau_n)$ is Markovian, then so is $X(t)$, and such a converse is given by the following theorem.

THEOREM 6. *Let $X(t)$ $(t \geqq 0)$ be a stochastic process on the countable state space S, continuous in probability and measurable, and let $Y_n = X(\tau_n)$ be the process obtained from it by Poisson sampling. If Y_n is a Markov process with stationary transition probabilities, then so is $X(t)$.*

PROOF. Let $p_{ij}(t) = \mathbf{P}\{X(t) = j \mid X(0) = i\}$. Then $p_{ij}(t)$ is continuous, and the n-step transition probabilities $\bar{p}_{ij}^{(n)}$ of $\{Y_n\}$ are given by

$$\bar{p}_{ij}^{(n)} = \mathbf{P}\{Y_n = j \mid Y_0 = i\} = \int_0^\infty p_{ij}(t)\, t^{n-1}\, e^{-t}\, dt/(n-1)!.$$

Substituting this in the Chapman-Kolmogorov equation

$$\bar{p}_{ij}^{(m+n)} = \sum_{k \varepsilon S} \bar{p}_{ik}^{(m)} \bar{p}_{kj}^{(n)},$$

we get

$$\int_0^\infty p_{ij}(t) t^{m+n-1}\, e^{-t}\, dt/(m+n-1)! = \sum_{k\varepsilon S} \int_0^\infty p_{ik}(u) u^{m-1}\, e^{-u}\, du/(m-1)!$$

$$\cdot \int_0^\infty p_{kj}(v) v^{n-1}\, e^{-v}\, dv/(n-1)!,$$

which can be written

$$\int_0^\infty\int_0^\infty \{p_{ij}(u+v) - \sum_{k\varepsilon S} p_{ik}(u) p_{kj}(v)\}\, u^{m-1}\, v^{n-1}\, e^{-u-v}\, du\, dv = 0.$$

Multiplying by $x^{m-1} y^{n-1}/(m-1)!(n-1)!$ and summing over $m, n \geqq 1$ shows that, for all $|x|, |y| < 1$,

$$\int_0^\infty\int_0^\infty \{p_{ij}(u+v) - \sum_{k\varepsilon S} p_{ik}(u)\, p_{kj}(v)\}\, e^{-(1-x)u-(1-y)v}\, du\, dv = 0,$$

and so, using the continuity of $p_{ij}(t)$, we get $p_{ij}(u+v) = \sum_{k\varepsilon S} p_{ik}(u) p_{kj}(v)$. Moreover, the continuity in probability shows that (30) holds, and hence there exists a Markov process $X_1(t)$, continuous in probability and measurable, with stationary transition probabilities $p_{ij}(t)$. An application of Theorem 5 to X and X_1 now completes the proof.

Another property preserved by Poisson sampling is that of stationarity. Thus let $Y_n = X(\tau_n)$ be obtained from $X(t)$ by Poisson sampling, and write, for any real numbers $x_1, \cdots, x_n$, $F(t_1, \cdots, t_n) = \mathbf{P}\{X(t_j) \leqq x_j ; j = 1, 2, \cdots, n\}$. $\Phi_m = \mathbf{P}\{Y_{m+j} \leqq x_j ; j = 1, 2, \cdots, n\}$. Then $X(t)$ is stationary if and only if (for all choices of n, t_j, x_j) $F(t_1, \cdots, t_n)$ depends only on the differences $t_j - t_{j-1}$, and Y_n is stationary if and only if (for all choices of n, x_j) Φ_m is independent of m. We shall show that these two conditions are equivalent.

We have, in fact, $\Phi_m = \mathbf{E}\{F(\tau_{m+1}, \cdots, \tau_{m+n})\}$. Now write $\tau_j' = \tau_{m+j} - \tau_m$. Then

$$\begin{aligned}\Phi_m &= \mathbf{E}\{F(\tau_m + \tau_1', \cdots, \tau_m + \tau_n')\}\\ &= \int_0^\infty \mathbf{E}\{F(T + \tau_1', \cdots, T + \tau_n')\}\, T^{m-1} e^{-T}\, dT/(m-1)!\\ &= \int_0^\infty \mathbf{E}\{F(T + \tau_1, \cdots, T + \tau_n)\}\, T^{m-1} e^{-T}\, dT/(m-1)!.\end{aligned}$$

It follows, by a now familiar argument, that Φ_m is independent of m if and only if $\mathbf{E}\{F(T + \tau_1, \cdots, T + \tau_n)\}$ is independent of T. That is, Y_n is stationary if and only if the process obtained from $X(T + t)$ by Poisson sampling has distributions independent of T. Hence, by Theorem 5, Y_n is stationary if and only if $X(t)$ is.

A rather similar theorem, proved in the same way, states that a random sequence of events is stationary (in the sense that its cumulative count process has stationary increments) if and only if its Poisson count process is stationary. A special case of this theorem is the remark made at the end of Section 2 that the Poisson count process of an equilibrium renewal sequence is stationary.

We have so far only considered Poisson sampling at one fixed rate $\lambda = 1$. To conclude this section we discuss briefly the relation between processes obtained by Poisson sampling at different rates. Let $X(t)$ be a measurable stochastic process, and let $\{\tau_n\}$ be a Poisson sequence of rate λ independent of X (in the strong sense described in the previous section). Now let us pick out at random a subsequence $\{\sigma_m\}$ of $\{\tau_n\}$, the probability of τ_n being chosen being p, independently of the other τ_k chosen. Then it is clear that $\{\sigma_m\}$ is a Poisson sequence of rate λp.

It follows that, if $Y_n^\lambda = X(\tau_n)$ is obtained by Poisson sampling at rate λ, then we can obtain a Poisson sample of rate λp by choosing a random subsequence of $\{Y_n^\lambda\}$, in the manner described above. In particular, if we know the joint distributions of $\{Y_n^\lambda\}$, we can deduce those of $\{Y_n^\mu\}$ for any $\mu < \lambda$. For example,

$$\mathbf{P}\{Y_1^\mu \leqq x\} = \sum_{n=1}^\infty (1 - (\mu/\lambda))(\mu/\lambda)^{n-1}\mathbf{P}\{Y_n^\lambda \leqq x\}. \tag{36}$$

There does not seem to be any analogous probabilistic procedure for going back from $\{Y_n^\mu\}$ to $\{Y_n^\lambda\}$, although there is of course a formal procedure based (as in the proof of Theorem 5) on analytic continuation.

7. Poisson counts in the theory of queues. In this section we indicate the way in which the Poisson count process arises in the theory of queues, and the consequences for that theory of the results established in the preceding sections.

Consider a queueing system at which successive customers $\mathcal{C}_n$ $(n = 1, 2, \cdots)$ arrive at instants T_n. Then the input process is specified by the random sequence of events $\mathfrak{I} = \{T_n\}$, about which we make no specific assumptions. Let the customers be served by a single server, and suppose that, if s_n is the service time of $\mathcal{C}_n$, then the s_n are independent of each other and of $\mathfrak{I}$, and are exponentially distributed with unit mean. In terms of the notation introduced by D. G. Kendall [7], this model might be denoted by $X/M/1$, the letter X indicating that no assumption has been made about the stochastic nature of the input process $\mathfrak{I}$.

It has been pointed out by Winsten [9] that the service time mechanism in this queue can be regarded in the following way. Let $\{\tau_n\}$ be a Poisson sequence of unit rate, independent of $\mathfrak{I}$. Then at each instant τ_n at which the queue is non-empty, the server completes a service period and a customer leaves the queue. In other words, the time-instants at which customers leave the queue form a subsequence Σ of a Poisson sequence $\{\tau_n\}$, τ_n belonging to Σ if and only if the queue is non-empty immediately prior to τ_n.

We remark in passing that this procedure can be modified for a many-server queue, and that all our remarks about the queue $X/M/1$ can be carried over, with only slight modifications, to the queue $X/M/k$ (where k is any positive integer).

The usual method of analyzing queues with exponential service-time distribution is to consider the queue size at the instant T_n of the arrival of $\mathcal{C}_n$. Here, however, we adopt the reverse approach, and look at the length of the queue at the instants of service, or more precisely, at the "potential service instants" τ_n. Thus let Q_n be the number of customers in the queue, including the one being served, immediately prior to τ_n. Then it is clear that

$$Q_n = N_n + \max (Q_{n-1} - 1, 0), \tag{37}$$

where N_n is the number of customers arriving in the interval $[\tau_{n-1}, \tau_n)$. We see that $\{N_n\}$ is exactly (except for an event of probability zero) the Poisson count process of the input sequence $\mathfrak{I}$. Hence, so far as the process $\{Q_n\}$ is concerned, the input process enters through its Poisson count process $\mathfrak{N}$. In fact, from (37) we can express $\{Q_n\}$ explicitly in terms of $\mathfrak{N}$; if $Q_0 = 0$ this expression takes the form

$$Q_n = \max_{1 \leq k \leq n} \left[\sum_{r=k}^{n} N_r - (n - k) \right]. \tag{38}$$

Suppose that, in some practical situation, we were faced with the problem of analyzing a queue of the type $X/M/1$, the input process being of some complex type. Then the above considerations suggest that one should try to fit a model for the input whose Poisson count process had a fairly simple structure, rather than to concentrate on the interarrival time process. Indeed, there are important cases in which the Poisson count process is easier to deal with than the interarrival time process. For example, Winsten [9] has considered a situation in which customers are scheduled to arrive at regular intervals, but are subject to delays. He finds that, if the delays are sufficiently long to allow customers to arrive in an order substantially different from the scheduled one, then the interarrival time process becomes unmanageably complicated. The Poisson count process does not consider order, but only the number of arrivals in different intervals, and it is in fact possible to compute the Poisson count process of an input of the Winsten type in several cases of interest. Similar remarks hold for inputs obtained by superposing several independent sequences of events.

The fact that the input process affects $\{Q_n\}$ through $\mathfrak{N}$ suggests that we should look for sequences whose Poisson count processes have simple stochastic structure. Obviously the very simplest case is that in which the N_n are independent and identically distributed, and this, by Theorem 4, implies that $\mathfrak{I}$ is a bulk Poisson sequence. In the language of queueing theory, this means that the input consists of batches of independent random sizes, arriving in a Poisson process. This result is disappointing; it might perhaps have been expected that more general inputs would have independent Poisson counts.

Notice that, if the input is a bulk Poisson sequence, with batch distribution $\{b_k\}$, then we can regard the batches as arriving in a Poisson sequence, and having total service time with probability density

$$\sum_{k=1}^{\infty} b_k t^{k-1} e^{-t} / (k-1)!. \tag{39}$$

Thus, in a sense, this queue is equivalent to a queue $M/G/1$, in which the service time distribution has the form (39). Conversely, a queue $M/G/1$ in which the service time distribution is given by (39) can be converted into a queue of the form $X/M/1$ whose input is a bulk Poisson sequence.

More generally, if we have any queue $X/G/1$ whose service time distribution has density of the form

$$e^{-at} \sum_{k=0}^{\infty} c_k t^k, \tag{40}$$

where $a > 0$, $c_k \geqq 0$, then we can convert it into a queue $X/M/1$ by regarding the customers as made up of a random number of units, each having an exponentially distributed service time with mean $1/a$. This device has been used by a number of authors (see the paper of Wishart [10] for references). It may be shown that any service time distribution whatsoever can be approximated by distributions having densities of the form (40).

Finally, let us consider the case in which the input process is a renewal sequence, i.e. the queue $GI/M/1$ studied by Kendall [7]. Then, according to Theorem 1, the Poisson count process takes the form

$$N_n = \epsilon_n \nu_n , \tag{41}$$

where $\{\epsilon_n = 1\}$ is a delayed recurrent event, and the ν_n are independent and geometrically distributed. It has long been known that the limiting distribution of Q_n as $n \to \infty$ is geometric, whatever the interarrival time distribution; it is no accident that the geometric distribution also arises in the distribution of ν_n.

In regarding $\{Q_n\}$ as a discrete time process, we are effectively considering a new (discrete) time n related to true time by $n \leftrightarrow \tau_n$. Then (37) can be regarded as exhibiting $\{Q_n\}$ as a discrete time queueing process, in which groups of customers $\mathcal{G}_1, \mathcal{G}_2, \cdots$ arrive at the instants of a (delayed) recurrent event. In terms of this discrete time, the groups have a service time which is geometrically distributed (having the same distribution as the ν_n). Thus we have a discrete-time analogue of the queue $GI/M/1$ with which we started. This "discretization" effect is a general property of the Poisson count process, and is one which might be useful from a computational point of view. Unlike most procedures for turning a continuous time process into one in discrete time, it is exact.

We have seen that the customers $\mathcal{C}$ are naturally collected into groups $\mathcal{G}$, each group containing a random number of customers having a geometric distribution on 1, 2, 3, $\cdots$. By repeating the procedure we can collect the groups into second-order groups $\mathcal{G}^{(2)}$, each $\mathcal{G}^{(2)}$ containing all the groups which arrive during the service period of a given group. The number of groups in any second-order group will have a geometric distribution, and hence, by a well-known reproducing property of the geometric distribution, the number of customers in each $\mathcal{G}^{(2)}$ will have a geometric distribution. Proceeding in this way, we obtain an infinite hierarchy of groups of customers $\mathcal{G}, \mathcal{G}^{(2)}, \mathcal{G}^{(3)}, \cdots$, and each $\mathcal{G}^{(r)}$ will contain a geometrically distributed number of customers. A fairly simple probabilistic argument shows that the limiting distribution of Q_n as $n \to \infty$ is exactly the same as the limit, as $r \to \infty$, of the distribution of the number of customers in $\mathcal{G}^{(r)}$, and is thus geometric. This accounts, not only for the occurrence of the geometric distribution in the solution of the queue $GI/M/1$, but also for the close formal connection of the theory of this queue with the theory of branching processes.

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