MULTIVARIATE BETA DISTRIBUTIONS AND INDEPENDENCE PROPERTIES OF THE WISHART DISTRIBUTION[1]

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1. Summary and introduction. If X and Y are independent random variables having chi-square distributions with n and m degrees of freedom, respectively, then except for constants, X/Y and $X/(X + Y)$ are distributed as F and Beta variables. In the multivariate case, the Wishart distribution plays the role of the chi-square distribution. There is, however, no single natural generalization of a ratio in the multivariate case. In this paper several generalizations which lead to multivariate analogs of the Beta or F distribution are given. Some of these distributions arise naturally from a consideration of the sufficient statistic or maximal invariant in various multivariate problems, e.g., (i) testing that k normal populations are identical [1], p. 251, (ii) multivariate analysis of variance tests [9], (iii) multivariate slippage problems [4], p. 321. Although several of the results may be known as folklore, they have not been explicitly stated. Other of the distributions obtained are new.

Intimately related to some of the distributional problems is the independence of certain statistics, and results in this direction are also given.

2. Notation and comments. If V and W are symmetric matrices, $V > W$ means that $V - W$ is positive definite. I_p denotes the identity of order p; the subscript is omitted when the dimensionality is clear from the context. We write etr A to mean exp tr A. $X \sim Y$ means that X and Y have the same distribution. $V \sim \mathcal{W}(\Sigma, p, n)$ means that V is a $p \times p$ symmetric matrix whose $p(p + 1)/2$ elements are random variables having a Wishart distribution with (non-degenerate) covariance matrix $\Sigma \equiv \Lambda^{-1}$ and n degrees of freedom ($n \geqq p$ assumed throughout), i.e., with density function.

$$(2.1)\qquad \begin{aligned} p(V) &= c(n, p)|\Lambda|^{n/2}|V|^{(n-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}\Lambda V), \qquad V > 0, \\ 1/c(n, p) &= 2^{pn/2}\pi^{p(p-1)/4}\prod_1^p \Gamma[\tfrac{1}{2}(n - i + 1)]. \end{aligned}$$

If $A > 0$, $A: m \times m$, then $A^{\frac{1}{2}}$ may be defined as $A^{\frac{1}{2}} = \Delta D_\alpha \Delta'$, where Δ is an orthogonal matrix, $D_\alpha = \operatorname{diag}(\alpha_1, \cdots, \alpha_m)$, and $\alpha_1^2, \cdots, \alpha_m^2$ are the characteristic roots of $A = \Delta D_\alpha^2 \Delta'$. We may also write $A = BB'$ and define $A^{\frac{1}{2}} = B$. With the first definition, $A^{\frac{1}{2}}$ is symmetric, but this need not be the case with the second definition. We adopt the convention that in an expression such as $A^{\frac{1}{2}}VA^{\frac{1}{2}}$, the

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postmultiplier is $(A^{\frac{1}{2}})'$. In cases where the non-symmetric square root is used, we will let the lower triangular matrix T, with $t_{ii} > 0$, be defined uniquely by $A = TT'$.

A curious phenomenon which occurs is that certain distributions are different depending upon which definition of $A^{\frac{1}{2}}$ is adopted. Indeed, some independence properties which intuitively seem reasonable do not hold if any square root is used.

We also need a symbolism for certain submatrices. If $A: p \times p$, we write $A^{[\alpha]} = (a_{ij}): i, j = 1, \cdots, \alpha$ and $A_{[\alpha]} = (a_{ij}): i, j = p - \alpha + 1, \cdots, p.$

If $Y = f(X)$ is a matrix transformation, the absolute value of the Jacobian $\partial(x_{ij})/\partial(y_{ij})$ is denoted by $J(X \to Y)$. The Jacobians required are obtained in [3] and [7]. In addition we need the following results.

LEMMA 2.1. *Let T be lower triangular and A symmetric. The Jacobian of the transformation $Y = TAT'$ from Y to T is* $J(Y \to T) \doteq 2^p \prod_1^p (t_{ii}^{p-i+1} |A^{[i]}|)$.

PROOF. Write $A = LL'$, where L is lower triangular, and let $M = TL$. Then $J(Y \to T) = J(Y \to M)J(M \to T) = (2^p \prod_1^p m_{ii}^{p-i+1})(\prod_1^p l_{ii}^{p-i+1}) = 2^p \prod_1^p (t_{ii}^{p-i+1} l_{ii}^{2(p-i+1)})$, ([3], [7]). The result follows by noting that $|A^{[i]}| = \prod_{j=1}^i l_{jj}^2$.‖[2]

LEMMA 2.2. *Let T be upper triangular and A symmetric. The Jacobian of the transformation $Y = TAT'$ from Y to T is* $J(Y \to T) = 2^p \prod_1^p (t_{ii}^i |A_{[i]}|)$.

The proof parallels that of Lemma 2.1.

3. Multivariate beta distributions. The first two theorems indicate the distinctions that arise by using different square roots. In the simplest case, let S_0, S_1 be independently distributed, $S_j \sim \mathcal{W}(I, p, n_j)$, $j = 0, 1$, and $(S_0^{\frac{1}{2}})^2 = S_0 = TT'$, where T is lower triangular. The distributions of $V = S_0^{-\frac{1}{2}} S_1 S_0^{-\frac{1}{2}}$ and of $U = T^{-1} S_1 T'^{-1}$ are

$$(3.1) \quad p(V) = c\,|V|^{(n_1-p-1)/2}\,|I + V|^{-(n_0+n_1)/2}, \qquad V > 0,$$

$$(3.2) \quad p(U) = c\,|U|^{(n_1-p-1)/2}\,|I + U|^{-(n_0+n_1+p+1)/2} \prod_1^p |(I + U)^{[j]}|, \qquad U > 0.$$

The result (3.2) holds for any Σ, whereas (3.1) does not hold for general Σ. The latter distribution is unknown, and in Section 5 we show wherein the difficulty lies. Both distributions are of interest in number theory, Bellman [2]. Related distributions have been considered by Olkin [8].

These distributions are now obtained for a more general case.

THEOREM 3.1. *Let $S_0, S_1, \cdots, S_k$ be independently distributed, $S_j \sim \mathcal{W}(I, p, n_j)$, $S_0 = (S_0^{\frac{1}{2}})^2$. The joint distribution of*

$$(3.3) \qquad V_j = S_0^{-\frac{1}{2}} S_j S_0^{-\frac{1}{2}}, \qquad j = 1, \cdots, k,$$

is

$$(3.4) \qquad p(V_1, \cdots, V_k) = c \prod_1^k |V_j|^{(n_j-p-1)/2} \left| I + \sum_1^k V_j \right|^{-n/2}, \qquad V_j > 0,$$

[2] The symbol ‖ denotes end of proof.

where $n = \sum_0^k n_j$, *and*

$$c = \frac{\prod_0^k c(n_j, p)}{c(n, p)} = \pi^{-\frac{1}{4}kp(p-1)} \frac{\prod_{i=1}^{p}\prod_{\alpha=0}^{k} \Gamma[\frac{1}{2}(n_\alpha - i + 1)]}{\Gamma[\frac{1}{2}(n - i + 1)]}. \tag{3.5}$$

PROOF. In the joint distribution of $S_0, \cdots, S_k$ obtained from (2.1) with $\Sigma = I$, make the transformation (3.3), the Jacobian being $J(S_1, \cdots, S_k \to V_1, \cdots, V_k) = |S_0|^{k(p+1)/2}$, and obtain the joint distribution of $S_0, V_1, \cdots, V_k$. Integrating over $S_0 > 0$ yields (3.4).||

THEOREM 3.2. *Let* $S_0, S_1, \cdots, S_k$ *be independently distributed,* $S_j \sim \mathcal{W}(\Sigma, p, n_j)$.

(i) *The joint distribution of* $U_j = T^{-1}S_jT'^{-1}$, $j = 1, \cdots, k$, *where* T *is a lower triangular matrix defined by* $S_0 = TT'$, *is*

$$p(U_1, \cdots, U_k) = c\,\frac{\prod_1^k |U_j|^{(n_j-p-1)/2} \left| I + \sum_1^k U_j \right|^{-(n-p-1)/2}}{\prod_{\alpha=1}^{p} \left| \left(I + \sum_1^k U_j\right)^{[\alpha]} \right|}, \qquad U_j > 0. \tag{3.6}$$

(ii) *The joint distribution of* $U_j = T^{-1}S_jT'^{-1}$, $j = 1, \cdots, k$, *where* T *is an upper triangular matrix defined by* $S_0 = TT'$, *is*

$$p(U_1, \cdots, U_k) = c\,\frac{\prod_1^k |U_j|^{(n_j-p-1)/2} \left| I + \sum_1^k U_j \right|^{-(n-p-1)/2}}{\prod_{\alpha=1}^{p} \left| \left(I + \sum_1^k U_j\right)_{[\alpha]} \right|}, \qquad U_j > 0. \tag{3.7}$$

In each case, $n = \sum_0^k n_j$, and $c = \prod_0^k c(n_j, p)/c(n, p)$.

PROOF. The proofs for (i) and (ii) parallel one another, and we present the details for (i). In the joint distribution of $S_0, \cdots, S_k$ obtained from (2.1), make the transformation $S_0 = TT'$, $S_j = TU_jT'$, $j = 1, \cdots, k$. The Jacobian is $J(S_0, S_1, \cdots, S_k \to T, U_1, \cdots, U_k) = 2^p |T|^{k(p+1)} \prod_1^p t_{ii}^{p-i+1}$. This yields

$$p(T, U_1, \cdots, U_k) = c\,|\Lambda|^{\frac{1}{2}n} \prod_1^k |U_j|^{\frac{1}{2}(n_j-p-1)} \prod_1^p t_{ii}^{n-i}$$
$$\operatorname{etr} - \tfrac{1}{2}\Lambda T \left(I + \sum_1^k U_j\right) T', \tag{3.8}$$

$$U_j > 0, \qquad 0 < t_{ii} < \infty, \qquad -\infty < t_{ij}(i > j) < \infty, \qquad c = 2^p \prod_1^p c(n_j, p).$$

Transform from T to Y by $Y = T(I + \Sigma U_j)T'$, using the Jacobian as given by Lemma 2.1. (Lemma 2.2 is used for (ii).) Thus we obtain the joint distribution of $(Y, U_1, \cdots, U_k)$. Integration over $Y > 0$ yields the result. ||

In the next two theorems it does not matter which square root is used.

THEOREM 3.3. *Let* $S_0, S_1, \cdots, S_k$ *be independently distributed,* $S_j \sim \mathcal{W}(\Sigma, p, n_j)$. *The joint distribution of*

$$W_j = \left(\sum_0^k S_i\right)^{-\frac{1}{2}} S_j \left(\sum_0^k S_i\right)^{-\frac{1}{2}'}, \qquad j = 1, \cdots, k, \tag{3.9}$$

for any square root, is

$$p(W_1, \cdots, W_k) = c\left(\prod_1^k |W_j|^{\frac{1}{2}(n_j-p-1)}\right)\left|I - \sum_1^k W_j\right|^{\frac{1}{2}(n_0-p-1)}, \tag{3.10}$$

$$W_j > 0, I - \sum W_j > 0,$$

where $c = \prod_0^k c(n_j, p)/c(n, p)$, $n = \sum_0^k n_j$.

PROOF. Let $Z = \sum_0^k S_j$, $W_j = Z^{-\frac{1}{2}} S_j Z^{-\frac{1}{2}}$, $j = 1, \cdots, k$; then $J(S_0, \cdots, S_k \to Z, W_1, \cdots, W_k) = |Z|^{k(p+1)/2}$, and we obtain the joint distribution of $Z, W_1, \cdots, W_k$. Integration over $Z > 0$ completes the proof. ||

THEOREM 3.4. *Let* $V_1, \cdots, V_k$ *be defined by* $V_j = S_0^{-\frac{1}{2}} S_j S_0^{-\frac{1}{2}}$, *as in* (3.3) *and define*

$$Z_j = \left(I + \sum_1^k V_j\right)^{-\frac{1}{2}} V_j \left(I + \sum_1^k V_j\right)^{-\frac{1}{2}}, \qquad j = 1, \cdots, k. \tag{3.11}$$

The joint distribution of $Z_1, \cdots, Z_k$ *is given by* (3.10).

PROOF. This proof requires more intricate transformations because of the successive square roots, i.e.,

$$Z_j = \left(S_0^{-\frac{1}{2}} \sum_0^k S_j\, S_0^{-\frac{1}{2}}\right)^{-\frac{1}{2}} S_0^{-\frac{1}{2}} S_j\, S_0^{-\frac{1}{2}} \left(S_0^{-\frac{1}{2}} \sum_0^k S_j\, S_0^{-\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

The result (3.11) follows by successively making the transformations $Q = I + \sum_1^k V_j$, $Z_j = Q^{-\frac{1}{2}} V_j Q^{-\frac{1}{2}}$, $j = 1, \cdots, k - 1$, from $(V_1, \cdots, V_k)$ to $(Z_1, \cdots, Z_{k-1}, Q)$, and then from Q to Z_k by $Z_k = Q^{-\frac{1}{2}} V_k Q^{-\frac{1}{2}} = I - Q^{-1} - \sum_1^{k-1} Z_j$. Alternatively, we note that for W_j defined by (3.9), $Z_j = A W_j A'$, where $A = (I + \sum_1^k V_j)^{-\frac{1}{2}} S_0^{-\frac{1}{2}} (\sum_0^k S_j)^{-\frac{1}{2}}$. Since $AA' = I$, $p(Z_1, \cdots, Z_k \mid A) = p(W_1, \cdots, W_k \mid A)$ for every such A, from which it follows that $p(Z_1, \cdots, Z_k)$ is given by (3.10). ||

Closely related to the distribution of Theorem 3.2 is the following.

THEOREM 3.5. *Let* S_1, S_2 *be independently distributed,* $S_j \sim \mathcal{W}(\Sigma, p, n_j)$.

(i) *The distribution of* $Y = T'(S_1 + S_2)^{-1} T$, *where* $S_2 = TT'$, T *is lower triangular, is*

$$p(Y) = \frac{c(n_1, p) c(n_2, p)}{c(n_1 + n_2, p)} \frac{|Y|^{\frac{1}{2}n_2} |I - Y|^{\frac{1}{2}(n_1-p-1)}}{\prod_1^p |Y_{[i]}|}, \qquad 0 < Y < I. \tag{3.12}$$

(ii) *The distribution of* $Y = T'(S_1 + S_2)^{-1} T$, *where* $S_2 = TT'$, T *is upper triangular, is*

$$p(Y) = \frac{c(n_1, p) c(n_2, p)}{c(n_1 + n_2, p)} \frac{|Y|^{\frac{1}{2}n_2} |I - Y|^{\frac{1}{2}(n_1-p-1)}}{\prod_1^p |Y^{[i]}|}, \qquad 0 < Y < I. \tag{3.13}$$

PROOF.

(i) From the joint distribution of S_1, S_2, make the transformation $W =$

$S_1 + S_2$, $TT' = S_2$, then $J(S_1, S_2 \to W, T) = 2^p \prod_1^p t_{ii}^{p-i+1}$. Thus

$$p(W, T) = c\,|W - TT'|^{\frac{1}{2}(n_1-p-1)} \prod_1^p t_{ii}^{n_2-i} \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}W).$$

Now transform from T to Y by $Y = T'W^{-1}T$. The Jacobian is given by Lemma 2.2, namely, $J(T \to Y) = 2^{-p}\prod\{t_{ii}^{-i}\,|(W^{-1})_{[i]}|^{-1}\}$, so that

$$p(Y, W) = c\,|W|^{\frac{1}{2}(n_1-p-1)}|I - Y|^{\frac{1}{2}(n_1-p-1)} \prod_1^p \{t_{ii}^{n_2-2i}\,|(W^{-1})_{[i]}|^{-1}\} \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}W).$$

Also $\prod_1^p t_{ii}^{2i} = |Y_{[i]}|/|(W^{-1})_{[i]}|$. Integration over $W > 0$ yields the result.

(ii) The proof parallels the above, using Lemma 2.1 instead of 2.2, and $\prod_1^p t_{ii}^{2(p-i+1)} = \prod_1^p |Y^{[i]}|/|(W^{-1})^{[i]}|$. ||

Remark. Alternative proofs can easily be suggested. One which is involved in other problems is to transform from A to A^{-1}. The Jacobian $J(A \to A^{-1}) = |A|^{p+1}$ and follows from the well-known equation $dA^{-1} = -A^{-1}(dA)A^{-1}$.

4. Central Studentized Wishart distributions. Let $X{:}k \times p$, $k \leqq p$, and $V{:}p \times p$ be independent random matrices, where the rows of X are independently and identically distributed as $\mathfrak{N}(0, \Sigma)$, and $V \sim \mathfrak{W}(\Sigma, p, n)$.

Theorem 4.1. *The distribution of* $G = XV^{-1}X'$ *is*

$$p(G) = c\,|G|^{(p-k-1)/2}|I + G|^{-(n+k)/2}, \qquad G > 0, \tag{4.1}$$

where

$$c = \pi^{-k(k-1)/4}\prod_1^k \Gamma[\tfrac{1}{2}(n + k - i + 1)]/ \{\Gamma[\tfrac{1}{2}(p - i + 1)]\Gamma[\tfrac{1}{2}(n + k - p - i + 1)]\}.$$

Proof. Because of the invariance of G under the transformation $X \to XA$, $V \to A'VA$, we can assume $\Sigma = I$. Using any square root, let $Y = XV^{-\frac{1}{2}}$ be a transformation from X to Y, with $J(X \to Y) = |V|^{k/2}$. This yields the joint distribution of Y and V, and after integration over $V > 0$ we obtain

$$p(Y) = (2\pi)^{-pk/2}[c(n, p)/c(n + k, p)]|I + Y'Y|^{-(n+k)/2}, \qquad -\infty < y_{ij} < \infty. \tag{4.2}$$

Noting that $|I_p + Y'Y| = |I_k + YY'|$, we can apply Hsu's lemma [1] p. 319, to obtain the distribution of $G = YY'$. ||

Theorem 4.2. *Let* $V = TT'$ *and* $H = XT'^{-1}$.

(i) *If T is lower triangular, then*

$$p(H) = c\,|I_p + H'H|^{-(n+k-p-1)/2} \prod_1^p |(I_p + H'H)^{[i]}|^{-1}. \tag{4.3}$$

(ii) *If T is upper triangular, then*

$$p(H) = c\,|I_p + H'H|^{-(n+k-p-1)/2} \prod_1^p |(I_p + H'H)_{[i]}|^{-1}, \tag{4.4}$$

where

$$c = (2\pi)^{-kp/2}\prod_1^p \Gamma[\tfrac{1}{2}(n + k - i + 1)]/\Gamma[\tfrac{1}{2}(n - i + 1)].$$

PROOF. Transforming from (X, V) to (X, T) and then to (H, T), we obtain an expression similar to (3.8). Now let $M = T(I + H'H)T'$. Using Lemmas 2.1 and 2.2, we obtain the joint distribution of (H, M) and integrating over $M > 0$ yields the result. ||

5. Square root transformation. If $V: p \times p$ is positive definite, we may wish to transform from V to $V^{\frac{1}{2}}$ by $V = (V^{\frac{1}{2}})^2$. The Jacobian is now evaluated and we see that this introduces the symmetric functions of the characteristic roots of V.

THEOREM 5.1. *The Jacobian of the transformation* $V = S^2$, *S symmetric, is* $J(V \to S) = \prod_{i \leq j}(\theta_i + \theta_j)$, *where* $\theta_1, \cdots, \theta_p$ *are the characteristic roots of* S.

PROOF. Taking differentials, we obtain

$$dV = S(dS) + (dS)S. \tag{5.1}$$

Write $S = \Gamma D_\theta \Gamma'$, where Γ is orthogonal, $D_\theta = \text{diag}(\theta_1, \cdots, \theta_p)$, so that (5.1) can be written as

$$\Gamma'(dV)\Gamma = D_\theta \Gamma'(dS)\Gamma + \Gamma'(dS)\Gamma D_\theta. \tag{5.2}$$

Let $W = \Gamma'(dV)\Gamma$, $R = \Gamma'(dS)\Gamma$, then $W = D_\theta R + RD_\theta$, and

$$J(V \to S) = J(dV \to dS) = J(dV \to W)J(W \to R)J(R \to dS).$$

An easy computation gives the result. ||

To make use of this transformation, $\prod_{i \leq j}(\theta_i + \theta_j) \equiv g(\theta)$ should be expressed as a function of S. This can be done for any p, but no general formula seems to be available. If we denote by a_k the kth elementary symmetric function of $\theta_1, \cdots, \theta_p$, then $g(\theta)$ is a function of the a_k's. But $a_k = \text{tr}_k S$, where $\text{tr}_k S$ is the sum of all kth order principal minors of S. For $p = 2, 3, 4$, $g(\theta)$ is equal to $2^2 a_2 a_1$, $2^3 a_3(a_1 a_2 - a_3)$, $2^4 a_4(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4)$, respectively. In particular, for the bivariate case, the density of $S = V^{\frac{1}{2}}$ is

$$p(S) = [c(n, 2)2^2/|\Sigma|^{\frac{1}{2}n}](s_{11}s_{22} - s_{12}^2)^{n-2}(s_{11} + s_{22})\text{etr}(-\tfrac{1}{2}\Sigma^{-1}S^2), \qquad S > 0.$$

We also compute the Jacobian of a related transformation.

COROLLARY 5.2. *The Jacobian of the transformation* $V = SAS$, *S and A symmetric, is* $J(V \to S) = \prod_{i \leq j}(\eta_i + \eta_j)$, *where* $\eta_1, \cdots, \eta_p$ *are the characteristic roots of* $A^{\frac{1}{2}}SA^{\frac{1}{2}}$.

PROOF. Write $Q = A^{\frac{1}{2}}VA^{\frac{1}{2}} = (A^{\frac{1}{2}}SA^{\frac{1}{2}})^2 \equiv B^2$. Then $J(V \to S) = J(V \to Q)$ $J(Q \to B)J(B \to S) = (|A|^{-(p+1)/2})\prod_{i \leq j}(\eta_i + \eta_j)(|A|^{(p+1)/2})$. ||

6. An independence property of the Wishart distribution. Let $V_1, \cdots, V_k$ be independently distributed, $V_j \sim \mathcal{W}(\Sigma; p, n_j)$. In this section we consider the problem of finding NASC for the independence of $\sum_1^k V_j$ and $g(V_1, \cdots, V_k)$, where g is a matrix function of matrix arguments, e.g., $g(V_1, V_2) = V_1^{-\frac{1}{2}} V_2 V_1^{-\frac{1}{2}}$. The genesis of the problem lies in a theorem of Laha [6] which treats the univariate case of the Gamma distribution, and which is related to some of the distributions obtained.

THEOREM 6.1. *Let* $V_j \sim \mathcal{W}(\Sigma, p, n_j), j = 1, \cdots, k$, *be independently distributed.*

(i) *If* $Z \equiv \sum_1^k V_j$ *and* $g(V_1, \cdots, V_k)$ *are independently distributed, then* $g(V_1, \cdots, V_k) \sim g(AV_1A', \cdots, AV_kA')$, *for all non-singular* $p \times p$ *matrices* A.

(ii) *If for each* $B > 0$, *there is an* M *with* $MM' = B$ *and* $g(V_1, \cdots, V_k) \sim g(MV_1M', \cdots, MV_kM')$, *then* Z *and* $g(V)$ *are independent.*

PROOF.

(i) Suppose Z and $g(V)$ are independently distributed, then, for $\Lambda \equiv \Sigma^{-1}$ and writing E_Λ for E_Σ,

$$E_\Lambda \operatorname{etr} i[TZ + Sg(V)] = E_\Lambda \operatorname{etr}(iTZ) E_\Lambda \operatorname{etr}[iSg(V)], \tag{6.1}$$

where T is symmetric, T and S are real. Since $E_\Lambda \operatorname{etr}[iSg(V)]$ is independent of T and since $E_\Lambda \operatorname{etr}(iTZ)$ is analytic for $\mathcal{R}(\Lambda - 2iT) > 0$, then $E_\Lambda \operatorname{etr} i[TZ + Sg(V)]$ is analytic for $\mathcal{R}(\Lambda - 2iT) > 0$ and Equation (6.1) holds for all T in this domain. By direct evaluation,

$$\begin{aligned} E_\Lambda \operatorname{etr}[iTZ + iSg(V)] &= |\Lambda|^{\frac{1}{2}n} |\Lambda - 2iT|^{-\frac{1}{2}n} E_{\Lambda - 2iT} \operatorname{etr}[iSg(V)], \\ E_\Lambda \operatorname{etr}[iTZ] &= |\Lambda|^{\frac{1}{2}n} |\Lambda - 2iT|^{-\frac{1}{2}n}, \end{aligned} \tag{6.2}$$

and hence by (6.1),

$$E_\Lambda \operatorname{etr}[iSg(V)] = E_{\Lambda - 2iT} \operatorname{etr}[iSg(V)], \tag{6.3}$$

for all T with $\mathcal{R}(\Lambda - 2iT) > 0$. By using the transformation $AV_jA' \to V_j$, we find that

$$E_\Lambda \operatorname{etr}[iSg(AVA')] = E_{A'^{-1}\Lambda A^{-1}} \operatorname{etr}[iSg(V)], \tag{6.4}$$

for all non-singular A. Now let $T = \frac{1}{2}iL$, L real symmetric, be defined by $\Lambda + L = A'^{-1}\Lambda A^{-1}$, so that (6.3) and (6.4) are equal, which proves (i).

(ii) Suppose $g(V) \sim g(MVM')$, then using (6.4),

$$E_\Lambda \operatorname{etr}[iSg(V)] = E_\Lambda \operatorname{etr}[iSg(MVM')] = E_{M'^{-1}\Lambda M^{-1}} \operatorname{etr}[iSg(V)]. \tag{6.5}$$

Now choose M so that $M'^{-1}\Lambda M^{-1} = \Lambda + L = \Lambda - 2iT$, L real symmetric. Then by (6.5) and (6.4) we obtain

$$E_\Lambda \operatorname{etr}[iSg(V)] = E_{\Lambda - 2iT} \operatorname{etr}[iSg(V)] = |\Lambda|^{-\frac{1}{2}n} |\Lambda - 2iT|^{\frac{1}{2}n} E_\Lambda \operatorname{etr}[iTZ + ISg(V)],$$

which is equivalent to the independence of Z and $g(V)$. ||

7. Special independence properties. In the following theorems the role of the two square roots is exhibited more clearly.

THEOREM 7.1. *If* $S_1, \cdots, S_k$ *are independently distributed,* $S_j \sim \mathcal{W}(\Sigma, p, n_j)$, *then the statistics* $W_j = (S_1 + \cdots + S_j)^{-\frac{1}{2}} S_{j+1} (S_1 + \cdots + S_j)^{-\frac{1}{2}}, j = 1, \cdots, k - 1$, *are independently distributed, where the square roots are defined by the triangular decomposition.*

PROOF. This theorem has been known and used for some time, and an explicit statement and proof is given by Khatri [5]. An alternative proof is based on the fact that with the triangular decomposition, W_1 and $S_1 + S_2$ are independently distributed, and hence W_1 is independent of W_j, $j = 2, \cdots, k$; similarly W_2 is independent of W_j, $j = 3, \cdots, k$, etc. ||

If the symmetric square root is used, this theorem no longer holds. We show this for the case $k = 2$.

THEOREM 7.2. *If S_1, S_2 are independently distributed, $S_j \sim \mathcal{W}(I, p, n_j)$, the statistics $Z = S_1 + S_2$, $W = S_1^{-\frac{1}{2}} S_2 S_1^{-\frac{1}{2}}$, where $(S_1^{\frac{1}{2}})^2 = S_1$, are not independent, and the joint distribution is given by* (7.1).

PROOF. From the joint distribution of S_1 and S_2, we wish to transform to Z and W. To accomplish this we make some intermediate transformations, (i): $(S_1, S_2) \to (X, W)$, where $X^2 = S_1$. $W = X^{-1} S_2 X^{-1}$, and then (ii): $(X, W) \to (Z, W)$, where $Z = X(I + W)X$. The Jacobian for (i) is given by Theorem 5.1 and [7], $J(S_1, S_2 \to X, W) = \prod_{i \leqq j}(\theta_i + \theta_j)|X|^{p+1}$, where $\theta_1, \cdots, \theta_p$ are the characteristic roots of X. For (ii), by Corollary 5.2, $J(X \to Z) = \prod_{i \leqq j}(\eta_i + \eta_j)^{-1}$, where $\eta_1, \cdots, \eta_p$ are the characteristic roots of $(I + W)^{\frac{1}{2}} X (I + W)^{\frac{1}{2}}$. Hence

$$
\begin{aligned}
p(W, Z) = \frac{c(n_1, p)c(n_2, p)}{|\Sigma|^{\frac{1}{2}(n_1+n_2)}} \frac{|W|^{\frac{1}{2}(n_2-p-1)}}{|I + W|^{\frac{1}{2}(n_1+n_2-p-1)}} \\
\cdot |Z|^{\frac{1}{2}(n_1+n_2-p-1)} e^{-\frac{1}{2}\operatorname{tr} Z} \prod_{i \leqq j} \frac{(\theta_i + \theta_j)}{(\eta_i + \eta_j)},
\end{aligned} \tag{7.1}
$$

$W > 0$, $Z > 0$. The independence of W and Z depends on whether $\prod_{i \leqq j}[(\theta_i + \theta_j)/(\eta_i + \eta_j)] \equiv g(\theta)/g(\eta)$ can be expressed as $h_1(W)h_2(Z)$, i.e., as the product of functions of W and Z. Even for $p = 2$, this cannot be done. To see this, we note that $g(\theta) = 4\,|X|\operatorname{tr} X$, $g(\eta) = 4\,|X|\,|I + W|\operatorname{tr} X(I + W)$. But

$$
X = (I + W)^{-\frac{1}{2}}\{(I + W)^{\frac{1}{2}} Z (I + W)^{\frac{1}{2}}\}^{\frac{1}{2}}(I + W)^{-\frac{1}{2}},
$$

and hence

$$
\frac{g(\theta)}{g(\eta)} = \frac{|I + W|^{-1}[\operatorname{tr}\,(I + W)^{-1}\{(I + W)^{\frac{1}{2}} Z (I + W)^{\frac{1}{2}}\}^{\frac{1}{2}}]}{\operatorname{tr}\,[(I + W)^{\frac{1}{2}} Z (I + W)^{\frac{1}{2}}]^{\frac{1}{2}}},
$$

and it is clear that we do not get a factorization. ||

In the next theorem the partial sums are so arranged that either square root may be used.

THEOREM 7.3. *If $S_1, \cdots, S_k$ are independently distributed, $S_j \sim \mathcal{W}(\Sigma, p, n_j)$, then the statistics*

$$
X_j = (S_1 + \cdots + S_{j+1})^{-\frac{1}{2}} S_{j+1} (S_1 + \cdots + S_{j+1})^{-\frac{1}{2}}, \quad j = 1, \cdots, k - 1,
$$

are independently distributed for any square root.

PROOF. The following steps are easily verified. X_1 is independent of $S_1 +$

S_2, and hence of $X_2, \cdots, X_{k-1}$; X_2 is independent of $S_1 + S_2 + S_3$, and hence of $X_3, \cdots, X_{k-1}$, etc. ||

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