RADON-NIKODYM DERIVATIVES OF STATIONARY GAUSSIAN MEASURES

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1. Introduction and summary. A problem of considerable importance in time series analysis is that of determining whether two Gaussian processes are equivalent, i.e., mutually absolutely continuous with respect to each other. If we are given $\{\Omega, \mathcal{A}, P_k\}$, $k = 0, 1$, where Ω is a set of real valued functions on some interval $[a, b]$, $\mathcal{A}$ is a Borel field of subsets of Ω and P_k is a Gaussian probability measure on $\mathcal{A}$, then the problem is to determine the conditions under which P_0 is equivalent to P_1. Feldman (1958) has shown that a certain dichotomy exists in this problem in the following sense. If Ω is a linear space, then either P_0 and P_1 are equivalent or they are perpendicular, i.e., mutually singular. In addition, Feldman (1958) has shown, using some results of Segal (1958), that if Λ is the linear span of Ω and the real constants, then P_0 and P_1 are equivalent iff the P_0-equivalence classes of Λ are the same as the P_1-equivalence classes of Λ and the identity correspondence between the $L_2(P_0)$ closure of Λ and the $L_2(P_1)$ closure of Λ is a bounded invertible operator T such that $(T^* T)^{\frac{1}{2}} - I$ is a Hilbert-Schmidt operator.

It is well known that a Gaussian process, and hence its probability measure, is uniquely determined by its covariance function, $R(s, t)$, assuming, of course, that the mean value function is zero. It should, therefore, be possible to determine when P_0 and P_1 are equivalent in terms of the covariance functions of the two processes, $R_0(s, t)$, $R_1(s, t)$. Indeed, it is desirable to have a theorem which states the conditions for equivalence of P_0 and P_1 in terms of the covariance functions, since in most applications the only information concerning the Gaussian processes consists of the covariance functions. In this sense, Feldman's (1958) theorem is in a form which is not too suitable for practical applications. This situation has been corrected by Feldman (1960) who, using his earlier results, has given the necessary and sufficient conditions for equivalence of P_0 and P_1 in terms of the covariance functions for a rather wide class of stationary Gaussian processes with zero mean value functions. However, Feldman (1960) has not computed the Radon-Nikodym derivative (RND) dP_1/dP_0 for pairs of equivalent Gaussian processes. The RND, or likelihood ratio, is of considerable importance in statistical inference for stochastic processes, as is well known.

Recently, Parzen (1961), (1962) has described a new approach to time series analysis which is based on the notion of a reproducing kernel Hilbert space (RKHS). The theory of RKHS has been given by Aronszajn (1950). Using some results due to Aronszajn (1950) and Hájek (1958a), (1958b) Parzen (1962) has derived a necessary and sufficient condition for equivalence of pairs of Gauss-

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ian processes which have the same covariance function but have different mean value functions. In addition, he gives an explicit expression for the RND associated with pairs of equivalent Gaussian processes.

The present work stems from a desire to derive the necessary and sufficient conditions for equivalence of pairs of stationary Gaussian processes with different covariance functions and zero mean value functions and to compute the RND for pairs of certain equivalent Gaussian processes. The method of proof is based on the RKHS approach used by Parzen. Thus, part of the results obtained will serve as an alternate derivation of Feldman's (1960) result. However, Feldman's proof involves a large number of complicated manipulations, while the present proof has the advantage of requiring only a few elementary manipulations. This simplification is achieved by employing convergence theorems of Aronszajn (1950), Hájek (1958a), (1958b) and Hörmander (1963) and an expression for an inner product due to Parzen (1961).

2. Definitions and assumptions. Let Ω be a set of real valued functions on some interval $[a, b]$, $\mathfrak{A}$ is a Borel field of subsets of Ω, and let $\{X(t), a \leqq t \leqq b\}$ and $\{Y(t), a \leqq t \leqq b\}$ be two stationary Gaussian processes, with zero mean value functions, with probability measures P_0 and P_1 determined by their respective covariance functions $R_0(s - t)$, $R_1(s - t)$, $a \leqq s, t \leqq b$. We assume throughout that $R_k(s - t)$ is continuous at $s = t$, $k = 0, 1$. We have by Khintchine's theorem

$$R_k(s - t) = (2\pi)^{-1} \int \exp(i\omega(s - t))\, dG_k, \tag{2.1}$$

where $G_k(\omega)$ is the spectral distribution function and is monotone non-decreasing and of bounded variation, and where the limits of integration of all unspecified integrals are $-\infty$ and ∞. It will be assumed that $G_k(\omega)$ has an absolutely continuous component, with density $g_k(\omega)$, where the spectral density function $g_k(\omega)$ is nonnegative. Since G_k has a component which is absolutely continuous, it follows that $R_k(s - t)$ is positive definite.

Let T denote the closed interval $[a, b]$, $t_k = a + k(b - a)/N$, $N = 2^n$, $n = 1, 2, \cdots$, $k = 0, \cdots, N$, and for any finite subset $T_N = \{t_1, \cdots, t_N\}$ of T let P_{0N} and P_{1N} denote the probability distributions of $\{X(t), t \in T_N\}$ under P_0, P_1, respectively. The assumption made concerning the continuity of the covariance functions assures that the processes under consideration, or some standard modifications, are separable and measurable, cf. Doob (1953), p. 518. Now, for Gaussian processes P_{0N}, P_{1N} are mutually absolutely continuous with RND denoted by

$$p_N = dP_{1N}/dP_{0N} = (2\pi)^{-N/2} |\mathfrak{R}_{1N}|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\mathfrak{X}_N' \mathfrak{R}_{1N}^{-1}\mathfrak{X}_N) / (2\pi)^{-N/2} |\mathfrak{R}_{0N}|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\mathfrak{X}_N'\mathfrak{R}_{0N}^{-1}\mathfrak{X}_N) \tag{2.2}$$

$$\ln p_N = \tfrac{1}{2}(\ln |\mathfrak{R}_{0N}|/|\mathfrak{R}_{1N}| + \mathfrak{X}_N' (\mathfrak{R}_{0N}^{-1} - \mathfrak{R}_{1N}^{-1})\mathfrak{X}_N), \tag{2.3}$$

where the matrices $\mathfrak{X}_N$, $\mathfrak{R}_{0N}$, $\mathfrak{R}_{1N}$ are defined as

$$\mathfrak{X}_N = \begin{bmatrix} X(t_1) \\ \vdots \\ X(t_N) \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}, \tag{2.4}$$

$$\mathfrak{R}_{kN} = \begin{bmatrix} R_k(t_1 - t_1) & \cdots & R_k(t_1 - t_N) \\ & \cdots & \\ R_k(t_N - t_1) & \cdots & R_k(t_N - t_N) \end{bmatrix}, \qquad k = 0, 1, \tag{2.5}$$

and $\mathfrak{R}_{kN}^{-1}$ denotes inverse matrix, $|\mathfrak{R}_{kN}|$ denotes determinant and $\mathfrak{X}_N'$ denotes transpose matrix.

The divergence between P_{1N} and P_{0N} is defined as

$$\begin{aligned} J_N &= E_1(\ln p_N) - E_0(\ln p_N) \\ &= \tfrac{1}{2}E_1(\mathfrak{X}_N'(\mathfrak{R}_{0N}^{-1} - \mathfrak{R}_{1N}^{-1})\mathfrak{X}_N) - \tfrac{1}{2}E_0(\mathfrak{X}_N'(\mathfrak{R}_{0N}^{-1} - \mathfrak{R}_{1N}^{-1})\mathfrak{X}_N). \end{aligned} \tag{2.6}$$

It was pointed out by Hájek (1958a) that $0 \leqq J_N \leqq J_{N^*}$, if $N \leqq N^*$. Consequently, the limit $J_T = \text{limit}_{N\to\infty} J_N$ exists and is finite or infinite. In addition, Hájek (1958b) has shown that P_0 and P_1 are equivalent or singular depending, respectively, on whether J_T is finite or infinite, and if P_0 and P_1 are equivalent the RND is given by $dP_1/dP_0 = \text{limit}_{N\to\infty} p_N$. We will use Hájek's results to determine necessary and sufficient conditions for equivalence of P_0, P_1 and to compute the RND.

Since both $R_0(s - t)$ and $R_1(s - t)$ are positive definite, the matrices $\mathfrak{R}_{0N}$, $\mathfrak{R}_{1N}$ are positive definite. Therefore, there exists a nonsingular matrix $\mathfrak{C}_N$ such that, cf. Anderson (1958), p. 341,

$$\mathfrak{C}_N'\mathfrak{R}_{0N}\mathfrak{C}_N = \begin{bmatrix} \lambda_{N1} & 0 & \cdots & 0 \\ 0 & \lambda_{N2} & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \lambda_{NN} \end{bmatrix} = \Lambda_N \tag{2.7}$$

$$\mathfrak{C}_N'\mathfrak{R}_{1N}\mathfrak{C}_N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_N, \tag{2.8}$$

where $\lambda_{N1} \leqq \cdots \leqq \lambda_{NN}$ are the eigenvalues of the matrix equation

$$\mathfrak{R}_{0N}\, \mathfrak{X} = \lambda\, \mathfrak{R}_{1N}\, \mathfrak{X}, \tag{2.9}$$

and are all positive.

Let $\mathcal{Y}_N = \mathcal{C}_N' \mathcal{X}_N$, so that

$$
\begin{aligned}
J_N &= \tfrac{1}{2}E_1(\mathcal{Y}_n' \mathcal{C}_N^{-1}(\mathcal{R}_{0N}^{-1} - \mathcal{R}_{1N}^{-1})\ (\mathcal{C}_N')^{-1}\mathcal{Y}_N) \\
&\quad -\tfrac{1}{2}E_0(\mathcal{Y}_N' \mathcal{C}_N^{-1}(\mathcal{R}_{0N}^{-1} - \mathcal{R}_{1N}^{-1})\ (\mathcal{C}_N')^{-1}\mathcal{Y}_N) \\
&= \tfrac{1}{2}E_1(\mathcal{Y}_N'(\Lambda_N^{-1} - I_N)\mathcal{Y}_N) - \tfrac{1}{2}E_0(\mathcal{Y}_N'(\Lambda_N^{-1} - I_N)\mathcal{Y}_N) \\
&= \tfrac{1}{2}\sum_{i=1}^{N} (\lambda_{Ni} - 1)^2/\lambda_{Ni}.
\end{aligned}
\tag{2.10}
$$

It is easily seen that J_N approaches a finite limit iff λ_{N1} is bounded uniformly away from zero, i.e.,

$$
\text{limit}_{N\to\infty}\, \lambda_{N1} > 0, \tag{2.11}
$$

and $\text{limit}_{N\to\infty} \sum_{i=1}^{N} (\lambda_{Ni} - 1)^2$ is finite. It should be noted that, due to the variational properties of eigenvalues, $\{\lambda_{N1}\}$ is a nonincreasing sequence so that $\text{limit}_{N\to\infty}$ λ_{N1} exists.

LEMMA 2A. *A sufficient condition for* $\text{limit}_{N\to\infty}\, \lambda_{N1} > 0$ *is* $g_0(\omega)/g_1(\omega) \to \delta > 0$, $\omega \to \infty$, *and* $dG_1 = d\omega/(1 + \omega^2)^u$, *u is an integer greater than or equal to unity.*

PROOF. Let $Y_{N1}, \cdots, Y_{NN}$ denote the elements of the matrix $\mathcal{Y}_N$. We have $E_1(Y_{N1}^2) = 1$, all N, and $E_0(Y_{N1}^2) = \lambda_{N1}$. However, since $Y_{N1} = \sum_{k=1}^{N} c_{1k} X_k$ we obtain

$$
E_1(Y_{N1}^2) = E_1 \sum_{j,k=1}^{N} c_{1j}c_{1k}X_jX_k = \sum_{j,k=1}^{N} c_{1j}c_{1k}R_1(t_j - t_k) = 1, \tag{2.12}
$$

and similarly

$$
\sum_{j,k=1}^{N} c_{1j}c_{1k}R_0(t_j - t_k) = \lambda_{N1}. \tag{2.13}
$$

We can rewrite Equations (2.12), (2.13) as

$$
(2\pi)^{-1} \int |K_N|^2\, dG_1 = 1, \tag{2.14}
$$

$$
(2\pi)^{-1} \int |K_N|^2\, dG_0 = \lambda_{N1}, \tag{2.15}
$$

where

$$
K_N(\omega) = \sum_{k=1}^{N} c_{1k} \exp(i\omega t_k). \tag{2.16}
$$

The c_{jk}'s depend on N, but for simplicity, this is not brought out by the notation.

Now, choosing A very large and positive we get

$$
(2\pi)^{-1} \int_{|\omega| \leq A} |K_N|^2\, dG_1 + (2\pi)^{-1} \int_{|\omega| > A} |K_N|^2\, dG_1 = 1, \tag{2.17}
$$

$$
(2\pi)^{-1} \int_{|\omega| \leq A} |K_N|^2\, dG_0 + (2\pi)^{-1} \int_{|\omega| > A} |K_N|^2\, dG_0 = \lambda_{N1}. \tag{2.18}
$$

Equation (2.18) can be written as

$$(2.19) \qquad (2\pi)^{-1} \int_{|\omega| \leqq A} |K_N|^2 \, dG_0 + \delta(2\pi)^{-1} \int_{|\omega| > A} |K_N|^2 \, dG_1 = \lambda_{N1}.$$

Thus, if $\lambda_{N1} \to 0$, we must have

$$(2.20) \qquad \int_{|\omega| \leqq A} |K_N|^2 \, dG_0 \to 0,$$

$$(2.21) \qquad (2\pi)^{-1} \int_{|\omega| \leqq A} |K_N|^2 \, dG_1 \to 1.$$

As a consequence of (2.20) and (2.14) there must be a subsequence K_{N_j}, uniformly bounded and equicontinuous on every compact set, Hörmander (1963), p. 38, converging to zero on a set of positive measure. However, K_{N_j} must converge to an analytic function, Hörmander (1963), p. 37, so that K_{N_j} converges to zero everywhere, $|\omega| \leqq A$. In addition, we can bound this sequence by a function integrable with respect to G_1, $|\omega| \leqq A$, so that by Lebesgue's theorem on dominated convergence $\int_{|\omega| \leqq A} |K_{N_j}|^2 \, dG_1 \to 0$, which is a contradiction of (2.21). Therefore $\text{limit}_{N \to \infty} \lambda_{N1} > 0$.

It should be noted that the results concerning the convergence of K_{N_j} to an analytic function could also be obtained from a proposition of Feldman (1960).

3. Reproducing kernel Hilbert spaces. We now introduce the concept of a RKHS which will be used extensively in the ensuing discussions. A more complete treatment of the definitions and concepts involved has been given by Parzen (1961). In our discussions we will speak only of separable Hilbert spaces which are complete with respect to a suitably defined norm.

DEFINITION. A Hilbert space H is said to be a RKHS with reproducing kernel (RK) R, if the members of H are functions on some set T, and if there is a kernel R on $T \otimes T$ having the following two properties; for every $t \in T$ $R(\cdot, t) \in H$, where $R(\cdot, t)$ is the function defined on T, with value at $s \in T$ equal to $R(s, t)$, and, in addition, there is a reproducing kernel inner product (RKIP) defined as $(g, R(\cdot, t))_R = g(t)$, for every g in H.

It is known that the covariance kernel R of a stochastic process generates a unique Hilbert space, $H(R)$, of which R is the RK.

Following Aronszajn (1950), pp. 357–362, we introduce the notion of a direct product space. Consider two identical RKHS $H(R)$, $H(R)$, and the norm $\| \ \|_R$ corresponding to the RK R, and form the direct product $H'(R)$ of $H(R)$ with $H(R)$, $H'(R) = H(R) \otimes H(R)$. This direct product is constructed in the following manner: We form the product set $T' = T \otimes T$ of all couples of points $\{s, t\}$, s, $t \in T$. We will only consider the case of infinite dimensional Hilbert space, the finite-dimensional case being similar and somewhat simpler.

In the set T' consider the class of all functions $g'(s, t)$ representable in the form

$$g'(s, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} \, g_i(s) g_k(t),$$

where $\{g_i\}$ is a complete orthonormal sequence in the space $H(R)$, and also define

$$\|g'\|_{R\otimes R}^2 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i'=1}^{\infty}\sum_{k'=1}^{\infty} \alpha_{ik}\alpha_{i'k'}\,(g_i, g_{i'})_R(g_k, g_{k'})_R = \sum_{i,k=1}^{\infty} \alpha_{ik}^2 < \infty .$$

It is shown by Aronszajn that the class of all functions $g'(s, t)$ with norm given above forms the direct product RKHS $H'(R)$ and $R'(s_1, s_2, t_1, t_2) = R(s_1, t_1)\cdot R(s_2, t_2)$ is the RK of $H'(R)$.

EXAMPLE. Let

$$R(s-t) = (2\pi)^{-1}\int \left|\sum_{k=0}^{m} a_k(i\omega)^{m-k}\right|^{-2} \exp\,(i\omega(s-t))\,d\omega, \tag{3.1}$$

where the polynomial $\sum_{k=0}^{m} a_k z^{m-k}$ has no zeros in the right half of the complex z-plane. It has been shown by Parzen (1961), p. 970, that the RKHS $H(R)$ contains all functions $g(t)$ on $a \leqq t \leqq b$ which are continuously differentiable of order m. The RKIP is given by

$$(h, g)_R = \int_a^b (L_t h)(L_t g)\,dt + \sum_{j,k=0}^{m-1} d_{jk}\,h^{(j)}(a)g^{(k)}(a), \tag{3.2}$$

where

$$L_t h = \sum_{k=0}^{m} a_k h^{(m-k)}(t), \tag{3.3}$$

and d_{jk} are the elements of the matrix which is the inverse of the matrix whose elements are

$$(\partial^{j+k}/\partial t^j \partial u^k)R(t-u)\,|_{t=u=a}, \quad j, k = 0, \cdots, m-1. \tag{3.4}$$

The RKHS $H'(R)$ consists of all functions $g(s, t)$ on $a \leqq s, t, \leqq b$ which are continuously differentiable of order m in s and in t. The norm in the direct product space is given by

$$\begin{aligned} \|g\|_{R\otimes R}^2 = & \int_a^b\int_a^b |L_s L_t g(s, t)|^2\,ds\,dt \\ & + \sum_{j,k=0}^{m-1} d_{jk}\int_a^b \frac{\partial^j}{\partial s^j} L_t g(s, t)\,\frac{\partial^k}{\partial s^k} L_t g(s, t)\Big|_{s=a}\,dt \\ & + \sum_{j,k=0}^{m-1} d_{jk}\int_a^b \frac{\partial^j}{\partial t^j} L_s g(s, t)\,\frac{\partial^k}{\partial t^k} L_s g(s, t)\Big|_{t=a}\,ds \\ & + \sum_{i,j,k,l=0}^{m-1} d_{jk}\,d_{il}\,\frac{\partial^{i+j}}{\partial s^i \partial t^j} g(s, t)\,\frac{\partial^{l+k}}{\partial s^l \partial t^k} g(s, t)\Big|_{s=t=a}. \end{aligned} \tag{3.5}$$

In passing we note that an explicit expression can be found for d_{jk} by noting that the RKHS of the time series $\{X(a+b-t),\ a \leqq t \leqq b\}$, is the same as that of $\{X(t), a \leqq t \leqq b\}$, so that

$$\int_a^b (L_t h)(L_t g)\, dt + \sum_{j,k=0}^{m-1} d_{jk} h^{(j)}(a) g^{(k)}(a) = \int_a^b (L_t^* h)(L_t^* g)\, dt + \sum_{j,k=0}^{m-1} d_{jk} h^{(j)}(b) g^{(k)}(b), \tag{3.6}$$

where L_t^* is the adjoint of the differential operator L_t. Integrating by parts we get

$$\begin{aligned}\int_a^b (L_t h)(L_t g)\, dt &- \int_a^b (L_t^* h)(L_t^* g)\, dt \\ &= 2 \sum_{\substack{j+k \text{ odd} \\ j<k}}^{m} \sum^{m} a_{m-k} a_{m-j} \sum_{i=0}^{k-j-1} (-1)^i (h^{(j+i)}(b) g^{(k-i-1)}(b) - h^{(j+i)}(a) g^{(k-i-1)}(a)) \\ &= \sum_{j+k \text{ even}}^{m-1} \sum^{m-1} (h^{(j)}(b) g^{(k)}(b) - h^{(j)}(a) g^{(k)}(a)) \\ &\qquad \cdot 2 \sum_{i=\max(0,j+k+1-m)}^{\min(j,k)} (-1)^{j-i} a_{m-i} a_{m+i-j-k-1}.\end{aligned}$$

Therefore, using Equation (3.6) we obtain

$$\begin{aligned} d_{jk} &= 2 \sum_{i=\max(0,j+k+1-m)}^{\min(j,k)} (-1)^{j-i} a_{m-i} a_{m+i-j-k-1}, \qquad j+k \text{ even}, \\ &= 0, j+k \text{ odd}. \end{aligned} \tag{3.7}$$

It is noted that the existence of the d_{jk}'s implies that the matrix $\{d_{jk}\}$ is positive definite, since it is the inverse of the covariance matrix of $X_0, \cdots, X_0^{(m-1)}$, so that the first two sums in Equation (3.5) are nonnegative. The fact that the third sum is also nonnegative follows easily from the definition of the norm in the product space.

The last result we need is a special form of a theorem on the limit of RK's due to Aronszajn (1950). Let $\{T_N\}$ be an increasing sequence of sets, T their sum

$$T = T_1 + T_2 + \cdots, T_1 \subset T_2 \subset \cdots.$$

Let F_N, $N = 1, 2, \cdots$, be a class of functions defined in T_N. For a function $f_N \varepsilon F_N$ we denote by f_{Nm}, $m \leqq N$, the restriction of f_N to the set $T_m \subset T_N$, $f_{NN} = f_N$. We assume that the classes F_N form a decreasing sequence in the sense that for every $f_N \varepsilon F_N$ and every $m \leqq N$, $f_{Nm} \varepsilon F_m$. We further assume that the norms $\| \ \|_{R_N}$ defined in F_N form an increasing sequence in the sense that for every $f_N \varepsilon F_N$ and every $m \leqq N$, $\|f_{Nm}\|_{R_m} \leqq \|f_N\|_{R_N}$. Finally, we assume that every F_N possesses a RK $R_N(s, t)$. Under these conditions we have the following theorem due to Aronszajn (1950), p. 362.

THEOREM. *The kernels R_N converge to a kernel $R(s, t)$ defined for all s, t in T. R is the* RK *of the* RKHS *$H(R)$ consisting of all functions f_0 defined in T such that their restrictions f_{0N} in T_N belong to F_N, $N = 1, 2, \cdots$, and* $\text{limit}_{N\to\infty} \|f_{0N}\|_{R_N} < \infty$. *The norm of $f_0 \varepsilon H(R)$ is given by* $\|f_0\|_R = \text{limit}_{N\to\infty} \|f_{0N}\|_{R_N}$.

This theorem is extremely important and forms the basis for the utility of the RK approach to the present problem.

4. Necessary and sufficient conditions for equivalence. It was shown previously that P_0 and P_1 are equivalent iff $\text{limit}_{N\to\infty} \lambda_{N1} > 0$, and

$$\text{limit}_{N\to\infty} \sum_{i=1}^{N} (\lambda_{Ni} - 1)^2 \tag{4.1}$$

is finite. It is desirable, therefore, to evaluate the limit given in (4.1).

We obtain from Equation (2.9) that

$$\sum_{i=1}^{N} \lambda_{Ni} = \text{Tr}\,\{\mathcal{R}_{1N}^{-1}\,\mathcal{R}_{0N}\} = (R_{0N}, R_{1N})_{R_{1N}\otimes R_{1N}}, \tag{4.2}$$

where the last expression is an inner product in the RKHS corresponding to the kernel $R_{1N} \otimes R_{1N}$, which is a function of four variables defined by

$$R_{1N} \otimes R_{1N}(s_i, s_j, t_k, t_l) = R_1(s_i, t_k)\,R_1(s_j, t_l), \qquad 1 \leqq i, j, k, l \leqq N. \tag{4.3}$$

Equation (4.2) is obtained as follows:

$$\begin{aligned}(R_{0N}, R_{1N})_{R_{1N}\otimes R_{1N}} &= \sum_{i,j,k,l=1}^{N} R_0(s_i, t_k)R_1^{-1}(s_j, t_i)R_1^{-1}(s_l, t_k)R_1(s_l, t_j)\\ &= \sum_{i,j,k=1}^{N} R_0(s_i, t_k)R_1^{-1}(s_j, t_i)\delta(t_j, t_k)\\ &= \sum_{i,k=1}^{N} R_0(s_i, t_k)R_1^{-1}(s_k, t_i)\\ &= \text{Tr}\,\{\mathcal{R}_{0N}\mathcal{R}_{1N}^{-1}\} = \text{Tr}\,\{\mathcal{R}_{1N}^{-1}\mathcal{R}_{0N}\},\end{aligned}$$

since for any two square matrices of same order $\text{Tr}\,\{\mathcal{A}\,\mathcal{B}\} = \text{Tr}\,\{\mathcal{B}\,\mathcal{A}\}$, and where we have used the notation $R_1^{-1}(s_i, t_j)$ for the elements of $\mathcal{R}_{1N}^{-1}$. Similarly,

$$\sum_{i=1}^{N} (1) = N = (R_{1N}, R_{1N})_{R_{1N}\otimes R_{1N}} \tag{4.4}$$

and

$$\sum_{i=1}^{N} \lambda_{Ni}^2 = \text{Tr}\,\{\mathcal{R}_{1N}^{-1}\mathcal{R}_{0N}\}^2 = (R_{0N}, R_{0N})_{R_{1N}\otimes R_{1N}}. \tag{4.5}$$

Using Equations (4.2), (4.4), (4.5), we have

$$\sum_{i=1}^{N} (\lambda_{Ni} - 1)^2 = \|R_{0N} - R_{1N}\|^2_{R_{1N}\otimes R_{1N}}. \tag{4.6}$$

Now it is easily seen that the conditions of Aronszajn's theorem for limits of RK's are satisfied and we have

$$\text{limit}_{N\to\infty} \sum_{i=1}^{N} (\lambda_{Ni} - 1)^2 = \|R_0 - R_1\|^2_{R_1\otimes R_1}. \tag{4.7}$$

This leads us to the following theorem.

THEOREM 4A. *P_0 and P_1 are equivalent* iff $\text{limit}_{N\to\infty} \lambda_{N1} > 0$, *and* $(R_0 - R_1) \varepsilon H(R_1) \otimes H(R_1)$, *in which case the* RND *is given by* $dP_1/dP_0 = \text{limit}_{N\to\infty} \exp \frac{1}{2} ((\hat{X}_N \hat{X}_N^*, R_{1N} - R_{0N})_{R_{0N} \otimes R_{1N}} + \sum_{i=1}^{N} \ln \lambda_{Ni})$, *where $\hat{X}_N \hat{X}_N^*$ is the restriction of $X(s)X(t)$ to T_N'. If* $\text{limit}_{N\to\infty} \prod_{i=1}^{N} \lambda_{Ni}^{\frac{1}{2}} = c$ *exists, then the* RND *can be written in the more convenient form* $dP_1/dP_0 = c \exp \frac{1}{2}(XX^*, R_1 - R_0)_{R_0 \otimes R_1}$.

PROOF. Follows immediately from (4.7) and Hájek's results stated previously.

A more convenient form of Theorem 4A is the following.

THEOREM 4B (Feldman (1960)). *Let $dG_1 = d\omega/(1 + \omega^2)^u$, where u is an integer greater than or equal to unity, and let G_0 be some other finite nonnegative measure on the real line which has a component that is absolutely continuous. Then, P_0 is equivalent to P_1* iff

$$\int_a^b \int_a^b \left| \int (1 + \omega^2)^u \exp (i\omega(s - t)) \, dG' \right|^2 ds \, dt < \infty, \tag{4.8}$$

where $G' = G_0 - G_1$, and the Fourier transform of $(1 + \omega^2)^u \, dG'$ in (4.8) *is interpreted in the sense of Schwartz* (1951) *distributions.*

PROOF. Using (3.5) and interpreting Fourier transforms in the sense of Schwartz (1951) distributions, we obtain

$$\begin{aligned} \|R_0 - R_1\|_{R_1 \otimes R_1}^2 = & \int_a^b \int_a^b \left| \int (1 + \omega^2)^u \exp (i\omega(s - t)) \frac{dG'}{2\pi} \right|^2 ds \, dt \\ & + 2 \sum_{j,k=0}^{u-1} d_{jk} \int_a^b \left(\int (i\omega)^j (1 - i\omega)^u \exp (i\omega(a - t)) \frac{dG'}{2\pi} \right) \\ & \cdot \left(\int (i\omega)^k (1 - i\omega)^u \exp (i\omega(a - t)) \frac{dG'}{2\pi} \right) dt \\ & + \sum_{i',j,k,l=0}^{u-1} (-1)^{j+k} d_{jk} \, d_{i'l} \int (i\omega)^{i'+j} \frac{dG'}{2\pi} \int (i\omega)^{k+l} \frac{dG'}{2\pi}. \end{aligned} \tag{4.9}$$

Now it is easy to see that $\|R_0 - R_1\|_{R_1 \otimes R_1}$ is finite iff (4.8) is true. In addition (4.8) implies $g_0/g_1 \to 1$, $\omega \to \infty$ so that $\text{limit}_{N\to\infty} \lambda_{N1} > 0$. This completes the proof.

The following theorem follows directly from Theorem 4B, by noting that if $P_0 \sim P_1 \sim P_2$, then $P_0 \sim P_2$ and if $P_0 \sim P_1$, $P_1 \perp P_2$, then $P_0 \perp P_2$, cf. Halmos (1950), p. 133.

THEOREM 4C. *If the spectral densities g_k, $k = 0, 1$, are of the following form*

$$|\omega|^{q_k} g_k(\omega) = c_k + O(\omega^{-2}), \qquad q_0 > 1, \; q_1 = 2, 4, \cdots, \; c_0 c_1 \neq 0,$$

then P_0 and P_1 are equivalent iff $g_0/g_1 \to 1$, $\omega \to \infty$, or what is the same thing, iff $(c_0, q_0) = (c_1, q_1)$.

It would be desirable to have a theorem, similar to Theorem 4C, in which g_1 would have arbitrary high frequency behaviour, i.e., q_1 need not be restricted to even integral values but can be any number greater than 1. The main difficulty in proving such a theorem with the present method is that of writing the RKIP for such kernels. Recently, Gladyshev (1961), using methods similar to those of Baxter (1956), has proved a theorem which shows the necessity of the condition $g_0/g_1 \to 1$, $\omega \to \infty$, for equivalence of P_0 and P_1 in the case when g_0 and g_1 have arbitrary high frequency behaviour. The sufficiency of this condition for equivalence of P_0 and P_1 still remains to be proved. It is worthwhile to point out that Theorem 4B has been stated in a different form, in terms of the covariance functions, by Yaglom (1963).

The following theorem, which has also been derived recently by Parzen (1963), is of some interest and follows from Equation (2.10) and computations similar to those given in Equations (4.2)–(4.6).

THEOREM 4D. *P_0 is equivalent to P_1 iff* $(R_0 - R_1) \varepsilon H(R_0) \otimes H(R_1)$.

This form of the theorem is inconvenient for the general treatment of stationary Gaussian processes since it is no longer possible to write the RKIP by introducing a single reference covariance function. This form can be used, however, to rederive a theorem of Varberg (1961), since there the RKIP in $H(R_0) \otimes H(R_1)$ can be written directly.

5. Computation of Radon-Nikodym derivatives. We will now compute the RND for pairs of equivalent autoregressive Gaussian schemes. These processes are of importance in many practical applications. In this case we have

$$R_0(s - t) = (2\pi)^{-1} \int \left| \sum_{k=0}^{m_0} a_k (i\omega)^{m_0-k} \right|^{-2} \exp(i\omega(s - t))\, d\omega \tag{5.1}$$

$$R_1(s - t) = (2\pi)^{-1} \int \left| \sum_{k=0}^{m_1} b_k (i\omega)^{m_1-k} \right|^{-2} \exp(i\omega(s - t))\, d\omega \tag{5.2}$$

where the polynomials $\sum_{k=0}^{m_0} a_k z^{m_0-k}$, $\sum_{k=0}^{m_1} b_k z^{m_1-k}$ have no zeros in the right half of the complex z-plane. We also define

$$L_t h = \sum_{k=0}^{m_0} a_k h^{(m_0-k)}(t), \tag{5.3}$$

$$M_t h = \sum_{k=0}^{m_1} b_k h^{(m_1-k)}(t), \tag{5.4}$$

and assume that $a = 0$, $b = T$. It follows from Theorem 4C that two such processes are equivalent iff $m_0 = m_1 = m$, $a_0 = b_0$.

We obtain from (3.2) that

$$\begin{aligned}(XX^*, R_1 - R_0)_{R_0 \otimes R_1} &= \sum_{j,k=1}^{m} (a_j + b_j)(a_k - b_k) \int_0^T X^{(m-j)}(t) X^{(m-k)}(t)\, dt \\ &\quad + 2a_0 \sum_{k=1}^{m} (a_k - b_k) \int_0^T X^{(m-k)}(t)\, dX^{(m-1)}(t) \\ &\quad + \sum_{j,k=0}^{m-1} (d_{jk}^0 - d_{jk}^1) X^{(j)}(0) X^{(k)}(0),\end{aligned} \tag{5.5}$$

where

$$d_{jk}^{0} = 2 \sum_{i=\max(0,j+k+1-m)}^{\min(j,k)} (-1)^{j-i} a_{m-i} a_{m+i-j-k-1}, \qquad j+k \text{ even} \tag{5.6}$$
$$= 0, \qquad j+k \text{ odd},$$

$$d_{jk}^{1} = 2 \sum_{i=\max(0,j+k+1-m)}^{\min(j,k)} (-1)^{j-i} b_{m-i} b_{m+i-j-k-1}, \qquad j+k \text{ even} \tag{5.7}$$
$$= 0, \qquad j+k \text{ odd}.$$

We will now evaluate

$$\text{limit}_{N\to\infty} \prod_{i=1}^{N} \lambda_{Ni}^{\frac{1}{2}} = \text{limit}_{N\to\infty} |\mathcal{R}_{0N}|^{\frac{1}{2}}/|\mathcal{R}_{1N}|^{\frac{1}{2}}. \tag{5.8}$$

A similar computation for the limits of these determinants has been given recently by Hájek (1962). The limit in (5.8) will be evaluated by noting

$$|\mathcal{R}_{0N}| = |E_0(X_jX_k \mid X_0, \cdots, X_0^{(m-1)})|/|d_{jk}^0|, \tag{5.9}$$

$$|\mathcal{R}_{1N}| = |E_1(X_jX_k \mid X_0, \cdots, X_0^{(m-1)})|/|d_{jk}^1|, \tag{5.10}$$

where $|a_{jk}|$ denotes the determinant of the matrix whose elements are a_{jk}, $j, k = 1, \cdots, N$. It is, however, more convenient to work with the conditional expectations of $X_j^{(m-1)}X_k^{(m-1)}$, which is possible since

$$\begin{aligned}&\text{limit}_{N\to\infty} |E_0(X_jX_k \mid X_0, \cdots, X_0^{(m-1)})|/|E_1(X_jX_k \mid X_0, \cdots, X_0^{(m-1)})| \\ &= \text{limit}_{N\to\infty} |E_0(X_j^{(m-1)}X_k^{(m-1)} \mid X_0, \cdots, X_0^{(m-1)})|/ \\ &\qquad\qquad |E_1(X_j^{(m-1)}X_k^{(m-1)} \mid X_0, \cdots, X_0^{(m-1)})|,\end{aligned} \tag{5.11}$$

if either limit exists. By putting the determinants in diagonal form, we can write

$$\frac{|E_0(X_j^{(m-1)}X_k^{(m-1)} \mid X_0, \cdots, X_0^{(m-1)})|}{|E_1(X_j^{(m-1)}X_k^{(m-1)} \mid X_0, \cdots, X_0^{(m-1)})|} = \prod_{j=1}^{N} \frac{E_0(X_j^{(m-1)} - X_{j,0}^{(m-1)})^2}{E_1(X_j^{(m-1)} - X_{j,1}^{(m-1)})^2} \tag{5.12}$$

where $X_{j,k}^{(m-1)}$, $k = 0, 1$, is the projection of $X_j^{(m-1)}$ on the subspace spanned by $X_0, \cdots, X_0^{(m-1)}, X_1^{(m-1)}, \cdots, X_{j-1}^{(m-1)}$, under P_k, i.e., the random variable in the subspace which is closest to $X_j^{(m-1)}$ in the mean square sense, cf. Doob (1953), p. 155.

It is simpler to find first the projection, under P_0, of $X_j^{(m-1)}$, say Z_{j0}, on the subspace spanned by $\{X(t), 0 \leqq t \leqq (j-1)T/N\}$. We obtain from (3.6) that

$$\begin{aligned}(h, g)_{R_0} &= \int_0^T (L_th)(L_tg)\,dt + \sum_{j,k=0}^{m-1} d_{jk}^0 h^{(j)}(0)g^{(k)}(0) \\ &= \int_0^T (L_t^*h)(L_t^*g)\,dt + \sum_{j,k=0}^{m-1} d_{jk}^0 h^{(j)}(T)g^{(k)}(T).\end{aligned} \tag{5.13}$$

Using (5.13) we may evaluate Z_{i0} by writing

$$(5.14)\qquad Z_{i0} = \int_0^{(i-1)T/N} (L_t^* X(t)(L_t^* E_0(X_i^{(m-1)} X(t))\, dt + \sum_{j,k=0}^{m-1} d_{jk}^0 X_{i-1}^{(j)} \frac{\partial^k}{\partial t^k} E_0(X_i^{(m-1)} X(t) \,|_{t=(i-1)T/N}$$

However, $L_t^* E_0(X_i^{(m-1)} X(t)) = L_t^* R_0^{(m-1)}(iT/N - t)$, and $L_t^* R_0(s - t) = 0$, $s > t$, therefore

$$L_t^* R_0^{(m-1)}(iT/N - t) = 0, \qquad 0 \leqq t \leqq (i-1)T/N,$$

so that (5.14) can be written as

$$(5.15)\qquad \begin{aligned} Z_{i0} &= \sum_{j,k=0}^{m-1} d_{jk}^0 X_{i-1}^{(j)} (\partial^k/\partial t^k)(\partial^{m-1}/\partial s^{m-1}) R_0(s-t)|_{s=iT/N, t=(i-1)T/N} \\ &= \sum_{j,k=0}^{m-1} d_{jk}^0 X_{i-1}^{(j)} (-1)^k R_0^{(m-1+k)}(T/N) \\ &= \sum_{j=0}^{m-1} X_{i-1}^{(j)} \sum_{k=0}^{m-1} d_{jk}^0 (-1)^k (R_0^{(m-1+k)}(0) \\ &\qquad + (T/N) R_0^{(m+k)}(0+) + O(N^{-2})). \end{aligned}$$

Now, we have

$$(5.16)\qquad \begin{aligned} \sum_{k=0}^{m-1} (-1)^k R_0^{(m-1+k)}(0)\, d_{jk}^0 &= 1, \qquad \text{if } j = m-1 \\ &= 0, \qquad \text{if } j \neq m-1, \end{aligned}$$

since $(-1)^k R_0^{(j+k)}(0)$ represent the elements of the covariance matrix of $X_0, \cdots, X_0^{(m-1)}$, cf. (3.4), and d_{jk}^0 is its inverse. Moreover, in view of $L_t^* R_0^{(k)}(s - t) = 0$, $t < s$, $k = 0, 1, 2, \cdots$, we have

$$R_0^{(m+k)}(0+) = -a_0^{-1} \sum_{h=0}^{m-1} a_{m-h} R_0^{(h+k)}(0+), \quad k = 0, 1, 2, \cdots,$$

so that

$$(5.17)\qquad \sum_{k=0}^{m-1} (-1)^k R_0^{(m+k)}(0+)\, d_{jk}^0 = -a_{m-j}/a_0, \qquad 0 \leqq j \leqq m-1.$$

If we use (5.16) and (5.17) in (5.15) we get

$$(5.18)\qquad Z_{i0} = X_{i-1}^{(m-1)} - (T/N) \sum_{j=0}^{m-1} (a_{m-j}/a_0) X_{i-1}^{(j)} + O(N^{-2}).$$

Now $X_{i-1}^{(j)}$ can be written as

$$(5.19)\qquad \begin{aligned} X_{i-1}^{(j)} &= \int_0^{(i-1)T/N} X^{(j+1)}(t)\, dt \\ &= X_0^{(j)} + \sum_{h=1}^{i-1} X_h^{(j+1)} T/N + \epsilon, \quad j = 0, \cdots, m-1, \end{aligned}$$

where $E_0(\epsilon^2) = O(N^{-1})$. Thus, using (5.19) in (5.18) we may evaluate $X_{i,0}^{(m-1)}$ as

$$X_{i,0}^{(m-1)} = X_{i-1}^{(m-1)} - (T/N) \sum_{j=0}^{m-1} (a_{m-j}/a_0) X_{i-1}^{(j)} + O(N^{-2}). \tag{5.20}$$

Hence

$$\begin{aligned} E_0(X_i^{(m-1)} - X_{i,0}^{(m-1)})^2 &= 2(1 - (-1)^m R_0^{(2m-2)}(T/N)) \\ &\quad + (2T/N) \sum_{j=0}^{m-1} (a_{m-j}/a_0)(-1)^j (R_0^{(m-1+j)}(T/N) - R_0^{(m-1+j)}(0)) \\ &\quad + (T/N)^2 \sum_{j,k=0}^{m-1} (a_{m-j}a_{m-k}/a_0^2)(-1)^j R_0^{(j+k)}(0+) + O(N^{-3}) \\ &= 2(-1)^m (T/N) R_0^{(2m-1)}(0) + (-1)^m (T/N)^2 R_0^{(2m)}(0+) \\ &\quad + 2(T/N)^2 \sum_{j=0}^{m-1} (a_{m-j}/a_0)(-1)^j R_0^{(m+j)}(0+) \\ &\quad + (T/N)^2 \sum_{j,k=0}^{m-1} (a_{m-j}a_{m-k}/a_0^2)(-1)^j R_0^{(j+k)}(0+) + O(N^{-3}) \qquad (5.21) \\ &= 2(-1)^m (T/N) R_0^{(2m-1)}(0+) \\ &\quad + (T/N)^2 \sum_{j=0}^{m-1} (a_{m-j}/a_0)(-1)^j R_0^{(m+j)}(0+) + O(N^{-3}) \\ &= 2(-1)^m (T/N) R_0^{(2m-1)}(0+) \\ &\quad - 2(-1)^m (T/N)^2 (a_1/a_0) R_0^{(2m-1)}(0+) + O(N^{-3}) \\ &= (T/N) a_0^{-2} (1 - (T/N)(a_1/a_0)) + O(N^{-3}), \end{aligned}$$

where use has been made of the well known relations

$$\begin{aligned} R_0^{(2j+1)}(0\pm) &= 0, & j &< m - 1 \\ R_0^{(2j)}(0+) &= R_0^{(2j)}(0-) = R_0^{(2j)}(0), & 0 &\leqq j \leqq m - 1 \\ R_0^{(2m-1)}(0+) &= -R_0^{(2m-1)}(0-) = (-1)^m \tfrac{1}{2} a_0^2, \\ (-1)^m R_0^{(m+k)}(0-) &= -\sum_{h=0}^{m-1} (-1)^h (a_{m-h}/a_0) R_0^{(h+k)}(0-). \end{aligned}$$

Thus, we have

$$\begin{aligned} \operatorname{limit}_{N\to\infty} \frac{|\mathcal{R}_{0N}|^{\frac{1}{2}}}{|\mathcal{R}_{1N}|^{\frac{1}{2}}} &= \frac{|d_{jk}^1|^{\frac{1}{2}}}{|d_{jk}^0|^{\frac{1}{2}}} \operatorname{limit}_{N\to\infty} \\ &\quad \cdot \prod_{i=1}^{N} \left[\frac{(T/N)a_0^{-2}(1 - (T/N)(a_1/a_0)) + O(N^{-3})}{(T/N)b_0^{-2}(1 - (T/N)(b_1/b_0)) + O(N^{-3})} \right]^{\frac{1}{2}} \qquad (5.22) \\ &= \frac{|d_{jk}^1|^{\frac{1}{2}}}{|d_{jk}^0|^{\frac{1}{2}}} \exp \frac{1}{2} T \left(\frac{b_1}{b_0} - \frac{a_1}{a_0} \right). \end{aligned}$$

The RND associated with pairs of equivalent Gaussian autoregressive schemes is obtained from (5.5) and (5.22) as

$$\frac{dP_1}{dP_0} = \frac{|d_{jk}^1|^{\frac{1}{2}}}{|d_{jk}^0|^{\frac{1}{2}}} \exp \frac{1}{2} \Big(T \Big(\frac{b_1}{b_0} - \frac{a_1}{a_0} \Big) + \sum_{j,k=1}^{m} (a_j + b_j)(a_k - b_k) \int_0^T X^{(m-k)}(t) X^{(m-j)}(t)\, dt + 2a_0 \sum_{k=1}^{m} (a_k - b_k) \int_0^T X^{(m-k)}(t)\, dX^{(m-1)}(t) + \sum_{j,k=0}^{m-1} (d_{jk}^0 - d_{jk}^1) X_0^{(j)} X_0^{(k)} \Big). \tag{5.23}$$

The expressions for d_{jk}^0, d_{jk}^1 are given in (5.6), (5.7), respectively.

Example 1. $m = 1$. Let

$$R_0(s-t) = \sigma_0^2 \exp -\beta_0 |s-t|, \qquad R_1(s-t) = \sigma_1^2 \exp -\beta_1 |s-t|,$$

$$2\sigma_0^2\beta_0 = 2\sigma_1^2\beta_1 = K, \qquad \sigma_k^2 > 0,\ \beta_k > 0,\ k = 0, 1.$$

The RND is given by

$$\frac{dP_1}{dP_0} = \Big(\frac{\beta_1}{\beta_0}\Big)^{\frac{1}{2}} \exp -\frac{1}{2K}\Big((\beta_1 - \beta_0)(X_0^2 + X_T^2 - KT) + (\beta_1^2 - \beta_0^2) \int_0^T X^2(t)\, dt\Big).$$

This result agrees with that given by Streibel (1959).

Example 2. $m = 2$. Let

$$g_0(\omega) = |-\omega^2 + 2\alpha_0 \omega i + \gamma_0^2|^{-2},$$
$$g_1(\omega) = |-\omega^2 + 2\alpha_1 \omega i + \gamma_1^2|^{-2}, \qquad \alpha_k > 0, \gamma_k^2 > 0, k = 0, 1.$$

The RND is given by

$$\frac{dP_1}{dP_0} = \frac{\alpha_1\gamma_1}{\alpha_0\gamma_0} \exp -\Big((\alpha_1 - \alpha_0)((X_0^{(1)})^2 + (X_T^{(1)})^2 - T) + (\alpha_1\gamma_1^2 - \alpha_0\gamma_0^2)(X_0^2 + X_T^2) + (\gamma_1^2 - \gamma_0^2)(X_T X_T^{(1)} - X_0 X_0^{(1)}) + (2\alpha_1^2 - 2\alpha_0^2 - \gamma_1^2 + \gamma_0^2) \int_0^T (X^{(1)}(t))^2\, dt + \frac{1}{2}(\gamma_1^4 - \gamma_0^4) \int_0^T X^2(t)\, dt\Big).$$

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