# A REMARK ON THE COIN TOSSING GAME[1]

By David Blackwell and David Freedman

*University of California, Berkeley*

Let $X_n : n \geqq 1$ be independent and identically distributed random variables, assuming the values $\pm 1$ with probability $\frac{1}{2}$ each. Let $S_n = X_1 + \cdots + X_n$. If $0 < c < \infty$, DeMoivre's (1718) central limit theorem implies $|S_n| > cn^{\frac{1}{2}}$ for a large enough $n$. How large? Let $\tau(N, c)$ be the least $n \geqq N$ with $|S_n| > cn^{\frac{1}{2}}$.

Theorem 1. *The mean waiting time for $|S_n|$ to exceed $n^{\frac{1}{2}}$ is infinite; that is,* $E[\tau(1, 1)] = \infty$.

Theorem 2. *If* $0 < c < 1$, *the mean waiting time for $|S_n|$ to exceed $cn^{\frac{1}{2}}$ is finite; that is,* $E[\tau(N, c)] < \infty$ *for all* $N$.

Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorem IIa (Blackwell and Girshick (1947)). The direct proof given here may be of interest.

Consider the Markov chain with state space $S$ of all pairs of integers, and stationary transition mechanism: from $(m, n)$ move to $(m + 1, n \pm 1)$ with probability $\frac{1}{2}$ each. Let $D$ be the set of $(m, n) \varepsilon S$ with $n^2 \leqq m$. Let $p_k(m, n)$ be the probability that, starting from $(m, n)$, the first $k = 1, 2, \cdots$ positions of the chain are in $D$. In particular, $\frac{1}{2}p_k(2, 0)$ underestimates the probability that $\tau(1, 1)$ exceeds $k + 1$. Therefore, Theorem 1 follows from

$$\sum_{k=1}^{\infty} p_k(2, 0) = \infty, \tag{1}$$

which will now be proved.

For $0 < \beta < 1$, let $U_\beta(m, n) = \sum_{k=1}^{\infty} \beta^{k-1} p_k(m, n)$, and let $T_\beta$ transform real-valued functions with domain $S$:

$$\begin{aligned}(T_\beta f)(m, n) &= 0, && \text{for } (m, n) \not\varepsilon D,\\ &= 1 + \tfrac{1}{2}\beta[f(m + 1, n + 1) + f(m + 1, n - 1)], && \text{for } (m, n) \varepsilon D.\end{aligned}$$

Plainly, $T_\beta$ is a uniformly strict contraction of the Banach space $B$ of bounded, real-valued functions on $S$, in the supremum norm: $\|T_\beta f - T_\beta g\| \leqq \beta\|f - g\|$. Since $U_\beta(\cdot, \cdot)$ is a fixed point of $T_\beta$, the sequence $T_\beta^n f$ converges to $U_\beta(\cdot\ \cdot)$ for any $f \varepsilon B$. Let $W_\beta$ be a real-valued function on $S$, bounded above, with $T_\beta W_\beta \geqq W_\beta$. Then $U_\beta \geqq W_\beta$, because $T_\beta^n \max[W_\beta, 0] \geqq T_\beta^n W_\beta \geqq W_\beta$ and converges to $U_\beta$. This argument has been used in (Blackwell (1954), (1964)), (Dubins (1962)), and considered under very general circumstances in (Dubins and Savage (1963)). It is stated here for ease of reference.

---

Received 3 January 1964.

[1] This paper was prepared with the partial support of the U. S. Army Research Office (Durham), Grant DA-ARO(D)-31-124-G476.

Here is an interesting $W_\beta$ :

$$W_\beta(m, n) = A_\beta(m)(m - n^2) \tag{2}$$

$$A_0 = 0, \qquad A_\beta(m) = \sum_{j=0}^{\infty} \beta^j/(m + j), \qquad m \geqq 1. \tag{3}$$

Clearly, $W_\beta$ is bounded above and $T_\beta W_\beta \geqq W_\beta$ at $(0, 0)$ and on $\{(m, n): (m, n) \varepsilon S \text{ and } n^2 > m\}$. For $m \geqq 1$ and $n^2 \leqq m$, Equation (2) implies

$$(T_\beta W_\beta)(m, n) = 1 + \beta A_\beta(m + 1)(m - n^2),$$

so $(T_\beta W_\beta)(m, n) - W_\beta(m, n) = 1 - (m - n^2)[A_\beta(m) - \beta A_\beta(m + 1)] = n^2/m \geqq 0$, by Equation (3).

The previous paragraph implies $U_\beta \geqq W_\beta$ ; in particular, $U_\beta(2, 0) \geqq 2 \sum_{j=0}^{\infty} \beta^j/(2 + j)$. Making $\beta \uparrow 1$ proves Equation (1), and with it Theorem 1

PROOF OF THEOREM 2. Let $1 < \sigma < \infty$, and choose $M$ so large that

$$(M + 1)^{\frac{1}{2}} - M^{\frac{1}{2}} < (\sigma^2 - 1)/2\sigma. \tag{4}$$

Consider the Markov process with state space $S$ of pairs $(m, y)$, where $m$ is a nonnegative integer and $y$ a real number; and stationary transition mechanism: from $(m, y)$ move to $(m + 1, y \pm \sigma)$ with probability $\frac{1}{2}$ each. Let $D$ be the set of all $(m, y) \varepsilon S$ with $m \geqq M$ and $y^2 \leqq m$. The last assertion of Theorem 2, with $c = 1/\sigma$ and $N \geqq M$, follows from: the mean waiting time $W(m, y)$ for the process to leave $D$, starting from $(m, y) \varepsilon D$, is finite; this will now be proved.

Let $D_1$ be the set of all $(m, y)$ in $S$ but not in $D$ which can be reached from $D$ in one move. Let $T$ transform real-valued functions with domain $S$:

$$\begin{aligned}(Tf)(m, y) &= 0, && \text{for } (m, y) \not\varepsilon D,\\ &= 1 + \tfrac{1}{2}[f(m + 1, y + \sigma) + f(m + 1, y - \sigma)], && \text{for } (m, y) \varepsilon D\end{aligned}$$

Let $V$ be a nonnegative, real-valued function on $D \cup D_1$ with $V \geqq TV$ on $D$. Then $V \geqq W$, because $V \geqq T^n V \geqq T^n 0 \uparrow W$. This argument was developed in the references already cited.

Here is an interesting $V$:

$$V(m, y) = (m - y^2 + 2\sigma m^{\frac{1}{2}} + \sigma^2 - 1)/A, \tag{5}$$

$$A = \sigma^2 - 1 - 2\sigma[(M + 1)^{\frac{1}{2}} - M^{\frac{1}{2}}]. \tag{6}$$

By Equation (4), $A > 0$, so $V \geqq 0$ on $D$. If $(m, y) \varepsilon D_1$ and $y > 0$, then $(m - 1, y - \sigma) \varepsilon D$, so $(y - \sigma)^2 \leqq m - 1$, and $y \leqq \sigma + (m - 1)^{\frac{1}{2}}$, implying $y^2 \leqq m - 1 + 2\sigma(m - 1)^{\frac{1}{2}} + \sigma^2$. By symmetry this also holds for $(m, y) \varepsilon D_1$ with $y < 0$. Therefore $V \geqq 0$ on $D_1$ . If $(m, y) \varepsilon D$, Equation (5) implies $V(m, y) - (TV)(m, y) = [\sigma^2 - 1 - 2\sigma((m + 1)^{\frac{1}{2}} - m^{\frac{1}{2}})]/A - 1 \geqq 0$, using Equation (6) and: $x \to (x + 1)^{\frac{1}{2}} - x^{\frac{1}{2}}$ decreases as $x$ increases through positive values. The previous paragraph implies $V \geqq W$; in particular, $W$ is finite. This completes the proof of Theorem 2.

## REFERENCES

Blackwell, David (1954). On optimal systems. *Ann. Math. Statist.* **25** 394–397.

Blackwell, David (1964). Probability bounds via dynamic programming. *Proc. Symp. Control Processes*. Amer. Math. Soc. (To appear.)

Blackwell, D. and Girshick, M. A. (1947). A lower bound for the variance of some unbiased sequential estimates. *Ann. Math. Statist.* **18** 277–280.

De Moivre, Abraham (1718). *The Doctrine of Chance.*

Dubins, L. E. (1962). Rises and upcrossings of non-negative martingales. *Illinois J. Math.* **6** 226–241.

Dubins, Lester E. and Savage, Leonard J. (1963). *How to Gamble If You Must.* Multilithed, Universities of California and Michigan.