NOTES

THE LAST RETURN TO EQUILIBRIUM IN A COIN-TOSSING GAME

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Let $X_i : i \leqq i < \infty$ be independent random variables with the common distribution: Prob $(X_i = \pm 1) = \frac{1}{2}$ for $1 \leqq i < \infty$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $1 \leqq n < \infty$. Of course, Prob $(S_n = 0) = 0$ unless n is even. Write $P(m, n)$ for the probability that $S_{2i} = 0$ for at least one i satisfying $m \leqq i < m + n$.

The purpose of this note is to prove

$$P(m, n) + P(n, m) = 1 \tag{1}$$

for $m \geqq 1$ and $n \geqq 1$. In particular: $P(m, m) = \frac{1}{2}$. Formula (1) is known for $m = 1$ and all n; see (Feller, 1960, Equations 4.4 and 4.5, pp. 74–75). The generalization to arbitrary m was suggested by an IBM 7090 calculation, performed mainly to check a program. The calculation gave $P(m, n)$ and $P(n, m)$ to four decimals for 120 pairs (m, n) with $100 \leqq m, n \leqq 500$, and (1) was true to four decimals for each pair.

Let L_{2j} be the largest $n \leqq 2j$ with $S_n = 0$. After some time, we realized (1) was equivalent to

$$\text{Prob}\,(L_{2j} = 2k) = \text{Prob}\,(L_{2j} = 2j - 2k) \tag{$[j.k]$}$$

for $1 \leqq j < \infty$ and $0 \leqq k \leqq j$, because $P(m, n) = \text{Prob}\,(L_{2m+2n-2} \geqq 2m)$ and $1 - P(n, m) = \text{Prob}\,(L_{2m+2n-2} \leqq 2m - 2)$. In particular, $[j.0]$ is known. But $[j.k]$ can be deduced from $[k.0]$ and $[k - j.0]$ by this easy calculation:

$$\begin{aligned}
\text{Prob}\,(L_{2j} = 2k) &= \text{Prob}\,(S_{2k} = 0 \text{ and } S_{2k+2i} \neq 0 \text{ for } 1 \leqq i \leqq j - k) \\
&= \text{Prob}\,(S_{2k} = 0)\,\text{Prob}\,(S_{2i} \neq 0 \text{ for } 1 \leqq i \leqq j - k) \\
&= \text{Prob}\,(S_{2i} \neq 0 \text{ for } 1 \leqq i \leqq k)\,\text{Prob}\,(S_{2j-2k} = 0) \\
&= \text{Prob}\,(S_{2j-2k} = 0 \text{ and } S_{2j-2k+2i} \neq 0 \text{ for } 1 \leqq i \leqq k) \\
&= \text{Prob}\,(L_{2j} = 2j - 2k).
\end{aligned}$$

REFERENCE

[1] Feller, William (1960). *An Introduction to Probability Theory and Its Applications*, (2nd ed.). Wiley, New York.

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