ON THE ASYMPTOTIC DISTRIBUTION OF THE AUTOCORRELATIONS OF A SAMPLE FROM A LINEAR STOCHASTIC PROCESS[1]

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1. Introduction. Let $x_1, x_2, \cdots, x_n$ be a sample of n consecutive observations from a stationary discrete-parameter stochastic process $\{x_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, with $E(x_t^2) < \infty$, and let r_i and r_i^*, $i = 1, 2, 3, \cdots, n-1$, be the autocorrelations, which we shall define as $r_i = C_i/C_0$, where

$$C_i = \sum_{t=1}^{n-i} (x_t - \mu)(x_{t+i} - \mu)/(n-i), \qquad i = 0, 1, \cdots, n-1, \tag{1.1}$$

when $\mu = E(x_t)$ is specified *a priori*, and as $r_i^* = C_i^*/C_0^*$, where

$$C_i^* = \sum_{t=1}^{n-i} (x_t - \bar{x})(x_{t+i} - \bar{x})/(n-i), \qquad i = 0, 1, \cdots, n-1. \tag{1.2}$$

Under certain conditions on $\{x_t\}$ the distribution of any finite set of sample autocorrelations is asymptotically normal; that is, the limit of the joint distribution of $n^{\frac{1}{2}}(r_i - \rho_i)$, $1 \leqq i \leqq s$ (or of $n^{\frac{1}{2}}(r_i^* - \rho_i)$, $1 \leqq i \leqq s$), where $\rho_i = \operatorname{cov}(x_t, x_{t+i})/\operatorname{var} x_t$, is the s-variate normal distribution with means 0 and nonsingular covariance matrix $W = (w_{ij})$ say, when $n \to \infty$. When $\{x_t\}$ is a linear process such that

$$x_t - \mu = \sum_{i=-\infty}^{\infty} \gamma_i \epsilon_{t-i}, \qquad t = 0, \pm 1, \pm 2, \cdots, \tag{1.3}$$

where $\sum_{i=-\infty}^{\infty} |\gamma_i| < \infty$ and $\{\epsilon_t\}$ is a set of independently and identically distributed random variables with $E(\epsilon_t) = 0$, the covariance is

$$w_{ij} = \sum_{v=-\infty}^{\infty} (\rho_v\rho_{v+i-j} + \rho_v\rho_{v+i+j} + 2\rho_i\rho_j\rho_v^2 - 2\rho_i\rho_v\rho_{v+j} - 2\rho_j\rho_v\rho_{v+i}); \tag{1.4}$$

thus W is a function of the ρ_i only. (See, for example, Hannan [5], pp. 40–41, or Parzen [8], pp. 982–983.) Several other definitions of r_i may be used. (For example, the divisor $n - i$ in (1.1) and (1.2) may be replaced by n.) However, they all lead to the same asymptotic distribution.

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This result has many applications in the asymptotic treatment of statistical inference problems where the data constitute a sample of consecutive observations from such a stationary process. (For a comprehensive account see Hannan [5], Chapters 2–4.) The assumption that $\{x_t\}$ is a stationary linear process will often be reasonable. In particular, it is satisfied when $\{x_t\}$ is either a linear autoregressive process of some finite order p, namely, the stationary solution of a set of stochastic difference equations of the form

$$x_t - \mu + \sum_{r=1}^{p} \beta_r(x_{t-r} - \mu) = \epsilon_t, \tag{1.5}$$

$\{\epsilon_t\}$ denoting (as above) a set of independently and identically distributed random variables and $\beta_1, \beta_2, \cdots, \beta_p$ being constants such that the roots of the equation $z^p + \sum_1^p \beta_r z^{p-r} = 0$ have moduli less than unity, or is a moving-average process of finite order q, so that

$$x_t - \mu = \sum_{r=0}^{q} \alpha_r \epsilon_{t-r}. \tag{1.6}$$

(If $\{x_t\}$ is not a linear process, an additional term involving fourth-order cumulants of $x_{t_1}, x_{t_2}, x_{t_3}, x_{t_4}$, which in general will be extremely complicated, has to be added to the right-hand side of (1.4).)

The purpose of this paper is to prove the asymptotic normal distribution of the autocorrelations of a linear process under the assumption of finite second-order moments, $E(\epsilon_t^2) < \infty$ (which we must make in any case to have $E(x_t^2) < \infty$). In previous work the existence of higher moments of ϵ_t has always been assumed. For example, Mann and Wald [7] suppose that $E\{|\epsilon_t|^r\} < \infty$ for all $r > 0$ when dealing with the autoregressive case (1.5). Diananda [3] supposes that $E(\epsilon_t^4) < \infty$ for the moving-average case (1.6), this being a weakening of the condition $E(\epsilon_t^6) < \infty$ previously used by Hoeffding and Robbins [6]; and Walker [9], in extending Diananda's method to obtain the result for any linear process (1.3), retains his condition $E(\epsilon_t^4) < \infty$ and also requires[4] that $\sum_{i=-\infty}^{\infty} |i\gamma_i| < \infty$ (though he considered a generalization of (1.3) in which $\{\epsilon_t\}$ is merely a "finitely-dependent" stationary process, such that there exists a finite integer, m say, with the property that any two sets of ϵ's are independent whenever the suffices of the members of one set differ from each of those of the members of the other set by at least $m + 1$). The fact that the contribution of $E(\epsilon_t^4)$ to (1.4) automatically vanishes, as was first noted by Bartlett [2], p. 29, makes it reasonable to expect that the finiteness of $E(\epsilon_t^4)$ can be dispensed with. Indeed Anderson, [1], Section 4, proved a result equivalent to the asymptotic normality of r_1 for the autoregressive case with $p = 1$ assuming only that $E(\epsilon_t^2) < \infty$, and the argument here is essentially a generalization of his method.

[4] The argument used by Walker can be modified when the ϵ_t are mutually independent to show that this additional condition can be omitted, it being sufficient that $\sum_{i=-\infty}^{\infty} |\gamma_i| < \infty$.

THEOREM. *Let* $\{x_t\}$ *be a linear stochastic process, defined by*

$$(1.3)\qquad x_t - \mu = \sum_{i=-\infty}^{\infty} \gamma_i \epsilon_{t-i}, \qquad t = 0, \pm 1, \pm 2, \cdots,$$

where $\sum_{i=-\infty}^{\infty} |\gamma_i| < \infty$ *and* $\sum_{i=-\infty}^{\infty} |i|\gamma_i^2 < \infty$ *and* $\{\epsilon_t\}$ *is a set of independently and identically distributed random variables with* $E(\epsilon_t) = 0$ *and* $E(\epsilon_t^2) = \sigma^2 < \infty$. *Let* $r_i = C_i/C_0$, $i = 1, 2, 3, \cdots, n - 1$, *where* C_i *and* C_0 *are defined by* (1.1). *Then the joint distribution of* $n^{\frac{1}{2}}(r_i - \rho_i)$, $1 \leqq i \leqq s$, *where*

$$\rho_i = \operatorname{cov}(x_t, x_{t+i})/\operatorname{var} x_t = \sum_{v=-\infty}^{\infty} \gamma_v\gamma_{v+i} \Big/ \sum_{v=-\infty}^{\infty} \gamma_v^2,$$

tends to $N(0, W)$ *when* $n \to \infty$, *where* $W = (w_{ij})$ *is given by* (1.4).

COROLLARY. *Under the conditions of the theorem the joint distribution of* $n^{\frac{1}{2}}(r_i^* - \rho_i)$, $1 \leqq i \leqq s$, *where* $r_i^* = C_i^*/C_0^*$ *and* C_i^* *and* C_0^* *are defined by* (1.2), *tends to* $N(0, W)$ *when* $n \to \infty$, *where* $W = (w_{ij})$ *is given by* (1.4).

2. Proof of Theorem. We shall first prove that $n^{\frac{1}{2}}(r_l - \rho_l)$ has $N(0, w_{ll})$ as its limiting distribution. This will follow from the fact that the limiting distribution of

$$(2.1)\qquad z_n^{(l)} = n^{-\frac{1}{2}} \sum_{t=1}^{n} y_t^2\{[(n - l)/n]r_l - \rho_l\} = n^{-\frac{1}{2}}\Big\{\sum_{t=1}^{n-l} y_t y_{t+l} - \rho_l \sum_{t=1}^{n} y_t^2\Big\},$$

where $y_t = x_t - \mu$ is $N(0, \sigma^4 v_l)$, where

$$(2.2)\qquad v_l = \Big(\sum_{i=-\infty}^{\infty} \gamma_i^2\Big)^2 w_{ll}.$$

Let

$$(2.3)\qquad y_{t,k} = \sum_{i=-k}^{k} \gamma_i \epsilon_{t-i},$$

$$(2.4)\qquad z_{n,k}^{(l)} = n^{-\frac{1}{2}}\Big\{\sum_{t=1}^{n-l} y_{t,k}y_{t+l,k} - \rho_{l,k}\sum_{t=1}^{n} y_{t,k}^2\Big\},$$

where

$$(2.5)\qquad \rho_{l,k} = \sum_{i=-k}^{k-l} \gamma_i\gamma_{i+l} \Big/ \sum_{i=-k}^{k} \gamma_i^2.$$

Substituting (2.3) into (2.4) we have

$$(2.6)\qquad z_{n,k}^{(l)} = n^{-\frac{1}{2}}\Big[\sum_{t=1}^{n-l} \sum_{i,j=-k}^{k} \gamma_i\gamma_j\epsilon_{t-i}\epsilon_{t+l-j} - \rho_{l,k}\sum_{t=1}^{n} \sum_{i,j=-k}^{k} \gamma_i\gamma_j\epsilon_{t-i}\epsilon_{t-j}\Big].$$

Let $z_{n,k}^{(l)*}$ denote the expression obtained from $z_{n,k}^{(l)}$ by omitting all terms containing $\epsilon_s^2(1 - k \leqq s \leqq n + k)$, namely,

$$(2.7)\qquad z_{n,k}^{(l)*} = n^{-\frac{1}{2}}\Big[\sum_{t=1}^{n-l} \sum_{i=-k}^{k} \sideset{}{'}\sum_{j=-k-l}^{k-l} \gamma_i\gamma_{j+l}\epsilon_{t-i}\epsilon_{t-j} - \rho_{l,k}\sum_{t=1}^{n} \sideset{}{'}\sum_{i,j=-k}^{k} \gamma_i\gamma_j\epsilon_{t-i}\epsilon_{t-j}\Big],$$

where $\sum'$ indicates that the terms in the summation with $j = i$ are to be omitted.

LEMMA 1. *The limiting distribution of* $z_{n,k}^{(l)*}$ *as* $n \to \infty$ *is* $N(0, \sigma^4 v_{l,k})$, *where*

$$v_{l,k} = \sum_{r=1}^{2k+l} \delta_{r,k}^{(l)2}, \tag{2.8}$$

$$\delta_{r,k}^{(l)} = \sum_{i=-k}^{k} [\gamma_i'\gamma_{i+l+r}' + \gamma_i'\gamma_{i+l-r}' - \rho_{l,k}(\gamma_i'\gamma_{i+r}' + \gamma_i'\gamma_{i-r}')], \tag{2.9}$$

and $\gamma_i' = \gamma_i$ *when* $|i| \leqq k$ *and* $\gamma_i' = 0$ *when* $|i| > k$.

PROOF. When we change the order of summation with respect to t and i, j in (2.7) and add and subtract a finite number of terms (the number not depending on n), we obtain

$$z_{n,k}^{(l)*} = n^{-\frac{1}{2}} \sum_{r=1}^{2k+l} \delta_{r,k}^{(l)} \sum_{t=1}^{n} \epsilon_t \epsilon_{t+r} + O_p(n^{-\frac{1}{2}}), \tag{2.10}$$

where $O_p(n^{-\frac{1}{2}})$ is a generic symbol for a random variable which is of order $n^{-\frac{1}{2}}$ in probability. For any set of constants $c_1, \cdots, c_m$, the sequence $\{\sum_{r=1}^{m} c_r \epsilon_t \epsilon_{t+r}\}$ constitutes a finitely-dependent stationary process having mean zero, variance $\sigma^4 \sum_{r=1}^{m} c_r^2$, and autocorrelations all equal to zero (since $E(\epsilon_t\epsilon_{t+r}\epsilon_{t'}\epsilon_{t'+s}) = 0$ when $t' \neq t$). Hence from a central limit theorem due to Diananda ([3], p. 241, Theorem 2), $n^{-\frac{1}{2}} \sum_{r=1}^{m} c_r \sum_{t=1}^{n} \epsilon_t \epsilon_{t+r}$ has the limiting distribution $N(0, \sigma^4 \sum_{r=1}^{m} c_r^2)$ when $n \to \infty$. Applying this result to (2.10) proves Lemma 1.

Let $z_n^{(l)*}$ denote the expression obtained by substituting (1.3) for $y_t = x_t - \mu$ in (2.1) and omitting all terms containing $\epsilon_s^2(-\infty < s < \infty)$, namely,

$$z_n^{(l)*} = n^{-\frac{1}{2}}\Big[\sum_{t=1}^{n-l} \sum_{i,j=-\infty}^{\infty}{}' \gamma_i\gamma_{j+l}\epsilon_{t-i}\epsilon_{t-j} - \rho_l \sum_{t=1}^{n} \sum_{i,j=-\infty}^{\infty}{}' \gamma_i\gamma_j\epsilon_{t-i}\epsilon_{t-j}\Big]. \tag{2.11}$$

LEMMA 2. *The limiting distribution of* $z_n^{(l)*}$ *as* $n \to \infty$ *is* $N(0, \sigma^4 v_l)$, *where* v_l *is given by* (2.2).

PROOF. The limit of $N(0, \sigma^4 v_{l,k})$ (which is the limiting distribution of $z_{n,k}^{(l)*}$ as $n \to \infty$) is $N(0, \sigma^4 v_l)$ as $k \to \infty$ since

$$\lim_{k\to\infty} v_{l,k} = \frac{1}{2} \sum_{r=-\infty}^{\infty} \Big\{ \sum_{i=-\infty}^{\infty} [\gamma_i\gamma_{i+l+r} + \gamma_i\gamma_{i+l-r} - \rho_l(\gamma_i\gamma_{i+r} + \gamma_i\gamma_{i-r})]\Big\}^2 \tag{2.12}$$

which is equivalent to (2.2). Lemma 2 will then follow from Anderson's form of a convergence theorem of Diananda ([1], p. 687, Theorem 4.5) when we show that

$$E|R_{n,k}^{(l)*}|^2 \leqq M_k, \tag{2.13}$$

$$\lim_{k\to\infty} M_k = 0, \tag{2.14}$$

where

$$R_{n,k}^{(l)*} = z_n^{(l)*} - z_{n,k}^{(l)*}. \tag{2.15}$$

Let

$$R_{n,k}^{(l)} = z_n^{(l)} - z_{n,k}^{(l)}. \tag{2.16}$$

If we define

$$u_{t,k} = y_t - y_{t,k} = \sum_{|i|>k} \gamma_i \epsilon_{t-i}, \tag{2.17}$$

then

$$\begin{aligned} n^{\frac{1}{2}} R_{n,k}^{(l)} &= \sum_{t=1}^{n-l} [(y_{t,k} + u_{t,k})(y_{t+l,k} + u_{t+l,k}) - y_{t,k} y_{t+l,k}] \\ &\qquad + \sum_{t=1}^{n} [-\rho_l (y_{t,k} + u_{t,k})^2 + \rho_{l,k} y_{t,k}^2] \\ &= \sum_{t=1}^{n-l} [u_{t,k} y_{t+l,k} + y_{t,k} u_{t+l,k} + u_{t,k} u_{t+l,k} \\ &\qquad + \sum_{t=1}^{n} (\rho_{l,k} - \rho_l) y_{t,k}^2 - 2\rho_l y_{t,k} u_{t,k} - \rho_l u_{t,k}^2] \\ &= \sum_{h=1}^{6} T_h, \end{aligned} \tag{2.18}$$

say, where

$$T_1 = \sum_{t=1}^{n-l} u_{t,k} y_{t+l,k}, \cdots, T_6 = -\rho_l \sum_{t=1}^{n} u_{t,k}^2. \tag{2.19}$$

Similarly, from (2.7), (2.11), and (2.15), we have

$$n^{\frac{1}{2}} R_{n,k}^{(l)*} = \sum_{h=1}^{6} T_h^*, \tag{2.20}$$

where T_h^* is obtained from T_h by expressing the latter as a linear combination of a finite or countably infinite number of terms of the form $\sum_{t=1}^{n-l} \epsilon_{t-i}\epsilon_{t-j}$, and omitting the contributions from all terms for which $i = j$. (These in fact occur only in T_3, T_4, T_6.)

Now we have

$$n^{-1} E(T_h^{*2}) \leqq M_{h,k}, \qquad \lim_{k \to \infty} M_{h,k} = 0. \tag{2.21}$$

For example

$$E(T_3^{*2}) = \sum_{\substack{|i|,|j+l|>k \\ i \neq j}} \gamma_i \gamma_{j+l} \sum_{\substack{|i'|,|j'+l|>k \\ i' \neq j'}} \gamma_{i'} \gamma_{j'+l} \sum_{t,t'=1}^{n-l} E(\epsilon_{t-i}\epsilon_{t-j}\epsilon_{t'-i'}\epsilon_{t'-j'}), \tag{2.22}$$

and if $E(\epsilon_{t-i}\epsilon_{t-j}\epsilon_{t'-i'}\epsilon_{t'-j'}) \neq 0$, we must have either $t - i = t' - i'$, $t - j = t' - j'$, and therefore $t' - t = i' - i = j' - j$, or $t - i = t' - j'$, $t - j = t' - i'$, and therefore $t' - t = j' - i = i' - j$, the value of t' for a nonzero contribution to the final summation in (2.22), if any, being thus uniquely determined by t. It follows that

$$E(T_3^{*2}) \leqq (n - l)\sigma^4 \sum_{i,j,i',j'} |\gamma_i| \, |\gamma_{i'}| \, |\gamma_{j+l}| \, |\gamma_{j'+l}| = (n - l)\sigma^4 \{\sum_{|i|>k} |\gamma_i|\}^4,$$

giving $n^{-1}E(T_3^{*2}) \leqq \sigma^4\{\sum_{|i|>k}|\gamma_i|\}^4$, which shows that (2.21) holds for $h = 3$. A similar type of argument applies for $h = 1, 2, 5, 6$, and for $h = 4$ we also use $\lim_{k\to\infty}\rho_{l,k} = \rho_l$. We then obtain (2.13) from (2.21) by using the inequality

$$E(\sum_{h=1}^{6} T_h^*)^2 \leqq 6\sum_{h=1}^{6} E(T_h^{*2}).$$

This proves Lemma 2.

LEMMA 3. *The limiting distribution of* $z_n^{(l)}$ *is* $N(0, \sigma^4 v_l)$.

PROOF. We have

$$\begin{aligned} z_n^{(l)} - z_n^{(l)*} &= n^{-\frac{1}{2}}[\sum_{t=1}^{n-l}\sum_{i=-\infty}^{\infty}\gamma_i\gamma_{i+l}\epsilon_{t-i}^2 - \rho_l\sum_{t=1}^{n}\sum_{i=-\infty}^{\infty}\gamma_i^2\epsilon_{t-i}^2] \\ &= n^{-\frac{1}{2}}[\sum_{i=-\infty}^{\infty}\gamma_i\gamma_{i+l}\sum_{u=1-i}^{n-l-i}\epsilon_u^2 - \rho_l\sum_{i=-\infty}^{\infty}\gamma_i^2\sum_{u=1-i}^{n-i}\epsilon_u^2] \\ &= n^{-\frac{1}{2}}[\sum_{i=-\infty}^{\infty}\gamma_i\gamma_{i+l}(\sum_{t=1}^{n}\epsilon_t^2 + T_{n,i}^{(l)}) - \rho_l\sum_{i=-\infty}^{\infty}\gamma_i^2(\sum_{t=1}^{n}\epsilon_t^2 + T_{n,i}^{(0)})] \\ &= n^{-\frac{1}{2}}[\sum_{i=-\infty}^{\infty}\gamma_i\gamma_{i+l}T_{n,i}^{(l)} - \rho_l\sum_{i=-\infty}^{\infty}\gamma_i^2T_{n,i}^{(0)}], \end{aligned}$$

where

$$T_{n,i}^{(l)} = \sum_{t=1-i}^{n-l-i}\epsilon_t^2 - \sum_{t=1}^{n}\epsilon_t^2. \tag{2.23}$$

Then

$$E|T_{n,i}^{(l)}| \leqq (2|i| + l)\sigma^2$$

and

$$E\{n^{\frac{1}{2}}|z_n^{(l)} - z_n^{(l)*}|\} \leqq \sigma^2[\sum_{i=-\infty}^{\infty}|\gamma_i|\,|\gamma_{i+l}|(2|i| + l) + |\rho_l|\sum_{i=-\infty}^{\infty}\gamma_i^2 2|i|] < \infty$$

since

$$\begin{aligned} [\sum_{i=-\infty}^{\infty}|i|\,|\gamma_i|\,|\gamma_{i+l}|]^2 &\leqq \sum_{i=-\infty}^{\infty}|i|\gamma_i^2\sum_{i=-\infty}^{\infty}|i|\gamma_{i+l}^2 \\ &= \sum_{i=-\infty}^{\infty}|i|\gamma_i^2\sum_{j=-\infty}^{\infty}|j - l|\gamma_j^2 < \infty. \end{aligned}$$

Thus

$$z_n^{(l)} - z_n^{(l)*} = O_p(n^{-\frac{1}{2}}). \tag{2.24}$$

This proves Lemma 3.

LEMMA 4. *The limiting distribution of* $n^{\frac{1}{2}}(r_l - \rho_l)$ *as* $n \to \infty$ *is* $N(0, w_{ll})$.

PROOF. This follows from Lemma 3, the fact that

$$n^{\frac{1}{2}}(r_l - \rho_l) = \left[z_n^{(l)} / n^{-1} \sum_{t=1}^{n} y_t^2 \right] + O_p(n^{-\frac{1}{2}}),$$

and

$$\text{p}\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} y_t^2 = E(y_t^2). \tag{2.25}$$

The last follows immediately by applying the law of large numbers for stationary stochastic processes (Doob [4], p. 465, Theorem 2.1), since the process $\{y_t^2\}$ is metrically transitive (see Doob [4], pp. 458, 460); in fact, this gives the stronger result that the limit is an almost certain one. A more elementary proof can also be given by writing

$$n^{-1} \sum_{t=1}^{n} y_t^2 = n^{-1} \sum_{t=1}^{n} y_{t,k}^2 + 2n^{-1} \sum_{t=1}^{n} y_{t,k} u_{t,k} + n^{-1} \sum_{t=1}^{n} u_{t,k}^2 ,$$

and using

$$n^{-1} E\left\{ \sum_{t=1}^{n} u_{t,k}^2 \right\} = \sigma^2 \sum_{|i|>k} \gamma_i^2 \to 0 \tag{2.26}$$

when $k \to \infty$,

$$\begin{aligned} n^{-1} E\left| \sum_{t=1}^{n} y_{t,k} u_{t,k} \right| \leqq E|y_{t,k} u_{t,k}| &\leqq [E(y_{t,k}^2) E(u_{t,k}^2)]^{\frac{1}{2}} \\ &= \sigma^2 \left\{ \sum_{|i|\leqq k} \gamma_i^2 \sum_{|i|>k} \gamma_i^2 \right\}^{\frac{1}{2}} \to 0 \end{aligned} \tag{2.27}$$

when $k \to \infty$, and

$$\begin{aligned} \text{p}\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} y_{t,k}^2 &= \sum_{|i|\leqq k} \gamma_i^2 \,\text{p}\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} \epsilon_{t-i}^2 \\ &+ \sum_{\substack{|i|,|j|\leqq k \\ i\neq j}} \gamma_i \gamma_j \,\text{p}\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} \epsilon_{t-i} \epsilon_{t-j} = \sigma^2 \sum_{|i|\leqq k} \gamma_i^2 \to E(y_t^2) \end{aligned} \tag{2.28}$$

when $k \to \infty$. The second step in (2.28) is the result of applying the weak law of large numbers for a sequence of independently and identically distributed random variables to the first set of terms, and the central limit theorem of Diananda referred to above (combined with Chebyshev's inequality) to the second set of terms. This proves Lemma 4.

To complete the proof of the Theorem we have only to observe that a similar argument can be carried through to show that the limiting distribution of $\sum_{l=1}^{s} c_l n^{\frac{1}{2}}(r_l - \rho_l)$, $c' = (c_1, c_2, \cdots, c_s)$ being an arbitrary set of constants, is $N(0, c'Wc)$; it will then follow, for example by means of the device based on the continuity theorem for characteristic functions which was employed by Walker in a similar context ([9], p. 64), that the joint limiting distribution of $n^{\frac{1}{2}}(r_l - \rho_l)$, $1 \leqq l \leqq s$, is $N(0, W)$.

The corollary is deduced in the usual way. We have

$$nE(\bar{x} - \mu)^2 = \sum_{i=-(n-1)}^{n-1} (1 - |i|/n) \operatorname{cov}(x_t, x_{t+i}) \to \operatorname{var} x_t \sum_{i=-\infty}^{\infty} \rho_i = \sigma^2\Big(\sum_{i=-\infty}^{\infty} \gamma_i\Big)^2$$

when $n \to \infty$; it is easily shown from this that the difference between C_i and C_i^* is $O_p(n^{-1})$, and hence that the difference between r_i and r_i^* is $O_p(n^{-1})$.

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