ON BIVARIATE RANDOM VARIABLES WHERE THE QUOTIENT OF THEIR COORDINATES FOLLOWS SOME KNOWN DISTRIBUTION

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1. Introduction. Let X_1, X_2 be a pair of independent random variables, symmetrical about the origin, having the same distribution function $F(x)$, and let the quotient

$$Z = X_1 : X_2 \tag{1.1}$$

follow the Cauchy law. It is known that $F(x)$ may be normal with zero mean. A number of authors investigated whether the normal distribution can be characterized by this property. Mauldon [9], Laha [4] and Steck [11] showed this supposition to be false; there exist distribution functions $F(x)$ differing from the normal, where the quotient (1.1) follows the Cauchy law. Denote by C the set of distribution functions $F(x)$ having the above-mentioned property; Laha [5], [6] and Kotlarski [2] undertook a study of characterizing the set C. Kotlarski [2] characterized the set C by the properties of the Mellin transform $h(s)$ of $F(x)$ given by the formula

$$h(s) = \int_{-\infty}^{+\infty} |x|^s \, dF(x). \tag{1.2}$$

On this subject see also [7] p. 324 and [8] p. 178.

In this paper we shall consider a bivariate random variable (X, Y) having distribution $F(x, y)$, where the coordinates (not necessarily independent) have identical marginal distributions $F(x, \infty) = F(\infty, x)$, $(-\infty < x < +\infty)$ and the quotient

$$Z = X : Y \tag{1.3}$$

follows the Cauchy law (Section 3). The set $\mathfrak{X}$ of such distribution functions $F(x, y)$ will be described by using their two-dimensional Mellin transforms (see Section 2).

With the same method we describe in Section 4, the set $\mathfrak{Y}$ of distribution functions $F(x, y)$ of bivariate random variables (X, Y) having positive coordinates (not necessarily independent or identically distributed), where the quotient (1.3) follows Snedecor's law. On this subject see also [3], [9].

Further in a similar way may be described the set of distribution functions $F(x, y)$ of bivariate random variables (X, Y) where X has a symmetrical distribution about the origin and Y takes positive values only, X and Y not necessarily being independent, where the quotient (1.3) follows Student's law.

2. The Mellin transforms of bivariate random variables whose coordinates take only positive values. We define the Mellin transform of a bivariate random

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variable (X, Y) having positive coordinates by the formula (see [1])

$$h_{X,Y}(u, v) = E[X^u Y^v] = \int_0^\infty \int_0^\infty x^u y^v \, dF(x, y), \tag{2.1}$$

where u and v are complex variables. The function $h(u, v)$ is always defined in the set of pairs (u, v) of complex variables

$$S = \{(u, v): a_1 < \operatorname{Re} u < a_2, b_1 < \operatorname{Re} v < b_2\}, \tag{2.2}$$

where

$$a_1 < 0 < a_2, b_1 < 0 < b_2. \tag{2.3}$$

The Mellin transform $h(u, v)$ defines the bivariate distribution function $F(x, y)$ of the random variable (X, Y) where X and Y take only positive values uniquely.

If the distribution function $F(x, y)$ has a density $f(x, y) = \partial^2 F/\partial x \partial y$, then it is given at every point of its continuity by the formula

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} x^{-u-1} y^{-v-1} h(u, v) \, du \, dv, \tag{2.4}$$

where

$$a_1 < a < a_2, b_1 < b < b_2 \tag{2.5}$$

and the integrals are taken in the sense of Cauchy.

Taking iu and iv instead of u and v we obtain the characteristic function of the bivariate random variable $(\log X, \log Y)$ because of

$$h_{(X,Y)}(iu, iv) = E[X^{iu} Y^{iv}] = E[e^{i(u \cdot \log X + v \cdot \log Y)}] = \varphi_{(\log X, \log Y)}(u, v). \tag{2.6}$$

From the properties of bivariate characteristic functions we see that for u, v real:

$$\begin{gathered} h(iu, iv) \text{ is continuous on the whole plane } (u, v); \\ h(0, 0) = 1, |h(iu, iv)| \leqq 1, h(-iu, -iv) = \overline{h(iu, iv)}; \\ h(iu, iv) \text{ is a positive definite function.} \end{gathered} \tag{2.7}$$

The one-dimensional Mellin transforms of marginal distributions of X and Y are given by

$$h_X(u) = h_{(X,Y)}(u, 0); \; h_Y(v) = h_{(X,Y)}(0, v). \tag{2.8}$$

For X and Y to be independent it is necessary and sufficient that

$$h(u, v) = h(u, 0) \cdot h(0, v). \tag{2.9}$$

The one-dimensional Mellin transform of the product

$$Q = a \cdot X^p \cdot Y^q \qquad (a \text{ positive, } p, q \text{ real}) \tag{2.10'}$$

is

$$h_Q(s) = E[Q^s] = E[a^s X^{ps} Y^{qs}] = h_{(X,Y)}(ps, qs) \cdot a^s. \tag{2.10''}$$

If we have r independent bivariate random variables having positive coordinates

$$(X_1, Y_1), (X_2, Y_2), \cdots, (X_r, Y_r) \tag{2.11'}$$

and their two-dimensional Mellin transforms

$$h_1(u, v), h_2(u, v), \cdots, h_r(u, v), \tag{2.11''}$$

then for arbitrary real numbers $p_1, p_2, \cdots, p_r; q_1, q_2, \cdots, q_r$ the two-dimensional Mellin transform of the pair of products

$$(P, Q) = (X_1^{p_1} \cdot X_2^{p_2} \cdot \cdots \cdot X_r^{p_r}, Y_1^{q_1} \cdot Y_2^{q_2} \cdot \cdots \cdot Y_r^{q_r}) \tag{2.12'}$$

is

$$h_{(P,Q)}(u, v) = h_1(p_1 u, q_1 v) \cdot h_2(p_2 u, q_2 v) \cdot \cdots \cdot h_r(p_r u, q_r v). \tag{2.12''}$$

From the theorem of Cramér and Lévy it follows that the necessary and sufficient condition of weak convergence of the sequence of distribution functions $F_k(x, y)$ of bivariate random variables (X_k, Y_k) whose coordinates take only positive values to a distribution function $F_0(x, y)$ is the convergence of the sequence of the corresponding Mellin transforms $h_k(u, v)$ to a function $h_0(u, v)$ continuous at the point $(0, 0)$. $F_0(x, y)$ corresponds in this case to $h_0(u, v)$. Hence we see that the integer r in the formulas (2.12) may tend to infinity.

More directly, the Mellin transform of the pair of products

$$(P, Q) = \left(\prod_{k=1}^{\infty} X_k^{p_k}, \quad \prod_{k=1}^{\infty} Y_k^{q_k}\right) \tag{2.13'}$$

is

$$h_{(P,Q)}(u, v) = \prod_{k=1}^{\infty} h_k(p_k u, q_k v), \tag{2.13''}$$

where the limes in (2.13′) is in the sense of weak convergence, and $h_{(P,Q)}(u, v)$ should be continuous at the point $(0, 0)$.

If $h(u, v)$ is the Mellin transform of the bivariate random variable (X, Y) having density $f(x, y)$, then $h(-u, -v)$ is also a Mellin transform of a bivariate random variable (X^{-1}, Y^{-1}), whose density is $x^{-2}y^{-2}f(x^{-1}, y^{-1})$.

3. Determining the set of bivariate random variables where the quotient of their coordinates follows the Cauchy law.

3.1. *Formulating the problem.* Let the set $\mathfrak{X}^*$ consists of distribution functions $F^*(x, y)$ of bivariate random variables (X^*, Y^*) satisfying the following conditions;

$$F^*(y, x) = F^*(x, y), \tag{3.1.1'}$$

$$F^*(-x, y) = F^*(\infty, y) - F^*(x, y), \tag{3.1.1''}$$

$$F^*_{X/Y}z = \pi^{-1} \text{ arc tg } z. \tag{3.1.1'''}$$

[In this paper the distribution function $F(x, y)$ of a bivariate random variable (X, Y) is defined as $F(x, y) = \frac{1}{4}[P(X < x, Y < y) + P(X \leqq x, Y < y) + P(X < x, Y \leqq y) + P(X \leqq x, Y \leqq y)]$ (see [10]).]

The condition (3.1.1′) means that the marginal distributions of X^* and Y^* are the same as well as the conditional distribution of one of them when the other takes a fixed number. The condition (3.1.1″) with (3.1.1′) means that the distributions of X^* and Y^* are symmetric about the origin as well as the distribution of (X^*, Y^*). The condition (3.1.1‴) means that the quotient $Z^* = X^* : Y^*$ follows the Cauchy law.

The problem is to characterize the set $\mathfrak{X}^*$. Let us take

$$X = |X^*|, \; Y = |Y^*|. \tag{3.1.2}$$

In this way we obtain instead of $\mathfrak{X}^*$ the set $\mathfrak{X}$ of distribution functions $F(x, y)$ of bivariate random variables (X, Y) having positive coordinates. The distribution functions $F^*(x, y)$ and $F(x, y)$ determine one another uniquely.

The conditions (3.1.1) take for $F(x, y)$ the form

$$F(0, y) = F(x, 0) = 0, \tag{3.1.3'}$$

$$F(x, y) = F(y, x), \tag{3.1.3''}$$

$$F_{X/Y}(z) = (2/\pi) \text{ arc tg } z \quad \text{for } z > 0, \qquad = 0 \quad \text{for } z \leqq 0 \tag{3.1.3'''}$$

It should be noted that it is enough to describe the set $\mathfrak{X}$ instead of $\mathfrak{X}^*$.

3.2. *Characterizing the set* $\mathfrak{X}$ *by the properties of Mellin transforms.* Taking Mellin transforms of both sides of equations (3.1.3″) and (3.1.3‴) we obtain conditions for Mellin transforms of distribution functions from $\mathfrak{X}$ in the form

$$h(u, v) = h(v, u), \tag{3.2.1'}$$

$$h(u, -u) = 1/\cos \tfrac{1}{2}\pi u \qquad (-1 < \text{Re } u < 1)\cdot \tag{3.2.1''}$$

Let us take iu and iv (u, v real) instead of u, v complex, and let us represent the unknown function $h(u, v)$ in the form

$$h(iu, iv) = [\text{ch}(\pi u/2) \cdot \text{ch}(\pi v/2)]^{\frac{1}{2}} e^{\alpha(u,v)+i\beta(u,v)}, \tag{3.2.2}$$

where $\alpha(u, v)$ and $\beta(u, v)$ are new unknown functions. Substituting (3.2.2) into (3.2.1) we obtain the following conditions for $\alpha(u, v)$, $\beta(u, v)$:

$$\alpha(u, v) = \alpha(v, u), \qquad \beta(u, v) = \beta(v, u), \tag{3.2.3'}$$

$$\alpha(u, -u) = 0, \qquad \beta(u, -u) = 0. \tag{3.2.3''}$$

Taking into account that (3.2.2) should be a characteristic function we see that it satisfies the conditions (2.7). Thus we obtain the following conditions:

(3.2.4′) $\alpha(-u, -v) = \alpha(u, v), \qquad \beta(-u, -v) = -\beta(u, v),$

(3.2.4″) $\alpha(u, v) \leqq \frac{1}{2}\log(\mathrm{ch}(\pi u/2)\cdot\mathrm{ch}(\pi v/2)),$

(3.2.4‴) $\alpha(u, v)$, $\beta(u, v)$ should be real and continuous on the whole plane (u, v).

So the result may be formulated as a theorem.

THEOREM 1. *For a distribution function $F(x, y)$ satisfying conditions* (3.1.3′), (3.1.3″) *to belong to the set $\mathfrak{X}$ it is necessary and sufficient that its Mellin transform* (2.1) *should be represented in form* (3.2.2), *where $\alpha(u, v)$ and $\beta(u, v)$ satisfy conditions* (3.2.3) *and* (3.2.4) *and the function* (3.2.2) *is positive definite.*

3.3. *Particular cases.* Let the distribution $F(x, y)$ be given by the density

$$f(x, y) = f_0(x^2 + y^2) \quad \text{for } x > 0, y > 0; \qquad = 0 \quad \text{otherwise} \tag{3.3.1′}$$

where the function $f_0(z)$ is in such a form, that (3.3.1′) is a density function. Its Mellin transform is

$$\begin{aligned} h(u, v) &= \int_0^\infty \int_0^\infty x^u\, y^v f_0(x^2 + y^2)\, dx\, dy \\ &= \int_0^\infty r^{u+v+1} f_0(r^2)\, dr \int_0^{\pi/2} \cos^u \varphi \sin^v \varphi\, d\varphi \\ &= \frac{1}{2}\int_0^\infty r^{(u+v)/2} f_0(r)\, dr\cdot\tfrac{1}{2}\, B\left(\frac{1+u}{2}, \frac{1+v}{2}\right) \\ &= H(u+v)\cdot\Gamma\left(\frac{1+u}{2}\right)\cdot\Gamma\left(\frac{1+v}{2}\right) \end{aligned} \tag{3.3.1″}$$

where

$$H(w) = \frac{\pi \int_0^\infty r^{w/2} f_0(r)\, dr}{4\Gamma[1 + w/2)]}, \qquad H(0) = 1/\pi. \tag{3.3.2}$$

We see that the Mellin transform (3.3.1″) satisfies the conditions (3.2.1) and that is why the distribution given by density (3.3.1′) belongs to $\mathfrak{X}$.

Taking for instance

$$f_0(z) = \text{const} \quad \text{for } a < z < b; \qquad = 0 \quad \text{otherwise} \tag{3.3.3}$$

where $(0 \leqq a < b)$ we see that the random variable having uniform distribution on the ring $a^2 < x^2 + y^2 < b^2$ belongs to $\mathfrak{X}^*$.

Taking the limit by $b \to a$ $(a > 0)$ we see that the random variable having uniform distribution on the periphery of the circle $x^2 + y^2 = a^2$ belongs to $\mathfrak{X}^*$.

Now let us take $f_0(z) = Az^p e^{-az^q}$ $a > 0$, $q \neq 0$. We obtain the density

$$(3.3.4) \quad \begin{aligned} f(x, y) &= A(x^2 + y^2)^p \exp - a(x^2 + y^2)^q \quad \text{for } x > 0, y > 0 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

for which the distribution belongs to $\mathfrak{X}$. Taking further $p = 0$, $q = 1$ we obtain the bivariate normal distribution, where the coordinates are noncorrelated, symmetric about the origin, and have equal standard deviations.

Denote

$$(3.3.5') \quad f(x, y) = f^+(x, y),\ h(u, v) = h^+(u, v)$$

and according to the end of Section 2

$$(3.3.5'') \quad h^-(u, v) = h^+(-u, -v),\ f^-(x, y) = (xy)^{-2} f^+(x^{-1}, y^{-1}).$$

Taking $f^+(x, y)$ and $h^+(u, v)$ as (3.3.1) we obtain the Mellin transform

$$(3.3.6') \quad \begin{aligned} h^-(u, v) = h^+(-u, -v) &= H(-u, -v)\Gamma[(1 - u)/2]\Gamma[[1 - v)/2] \\ &= H^-(u + v)\Gamma[(1 - u)/2]\Gamma[(1 - v)/2] \end{aligned}$$

satisfying the conditions (3.2.1). Thus the corresponding distribution given by the density

$$(3.3.6'') \quad \begin{aligned} f^-(x, y) &= (xy)^{-2} f^+(x^{-1}, y^{-1}) = (xy)^{-2} f_0(x^{-2} + y^{-2}) \\ &\qquad \text{for } x > 0, y > 0 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

belongs to $\mathfrak{X}$.

3.4. *A method of determining other distributions belonging to* $\mathfrak{X}$. Let us represent the Mellin transform (3.3.1″) as a product of a finite or an infinite number of Mellin transforms

$$(3.4.1) \quad \begin{aligned} h^+(u, v) &= H^+(u + v)\Gamma((1 + u)/2)\Gamma((1 + v)/2) \\ &= \prod_{k \varepsilon K} H_k^+(u + v)\gamma_k(u)\gamma_k(v) = \prod_{k \varepsilon K} h_k^+(u, v). \end{aligned}$$

Taking $H_k^+(u + v)$, $\gamma_k(u)$ in such a way that $h_k^+(u, v)$ are Mellin transforms of bivariate random variables having positive coordinates, we see that $h_k^-(u, v) = h_k^+(-u, -v)$ are also Mellin transforms of such distributions.

Let us devide the set K of integers k into two mutually exclusive and exhaustive subsets K_1 and K_2. The product

$$(3.4.2) \quad \begin{aligned} h(u, v) &= \prod_{k \varepsilon K_1} h_k^+(u, v) \prod_{k \varepsilon K_2} h_k^-(u, v) \\ &= \prod_{k \varepsilon K_1} H_k^+(u + v)\gamma_k(u)\gamma_k(v) \prod_{k \varepsilon K_2} H_k^-(u + v)\gamma_k(-u)\gamma_k(-v) \end{aligned}$$

satisfies the conditions (3.2.1) and that is why (3.4.2) is the Mellin transform of a distribution from $\mathfrak{X}$.

Using for instance the known formula for gamma function

$$(3.4.3)\qquad \Gamma(nw) = [n^{nw-\frac{1}{2}}/(2\pi)^{(n-1)/2}]\prod_{k=1}^{n}\Gamma[w + (k-1)/n].$$

we can expend the Mellin transforms (3.3.1″), (3.3.6′) in the form

$$(3.4.4')\qquad \begin{aligned} h^{+}(u, v) &= \prod_{k=1}^{n} h_k^{+}(u, v) \\ &= \prod_{k=1}^{n} H_k^{+}(u+v)\Gamma[(2k-1+u)/2n]\Gamma[(2k-1+v)/2n], \end{aligned}$$

$$(3.4.4'')\qquad \begin{aligned} h^{-}(u, v) &= \prod_{k=1}^{n} h_k^{-}(u, v) \\ &= \prod_{k=1}^{n} H_k^{-}(u+v)\Gamma[(2k-1-u)/2n]\Gamma[(2k-1-v)/2n]. \end{aligned}$$

The distribution functions $F_k^{+}(x, y)$, $F_k^{-}(x, y)$ of the pairs (X_k^{+}, Y_k^{+}), (X_k^{-}, Y_k^{-}) corresponding to the factors $h_k^{+}(u, v)$, $h_k^{-}(u, v)$ have densities given by formulas

$$(3.4.5')\qquad \begin{aligned} f_k^{+}(x, y) &= x^{2k-2}y^{2k-2}g_k(x^{2n} + y^{2n}) && \text{for } x > 0,\ y > 0 \\ &= 0 && \text{otherwise,} \end{aligned}$$

$$(3.4.5'')\qquad \begin{aligned} f_k^{-}(x, y) &= x^{-2k}y^{-2k}g_k(x^{-2n} + y^{-2n}) && \text{for } x > 0,\ y > 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Thus the function

$$(3.4.6')\qquad \begin{aligned} h(u, v) &= \prod_{k\varepsilon K_1} h_k^{+}(u, v)\prod_{k\varepsilon K_2} h_k^{-}(u, v) \\ &= \prod_{k\varepsilon K_1} H_k^{+}(u+v)\Gamma[(2k-1+u)/2n]\Gamma[(2k-1+v)/2n] \\ &\quad\cdot\prod_{k\varepsilon K_2} H_k^{-}(u+v)\Gamma[(2k-1-u)]/2n]\Gamma[(2k-1-v)/2n] \end{aligned}$$

satisfies the conditions (3.2.1) and hence the corresponding distribution belongs to $\mathfrak{X}$. The bivariate random variable corresponding to (3.4.6′) may be presented in the form

$$(3.4.6'')\qquad (X, Y) = \Big(\prod_{k\varepsilon K_1} X_k^{+}\prod_{k\varepsilon K_2} X_k^{-},\ \prod_{k\varepsilon K_1} Y_k^{+}\prod_{k\varepsilon K_2} Y_k^{-}\Big),$$

where the pairs (X_k^{+}, Y_k^{+}) $k \varepsilon K_1$, (X_k^{-}, Y_k^{-}) $k \varepsilon K_2$ are independent and their distributions are given by densities (3.4.5′) and (3.4.5″) respectively. Taking for instance $n = 2$, (3.4.5″) for $k = 1$ and (3.4.5′) for $k = 2$ we obtain densities

$$(3.4.7')\qquad f_1^{-}(x, y) = x^{-2}y^{-2}g_1^{-}(x^{-4} + y^{-4}) \qquad \text{for } (X_1^{-}, Y_1^{-}),$$

$$(3.4.7'')\qquad f_2^{+}(x, y) = x^{2}y^{2}g_2^{+}(x^{4} + y^{4}) \qquad \text{for } (X_2^{+}, Y_2^{+}),$$

and the distribution of $(X_1^{-}X_2^{+}, Y_1^{-}Y_2^{+})$ belonging to $\mathfrak{X}$.

4. Determining the set of bivariate random variables where the quotient of their coordinates follows the Snedecor's law.

4.1. *Formulating the problem.* Let X_1 and X_2 be a pair of independent random variables having gamma distributions given by densities

$$(4.1.1)\qquad \begin{aligned} f_k(x) &= [a^{p_k}/\Gamma(p_k)]x^{p_k-1}e^{-ax} && \text{for } x > 0 \\ &= 0 && \text{for } x \leqq 0 \end{aligned}$$

respectively, where the constants a, p_k are positive. It is then known that the quotient (1.1) follows the Snedecor's distribution given by density

$$(4.1.2)\qquad \begin{aligned} g(z) &= [B(p_1, p_2)]^{-1}\cdot z^{p_2-1}/(1+z)^{p_1+p_2} && \text{for } x > 0 \\ &= 0 && \text{for } x \leqq 0. \end{aligned}$$

In this section we shall consider a bivariate random variable (X, Y) having distribution $F(x, y)$, whose coordinates take only positive values and are not necessarily independent, and where the quotient (1.3) follows the distribution given by density (4.1.2).

We denote the set of such distribution functions $F(x, y)$ by $\mathcal{Y}$. This set will be characterized by the properties of the Mellin transforms (2.1).

4.2. *Characterizing the set $\mathcal{Y}$ by the properties of Mellin transforms.* Taking on both sides of equations (4.1.1) and (4.1.2) the one-dimensional Mellin transforms, we obtain the Mellin transforms of X_k $(k = 1; 2)$, and Z

$$(4.2.1)\qquad \begin{aligned} h_{X_k}(s) &= \Gamma(p_k+s)/\Gamma(p_k), && \operatorname{Re} s > -p_k \\ h_Z(s) &= [\Gamma(p_1+s)/\Gamma(p_1)]\cdot[\Gamma(p_2-s)/\Gamma(p_2)] && -p_1 < \operatorname{Re} s < p_2 . \end{aligned}$$

Taking on both sides of (1.3) the bivariate Mellin transforms and using (4.2.1) we obtain the following equation

$$(4.2.2)\qquad h(u, -u) = [\Gamma(p_1+u)/\Gamma(p_1)]\cdot[\Gamma(p_2-u)/\Gamma(p_2)] \qquad (-p_1 < \operatorname{Re} u < p_2).$$

Hence we see that for characterizing the set $\mathcal{Y}$ it is enough to solve the equation (4.2.2) in terms of Mellin transforms of bivariate random variables having positive coordinates. In order to do this let us take iu and iv (u, v real) instead of u, v complex, and let us represent the unknown function $h(u, v)$ in the form

$$(4.2.3)\qquad h(iu, iv) = [\Gamma(p_1+iu)/\Gamma(p_1)]\cdot[\Gamma(p_2+iv)/\Gamma(p_2)]\cdot e^{\alpha(u,v)+i\beta(u,v)},$$

where $\alpha(u, v)$ and $\beta(u, v)$ are new unknown functions.

Substituting (4.2.3) into (4.2.2) we obtain the following conditions for $\alpha(u, v)$, $\beta(u, v)$,

$$(4.2.4)\qquad \alpha(u, -u) = 0, \qquad \beta(u, -u) = 0.$$

Taking into account that (4.2.3) should be a characteristic function we obtain new conditions for $\alpha(u, v)$, $\beta(u, v)$ in the form

$$\alpha(-u, -v) = \alpha(u, v), \qquad \beta(-u, -v) = -\beta(u, v)$$

$$\alpha(u, v) \leqq \log |[\Gamma(p_1)/\Gamma(p_1 + iu)] \cdot [\Gamma(p_2)/\Gamma(p_2 + iv)]| \tag{4.2.5}$$

$\alpha(u, v)$, $\beta(u, v)$ should be real and continuous on the whole plane (u, v).

The result may be formulated as a theorem.

THEOREM 2. *For a distribution function $F(x, y)$ to belong to the set $\mathcal{Y}$ it is necessary and sufficient that its Mellin transform* (2.1) *should be represented in the form* (4.2.3), *where $\alpha(u, v)$ and $\beta(u, v)$ satisfy conditions* (4.2.4), (4.2.5), *and the function* (4.2.3) *is positive definite.*

4.3. *Particular cases.* Let the distribution function $F(x, y)$ be given by the density

$$\begin{aligned} f(x, y) &= x^{p_1-1} y^{p_2-1} f_0(x + y) \qquad &&\text{for } x > 0,\ y > 0; \\ &= 0 &&\text{otherwise,} \end{aligned} \tag{4.3.1$'$}$$

where the function $f_0(z)$ is in such a form that (4.3.1) is a density function. The Mellin transform of (4.3.1′) is

$$\begin{aligned} h(u, v) &= \int_0^\infty \int_0^\infty x^u y^v f(x, y)\, dx\, dy \\ &= \int_0^\infty \int_0^\infty x^{p_1+u-1} y^{p_2+v-1} f_0(x + y)\, dx\, dy. \end{aligned} \tag{4.3.1$''$}$$

Substituting

$$x = \rho \cos^2 \vartheta, \qquad y = \rho \sin^2 \vartheta, \tag{4.3.2$'$}$$

we see the area of integration will be changed into

$$0 < \rho < \infty, \qquad 0 < \vartheta < \pi/2, \tag{4.3.2$''$}$$

and the Jacobian will be

$$J = 2\rho \sin \vartheta \cos \vartheta. \tag{4.3.2.$'''$}$$

Hence we obtain the Mellin transform (4.3.1″) in the form

$$\begin{aligned} h(u, v) &= \int_0^\infty \rho^{p_1+p_2+u+v-1} f_0(\rho)\, d\rho \int_0^{\pi/2} \cos^{2(p_1+u)-1} \vartheta \sin^{2(p_2+v)-1} \vartheta\, d\vartheta \\ &= \int_0^\infty \rho^{p_1+p_2+u+v-1} f_0(\rho)\, d\rho \cdot \frac{1}{2} B(p_1 + u, p_2 + v) \\ &= \frac{\int_0^\infty \rho^{p_1+p_2+u+v-1} f_0(\rho)\, d\rho}{2\Gamma(p_1 + p_2 + u + v)} \Gamma(p_1 + u)\Gamma(p_2 + v). \end{aligned} \tag{4.3.1$'''$}$$

Finally we obtain

$$h(u, v) = H(u + v)\Gamma(p_1 + u)\Gamma(p_2 + v),$$

where

$$(4.3.3') \qquad H(w) = \frac{\int_0^\infty \rho^{p_1+p_2+w-1} f_0(\rho)\, d\rho}{2\Gamma(p_1 + p_2 + w)}$$

and

$$(4.3.3'') \qquad H(0) = [\Gamma(p_1)\Gamma(p_2)]^{-1}.$$

It is easy to see that the Mellin transform (4.3.1) satisfies the equation (4.2.2). Thus the distribution function given by the density (4.3.1′) belongs to $\mathcal{Y}$.

Taking for instance $f_0(z)$ given by

$$(4.3.4) \qquad \begin{aligned} f_0(z) &= \text{const} \quad &&\text{for } x > 0, y > 0, x + y < a \\ &= 0 &&\text{otherwise} \end{aligned}$$

we see that the distribution function given by density

$$(4.3.5) \qquad \begin{aligned} f(x, y) &= Ax^{p_1-1}y^{p_2-1} \quad &&\text{for } x > 0, y > 0, x + y < a \\ &= 0 &&\text{otherwise} \end{aligned}$$

belongs to $\mathcal{Y}$.

Also the distribution function of a random variable, taking its values on the line $x > 0$, $y > 0$, $x + y = a$ with density

$$(4.3.6) \qquad \begin{aligned} f(x, y) &= Ax^{p_1-1}y^{p_2-1} \quad &&\text{for } 0 \leqq x < a, x + y = a \\ &= 0 &&\text{otherwise,} \end{aligned}$$

belongs to $\mathcal{Y}$ (we see that it is a degenerate two-dimensional random variable).

Now let us take $f_0(z) = Az^p e^{-az^q}$ $a > 0$, $q \neq 0$. We obtain the density

$$(4.3.7) \qquad \begin{aligned} f(x, y) &= A(x + y)^p e^{-a(x+y)^q} x^{p_1-1}y^{p_2-1} \quad &&\text{for } x > 0, y > 0 \\ &= 0 &&\text{otherwise} \end{aligned}$$

for which the distribution function belongs to $\mathcal{Y}$. Taking this further $p = 0$, $q = 1$ we obtain the bivariate distribution where the coordinates are independent and have gamma distributions.

Let us take in (4.3.1),

$$(4.3.8') \qquad h^*(u, v) = h(-v, -u) = H(-u - v)\Gamma(p_2 - u)\Gamma(p_1 - v).$$

Thus we also obtain a Mellin transform satisfying equation (4.2.2) and the distribution function corresponding to (4.3.8′) belongs also to $\mathcal{Y}$. This distribution is given by density

$$
\begin{aligned}
f^*(x, y) &= (xy)^{-2} f(x^{-1}, y^{-1}) = (xy)^{-2} x^{-p_2+1} y^{-p_1+1} f_0(x^{-1} + y^{-1}) \\
&= x^{-p_2-1} y^{-p_1-1} f_0[(x + y)/xy] \qquad \text{for } x > 0,\ y > 0 \\
&= 0 \qquad \text{otherwise.}
\end{aligned} \tag{4.3.8''}
$$

4.4. *A method of determining other distributions belonging to $\mathcal{Y}$.* Let us represent the Mellin transform (4.3.1) as a product of a finite or an infinite number of Mellin transforms

$$
\begin{aligned}
h(u, v) &= H(u + v)\Gamma(p_1 + u)\Gamma(p_2 + v) \\
&= \prod_{k \varepsilon K} H_k(u + v)\varphi_k(u)\psi_k(v) = \prod_{k \varepsilon K} h_k(u, v).
\end{aligned} \tag{4.4.1}
$$

Because every $h_k(u, v)$ is a Mellin transform of a bivariate distribution, then $h_k^*(u\ v) = h_k(-v, -u)$ is also a Mellin transform of such a distribution. Let us divide the set K of integers k into two mutually exclusive and exhaustive subsets K_1 and K_2. The product

$$
\begin{aligned}
h(u, v) &= \prod_{k \varepsilon K_1} h_k(u, v) \cdot \prod_{k \varepsilon K_2} h_k^*(u, v) \\
&= \prod_{k \varepsilon K_1} H_k(u + v)\varphi_k(u)\psi_k(v) \prod_{k \varepsilon K_2} H_k(-u - v)\psi_k(-u)\varphi_k(-v)
\end{aligned} \tag{4.4.2}
$$

satisfies the equation (4.2.2) and that is why (4.4.2) is the Mellin transform of a distribution from $\mathcal{Y}$.

Using for instance formula (3.4.3) we can expend the Mellin transforms (4.3.1) and (4.3.8′) in the form

$$
\begin{aligned}
h(u, v) &= \prod_{k=1}^{n} h_k(u, v) \\
&= \prod_{k=1}^{n} H_k(u + v)\Gamma\left(\frac{p_1 + k - 1 + u}{n}\right)\Gamma\left(\frac{p_2 + k - 1 + v}{n}\right)
\end{aligned} \tag{4.4.3'}
$$

$$
\begin{aligned}
h^*(u, v) &= \prod_{k=1}^{n} h_k^*(u, v) \\
&= \prod_{k=1}^{n} H_k(-u - v)\Gamma\left(\frac{p_2 + k - 1 - u}{n}\right)\Gamma\left(\frac{p_1 + k - 1 - v}{n}\right).
\end{aligned} \tag{4.4.3''}
$$

The distributions $F_k(x, y)$ and $F_k^*(x, y)$ of the pairs (X_k, Y_k), (X_k^*, Y_k^*) corresponding to the factors $h_k(u, v)$, $h_k^*(u, v)$ have densities given by formulas

$$
\begin{aligned}
f_k(x, y) &= x^{p_1+k-2} y^{p_2+k-2} g_k(x^n + y^n) \quad \text{for } x > 0,\ y > 0, \\
&= 0 \qquad \text{otherwise,}
\end{aligned} \tag{4.4.4'}
$$

$$
\begin{aligned}
f_k^*(x, y) &= x^{-p_2-k} y^{-p_1-k} g_k^*(x^{-n} + y^{-n}) \quad \text{for } x > 0,\ y > 0, \\
&= 0 \qquad \text{otherwise.}
\end{aligned} \tag{4.4.4''}
$$

Thus the function

$$(4.4.5') \qquad \begin{aligned} h(u, v) &= \prod_{k \varepsilon K_1} h_k(u, v) \prod_{k \varepsilon K_2} h_k^*(u, v) \\ &= \prod_{k \varepsilon K_1} H_k(u + v)\Gamma\left(\frac{p_1 + k - 1 + u}{n}\right)\Gamma\left(\frac{p_2 + k - 1 + v}{n}\right) \\ &\cdot \prod_{k \varepsilon K_2} H_k(-u - v)\Gamma\left(\frac{p_2 + k - 1 - u}{n}\right)\Gamma\left(\frac{p_1 + k - 1 - v}{n}\right) \end{aligned}$$

satisfies the equation (4.2.2) and hence the corresponding distribution belongs to $\mathcal{Y}$. The bivariate random variable corresponding to (4.4.5′) may be presented in the form

$$(4.4.5''] \qquad (X, Y) = (\prod_{k \varepsilon K_1} X_k \prod_{k \varepsilon K_2} X_k^*, \prod_{k \varepsilon K_1} Y_k \prod_{k \varepsilon K_2} Y_k^*),$$

where the pairs (X_k, Y_k) $k \varepsilon K_1$, (X_k, Y_k) $k \varepsilon K_2$ are independent and their distributions are given by densities (4.4.4) respectively.

Taking for instance $n = 2$, (4.4.4″) for $k = 1$ and (4.4.4′) for $k = 2$ we obtain

$$(4.4.6') \quad f_1^*(x, y) \begin{cases} = x^{-p_2-1} y^{-p_1-1} g_1^*(x^{-2} + y^{-2}) & x > 0, y > 0 \\ = 0 & \text{otherwise} \end{cases} \quad \text{for } (X_1^*, Y_1^*),$$

$$(4.4.6'') \quad f_2(x, y) \begin{cases} = x^{p_1} y^{p_2} g_2(x^2 + y^2) & \text{for } x > 0, y > 0 \\ = 0 & \text{otherwise} \end{cases} \quad \text{for } (X_2, Y_2)$$

and the distribution of $(X_1^* X_2, Y_1^* Y_2)$ belonging to $\mathcal{Y}$.

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