# A BAYES SEQUENTIAL SAMPLING INSPECTION PLAN[1]

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**1. Introduction and summary.** Given a lot of size $N$ whose items are obtained from a statistically controlled process with an unknown probability $p$, $0 < p < 1$, of an item being defective, a rectifying sampling inspection problem arises when the loss involved in sending out the lot with $d$, $0 \leqq d \leqq N$, defectives in it is $kd$ where $k > 0$ is the loss involved in sending out a defective item instead of replacing it with a good one. The cost of inspecting $n$ items (defective items detected during inspection are replaced by good ones) is $cn$, $c > 0$. An inspection plan is to be devised that minimizes (in some suitable sense) the risk (expected loss plus inspection cost).

Let $\psi$ denote any sequential inspection plan, which determines sequentially the (random) number $\mathfrak{N}$ of items to be inspected. If the lot is sent out after inspecting $n$ items, the cost of inspection incurred is $nc$ and the expected loss due to the defectives remaining in the lot is $(N - n)pk$. Then the risk is

$$(N - n)pk + nc = Npk + nk(c/k - p),$$

and hence the risk associated with the plan $\psi$ is given by

$$R(p, \psi) = Npk + k(c/k - p)E_p(\mathfrak{N}). \tag{1.1}$$

Since $E_p(\mathfrak{N}) \geqq 0$ for every $p \in (0, 1)$, it follows from (1.1) that if $p$ were known then the optimal plan (which we denote by $\psi_p$) would be to inspect the whole lot or not to inspect at all according as $p >$ or $\leqq p_0 \equiv c/k$. The risk $R(p, \psi_p)$ of this plan is given by

$$\begin{aligned} R(p, \psi_p) &= Npk, && \text{for } p \leqq p_0, \\ &= Nc, && \text{for } p > p_0 \end{aligned}$$

(see Figure 1).

Let $d$ be the number of defectives observed in the first $n$ items inspected when $p$ is unknown. Since $d$ is a sufficient statistic for $p$ based on "the past history," the decision to stop or continue inspection may be made to depend on $d$. Thus we may restrict our attention to plans defined by two mutually exclusive and exhaustive sets in the $(n, d)$ plane called the *continuation* and *stopping* sets. If at any stage the current position $(n, d)$ belongs to the continuation set we inspect another item from the lot. If it belongs to the stopping set we stop inspection and send out the lot. Obviously all the points with $n \geqq N$ or with $d \geqq N$ are stopping points. We now define the *boundary set* as the set of stopping points

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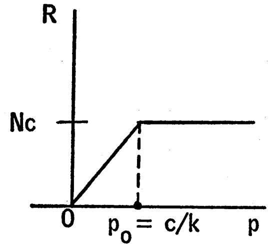


FIG. 1. Risk for optimal plan where $p$ is known.

which are accessible from some continuation point. Because of the monotonic nature of the two types of cost associated with a plan one would expect that for an admissible plan, the boundary set consists of points $\{(n, d_n)\}$ with $d_n$ monotonically nondecreasing in $n$ and the points $(n, d)$ are stopping or continuation points according as $d \leqq d_n$ or $d > d_n$. Such plans are thus defined by a set of boundary points $\{(n, d_n)\}$ or equivalently $\{(d, n_d)\}$.

Anscombe [1] considers a class of linear plans where $n_d$ is defined by

$$n_d = n_0 + d\omega, \qquad d = 0, 1, 2, \cdots, [(N - n_0)/(\omega + 1)], \tag{1.2}$$

([ ] denoting the greatest integer part). He considers the problem of choosing the integers $n_0$ and $\omega$ so that the following conditions are satisfied:

(1) Whatever the unknown number of defectives in the lot, there is at most a preassigned risk that after inspection the lot will contain more than a specified number of defectives.

(2) The average number of items inspected is as small as possible for some range of values (or simply for a fixed value) of the unknown number of defectives.

It is to be noted that Anscombe's problem as formulated above, is typical of the classical Neyman-Pearson theoretic formulation and as such differs from our decision theoretic formulation of minimizing (in some suitable sense) the risk associated with a plan. In fact, Wurtele, was first to consider this formulation in her thesis [7]. She gives a characterization of the Bayes sequential plan $\{(d, \tilde{n}_d)\}$ as a multiple sampling plan which is a consequence of the fact that $\tilde{n}_d$ is an increasing function of $d$. She presents algorithms for obtaining the large sample behavior of the optimal (Bayes) boundary in the limiting case appropriate for the Poisson approximation.

We shall concern ourselves with the Bayes sequential plans for beta prior distributions. To study the large sample behavior of the optimal boundary and associated risk we formulate the following associated normal version of the sampling inspection problem.

We observe $X_t$, a Wiener process with unknown drift $\mu$ per unit time and variance one per unit time. Let $T$ be the time at which observation is stopped. Subject to $T \leqq t_0$, it is desired to locate a stopping set which minimizes $E(T\mu)$ (where $E$ denotes the expectation operator over the marginal distribution of the data with a normal prior distribution for $\mu$). If $\mu$ were known, an optimal procedure would be to stop immediately if $\mu \geqq 0$ and to continue as long as possible if $\mu > 0$.

The associated problem is then tackled with the help of techniques developed by Chernoff [3], [4], [5] in connection with the sequential testing for the sign of $\mu$. The results are then related to the original sampling inspection problem. Numerical computations are given comparing the boundary and the risk calculated using the asymptotic formulae with the exact ones.

Lastly, the optimal (Bayes) plan is compared with the optimal plan among the class of linear plans (1.2). Here the optimum $n_0$ and $\omega$ are approximated by the solution of the corresponding Wiener process problem.

**2. Bayes solution.** It is well known [2] how to obtain the Bayes procedure for a truncated sequential decision problem. In our particular problem, the Bayes risk with respect to a beta prior density with parameters $(a_0, b_0)$ is given by $k\rho_N(a_0, b_0)$ where $\rho_N(a, b)$ is computed recursively from the equations:

$$\rho_0(a, b) = 0$$

and

$$\rho_i(a, b) = \min [\rho_i^*(a, b), [a/(a + b)]\rho_{i-1}(a + 1, b) + [b/(a + b)]\rho_{i-1}(a, b + 1) + p_0]$$

for $i \geqq 1$, where $\rho_i^*(a, b) = ia/(a + b)$. There is a Bayes stopping rule for the above prior distribution which calls for stopping after $n$ observations if and only if

$$\rho_{N-n}(a_0 + d, b_0 + r) = \rho_{N-n}^*(a_0 + d, b_0 + r)$$

where $d$ and $r$ are respectively the number of defectives and the number of non-defectives in the $n$ items inspected.

It is to be noted that the Bayes risk and hence the Bayes stopping rule depend on $a_0, b_0, d, r$, and $N$ and this dependence can be expressed in terms of $a = a_0 + d, b = b_0 + r$ and $M = N + a_0 + b_0$ alone since $N - n = M - a - b$. Thus a single set of stopping points in the $(a, b)$-plane calculated for a *fixed* $M$ suffices to give the stopping rules for various combinations of beta prior densities with parameters $(a_0', b_0')$ and lot sizes $N'$ such that $a_0' + b_0' + N' = M$. In other words, the Bayes risk with respect to a beta prior density with parameters $(a, b)$ for the problem with the lot size $M - a - b$ is given by $k\rho(a,b)$ where $\rho(a, b)$ satisfies the recursion relation $\rho(a, b) = 0$ for $a + b = M$,

$$\rho(a, b) = \min [\rho^*(a, b), [a/(a + b)]\rho(a + 1, b) + [b/(a + b)]\rho(a, b + 1) + p_0] \quad \text{for } a + b \leqq M - 1, \tag{2.1}$$

and where $\rho^*(a, b) = (M - a - b)a/(a + b)$ corresponds to stopping. Furthermore the point $(a, b)$ is in the stopping set if and only if $\rho(a, b) = \rho^*(a, b)$.

A set of stopping points in the $(a, b)$-plane may easily be reinterpreted as a set of stopping points in the $(a, m)$-plane where $m = a + b$. In other words, shifting the origin in the $(d, n)$-plane to a point $(a_0, a_0 + b_0)$ in the $(a, m)$-plane, the boundary points $\{(a, \tilde{m}_a)\}$ for a fixed $M$ give the set of boundary points $\{(d, \tilde{n}_d)\}$ for the problem with lot size $N \equiv M - a_0 - b_0$.

For some of the details of the characterization of the optimal boundary the interested reader is referred to Wurtele's thesis (loc. cit.). We shall, however, indicate certain facts which are useful in programming efficiently the computation of the optimal boundary and the optimal risk.

Let

$$\begin{aligned} a^* = M - b^* &= Mp_0 \qquad \text{if } Mp_0 \text{ is an integer} \\ &= [Mp_0] + 1 \qquad \text{otherwise.} \end{aligned}$$

It can be shown that

$$\begin{aligned} \rho(a, b) &= (M - a - b)a/(a + b) \qquad \text{for } b \geqq a^* + 1 \\ &= (M - a - b)p_0 \qquad \text{for } a \geqq a^*. \end{aligned}$$

Thus the recurrence relation (2.1) is needed for computing $\rho(a, b)$ only for $(a, b)$ in the rectangle $a \leqq a^* - 1, b \leqq b^*$.

Furthermore, it can be shown that the optimal boundary lies below the line $a/b = p_0/(1 - p_0)$ and to the left of $b = b^*$. (The $a$-axis is vertical and the $b$-axis is horizontal.)

**3. A free boundary problem related to the Bayes solution for large $M$.** In this section we indicate heuristically that for large $M$, the problem of obtaining the optimal boundary and the Bayes risk, reduces to a free boundary problem (FBP) involving a diffusion equation.

Let

$$\begin{aligned} (3.1) \quad \rho'(a, b) &= \rho(a, b) + a - Ma/(a + b) \\ &= \rho(a, b) - \rho^*(a, b), \quad 0 \leqq a + b \leqq M. \end{aligned}$$

It follows from (2.1) that $\rho'(a, b)$ satisfies the following recursion relation in the continuation region:

$$\begin{aligned} (3.2) \quad \rho'(a, b) = \min [0, &[a/(a + b)]\rho'(a + 1, b) \\ &+ [b/(a + b)]\rho'(a, b + 1) + p_0 - a/(a + b)]. \end{aligned}$$

We now normalize with the following transformation:

$$(3.3) \qquad \begin{aligned} z &= [a(1 - p_0) - bp_0]/[Mp_0(1 - p_0)]^{\frac{1}{2}} \\ &= [a - (a + b)p_0]/[Mp_0(1 - p_0)]^{\frac{1}{2}}, \\ t &= (a + b)/M, \\ B'(z, t) &= \rho'(a, b)/[Mp_0(1 - p_0)]^{\frac{1}{2}}. \end{aligned}$$

In the continuation region, we have

$$
\begin{aligned}
B'(z, t) &= \tfrac{1}{2}[B'(z + \delta_1 M^{-\frac{1}{2}}, t + M^{-1}) + B'(z + \delta_2 M^{-\frac{1}{2}}, t + M^{-1})] \\
&+ [(2p_0 - 1) + 2[p_0(1 - p_0)]^{\frac{1}{2}}(z/t)M^{-\frac{1}{2}}]\tfrac{1}{2}[B'(z + \delta_1 M^{-\frac{1}{2}}, t + M^{-1}) \\
&\qquad - B'(z + \delta_2 M^{-\frac{1}{2}}, t + M^{-1})] - (z/t)M^{-1},
\end{aligned} \tag{3.4}
$$

where

$$\delta_1 = (1 - p_0)/[p_0(1 - p_0)]^{\frac{1}{2}}, \qquad \delta_2 = -p_0/[p_0(1 - p_0)]^{\frac{1}{2}}.$$

Now assuming $B'(z, t)$ smooth enough to be differentiated with respect to both of its arguments the first few times, (3.4) may be simplified to

$$
\begin{aligned}
\tfrac{1}{2}B'_{zz} + B_t' + (z/t)B_z' - (z/t) &= \{(2p_0 - 1)/2[p_0(1 - p_0)]^{\frac{1}{2}}\}((z/t)B'_{zz} \\
&\qquad + \tfrac{1}{3}B'_{zzz})M^{-\frac{1}{2}} + O(M^{-1}).
\end{aligned} \tag{3.5}
$$

Thus in the limit as $M \to \infty$, $B'(z, t)$ satisfies the following partial differential equation:

$$\tfrac{1}{2}B'_{zz} + B_t' + (z/t)B_z' - (z/t) = 0 \tag{3.6}$$

in the continuation region in the $(z, t)$-plane,

$$z \geqq \tilde{z}(t), \qquad 0 \leqq t \leqq 1 \tag{3.7}$$

where $\tilde{z}(t)$ is the unknown boundary to be determined along with $B'(z, t)$. (See Figure 2.)

A boundary condition at $z = \tilde{z}(t)$ is given by the continuity requirement as

$$B' |_{z=\tilde{z}(t)} = 0. \tag{3.8}$$

Another boundary condition is obtained by the following argument: Suppose $(a, b)$ is on the boundary, while $(a, b + 1)$ is outside and $(a + 1, b)$ is inside the continuation region. Then $\rho'(a, b) = 0 = \rho'(a, b + 1)$ so that (3.2) suggests that

$$[a/(a + b)]\rho'(a + 1, b) \approx a/(a + b) - p_0 . \tag{3.9}$$

Using the transformation (3.3), expanding $B'$ about $(z, t)$ and remembering that $B'(z, t) = 0$ for $(a, b)$ on the boundary, the formal application of (3.9) as

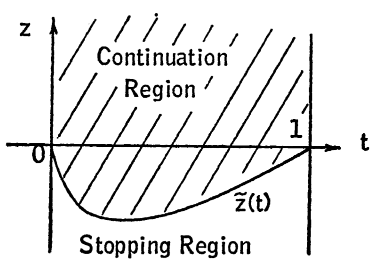


FIG. 2.

an equality leads to

$$B_z' \big|_{z=\tilde{z}(t)} = (z/t)[p_0(1 - p_0)]^{-\frac{1}{2}} M^{-\frac{1}{2}} + O(M^{-1}), \tag{3.10}$$

which suggests that in the limit as $M \to \infty$, we expect

$$B_z' \big|_{z=\tilde{z}(t)} = 0. \tag{3.11}$$

Thus the problem of obtaining the optimum boundary with the corresponding risk reduces in the limit to the following free boundary problem:

To solve for $\tilde{z}(t)$ and $B'(z, t)$ where $B'(z, t)$ satisfies the partial differential equation (3.6) in the region (3.7) subject to the boundary conditions (3.8) and (3.11).

It should be noted however that (3.3) is not the only transformation for which we arrive in the limit to the above mentioned free boundary problem. If, for example, we define $t$ slightly differently as

$$t = [a(1 - p_0) + bp_0]/2Mp_0(1 - p_0)$$

instead of by (3.3) [both are the same when $p_0 = \frac{1}{2}$] it may be verified by going through the same type of algebra that (3.10) remains unaltered whereas the right hand side of (3.5) is changed to

$$\{(2p_0 - 1)/2[p_0(1 - p_0)]^{\frac{1}{2}}\}[\tfrac{1}{3}B'_{zzz} + (z/t)(B'_{zz} + B_t' + (z/t)B_z' - z/t)]M^{-\frac{1}{2}} + O(M^{-1}).$$

Thus up to the first order of approximation either definition of $t$ yields the same equation. It is an interesting question, however, to ask which normalization gives a better fit to the exact solutions in the discrete case. However, we shall henceforth use the first normalization because of computational simplicity.

For the sake of conformity with the notations of [3], [4], [5], we transform to

$$x = -z, \tag{3.12}$$

$$B(x, t) = B'(z, t) + x, \tag{3.13}$$

and restate the free boundary problem as follows:

To solve for $\tilde{x}(t)$ and $B(x, t)$ where $B(x, t)$ satisfies

$$\tfrac{1}{2}B_{xx} + (x/t)B_x + B_t = 0 \tag{3.14}$$

in the continuation region,

$$x \leqq \tilde{x}(t), \qquad 0 \leqq t \leqq 1 \tag{3.15}$$

subject to the boundary conditions on $x = \tilde{x}(t)$:

$$B = x, \tag{3.16}$$

$$B_x = 1. \tag{3.17}$$

We notice that this free boundary problem differs from that of [4] in having the

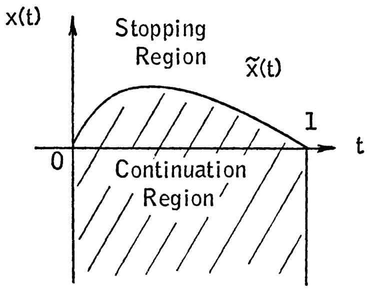


FIG. 3.

cost rate $c(x, t) = 0$, the stopping risk $D(x, t) = x$, and a single boundary. In so far as the techniques developed in [3], [4], [5] do not depend on the particular cost and risk functions they are equally applicable to this problem.

The similarity of the two free boundary problems immediately prompts us to notice that the FBP stated above also arises in the following Wiener process problem mentioned in the introduction.

We observe a Wiener process with unknown drift $\mu$ per unit time and variance 1 per unit time. Given that $\mu$ has a prior normal distribution with mean $\mu_0$ and variance $\sigma_0^2$, it is convenient to originate the process from the point $(x_0, t_0)$, $x_0 = \mu_0/\sigma_0^2$ and $t_0 = 1/\sigma_0^2$. Then the posterior probability distribution of $\mu$ given $X_t = x$ is normal with mean $x/t$ and variance $1/t$. It is desired to minimize the "risk" $E(T\mu)$ where $T$ is the stopping time, subject to $t_0 \leqq T \leqq 1$. If we let $B(x_0, t_0)$ represent the risk associated with the optimal plan, the equation $B(x, t) = E\{B(X_{t+\delta}, t + \delta)\} + o(\delta)$ is valid in the continuation region and implies Equation (3.14). The boundary conditions (3.16) and (3.17) follow as in [4]. The first condition is also valid for non-optimal procedures.

Since the free boundary problem is exact and not approximate for the Wiener process problem, we shall describe asymptotic behavior of the solution of the Wiener process problem. These will then be translated to approximate solutions of the sampling inspection problem.

**4. Asymptotic behavior of the solution of the free boundary problem near $t = 1$.** In this section we formally develop asymptotic expansions for the stopping boundary $\tilde{x}(t)$ and the risk $B(x, t)$ for $t$ near 1 by using techniques similar to those used by Breakwell and Chernoff [3]. It can be proved as in [3] that these formal expansions do represent asymptotic approximations.

We use the following transformation of variables on the FBP of (3.14)–(3.17):

$$\begin{aligned} y &= x/t, \\ s &= 1/t - 1, \qquad 0 \leqq s < \infty, \\ u(y, s) &= B(x, t) - y. \end{aligned} \tag{4.1}$$

Then $u(y, s)$ satisfies the heat equation

$$\tfrac{1}{2}u_{yy} = u_s \tag{4.2}$$

in the region $y \leqq \tilde{y}(s)$ with the boundary conditions on the boundary $\tilde{y}(s)$:

$$u = y(1/(s+1) - 1), \tag{4.3}$$

$$u_y = 1/(s+1) - 1, \tag{4.4}$$

for $t$ near 1, i.e., for small $s$, the right hand sides of (4.3) and (4.4) may be expanded in a power series of $s$. We are thus led to seek separable solutions for $u$ of the form

$$s^{n/2}G_n(y/s^{\frac{1}{2}}), \qquad n = 1, 2, 3, \cdots, \tag{4.5}$$

where $G_n(\tilde{y}/s^{\frac{1}{2}})$, may itself be expanded in a power series of $\tilde{y}/s^{\frac{1}{2}}$, for small $s$. Solutions of the heat equation (4.2) of the form (4.5) may be obtained by considering a distribution $y^n/n!$ of heat sources along the positive $y$-axis giving rise to

$$\int_0^\infty s^{-\frac{1}{2}} \varphi\left((y - y')/s^{\frac{1}{2}}\right) (y'^n/n!)\, dy' = (s^{n/2}/n!) \int_{-\beta}^\infty (\beta + \epsilon)^n \varphi(\epsilon)\, d\epsilon, \tag{4.6}$$

where

$$\varphi(\epsilon) = (2\pi)^{-\frac{1}{2}} \exp(-\epsilon^2/2), \tag{4.7}$$

$$\beta = y/s^{\frac{1}{2}}.$$

Thus

$$G_n(\beta) = (1/n!) \int_{-\beta}^\infty (\beta + \epsilon)^n \varphi(\epsilon)\, d\epsilon. \tag{4.8}$$

Note that $G_n'(\beta) = G_{n-1}(\beta)$ for $n \geqq 1$ and that $G_n(\beta)$ is meaningful for real (not necessarily integer) $n > -1$. While the integral (4.8) fails to exist for $n \leqq -1$, we may extend the definition to negative $n$ by the above differentiation property, obtaining

$$G_n(\beta) = P_n(\beta)\Phi(\beta) + Q_n(\beta)\varphi(\beta), \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{4.9}$$

where $\varphi(\beta)$ and $\Phi(\beta)$ are respectively the standard normal density and the corresponding cdf and $P_n(\beta)$ and $Q_n(\beta)$'s are certain polynomials in $\beta$. We write below the first few of these polynomials to be used later.

$$\begin{aligned}
&P_0 = 1, && Q_0 = 0,\\
&P_1 = \beta, && Q_1 = 1,\\
&P_2 = \beta^2/2 + \tfrac{1}{2}, && Q_2 = \beta/2,\\
&P_3 = \beta^3/6 + \beta/2, && Q_3 = \beta^2/6 + \tfrac{1}{3},\\
&P_4 = \beta^4/24 + \beta^2/4 + \tfrac{1}{8}, && Q_4 = \beta^3/24 + (5/24)\beta,\\
(4.10)\quad &P_5 = \beta^5/120 + \beta^3/12 + \beta/8, &&\\
& && Q_5 = \beta^4/120 + (3/40)\beta^2 + 1/15,
\end{aligned}$$

$$P_6 = \beta^6/720 + \beta^4/48 + \beta^2/16 + 1/48,$$
$$Q_6 = \beta^5/720 + (14/720)\beta^3 + (11/240)\beta,$$
$$P_7 = \beta^7/5040 + \beta^5/240 + \beta^3/48 + \beta/48,$$
$$Q_7 = \beta^6/5040 + \beta^4/252 + (87/5040)\beta^2 + 1/105.$$

We now take

$$u(y, s) = \textstyle\sum_{n=0}^{\infty} a_n s^{n/2} G_n(\beta), \tag{4.11}$$

$$\tilde{\beta}(s) = \textstyle\sum_{n=0}^{\infty} c_n s^n, \tag{4.12}$$

where

$$\tilde{y}(s) = \tilde{\beta}(s) s^{\frac{1}{2}}, \tag{4.13}$$

and the coefficients $a_n$ and $c_n$ are obtained alternately by matching the coefficients of equal powers of $s$ in the following equations obtained by substituting (4.11) and (4.12) in the boundary conditions (4.3) and (4.4):

$$\textstyle\sum_0^{\infty} a_n s^{n/2} G_n[\sum_0^{\infty} c_m s^m] = (\sum_0^{\infty} c_m s^m) s^{\frac{1}{2}}[-s + s^2 - s^3 + \cdots], \tag{4.14}$$

$$\textstyle\sum_0^{\infty} a_n s^{n/2} G_{n-1}[\sum_0^{\infty} c_m s^m] = s^{\frac{1}{2}}[-s + s^2 - s^3 + \cdots]. \tag{4.15}$$

Expanding the $G_n$'s around $c_0$ and matching coefficients we arrive at the following equations:

$$a_1 = 0 = a_{2n}, \qquad n = 0, 1, 2, \cdots. \tag{4.16}$$

$c_0$ is the unique solution of the equation

$$G_3(c_0) - c_0 G_2(c_0) = 0. \tag{4.17}$$

$$\begin{aligned} c_1 &= 2c_0/(5 + c_0^2), \\ c_2 &= (1/2c_0)[1 + (9 + c_0^2)c_1^2 \\ &\qquad - [7(5 + 10c_0^2 + c_0^4)/(35 + 16c_0^2 + c_0^4)](1 + c_1^2)], \\ a_3 &= -2(1 - c_0^2)/\Phi(c_0), \\ a_5 &= 40(1 - c_0^2)/(5 + c_0^2)\Phi(c_0), \\ a_7 &= [-1680(1 + c_1^2)/(c_0^4 + 16c_0^2 + 35)]\cdot(1 - c_0^2)/\Phi(c_0). \end{aligned} \tag{4.18}$$

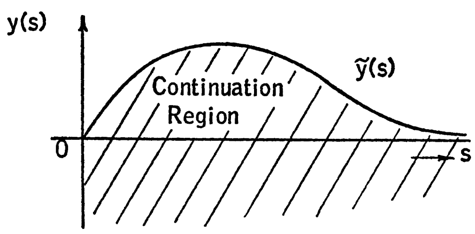


FIG. 4.

Numerical values of these coefficients, correct to 5 significant figures are as follows:

$$
\begin{aligned}
c_0 &= 0.63883, & a_3 &= -1.6029, \\
c_1 &= 0.23625, & a_5 &= 5.9278, \\
c_2 &= -0.089257, & a_7 &= -3.4094.
\end{aligned} \tag{4.19}
$$

Thus we have,

THEOREM 4.1. *The asymptotic expansions for the optimal boundary* $\tilde{x}(t)$ *and the optimal risk* $B(x, t)$ *within the continuation set are*

$$\tilde{x}(t) = [t(1-t)]^{\frac{1}{2}}[c_0 + c_1(1-t)/t + c_2((1-t)/t)^2 + \cdots], \tag{4.20}$$

$$
\begin{aligned}
B(x, t) = x/t &+ a_3((1-t)/t)^{\frac{3}{2}}G_3(x/[t(1-t)]^{\frac{1}{2}}) \\
&+ a_5((1-t)/t)^{\frac{5}{2}}G_5(x/[t(t-1)]^{\frac{1}{2}}) + \cdots
\end{aligned} \tag{4.21}
$$

*where the* $c_n$, $a_n$, $G_n$ *are determined by Equations* (4.14) *and* (4.15).

Translating this theorem in terms of the original sampling inspection problem, we refer to the equations at the beginning of Section 3, and (3.12) and (3.13). Denoting the set of optimal boundary points in the $(m, a) = (a + b, a)$ plane by $\{m, \tilde{a}_m\}$, we have for $m/M$ close to one,

$$
\begin{aligned}
\tilde{a}_m \approx mp_0 + [mp_0(1-p_0)]^{\frac{1}{2}}[1 - m/M]^{\frac{1}{2}}[c_0 &+ c_1(M/m - 1) \\
&+ c_2(M/m - 1)^2 + \cdots],
\end{aligned} \tag{4.22}
$$

while the risk $k\rho(a, b) = k\rho(a, m - a)$ is given by

$$\rho(a, b) = a(M/m - 1) - [Mp_0(1-p_0)]^{\frac{1}{2}}[B(x, t) - x] \tag{4.23}$$

for $(a, b)$ on the continuation set. In particular, along the critical line $a/b = p_0/(1 - p_0)$, $x = 0$ and

$$\rho(a, [(1-p_0)/p_0]a) = Mp_0 - a + [Mp_0(1-p_0)]^{\frac{1}{2}}B(0, a/Mp_0) \tag{4.24}$$

which is easily approximated using (4.21).

**5. Transformations of the free boundary problem.** Some remarks are in order concerning the choice of the transformations in Section 4 which led to the heat equation. It has been noted [3] that the prior distribution of $\mu$ makes $Y = X/t$ a Wiener process in the $-1/t$ scale. From this it naturally follows that $B(x, t)$ satisfies the heat equation in the $(y, 1/t)$ variables. For convenience in Section 4 we translated the $1/t$ scale so that the $t = 1$ would correspond to $s = 1/t - 1 = 0$. The translation does not affect the heat equation. The boundary condition $B = x$ along $t = 1$ could be simplified by subtracting a solution of the heat equation for which $B = x$ when $t = 1$ and $x < 0$. Such a solution is not unique and the term $y$ (which coincides with $x$ along $t = 1$) was selected as a matter of convenience. It would have been quite natural to use the solution generated by the heat sources equal to $x$ along $s = 0$; i.e.,

$$-s^{\frac{1}{2}}\psi^{+}(y/s^{\frac{1}{2}}) = \int_{-\infty}^{0} s^{-\frac{1}{2}}\varphi((y - y')/s^{\frac{1}{2}})y'\,dy', \qquad s = t^{-1} - 1. \tag{5.1}$$

These remarks are pertinent to the study, in Section 6, of $t \to 0$. Here the untranslated $1/t$ scale is algebraically slightly simpler to deal with. While subtracting $y$ would make the boundary condition zero along the negative part of the vertical axis $t = 1$, the $\psi^+$ term has an advantage in that it does not get very large as $x \to +\infty$. What is important, is to make sure that we subtract from $B$ a solution of the heat equation such that the difference approaches zero as $x \to -\infty$ along $t = 1$. Otherwise the boundary condition along $t = 1$ could transmit a large effect to $t \to 0(t^{-1} \to \infty)$. While subtracting (5.1) seems most appropriate $-t^{-\frac{1}{2}}\psi^+(yt^{\frac{1}{2}})$ is more convenient to deal with algebraically and is also adequate, since as $y \to -\infty$ along $t = 1$, $y + \psi^+(y) = \varphi(y) + y\Phi(y) \to 0$.

This correction term has a statistical interpretation. The prior distribution of $\mu$ is given by $\mathcal{L}(\mu) = \mathfrak{N}(y, t^{-1})$. When $t$ is small, $\mu$ has large variance, $|\mu|$ is almost certainly large and little observation time is required to decide the sign of $\mu$. Then the Bayes risk is approximately that associated with the action which would be correct if the "random" value of $\mu$ were given to the statistician. This risk is

$$(5.2) \quad \int_0^\infty t\mu\varphi[(\mu - y)t^{\frac{1}{2}}]t^{\frac{1}{2}}\, d\mu + \int_{-\infty}^0 \mu\varphi[(\mu - y)t^{\frac{1}{2}}]t^{\frac{1}{2}}\, d\mu = t^{\frac{1}{2}}\psi^-(yt^{\frac{1}{2}}) - t^{-\frac{1}{2}}\psi^+(yt^{\frac{1}{2}}).$$

The latter part of this expression is dominant and is our correction term.

These remarks are intended to point out that there is a certain amount of freedom in choosing appropriate transformations of scale and risk. At the same time we may recall that the asymptotic solution for $t$ near 1 involves a distribution of heat sources for positive $y$ at $t = 1$. We may also anticipate that, as in [5], the asymptotic solution for $t$ near 0 will invoke a distribution of heat sources at $t^{-1} = 0$. In the present application the sources will be restricted to positive $y$.

**6. Asymptotic behavior of the Bayes solution for small $t$.** Here we transform to

$$(6.1) \qquad y = xt^{-1}, \qquad s = t^{-1}, \qquad \alpha = y/s^{\frac{1}{2}},$$

$$(6.2) \qquad u^*(y, s) = B(x, t) + s^{\frac{1}{2}}\psi^+(y/s^{\frac{1}{2}}).$$

Then the free boundary problem is given by

$$(6.3) \qquad u_s^* = \tfrac{1}{2}u_{yy}^*,$$

$$(6.4) \qquad u^* = \alpha s^{-\frac{1}{2}} + s^{\frac{1}{2}}\psi^+(\alpha) \qquad \text{on the boundary } \tilde{\alpha}(s),$$

$$(6.5) \qquad u_\alpha^* = s^{-\frac{1}{2}} - s^{\frac{1}{2}}[1 - \Phi(\alpha)] \qquad \text{on the boundary } \tilde{\alpha}(s).$$

Following Chernoff [5] we consider solutions of the heat equation of the form

$$(6.6) \qquad u^* = K_0 s^{-\frac{1}{2}}\varphi(\alpha) + g(\alpha, s)$$

where

$$(6.7) \qquad g(\alpha, s) = \int_0^\infty [\varphi(\alpha - b)]f(s^{\frac{1}{2}}b)\, db,$$

while the boundary will be represented by an expansion of the form

$$\tilde{\beta} \equiv \log \tilde{s} = \alpha^2/2 + \log \alpha + k_0 + k_1\alpha^{-2} + k_2\alpha^{-4} + \cdots = \alpha^2/2 + \log \alpha + k_0 + \eta \tag{6.8}$$

where the unknown coefficients $k_i$ are to be determined along with the function $f$ with the help of the boundary conditions (6.4) and (6.5).

Note that for small $t$ we expect $\tilde{\alpha}$ to be large. Then along the boundary the term $K_0 s^{-\frac{1}{2}}\varphi(\alpha)$ is relatively negligible. Also in the integral (6.7) the dominant part comes when $b$ is close to $\alpha$. Using the expansions for large $\alpha$, of $g(\alpha, s)$, $g_\alpha(\alpha, s)$, $\psi^+(\alpha)$, $1 - \Phi(\alpha)$, as given in Chernoff [5] the boundary conditions (6.4) and (6.5) reduce to

$$\begin{aligned} f(s^{\frac{1}{2}}\alpha) + (s/2!)f^{(2)}(s^{\frac{1}{2}}\alpha) + (1\cdot 3/4!)s^2f^{(4)}(s^{\frac{1}{2}}\alpha) \\ + (1\cdot 3\cdot 5/6!)s^3f^{(6)}(s^{\frac{1}{2}}\alpha) + \cdots \\ \approx s^{-\frac{1}{2}}\alpha^{-1}\{\alpha^2 + (2\pi)^{-\frac{1}{2}}e^{k_0}e^{\eta}[1 - 3\alpha^{-2} + 15\alpha^{-4} - 105\alpha^6 + \cdots \}, \end{aligned} \tag{6.9}$$

$$\begin{aligned} s^{\frac{1}{2}}f^{(1)}(s^{\frac{1}{2}}\alpha) + (1\cdot 3/3!)s^{\frac{3}{2}}f^{(3)}(s^{\frac{1}{2}}\alpha) + (1\cdot 3\cdot 5/5!)s^{\frac{5}{2}}f^{(5)}(s^{\frac{1}{2}}\alpha) + \cdots \\ \approx -s^{-\frac{1}{2}}\{-1 + (2\pi)^{-\frac{1}{2}}e^{k_0}e^{\eta}[1 - \alpha^{-2} + 3\alpha^{-4} - 15\alpha^6 + \cdots]\}. \end{aligned} \tag{6.10}$$

Starting with the approximation $f_0(x) = 2 \log x^2/x$ the main terms match in Equations (6.9) and (6.10) if $(2\pi)^{-\frac{1}{2}} \exp(k_0) = 2$ or $k_0 = (\log 8\pi)/2$. We apply the resulting approximation

$$\tilde{\beta} \approx \beta_0 = \alpha^2/2 + \log \alpha + (\log 8\pi)/2 \tag{6.11}$$

to (6.9), substituting $s^{\frac{1}{2}}\alpha$ for $x$ in the argument of $f$ and we obtain a discrepancy [right side of (6.9) minus left side] which is

$$-(s^{\frac{1}{2}}\alpha)^{-1}\{3 \log \alpha^2 + \log 8\pi - 1 + O^*(\alpha^{-2})\}$$

where $O^*(\alpha^{-2r})$ is used to represent an expression which is bounded by some power of $\log \alpha^2$ divided by $\alpha^{2r}$ as $\alpha \to \infty$. To compensate for this discrepancy we apply a correction to $f_0$ making use of the approximations $s\alpha^2 \approx x^2$, $\log \alpha^2 \approx \log [2 \log x^2]$. This gives

$$f_1(x) = x^{-1}\{2 \log x^2 - 3 \log [2 \log x^2] - \log 8\pi + 1\}. \tag{6.12}$$

This approximation combined with (6.11) yields a certain discrepancy in (6.10), the main part of which is compensated for by the approximation

$$\tilde{\beta} \approx \beta_1 = \alpha^2/2 + \log \alpha + (\log 8\pi)/2 + \alpha^{-2}. \tag{6.13}$$

With one more step in the iteration we obtain the following formal expansions for $\tilde{\beta}$ and $f$:

$$\tilde{\beta} = \log \tilde{s} \sim \alpha^2/2 + \log \alpha + \tfrac{1}{2} \log 8\pi + \alpha^{-2} + \tfrac{1}{2}\alpha^{-4} + \cdots, \tag{6.14}$$

$$\begin{aligned} f(x) \sim x^{-1}\{2 \log x^2 - 3 \log (2 \log x^2) - \log 8\pi + 1 \\ + (2 \log x^2)^{-1}[9 \log (2 \log x^2) + 3 \log 8\pi - 4] + \cdots. \end{aligned} \tag{6.15}$$

Using this expansion for $f$ in (6.7), we may obtain a formal expansion for $g(\alpha, s)$ and hence for the optimal risk $B(x, t)$ as

$$B(x,t) \sim -s^{\frac{1}{2}}\psi^{+}(\alpha) + K_0 s^{-\frac{1}{2}}\varphi(\alpha) + \int_0^\infty \varphi(\alpha - b) f(sb^2 + 2)^{\frac{1}{2}}\, db \tag{6.16}$$

where $\alpha = x/t^{\frac{1}{2}}$ and $s = t^{-1}$.

As in [5] it may be proved that the formal expansions (6.14) and (6.16) yield asymptotic approximations as $t \to 0$. Also the main term in (6.16) after $-s^{\frac{1}{2}}\psi^{+}(\alpha)$ is asymptotically

$$s^{-\frac{1}{2}}(\log s)^2\varphi(\alpha)/2. \tag{6.17}$$

Finally, it may be verified that the $K_0$ term in (6.16) depends on the order of the approximation. Thus if we use $f_r$ in place of $f$ and $u_r^*(y, s) = g_r(ys^{\frac{1}{2}}, s)$ in place of $g(\alpha, s)$, $K_0$ is replaced by

$$K_{0r} = \int_B \tfrac{1}{2}[u_y^*(\tilde{y}, s) - u_{ry}^*(\tilde{y}, s)]\, ds + [u^*(\tilde{y}, s) - u_r^*(\tilde{y}, s)]\, d\tilde{y} \tag{6.18}$$

where $B$ is the optimal boundary including the vertical section from $y = -\infty$ to 0 along $s = 1$. (This dependence on $r$ could be evaluated by noting that for $r \geqq 3$, $K_{0r} - K_{02} = \int_0^\infty [f_r(x^2 + 2)^{\frac{1}{2}} - f_2(x^2 + 2)^{\frac{1}{2}}]\, dx$ where $f_r$ contains the terms of the expansion through those of order $(x^2 + 2)^{-\frac{1}{2}}[\log (x^2 + 2)]^{r+1}$ and $g_r$ is the corresponding integral as in (6.7).)

It is noteworthy that the main term in (6.16) corresponds to the main term obtained in the case of certainty, i.e., where the statistician is informed of the value of $\mu$ after it is selected according to the prior probability distribution with mean $x/t$ and variance $1/t$, (see (5.2)).

These results may be interpreted for the sampling inspection problem by transforming variables. Applying the transformation (3.3), (3.12) and (6.1), Equation (6.14) expresses the optimal boundary points $(\tilde{a}_m, m)$ in the $(a, m)$-plane, for $m/M$ small, by

$$\begin{aligned} &\log (m/M)^2 + (\tilde{a}_m - mp_0)^2/mp_0(1 - p_0) \\ &\quad + \log [(\tilde{a}_m - mp_0)^2/mp_0(1 - p_0)] + \log 8\pi \\ &\quad + 2mp_0(1 - p_0)/(\tilde{a}_m - mp_0)^2 + [mp_0(1 - p_0)]^2/[\tilde{a}_m - mp_0]^4 \approx 0 \end{aligned} \tag{6.19}$$

from which follows the very rough approximation

$$\tilde{a}_m = mp_0 - [2mp_0(1 - p_0) \log (M/m)]^{\frac{1}{2}}. \tag{6.20}$$

Of special interest is the cut-off point $m_0$, the value of $m$ for which $a = 0$ would suffice to stop sampling.

Substituting $\tilde{a}_m = 0$ in Equation (6.19) leads to the simple approximation

$$\begin{aligned} m_0 \approx [(1 - p_0)/p_0][&\log (M^2p_0^2/8\pi(1 - p_0)^2) \\ &- 3 \log \{\log (M^2p_0^2/8\pi(1 - p_0)^2)\}]. \end{aligned} \tag{6.21}$$

**7. The optimal linear boundary.** An unpublished result of M. V. Johns involves the use of a linear boundary in the sampling inspection problem. Since one would expect a good linear boundary in the $(n, d)$ scale to go from some cut-off point $(n_0, 0)$ through a point close to $(N, Np_0)$, Johns computes an asymptotic approximation to the best among all linear boundaries through $(N, Np_0)$. This computation invokes a prior distribution for $p$, large $N$, and the Poisson approximation to the binomial process. The result is "peculiar" in that the optimal linear boundary is quite sensitive to the prior distributions. Thus if the optimal linear boundary were recomputed on the basis of the posterior distribution after a single observation, it would be changed substantially. Presumably this effect occurs because (1) linear boundaries are not optimal among all boundaries and (2) the risk is relatively insensitive to variations in the linear boundary.

In this section we evaluate the optimal linear boundary for the normal version of the problem as $t \to 0$ and compare the risk with that of the optimal (non-linear) boundary. Note that a straight line through $(N, Np_0)$ in the $(n, d)$-plane corresponds to a straight line in the $(x, t)$-plane, through $(0, 1)$.

Let the continuation set be given by the points in the $(x, t)$-plane below the linear boundary

$$\hat{x}(t) = \theta(1 - t) \qquad 0 < t \leqq 1, \theta > 0. \tag{7.1}$$

The corresponding risk $\hat{B}_\theta(x, t)$ satisfies the partial differential equation (3.14) and the boundary condition

$$\hat{B}_\theta(x, t) = x \quad \text{on the boundary.} \tag{7.2}$$

Applying

$$s = 1/t - 1, \qquad y = x/t \tag{7.3}$$

the straight line boundary transforms into

$$\hat{y}(s) = \theta s \qquad 0 \leqq s < \infty, \theta > 0 \tag{7.4}$$

and the transformed risk

$$\hat{u}_\theta(y, s) = \hat{B}_\theta(x, t) - y, \tag{7.5}$$

satisfies the heat equation (4.2) on the continuation set with the boundary condition

$$\begin{aligned} \hat{u}_\theta(y, s) &= 0 && \text{for} \quad s = 0, y \leqq 0 \\ &= -s\hat{y}(s)/(s + 1) = -\theta s^2/(s + 1) && \text{for} \quad 0 < s < \infty. \end{aligned} \tag{7.6}$$

Since $Y$ is a Wiener process in the $-s$ scale, it follows (see [6]) that

$$\hat{u}_\theta(y, s) = E\hat{u}_\theta(Y, S) \tag{7.7}$$

where $(Y, S)$ is the random point at which the path of a Wiener process going backwards in the $s$ scale from $(y, s)$ first crosses the boundary and $E$ denotes the expectation operator over the collection of paths starting from $(y, s)$.

It is well known (see Chernoff [5]) that the first passage time for a Wiener process from $(0, 0)$ to cross a straight line $y = \gamma' - \theta s, \gamma' \geqq 0, \theta \geqq 0$ has the following probability density

$$(7.8) \quad f_{\gamma',\theta}(s') = [\gamma'/(2\pi)^{\frac{1}{2}}]s'^{-3/2} \exp\{-\tfrac{1}{2}(\gamma'/s'^{\frac{1}{2}} - \theta s'^{\frac{1}{2}})^2\}, \qquad 0 < s' < \infty.$$

Using (7.6), (7.7) and (7.8) it follows that

$$(7.9) \quad -\hat{u}_\theta(y, s) = [\gamma\theta/(2\pi)^{\frac{1}{2}}]\int_0^s [(s - s')^2/(s - s' + 1)]s'^{-3/2} \exp\{-\tfrac{1}{2}(\gamma/s'^{\frac{1}{2}} - \theta s'^{\frac{1}{2}})^2\}\, ds'$$

where

$$(7.10) \qquad \gamma = \theta s - y > 0.$$

Equation (7.9) gives the risk $\hat{u}_\theta(y, s)$ associated with the linear boundary (7.4), as a function of the slope $\theta$ and the starting point $(y, s)$. The optimum slope, $\theta^*(y, s)$, if there is any, maximizing the right hand side of (7.9) will in general depend, on the starting point $(y, s)$, i.e., on the particular prior distribution one starts with. Since the closed form evaluation of the integral in (7.9) seems intractable we obtain below an asymptotic expansion (for large $s$ and hence for small $t$) for the integral in inverse powers of $s$. The expansion, to be useful for our purpose, has to be valid uniformly for $\theta$ in some appropriate range covering the optimum $\theta^*$. Heuristic considerations based on the original problem lead us to expect $\theta^*$ to be small for large $s$. We proceed below with a formal expansion from which the maximizing $\theta^*$ is obtained by differentiating and then we check to determine that the expansion is valid for that order of magnitude of $\theta^*$. It turns out that the proper parameter in inverse powers of which the expansion should be developed is given by

$$(7.11) \qquad \rho = \gamma\theta = (\theta s - y)\theta > 0.$$

Let

$$v' = (\theta/\gamma)s' = \theta s'/(\theta s - y),$$
$$v = (\theta/\gamma)s = \theta s/(\theta s - y) = 1 + y/\gamma.$$

With this transformation, (7.9) becomes

$$(7.12) \quad -\hat{u}_\theta(y, s) = (\gamma/\theta)^{\frac{1}{2}}[\rho/(2\pi)^{\frac{1}{2}}]\int_0^v [(v - v')^2/(v - v' + \theta/\gamma)] \cdot v'^{-3/2} \exp\{-\tfrac{1}{2}\rho(1 - v')^2/v'\}\, dv'.$$

We write the first factor in the integrand above as the sum of three components as follows:

$$(7.13) \quad (v - v')^2/(v - v' + \theta/\gamma) = [v - v' - \theta/\gamma + (\theta^2/\gamma^2)/(v - v' + \theta/\gamma)]$$
$$= 1 - v' + (y - \theta)/\gamma + (\theta^2/\gamma^2)/[(1 - v') + (y + \theta)/\gamma].$$

Finally, using the transformation

$$w' = \rho^{\frac{1}{2}}(1 - v')/v'^{\frac{1}{2}},$$
$$w = \rho^{\frac{1}{2}}(1 - v)/v^{\frac{1}{2}} = -y/s^{\frac{1}{2}}$$

from which it follows that

$$|dw'/dv'| = (\rho^{\frac{1}{2}}/2)(v')^{-3/2}(1 + v'),$$
$$(v')^{1/2} = (1 + w'^2/4\rho)^{\frac{1}{2}} - w'/2\rho^{\frac{1}{2}},$$
$$2/(1 + v') = 1 + (w'/2\rho^{\frac{1}{2}})(1 + w'^2/4\rho)^{-\frac{1}{2}},$$
$$(1 - v')/(1 + v') = -1 + 2/(1 + v') = (w'/2\rho^{\frac{1}{2}})(1 + w'^2/4\rho)^{-\frac{1}{2}},$$

and

$$1 - v' = w'v'^{\frac{1}{2}}/\rho^{\frac{1}{2}} = (w'/\rho^{\frac{1}{2}})[(1 + w'^2/4\rho)^{\frac{1}{2}} - w'/2\rho^{\frac{1}{2}}.$$

Then (7.12) may be written as

$$(7.14)\quad -\hat{u}_\theta(y, s) = (\gamma/\theta)^{\frac{1}{2}}[I_1(w) + [\rho^{\frac{1}{2}}(y - \theta)/\gamma]I_2(w) + \rho(\theta/\gamma)^2 I_3(w)]$$

where

$$(7.15)\quad I_1(w) = \textstyle\int_w^\infty w'(1 + w'^2/4\rho)^{-\frac{1}{2}}\varphi(w')\,dw',$$

$$(7.16)\quad I_2(w) = \textstyle\int_w^\infty [1 + (w'/2\rho^{\frac{1}{2}})(1 + w'^2/4\rho)^{-\frac{1}{2}}]\varphi(w')\,dw'$$

and

$$(7.17)\quad I_3(w) = \textstyle\int_w^\infty [1 + (w'/2\rho^{\frac{1}{2}})(1 + w'^2/4\rho)^{-\frac{1}{2}}] \cdot \{w'[(1 + w'^2/4\rho)^{-\frac{1}{2}} - w'/2\rho^{\frac{1}{2}}] + \rho^{\frac{1}{2}}(y + \theta)/\gamma\}^{-1}\varphi(w')\,dw'.$$

Expanding in inverse powers of $\rho$ and retaining only the first few terms we have from (7.11)

$$(7.18)\quad \theta = s^{-\frac{1}{2}}\rho^{\frac{1}{2}}[1 - w/2\rho^{\frac{1}{2}} + \tfrac{1}{8}w^2/\rho - (1/128)w^4/\rho^2 + \cdots]$$

and hence

$$(7.19)\quad \gamma = \theta s - y = s^{\frac{1}{2}}\rho^{\frac{1}{2}}[1 - w/2\rho^{\frac{1}{2}} + \tfrac{1}{8}w^2/\rho - (1/128)w^4/\rho^2 + \cdots] + ws^{\frac{1}{2}}.$$

Applying (7.18) and (7.19) we have

$$(7.20)\quad (\gamma/\theta)^{\frac{1}{2}} = s^{\frac{1}{2}}[1 + w/2\rho^{\frac{1}{2}} + w^2/8\rho - \cdots]$$

and

$$(7.21)\quad \rho^{\frac{1}{2}}(y - \theta)/\gamma = -w + w^2/2\rho^{\frac{1}{2}} - w^3/8\rho - \rho^{\frac{1}{2}}s^{-1} + \cdots.$$

Also we have

$$(7.22)\quad I_1(w) = J_1(w) - (1/8\rho)J_3(w) + (3/128\rho^2)J_5(w) + \cdots$$

and

$$(7.23) \quad I_2(w) = J_0(w) + (1/2\rho^{\frac{1}{2}})J_1(w) - (1/16\rho^{\frac{3}{2}})J_3(w) + \cdots$$

where

$$(7.24) \quad J_n(w) = \int_w^\infty w'^n \varphi(w')\, dw'.$$

By repeated integration by parts we have, for the first few $n$,

$$(7.25) \quad \begin{aligned} J_0(w) &= 1 - \Phi(w), \\ J_1(w) &= \varphi(w), \\ J_2(w) &= w\varphi(w) + [1 - \Phi(w)], \\ J_3(w) &= (w^2 + 2)\varphi(w). \end{aligned}$$

Ignoring the $I_3$ term which will turn out to be negligible, we substitute the expansions (7.20) to (7.23) in (7.14) and using (7.25) we have

$$(7.26) \quad -\hat{u}_\theta(y, s) = s^{\frac{1}{2}}\{\psi^-(y/s^{\frac{1}{2}}) - (1/4\rho)\varphi(y/s^{\frac{1}{2}}) - (\rho^{\frac{1}{2}}/s)\Phi(y/s^{\frac{1}{2}}) + \cdots\}$$

where

$$(7.27) \quad \psi^-(\alpha) = \varphi(\alpha) + \alpha\Phi(\alpha).$$

Setting the derivative of the first few terms of (7.26), equal to zero, we find the optimal value of $\rho$ to be

$$(7.28) \quad \rho^*(y, s) \approx [\varphi(\alpha)/2\Phi(\alpha)]^{\frac{2}{3}} s^{\frac{2}{3}}$$

where

$$(7.29) \quad \alpha = y/s^{\frac{1}{2}}.$$

The corresponding value of $\theta$ is given by

$$(7.30) \quad \theta^*(y, s) \approx [\varphi(\alpha)/2\Phi(\alpha)]^{\frac{1}{3}} s^{-\frac{1}{6}},$$

and the associated risk by

$$(7.31) \quad \hat{u}_{\theta^*}(y, s) \approx -\psi^-(\alpha)s^{\frac{1}{2}} + \tfrac{3}{4}[4\varphi(\alpha)\Phi^2(\alpha)]^{\frac{1}{3}} s^{-\frac{1}{6}}.$$

To establish these approximations rigorously, we review the expansions from Equation (7.20) on. Taking $\alpha = y/s^{\frac{1}{2}} = -w$ bounded, these expansions apply for large $\rho$ uniformly in $s$ and $w$ (with $w$ bounded and $s$ large in which case $\theta \approx (\rho/s)^{\frac{1}{2}}$). Thus it suffices to show that (i) $\rho(\theta/\gamma)^2 I_3 = o(s^{-\frac{2}{3}})$ for $\rho s^{-\frac{2}{3}}$ bounded away from 0 and $\infty$, (ii) when $\rho s^{-\frac{2}{3}} = o(1)$, replacing (7.13) by the larger quantity $v - v'$ gives $-\hat{u}_\theta$ a smaller value than $-\hat{u}_{\theta^*}$, and (iii) when $\rho s^{\frac{2}{3}} \to \infty$, replacing (7.13) which is equal to $v - v' - (v - v')(\theta/\gamma)/((v - v') + (\theta/\gamma))$ by the larger quantity $v - v' - \frac{1}{2}\min(v - v', \theta/\gamma)$ gives $-\hat{u}_\theta$ a smaller value than $-\hat{u}_{\theta^*}$. These details are tedious and will be omitted.

In terms of the $(x, t)$ variables of (7.3) and (7.5), (7.30) and (7.31) become

$$\theta^* \approx \{\varphi[x/(t(1-t))^{\frac{1}{2}}]2\Phi[x/(t(1-t))^{\frac{1}{2}}]\}^{\frac{1}{3}}[t/(1-t)]^{\frac{1}{6}} \tag{7.32}$$

and

$$\begin{aligned}\hat{B}_{\theta^*}(x, t) \approx & -[(1-t)/t]^{\frac{1}{2}}\psi^+(x/[t(1-t)]^{\frac{1}{2}}) \\ & + \tfrac{3}{4}[4\varphi(x/[t(1-t)]^{\frac{1}{2}})\Phi^2(x/[t(1-t)]^{\frac{1}{2}})]^{\frac{1}{3}}[t/(1-t)]^{\frac{1}{6}}\end{aligned} \tag{7.33}$$

where $\psi^+(\alpha) = \varphi(\alpha) - \alpha[1 - \Phi(\alpha)]$. It should be noted that the main term of the risk function (7.26) does not depend on $\rho$ but does depend on $\alpha$ as does the approximation to $\rho^*$. Comparing (7.33) with the risk for the optimal procedure given by (6.14) we note that the main terms (risk when the statistician is told the value of $\mu$) coincide. The next term for the linear boundary is of the order $t^{\frac{1}{6}}$ which compares with $t^{\frac{1}{2}}(-\log t)$ for the optimal nonlinear boundary.

Translating in terms of the sampling inspection problem for $a_0 + b_0$ small compared to $M$ we have the linear boundary

$$\hat{a}^* \approx mp_0 - \theta^*(1 - m/M)[Mp_0(1-p_0)]^{\frac{1}{2}}, \tag{7.34}$$

where $\theta^* \sim M^{-\frac{1}{6}}$ is obtained from (7.32) by substituting

$$x = [(a_0 + b_0)p_0 - a_0]/[Mp_0(1-p_0)]^{\frac{1}{2}}$$

and

$$t = (a_0 + b_0)/M.$$

The cut-off point $m_0^*$ corresponding to the best line is given by

$$m_0^* = \theta^*[M(1-p_0)/p_0]^{\frac{1}{2}}/(1 + \theta^*[(1-p_0)/Mp_0]^{\frac{1}{2}}) \sim M^{\frac{1}{3}}. \tag{7.35}$$

which compares with $m_0 \sim \log M$ for the optimal Bayes procedure.

**8. Computational results.** An ALGOL program is available for computing the exact boundary and the exact risk function for the sampling inspection problem. This program has been used with a Burroughs B5000 to obtain the exact boundary points $\{(m, \tilde{a}_m), 1 \leqq m \leqq M\}$ and the exact risk $\{\rho(a, b), a \leqq a^* - 1, b \leqq b^*\}$ for the following values of $M$ and $p_0$ :

$$M = 100, \quad 200, \quad 500;$$

$$p_0 = \quad 0.1, \quad 0.2, \quad 0.5.$$

We recall that

$$\tilde{a}_m \approx mp_0 - \tilde{x}(t)[Mp_0(1-p_0)]^{\frac{1}{2}} \tag{8.1}$$

where $\tilde{x}(t)$ is the optimal boundary for the associated Wiener process problem. It is possible to derive inner and outer bounds for $\tilde{x}(t)$. It is also possible to calculate arbitrarily good approximations to $\tilde{x}(t)$ by a backward induction technique. Such computations have not yet been carried out. Taking these we

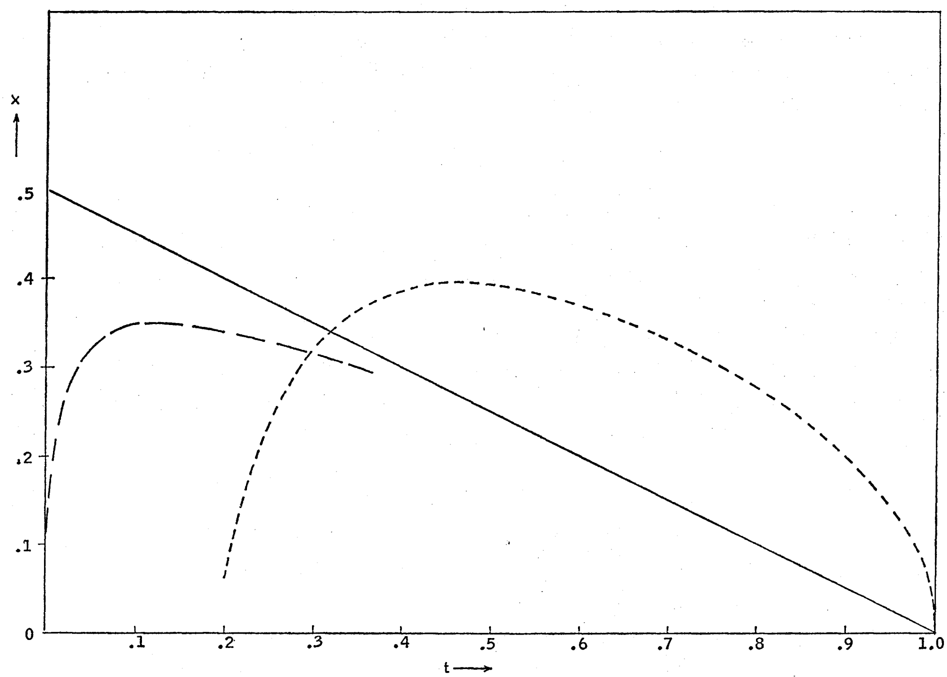


FIG. 5. - - - - Large $t$ approximation to optimal boundary. — — — Small $t$ approximation to optimal boundary. —— Best linear boundary for a starting point at $x = 0$, $t = 0.1$.

present Figure 5 which illustrates the two approximations of $\tilde{x}(t)$ developed in Sections 4 and 6. The dotted curve shows the large $t$ (i.e., $t$ close to one) approximation given by (4.20), i.e.,

$$\tilde{x}_1(t) \sim [t(1-t)]^{\frac{1}{2}}[0.63886 + 0.23625(1-t)/t - 0.089257((1-t)/t)^2] \tag{8.2}$$

for $t$ near 1. The small $t$ approximation given by the dashed curve, is based on the first three terms of the asymptotic expansion given in (6.14). One reason for curtailing the above expansion is that the curtailed expansion appears to give a better fit to the optimal boundary when $t$ is not very small. A point worth mentioning is that the incorporation of more terms of an asymptotic expansion does not necessarily lead to a better fit everywhere. Specifically, the small $t$ approximation is given by

$$\tilde{x}_2(t) = \tilde{\alpha}(t)t^{\frac{1}{2}} \tag{8.3a}$$

where $\tilde{\alpha}(t)$ is related to $t$ by

$$t = (1/2\tilde{\alpha})\varphi(\tilde{\alpha}) = (1/2\tilde{\alpha})\exp(-\tilde{\alpha}^2/2)/(2\pi)^{\frac{1}{2}}. \tag{8.3b}$$

It may be noticed that the two curves intersect each other at about $t = 0.3$.

This is a critical value of $t$ below which the small $t$ approximation given by (8.3) seems to be better and above which the large $t$ approximation given by (8.2) is apparently better. Finally, the straight line in the figure gives the best linear boundary

$$\hat{x}^*(t) = \theta^*(1 - t), \qquad 0 < t \leqq 1, \tag{8.4}$$

for a prior probability corresponding to the starting point $x = 0$, $t = 0.1$. From (7.32), $\theta^*$ for this starting point is 0.502. The influence of the magnitude of $t$ on the slope is rather small. For example, reducing $t$ from .1 to .01 reduces $\theta^*$ from .502 to .342.

The approximations (8.2) and (8.3) are now applied in (8.1) in their respective ranges of $t$ to obtain the approximate boundaries of the sampling inspection problem in the $(m, a)$-plane. The exact boundary when plotted is almost linear except for a slight curvature upwards near $t = 1$ and a slight

TABLE 1

*Exact and approximate optimal boundaries for the sequential sampling inspection problem*
$M = 100, \quad p_0 = .5, .2, .1$
$\tilde{a}$ = *exact optimal boundary*
$\tilde{a}_1$ = *approximation for* $t = m/M$ *near* 1
$\tilde{a}_2$ = *approximation for* $t = m/M$ *near* 0
$\hat{a}^*$ = *best straight line boundary corresponding to* $(x = 0, t = .1)$

| $m$ | $p_0 = .5$ | | | | $p_0 = .2$ | | | | $p_0 = .1$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\tilde{a}$ | $\tilde{a}_1$ | $\tilde{a}_2$ | $\hat{a}^*$ | $\tilde{a}$ | $\tilde{a}_1$ | $\tilde{a}_2$ | $\hat{a}^*$ | $\tilde{a}$ | $\tilde{a}_1$ | $\tilde{a}_2$ | $\hat{a}^*$ |
| 1 | 0 | — | 0.0 | 0.0 | 0 | — | 0.0 | 0.0 | 0 | — | 0.0 | 0.0 |
| 5 | 1 | — | 0.8 | 0.1 | 0 | — | 0.0 | 0.0 | 0 | — | 0.0 | 0.0 |
| 10 | 3 | — | 3.0 | 2.3 | 0 | — | 0.7 | 0.2 | 0 | — | 0.0 | 0.0 |
| 15 | 5 | — | 5.5 | 5.4 | 1 | — | 1.5 | 1.3 | 0 | — | 0.5 | 0.2 |
| 20 | 8 | 9.7 | 8.5 | 8.0 | 2 | 3.8 | 2.5 | 2.4 | 0 | 1.8 | 0.9 | 0.8 |
| 25 | 10 | 11.3 | 10.8 | 10.6 | 3 | 4.1 | 3.6 | 3.5 | 1 | 1.8 | 1.4 | 1.4 |
| 30 | 13 | 13.4 | 13.5 | 13.3 | 4 | 4.7 | 4.6 | 4.6 | 1 | 2.0 | 1.9 | 2.0 |
| 35 | 15 | 15.5 | 16.0 | 15.9 | 5 | 5.4 | — | 5.7 | 2 | 2.4 | — | 2.5 |
| 40 | 18 | 18.1 | — | 18.5 | 6 | 6.5 | — | 6.8 | 2 | 2.8 | — | 3.1 |
| 45 | 20 | 20.4 | — | 21.1 | 7 | 7.3 | — | 7.9 | 3 | 3.4 | — | 3.7 |
| 50 | 23 | 23.0 | — | 23.8 | 8 | 8.4 | — | 9.0 | 3 | 3.8 | — | 4.3 |
| 55 | 25 | 25.4 | — | 26.4 | 9 | 9.6 | — | 10.1 | 4 | 4.4 | — | 4.8 |
| 60 | 28 | 28.2 | — | 29.0 | 10 | 10.5 | — | 11.2 | 5 | 4.9 | — | 5.4 |
| 65 | 30 | 30.6 | — | 31.6 | 11 | 11.6 | — | 12.3 | 5 | 5.5 | — | 6.0 |
| 70 | 33 | 33.3 | — | 34.3 | 12 | 12.7 | — | 13.4 | 6 | 6.0 | — | 6.6 |
| 75 | 36 | 36.0 | — | 36.9 | 13 | 13.8 | — | 14.5 | 6 | 6.6 | — | 7.1 |
| 80 | 38 | 38.6 | — | 39.5 | 15 | 14.9 | — | 15.6 | 7 | 7.2 | — | 7.7 |
| 85 | 41 | 41.2 | — | 42.1 | 16 | 15.9 | — | 16.7 | 7 | 7.6 | — | 8.3 |
| 90 | 44 | 44.0 | — | 44.8 | 17 | 17.2 | — | 17.8 | 8 | 8.4 | — | 8.8 |
| 95 | 47 | 47.0 | — | 47.4 | 18 | 18.6 | — | 18.9 | 9 | 9.2 | — | 9.4 |
| 100 | 49 | 50.0 | — | 50.0 | 20 | 20.0 | — | 20.0 | 9 | 10.0 | — | 10.0 |

curvature towards the horizontal near the cut-off point, i.e., the point where the boundary hits the $m$-axis. These features are retained by the two approximations that follow the exact boundary closely in their respective ranges of validity. In fact, the exact boundary, its approximations and the best linear boundary

$$\hat{a}^*(m) = mp_0 - \hat{x}^*(t)[Mp_0(1 - p_0)]^{\frac{1}{2}}$$

are so close to each other, even for $M = 100$, that it has been decided to present Table 1 in place of the corresponding graphs.

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