ADDITIONAL LIMIT THEOREMS FOR INDECOMPOSABLE MULTI-DIMENSIONAL GALTON-WATSON PROCESSES[1]

By H. Kesten and B. P. Stigum

Cornell University

1. Introduction. In this paper we consider an indecomposable, non-singular, vector-valued Galton-Watson process. Specifically we consider a temporally homogeneous, k-vector-valued Markov chain, $\{Z_n ; n = 0, 1, \cdots\}$, with among others the following properties, assumed throughout this paper.

(1) Z_0 is taken to be one of the vectors,

$$e_i = (\delta_{i,1}, \cdots, \delta_{i,k}), \qquad 1 \leqq i \leqq k;$$

(2) If P denotes the probability measure of the process, if $Z_n = (Z_n^1, \cdots, Z_n^k)$, $n = 0, 1, \cdots$, and if for each n

$$F_{i,j}(x) = P\{Z_{n+1}^j \leqq x \mid Z_n = e_i\}, \qquad 1 \leqq i, j \leqq k; x \geqq 0,$$

then Z_n^j, $1 \leqq j \leqq k$, $0 \leqq n < \infty$, takes on only non-negative integer values and

$$P\{Z_{n+1}^j \leqq x \mid Z_0, \cdots, Z_n\} = F_{1,j}^{Z_n^1} * F_{2,j}^{Z_n^2} * \cdots * F_{k,j}^{Z_n^k}(x),$$

where the right hand side is the convolution of Z_n^i times $F_{i,j}$ for $i = 1, \cdots, k$;

(3) If E denotes the expectation functional, if $m_{i,j} = E\{Z_1^j \mid Z_0 = e_i\}$, $1 \leqq i$, $j \leqq k$, and if M denotes the matrix, $(m_{i,j})$, then

$$m_{i,j} = \int_0^\infty x \, dF_{i,j}(x) < \infty, \qquad 1 \leqq i, j \leqq k, \tag{1.1}$$

and for each pair, i, j, there exists an integer $t = t(i, j) \geqq 1$ such that

$$(M^{t(i,j)})_{i,j} > 0. \tag{1.2}$$

(4) If ρ denotes the largest positive characteristic root associated of M, then

$$\rho > 1. \tag{1.3}$$

We call a branching process satisfying (1.2) *indecomposable*. Whenever the integer t in (1.2) is independent of the pair i, j, then both M and the Z-process are called positively regular. We will extend the results obtained in [4] to processes that are indecomposable but not positively regular. We will also for indecomposable processes present several limit theorems of a type that has received little attention so far (see, however, the acknowledgement at the end of the paper). In a forthcoming paper [5] we shall show how to extend many of the results obtained here to the case of decomposable Galton-Watson processes, i.e. to processes that do not satisfy (1.2), but otherwise satisfy conditions (1)–(4).

Since M is non-negative and finite, it follows from the Perron-Frobenius

Received 23 May 1966.

[1] The second author was supported by a grant from the National Science Foundation.

theorem (see [2], v. 2, p. 53) that (1.2) implies that M has a positive eigenvalue ρ such that no other eigenvalue of M exceeds ρ in absolute value. Hence assumption 4 makes sense. It also follows from the Perron-Frobenius theorem that ρ is simple and that there exist right and left eigenvectors of M, denoted u and v, such that

$$vM = \rho v, \qquad v_i > 0, \qquad 1 \leqq i \leqq k, \tag{1.4}$$

$$Mu' = \rho u', \qquad u_i > 0, \qquad 1 \leqq i \leqq k, \quad \text{and} \tag{1.5}$$

$$vu' = \textstyle\sum_{i=1}^{k} u_i v_i = 1, \tag{1.6}$$

where u' denotes the transpose of u. Finally we note that if the integer t in (1.2) does not depend on (i, j), then ρ is larger than the absolute value of any other eigenvalue of M. These properties of M will be used repeatedly in the remaining parts of the paper.

In our previous paper [4] we proved Theorem 1.1. which is stated below for the convenience of the reader. In the statement of the theorem,

$$q_i \equiv P\{Z_n = 0 \quad \text{eventually} \mid Z_0 = e_i\}, \qquad 1 \leqq i \leqq k. \tag{1.7}$$

This agrees with (2.29) of [4] because of Theorem II.7.1 of [3].

THEOREM 1.1. *If the integer t in (1.2) is independent of i, j, then there exists a random vector W and a one-dimensional random variable w such that*[2]

$$\lim_{n\to\infty} Z_n/\rho^n = W \qquad \text{wp } 1 \tag{1.8}$$

and

$$W = w \cdot v \qquad \text{wp } 1. \tag{1.9}$$

If

$$E\{Z_1^j \log Z_1^j \mid Z_0 = e_i\} < \infty \quad \textit{for all} \quad 1 \leqq i, j \leqq k, \tag{1.10}$$

then

$$E\{W \mid Z_0 = e_i\} = u_i, \qquad 1 \leqq i \leqq k, \tag{1.11}$$

and if (1.10) fails to hold for some pair i, j, then

$$w = 0 \qquad \text{wp } 1. \tag{1.12}$$

Finally, if $Z_0 = e_{i_0}$, $1 \leqq i_0 \leqq k$, if (1.10) holds, and if there is at least one j_0 such that

$$\textstyle\sum_{m=1}^{k} Z_1^m u_m \quad \textit{can take at least two values with positive probability, given} \quad Z_0 = e_{j_0}, \tag{1.13}$$

then the distribution of w has a jump of magnitude $q_{i_0} < 1$ at the origin and a continuous density function on the set of positive real numbers. If (1.13) fails to hold for all j_0, $1 \leqq j_0 \leqq k$, then the distribution of w is concentrated at one point.

[2] "wp 1" stands for "with probability one".

If follows immediately from this theorem that if a is a vector such that

$$va' \neq 0, \tag{1.14}$$

and if the Z_n's satisfy the conditions of the theorem, then

$$\lim_{n\to\infty} P\{\rho^{-n}(Z_n a') \leqq x\} = P\{w(va') \leqq x\}.$$

Thus if the integer t in (1.2) does not depend on i, j, then for any vector satisfying condition, (1.14), the limit distribution of $\rho^{-n}(Z_n a')$ is "known." In Part 2 of this paper we will study limit distributions associated with the random variables $Z_n a'$, when a is a real vector that satisfies the condition

$$va' = 0, \qquad a \neq 0. \tag{1.15}$$

Of course this can arise only if $k \geqq 2$. In this case the sequence of normalizing constants and the form of the limit laws will depend, not only on ρ, u, and v, but on the full spectral representation of M. In fact, as we will show, under the additional assumption that the Z_1^j's all have finite second moments there exists an eigenvalue ρ_b of M and an exponent θ, both depending on a, such that if $|\rho_b|^2 \leqq \rho$, then the limit distribution of $[n^{-\theta}\rho^{-n/2}(Z_n a')]$ is a mixture of normal distributions, and if $|\rho_b|^2 > \rho$ and β denotes the number of distinct eigenvalues of M with absolute value equal to $|\rho_b|$, then there exist real numbers φ_δ and random variables X_δ, $1 \leqq \delta \leqq \beta$, such that for some γ

$$\lim_{n\to\infty} \{Z_n a'/n^\gamma |\rho_b|^n - \textstyle\sum_{\delta=1}^{\beta} \exp(\text{in}\,\varphi_\delta)\cdot X_\delta\} = 0 \qquad \text{wp } 1.$$

An example is given at the end of the section.

In Part 3 of this paper we extend Theorem 1.1 to cover the case when the integer t in (1.2) does depend on i, j. It is a simple consequence of the definition of M that in this case the components of Z can be divided into equivalence classes, $\{D_a\}$, $1 \leqq a \leqq h$, and reordered in such a way that a particle of type $i \varepsilon D_a$ only has descendants of types in D_{a+1} (i.e. in D_1 if $a = h$). We will show that if $Z_0 = e_i$ for some $i \varepsilon D_a$ and if $b \equiv a + m \pmod h$, then the subprocess, $\{Z^j_{nh+m}; j \varepsilon D_b\}$, $n = 0, 1, \cdots$, behaves like a positively regular process. In particular,

$$\lim_{n\to\infty} \rho^{-nh+m} Z^j_{nh+m} \quad \text{exists wp 1 for all} \quad j \varepsilon D_b$$

and analogues of (1.9), (1.11), and (1.12) hold. We will also briefly indicate how the results obtained in Part 2 can be extended to cover the case when the Z-process is indecomposable but not positively regular.

2. Limit theorems for positively regular branching processes. In this part we will study the limit distributions of random variables of the form $Z_n a'$, where the vector a' satisfies condition (1.15), as stated above. *We assume throughout that there exists an integer t independent of i, j, such that* $(M^t)_{i,j} > 0$. Thus ρ is simple and, as pointed out above, the absolute value of any other eigenvalue of M is smaller than ρ. We will also assume that

$$E\{(Z_1^j)^2 \mid Z_0 = e_i\} = \int_0^\infty x^2\, dF_{i,j}(x) < \infty, \qquad 1 \leqq i, j \leqq k. \tag{2.1}$$

This is a minimal assumption for Theorems 2.2 and 2.3 below. It could be slightly relaxed for Theorem 2.1. but we shall not insist on that.

The basic representation to be used for M in this part is the Jordan normal form,

$$(2.2)\qquad M = B\begin{pmatrix} E_1 & & & & 0 \\ & E_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & E_m \end{pmatrix} B^{-1},$$

where B is a non-singular (complex) matrix and the E_i's denote square matrices of size $d_i \times d_i$ and form

$$E_i = \begin{pmatrix} \rho_i & 1 & & & & 0 \\ & \rho_i & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ 0 & 1 & & & & \rho_i \end{pmatrix};$$

ρ_i is an eigenvalue of M. Thus if I_i and F_i are the matrices obtained by replacing E_j in (2.2) by 0 for $j \neq i$ and by the matrices

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & 1 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \text{ respectively}$$

for $j = i$, then

$$(2.3)\qquad M = \textstyle\sum_{i=1}^{m} B(\rho_i I_i + F_i)B^{-1}.$$

Moreover standard computations show that

$$(2.4)\qquad \begin{aligned} &F_i F_j = I_i F_j = F_j I_i = I_i I_j = 0, \qquad i \neq j,\ I_i^2 = I_i, \\ &I_i F_i = F_i I_i = F_i, \qquad F_i^{d_i - 1} \neq 0, \quad \text{and} \quad F_i^{d_i} = 0. \end{aligned}$$

Consequently, if $d = \max_i d_i$, then

$$(2.5)\qquad \begin{aligned} M^n = {} & \textstyle\sum_{i=1}^{m} \rho_i^{\,n} B I_i B^{-1} \\ & + \tbinom{n}{1} \textstyle\sum_{i=1}^{m} \rho_i^{\,n-1} B F_i B^{-1} + \cdots + \tbinom{n}{d-1} \textstyle\sum_{i=1}^{m} \rho_i^{\,n-d+1} B F_i^{d-1} B^{-1}. \end{aligned}$$

Without loss of generality we assume the eigenvalues of M so numbered that $\rho = \rho_1$ and $\rho_1 > |\rho_2| \geqq \cdots \geqq |\rho_m|$. It is easy to show that in this case B can be chosen so as to satisfy the conditions $(B^{-1})_{1,j} = v_j$ and $B_{j,1} = u_j$ for all $1 \leqq j \leqq k$. Thus, since $F_1 = 0$ as a consequence of the simplicity of ρ and of (2.4), the highest order term in (2.5) becomes $\rho_1{}^n BI_1B^{-1} = \rho^n u'v$. It follows from this fact that

$$M^n = \rho^n u'v + O(n^{d-1}|\rho_2|^n). \tag{2.6}$$

More generally, if $|\rho_{\alpha-1}| > |\rho_\alpha| = |\rho_{\alpha+1}| = \cdots = |\rho_\beta| > |\rho_{\beta+1}|$, and if we let $\rho_\delta = |\rho_\delta| \exp(i\varphi_\delta)$, $\alpha \leqq \delta \leqq \beta$, we can combine terms of order $n^\gamma |\rho_\alpha|^{n-\gamma} (n \to \infty)$ in (2.5) and observe that M^n can be written as a sum of expressions of the form,

$$\binom{n}{\gamma}|\rho_\alpha|^{n-\gamma}\{\exp(i(n-\gamma)\varphi_\alpha)\cdot BF_\alpha{}^\gamma B^{-1} + \cdots + \exp(i(n-\gamma)\varphi_\beta)BF_\beta{}^\gamma B^{-1}\}. \tag{2.7}$$

Here, and in the rest of this section we interpret $F_\alpha{}^0$ as I_α. If the resulting expansion of M^n is broken off after the term (2.7), then the error is $O(n^{\gamma-1}|\rho_\alpha|^n)$. We also note that since M^n is real, all sums of the form (2.7) must be real. In particular if $\alpha = \beta$, i.e. if $|\rho_\alpha|$ occurs only once, then $\varphi_\alpha = 0$ or π.

Let a be a fixed vector that satisfies condition (1.15). It follows from (2.6) that the highest order term in the expansion of M^n annihilates a. Clearly, there may be many terms in the expansion of M^n that have this property. Let (2.7) define the term of highest order which for infinitly many n does not annihilate a, provided such a term exists. Then

$$M^n a' = \binom{n}{\gamma}|\rho_\alpha|^{n-\gamma}\{\exp(i(n-\gamma)\varphi_\alpha)\cdot BF_\alpha{}^\gamma B^{-1}a' + \cdots + \exp(i(n-\gamma)\varphi_\beta)\cdot BF_\beta{}^\gamma B^{-1}a'\} + O(n^{\gamma-1}|\rho_\alpha|^n). \tag{2.8}$$

If all terms annihilate a for all large n, then for some n_0 $M^n a' = 0$ for $n \geqq n_0$. In particular M must have zero as an eigenvalue and we take $\gamma = 0$ and $|\rho_\alpha| = 0$ in this case. This again will make (2.8) valid.

To simplify our notation below we let

$$a'(\delta) \equiv BF_\delta{}^\gamma B^{-1}a', \qquad \alpha \leqq \delta \leqq \beta. \tag{2.9}$$

With this notation

$$M^n a' = \binom{n}{\gamma}|\rho_\alpha|^{n-\gamma}\textstyle\sum_{\delta=\alpha}^{\beta}\exp(i(n-\gamma)\varphi_\delta)\cdot a'(\delta) + O(n^{\gamma-1}|\rho_\alpha|^n) \tag{2.10}$$

and (since the second member of (2.11) below is a telescoping series)

$$\begin{aligned}(Z_n - Z_0M^n)a' &= \textstyle\sum_{r=0}^{n-1}(Z_{r+1} - Z_rM)M^{n-r-1}a' \\ &= \textstyle\sum_{r=0}^{n-1}(Z_{r+1} - Z_rM)\binom{n-r-1}{\gamma}|\rho_\alpha|^{n-r-\gamma-1} \\ &\quad\cdot\textstyle\sum_{\delta=\alpha}^{\beta}\exp(i(n-r-1-\gamma)\varphi_\delta)\cdot a'(\delta) + \sum_{r=0}^{n-1}(Z_{r+1} - Z_rM)C'(n,r),\end{aligned} \tag{2.11}$$

where $C(n, r)$ is a vector of order $(n-r)^{\gamma-1}|\rho_\alpha|^{n-r-1}$ as $n - r \to \infty$. Also

$$Z_0M^n a' = \binom{n}{\gamma}|\rho_\alpha|^{n-\gamma}\textstyle\sum_{\delta=\alpha}^{\beta}\exp(i(n-\gamma)\varphi_\delta)\cdot Z_0a'(\delta) + O(n^{\gamma-1}|\rho_\alpha|^n). \tag{2.12}$$

We will use these expansions repeatedly below.

Before stating our results we remark that it is possible that

$$(Z_{r+1} - Z_rM)\binom{n-r-1}{\gamma}|\rho_\alpha|^{n-r-\gamma-1}\textstyle\sum_{\delta=\alpha}^{\beta}\exp(i(n-r-1-\gamma)\varphi_\delta)\cdot a'(\delta) = 0 \qquad \text{wp } 1$$

for all $n - r - \gamma - 1 \geqq 0$. This can be true even if the main term in (2.8) is not zero. If this phenomenon occurs, then $Z_na' - Z_0M^na'$ will in general have a smaller normalization constant than Z_na'. We will return to this problem in Remark 2.2.

The study of the limit laws associated with the random variables Z_na', in a natural way can be broken into three separate parts according as $|\rho_\alpha|^2 > \rho$, $|\rho_\alpha|^2 = \rho$, or $|\rho_\alpha|^2 < \rho$. We remind the reader that $\rho > 1$ is always assumed.

THEOREM 2.1. *If* (2.1) *and* (2.8) *hold, if the integer t in* (1.2) *does not depend on i, j, and if $|\rho_\alpha|^2 > \rho$, then there exist random variables $X_\alpha, X_{\alpha+1}, \cdots, X_\beta$ such that for $Z_0 = e_i$,*

$$(2.13) \quad \lim_{n\to\infty} [Z_na'/n^\gamma\, |\rho_\alpha|^n - \textstyle\sum_\delta)\cdot X_\delta] = {}_\alpha\exp {}^\delta_\beta(in\varphi = 0 \qquad \text{wp } 1.$$

REMARK 2.1. The simplest case is of course the case when $\alpha = \beta$, and $\varphi_\alpha = 0$. Our result then reduces to the statement that the random variables $n^{-\gamma}\, |\rho_\alpha|^{-n}\, (Z_na')$ converge with probability one to a random variable.

PROOF. By (2.12)

$$(2.14) \quad \lim_{n\to\infty} \{Z_0M^na'/n^\gamma\, |\rho_\alpha|^n - [1/\gamma!\, |\rho_\alpha|^\gamma] \textstyle\sum_{\delta=\alpha}^\beta \exp(in\varphi_\delta) \exp(-i\gamma\varphi_\delta) Z_0a'(\delta)]\} = 0$$

and by (2.11)[3]

$$(2.15) \quad (Z_na' - Z_0M^na')/n^\gamma\, |\rho_\alpha|^n = O(n^{-1} \textstyle\sum_{r=0}^{n-1} |(Z_{r+1} - Z_rM)/|\rho_\alpha|^{r+1}|) + \sum_{\delta=\alpha}^\beta \exp(in\varphi_\delta)\cdot|\rho_\alpha|^{-\gamma} \sum_{r=0}^{n-1} n^{-\gamma}\binom{n-r-1}{\gamma} \exp(-i(r+1+\gamma)\varphi_\delta) \cdot[(Z_{r+1} - Z_rM)/\, |\rho_\alpha|^{r+1}]a'(\delta).$$

Thus to prove the theorem we need only show that

$$(2.16) \qquad \textstyle\sum_{r=0}^\infty |(Z_{r+1} - Z_rM)/|\rho_\alpha|^{r+1}| < \infty \qquad \text{wp } 1$$

since then (2.14) and (2.15) imply that

$$\lim_{n\to\infty} [Z_na'/n^\gamma\, |\rho_\alpha|^n - \textstyle\sum_{\delta=\alpha}^\beta \exp(in\varphi_\delta) \cdot\{Z_0 + \sum_{r=0}^\infty [(Z_{r+1} - Z_rM)/|\rho_\alpha|^{r+1}] \exp(-i(r+1)\varphi_\delta)\} \cdot\exp(-i\gamma\varphi_\delta)\cdot a'(\delta)/\gamma!\, |\rho_\alpha|^\gamma] = 0 \qquad \text{wp } 1.$$

To prove (2.16) we observe that

$$\begin{aligned} E\{|Z_{r+1} - Z_rM| \mid Z_0 = e_i\} &\leqq [E\{|Z_{r+1} - Z_rM|^2 \mid Z_0 = e_i\}]^{\frac12} \\ &= [E\{E\{|Z_{r+1} - Z_rM|^2 \mid Z_0 = e_i,\, Z_r\} \mid Z_0 = e_i\}]^{\frac12} \\ &= O([E\{|Z_r| \mid Z_0 = e_i\}]^{\frac12}) = O(\rho^{r/2}) \end{aligned}$$

(compare (2.13) of [4]). Thus for $|\rho_\alpha| > \rho^{\frac12}$

$$\textstyle\sum_{r=0}^\infty E\{|(Z_{r+1} - Z_rM)/|\rho_\alpha|^{r+1}| \mid Z_0 = e_i\} < \infty.$$

(2.16) is an immediate consequence. Q.E.D.

[3] For a k-vector we define $|c| = (\sum_{i=1}^k |c_i|^2)^{1/2}$.

Before stating the next theorem we note that with the possible exception of a set of P-measure zero $\lim_{n\to\infty} Z_n = 0$ on the set $\{w = 0\}$. In fact

$$\{\lim_{n\to\infty} Z_n = 0\} \subset \{w = 0\}$$

and by (2.29) and (2.30) of [4] and Theorem II.7.1 of [3]

$$q_i = P\{\lim_{n\to\infty} Z_n = 0 \mid Z_0 = e_i\} = P\{w = 0 \mid Z_0 = e_i\}.$$

Then to study the joint distribution of $Z_n a'$ and w for large n we need only consider the behavior of these random variables on the set $\{w > 0\}$. The following theorems, therefore, determine completely the joint limiting distribution of $Z_n a'$ and w. The reader should also note that since w has a continuous distribution function on $(0, \infty)$, the results stated in Theorems 2.2 and 2.3 hold for all $0 < x_1 < x_2$ and all $y \neq 0$. In both these theorems we make the convention that if $\sigma^2 = 0$, then

$$\begin{aligned}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y(x\sigma^2)^{-\frac{1}{2}}} e^{-t^2/2}\, dt &= 1 \qquad \text{if} \quad y > 0, \quad \text{and} \\ &= 0 \qquad \text{if} \quad y < 0.\end{aligned}$$

(See also Remark 2.2)

THEOREM 2.2. *If* (2.1) *and* (2.8) *hold and the integer* t *in* (1.2) *is independent of* i, j, *if* $|\rho_\alpha|^2 = \rho$, *and if* $I = [x_1, x_2]$, $0 < x_1 < x_2$, *then for all* $y \neq 0$ *and* $1 \leqq i \leqq k$

$$\begin{aligned}(2.17)\quad \lim_{n\to\infty} P\{w \in I,\ Z_n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} \leqq y \mid Z_0 = e_i\} \\ = \int_{x_1}^{x_2} d_x P\{w \leqq x \mid Z_0 = e_i\}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y(x\sigma^2)^{-\frac{1}{2}}} e^{-t^2/2}\, dt,\end{aligned}$$

where σ^2 *is as defined in* (2.31) *and* (2.32).

PROOF. Since $n^{\gamma+\frac{1}{2}}\rho^{n/2} = n^{\gamma+\frac{1}{2}}|\rho_\alpha|^n$, it follows from (2.12) that

$$(2.18)\qquad \lim_{n\to\infty} Z_0 M^n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} = 0.$$

We also note that the last term on the right-hand side of (2.11), after division by $n^{\gamma+\frac{1}{2}}\rho^{n/2}$, becomes

$$(2.19)\qquad n^{-\frac{3}{2}} \sum_{r=0}^{n-1} [(Z_{r+1} - Z_r M)/\rho^{r/2}]C'(n, r)/n^{\gamma-1} |\rho_\alpha|^{n-r}.$$

Since the vectors, $[n^{\gamma-1} |\rho_\alpha|^{n-r}]^{-1} C(n, r)$ are bounded and since the random vectors $Z_{r+1} - Z_r M$ satisfy the relation

$$E\{Z_{r+1} - Z_r M \mid Z_1, \cdots, Z_r\} = 0,$$

$$\begin{aligned}E\{|n^{-\frac{3}{2}} \textstyle\sum_{r=0}^{n-1} [(Z_{r+1} - Z_r M)/\rho^{r/2}]C'(n, r)/n^{\gamma-1} |\rho_\alpha|^{n-r}|^2 \mid Z_0 = e_i\} \\ = O(n^{-3} \textstyle\sum_{r=0}^{n-1} \rho^{-r} E\{E\{|Z_{r+1} - Z_r M|^2 \mid Z_0 = e_i, Z_r\} \mid Z_0 = e_i\}) \\ = O(n^{-3} \textstyle\sum_{r=0}^{n-1} \rho^{-r} E\{|Z_r| \mid Z_0 = e_i\}) = O(n^{-2}).\end{aligned}$$

Consequently, the random variables in (2.19) tend to zero wp 1 as $n \to \infty$. Thus, if we use the abbreviation

$$(2.20)\quad d(n, r) = \sum_{\delta=\alpha}^{\beta} a(\delta)\binom{n-r-1}{\gamma}(1/|\rho_\alpha|^{\gamma+1} n^\gamma) \exp(i(n - r - 1 - \gamma)\varphi_\delta,$$

then, by (2.11) and (2.18),

$$\text{(2.21)}\quad \lim_{n\to\infty} [Z_n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} - n^{-\frac{1}{2}} \textstyle\sum_{r=0}^{n-1} [(Z_{r+1} - Z_r M)/\rho^{r/2}]\, d'(n, r)] = 0 \qquad \text{wp } 1.$$

Clearly, (2.21) remains valid if we replace $\sum_{r=0}^{n-1}$ by $\sum_{r=R}^{n-1}$ for any fixed R. To prove the theorem it will also be convenient to replace $Z_{r+1} - Z_r M$ in (2.21) by another random variable which we shall now define. Let $Z_{r+1}^{i,j}$ denote the number of particles of type j in the $(r + 1)$st generation, which are descendants of particles of type i in the rth generation. If $X_t^{i,j}(r)$ denotes the number of descendants of type j of the tth particle of type i in the rth generation, then

$$Z_{r+1}^{i,j} = \textstyle\sum_{t=1}^{Z_r^i} X_t^{i,j}(r).$$

Let $X_t^{i,j}(r)$ for $t > Z_r^i$ be additional random variables, such that each vector

$$X_t^i(r) = (X_t^{i,1}(r), \cdots, X_t^{i,k}(r))$$

has the same distribution as Z_1, given $Z_0 = e_i$. We think of $X_t^i(r)$ for $t > Z_r^i$ as counting the descendants of fictitious particles of type i. In particular, we want the family $\{X_t^i(r) : t = 1, 2, \cdots, r = 0, 1, \cdots, i = 1, \cdots, k\}$ of random vectors to be completely independent (this is true for the subfamily of descendants of honest particles, i.e. with $t \leqq Z_r^i$).

For some fixed R and $r \geqq R$ we now let

$$V_{r+1}^{i,j} = \textstyle\sum_{t=1}^{(Z_R M^{r-R})_i} X_t^{i,j}(r).$$

If $(Z_R M^{r-R})_i \leqq Z_r^i$, then $V_{r+1}^{i,j}$ is obtained from $Z_{r+1}^{i,j}$ by killing off descendants of a number of particles, whereas for $(Z_R M^{r-R})_i > Z_r^i$ descendants of fictitious particles are added. At any rate $|V_{r+1}^{i,j} - Z_{r+1}^{i,j}|$ is the sum of $|Z_r^i - (Z_R M^{r-R})_i|$ variables $X_t^{i,j}(r)$ and

$$\text{(2.22)}\quad E\{V_{r+1}^{i,j} - (Z_R M^{r-R})_i m_{i,j} - Z_{r+1}^{i,j} + Z_r^i m_{i,j} \mid Z_1, \cdots, Z_r, V_{R+1}, \cdots, V_r\} = 0$$

and thus

$$\text{(2.23)}\quad \begin{aligned} &E\{|V_{r+1}^{i,j} - (Z_R M^{r-R})_i m_{i,j} - Z_{r+1}^{i,j} + Z_r^i m_{i,j}|^2 \mid Z_1, \cdots, Z_r, V_{R+1}, \cdots, V_r\} \\ &= |Z_r^i - (Z_R M^{r-R})_i| \cdot \sigma^2(X_t^{i,j}(r)) \leqq |Z_r^i - (Z_R M^{r-R})_i| \textstyle\int_0^\infty x^2\, dF_{i,j}(x). \end{aligned}$$

Finally, if we define the random vector $V_{r+1} = (V_{r+1}^1, \cdots, V_{r+1}^k)$ by $V_{r+1}^j = \sum_{i=1}^k V_{r+1}^{i,j}$, and take into account that $d(n, r)$ in (2.20) is bounded in n and r, we find from (2.22) and (2.23) that

$$\text{(2.24)}\quad \begin{aligned} &E\{|\textstyle\sum_{r=R}^{n-1} \rho^{-r/2}(Z_{r+1} - Z_r M)\, d'(n, r) - \rho^{-r/2}(V_{r+1} - Z_R M^{r-R+1})\, d'(n, r)|^2 \mid Z_0 = e_j\} \\ &= O(\textstyle\sum_{i=1}^k \sum_{r=R}^{n-1} \rho^{-r} E\{|Z_r^i - (Z_R M^{r-R})_i| \mid Z_0 = e_j\}. \end{aligned}$$

On the other hand

$$E\{|Z_r^i - (Z_R M^{r-R})_i| \mid Z_0, \cdots, Z_R\} \leqq [E\{|Z_r^i - (Z_R M^{r-R})_i|^2 \mid Z_0, \cdots, Z_R\}]^{\frac{1}{2}}$$
$$= O(\rho^{r-R}|Z_R|^{\frac{1}{2}}) \quad (\text{by formula II.9.2 in [3]})$$

and

$$E\{|Z_R|^{\frac{1}{2}} \mid Z_0 = e_i\} = O(\rho^{R/2}).$$

Thus, the right hand side of (2.24) is $O(\sum_{r=R}^{n-1} \rho^{-R/2}) = O(n\rho^{-R/2})$. Together with (2.21) this implies that

$$(2.25) \quad Z_n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} - n^{-\frac{1}{2}}\sum_{r=R}^{n-1} \rho^{-r/2}(V_{r+1} - Z_R M^{r-R+1})\, d'(n, r) \to 0$$

in probability, if $n \to \infty$ and R tends to infinity with n sufficiently slowly.

The remaining parts of the proof of the theorem can now be obtained relatively easy. Indeed, since (see (1.8) and (1.9)) $\lim_{n\to\infty} Z_n/\rho^n = wv$, one has for each $\epsilon > 0$

$$(2.26) \qquad \lim_{R\to\infty} P\{|Z_R/\rho^R - wv| \geqq \epsilon\} = 0$$

and thus for some sequence δ_R which decreases to zero sufficiently slowly,

$$\lim_{n\to\infty} P\{w \in I, Z_n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} \leqq y \mid Z_0 = e_i\}$$
$$= \lim_{R\to\infty} \lim_{n\to\infty} P\{\inf_{\lambda \in I} |Z_R/\rho^R - \lambda v| < \delta_R,$$
$$(2.27) \qquad n^{-\frac{1}{2}}\sum_{r=R}^{n-1} \rho^{-r/2}(V_{r+1} - Z_R M^{r-R+1})\, d'(n, r) \leqq y \mid Z_0 = e_i\}$$
$$= \lim_{R\to\infty} \int_{A(I,R)} P\{Z_R/\rho^R \in d\xi \mid Z_0 = e_i\}$$
$$\cdot \lim_{n\to\infty} P\{n^{-\frac{1}{2}}\sum_{r=R}^{n-1} \rho^{-r/2}(V_{r+1} - Z_R M^{r-R+1})\, d'(n, r) \leqq y \mid Z_R/\rho^R = \xi\},$$

where the integral is over the k dimensional region,

$$A(I, R) = \{\xi: \inf_{\lambda \in I} |\xi - \lambda v| < \delta_R\}.$$

Of course, for the first equality in (2.27) we use not just (2.26) but also (2.25).

Now

$$V_{r+1}^j - (Z_R M^{r-R+1})_j = \sum_{i=1}^k \sum_{t=1}^{(Z_R M^{r-R})_i} (X_t^{i,j}(r) - m_{i,j})$$

and by construction, all the variables $X_t^{i,j}(r)$ are independent for fixed j. Moreover when Z_R is known, $(Z_R M^{r-R})_i$ is fixed. Thus, conditionally on the value of Z_R, the V_{r+1}^j's are independent. Also, for a given Z_R,

$$(V_{r+1} - Z_R M^{r-R+1}) d'(n, r) = \sum_{i,j=1}^k \sum_{t=1}^{(Z_R M^{r-R})_i} (X_t^{i,j}(r) - m_{i,j})\, d_j(n, r)$$

has zero mean and

$$\sigma^2((V_{r+1} - Z_R M^{r-R+1})\, d'(n, r)) = \sum_{i=1}^k (Z_R M^{r-R})_i \sigma_i^2(n, r),$$

where

$$(2.28) \quad \sigma_i^2(n, r) = \sigma^2(\sum_{j=1}^k (X_t^{i,j}(r) - m_{i,j})\, d_j(n, r))$$
$$= \sigma^2(Z_1\, d'(n, r) \mid Z_0 = e_i).$$

Since all the $X_t^i(r)$'s (for i fixed) have the same distribution we conclude from the central limit theorem, that

$$(2.29)\quad \lim_{n\to\infty} P\{n^{-\frac{1}{2}}\textstyle\sum_{r=R}^{n-1} \rho^{-r/2}(V_{r+1} - Z_R M^{r-R+1})\, d'(n, r) \leqq y \mid Z_R/\rho^R = \xi\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y\tau^{-1}} e^{-t^2/2}\, dt = \Phi(y\tau^{-1}),$$

where

$$(2.30)\quad \begin{aligned}\tau^2 &= \lim_{n\to\infty} n^{-1}\textstyle\sum_{r=R}^{n-1} \rho^{-r} \sum_{i=1}^k (Z_R M^{r-R})_i \sigma_i^2(n, r) \\ &= \lim_{n\to\infty} n^{-1}\textstyle\sum_{r=R}^{n-1} \sum_{s=1}^k \rho^{-R} Z_R^s u_s \sum_{i=1}^k v_i \sigma_i^2(n, r) \\ &= \textstyle\sum_{s=1}^k \xi_s u_s \sum_{i=1}^k v_i \lim_{n\to\infty} n^{-1}\sum_{r=R}^{n-1} \sigma_i^2(n, r)\end{aligned}$$

(the second equality is based on (2.6)). We point out, that in view of (2.20) and (2.28) one has

$$(2.31)\quad \begin{aligned}\sigma_i^2 &= \lim_{n\to\infty} n^{-1}\textstyle\sum_{r=R}^{n-1} \sigma_i^2(n, r) \\ &= (\gamma!)^{-2}|\rho_\alpha|^{-2\gamma-2} \lim_{n\to\infty} n^{-1}\textstyle\sum_{s=0}^{n-1} (s/n)^{2\gamma} \\ &\quad \cdot\sigma^2[\textstyle\sum_{\delta=\alpha}^{\beta} Z_1 a'(\delta) \exp\, (i(s - \gamma)\varphi_\delta) \mid Z_0 = e_i].\end{aligned}$$

We can, therefore, write $\tau^2 = \xi u'\sigma^2$ where

$$(2.32)\quad \sigma^2 = \textstyle\sum_{i=1}^k v_i \sigma_i^2.$$

If (2.29)–(2.32) are substituted in (2.27) one obtains, in view of (2.26) and the fact that $vu' = 1$,

$$\begin{aligned}\lim_{n\to\infty} P\{w \in I,\ Z_n a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} \leqq y \mid Z_0 = e_i\} &= \lim_{R\to\infty} \int_{A(I,R)} P\{Z_R/\rho^R \in d\xi \mid Z_0 = e_i\}\Phi(y/\sigma(\xi u')^{\frac{1}{2}}) \\ &= \int_{x_1}^{x_2} d_x P\{w \leqq x \mid Z_0 = e_i\}\Phi(y/\sigma x^{\frac{1}{2}}).\end{aligned}$$

This is precisely the statement of the theorem. Q.E.D.

Finally we turn to the last case of our trichotomy.

THEOREM 2.3. *If (2.1) and (2.8) hold, if the integer t in (1.2) is independent of i, j, if $|\rho_\alpha|^2 < \rho$, and if $I = [x_1, x_2]$, $0 < x_1 < x_2$, then for all $y \neq 0$ and $1 \leqq i \leqq k$,*

$$\lim_{n\to\infty} P\{w \in I,\ Z_n a'/\rho^{n/2} \leqq y \mid Z_0 = e_i\} = \int_{x_1}^{x_2} d_x P\{w \leqq x \mid Z_0 = e_i\}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y(x\sigma^2)^{-\frac{1}{2}}} e^{-t^2/2}\, dt$$

where σ^2 is defined in (2.34)–(2.36). $\sigma^2 = 0$ occurs if and only if for all $n \geqq 0$

$$(2.33)\quad Z_n a' = Z_0 M^n a' = O(n^\gamma|\rho_\alpha|^n) \qquad \text{wp } 1.$$

PROOF. Almost exactly as in the proof of Theorem 2.2 one shows that

$$\begin{aligned}\lim_{n\to\infty} P\{w \in I,\ Z_n a'/\rho^{n/2} \leqq y \mid Z_0 = e_i\} &= \lim_{R\to\infty} \int_{A(I,R)} P\{Z_R/\rho^R \in d\xi \mid Z_0 = e_i\} \\ &\quad \cdot\lim_{n\to\infty} P\{\textstyle\sum_{r=R}^{n-1} [(V_{r+1} - Z_R M^{r-R+1})/\rho^{r/2}]f'(n - r) \leqq y \mid Z_R/R = \xi\},\end{aligned}$$

where V_{r+1} and $A(I, R)$ have the same meaning as in Theorem 22. In the place of

$d(n, r)/n^{\frac{1}{2}}$ we have put the vector,

$$f(n - r) = M^{n-r-1}a'/\rho^{(n-r)/2}. \tag{2.34}$$

With this definition the argument leading from (2.19) to (2.21) becomes unnecessary because we do not (and can not) ignore the last term of (2.11). Actually the f's are better behaved than the d's, for they are not only bounded but (by (2.8)) $\sum_{s=1}^{\infty} |f(s)| < \infty$. This is used to go from the analogue of (2.24) to the following replacement of (2.25),

$$Z_n a'/\rho^{n/2} - \sum_{r=R}^{n-1} [(V_{r+1} - Z_R M^{r-R+1})/\rho^{r/2}] f'(n - r) \to 0$$

in probability if $n \to \infty$ and $R \to \infty$ with n sufficiently slowly. If $d(n, r)/n^{\frac{1}{2}}$ is replaced by $f(n - r)$ in (2.29) and if one defines

$$\begin{aligned} \sigma_i^2(n - r) &= \sigma^2\left(\sum_{j=1}^{k} (X_t^{i,j}(r) - m_{i,j}) f_j(n - r)\right) \\ &= \sigma^2(Z_1 f'(n - r) \mid Z_0 = e_i\} \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \tau^2 &= \lim_{n\to\infty} \sum_{r=R}^{n-1} \rho^{-r} \sum_{i=1}^{k} (Z_R M^{r-R})_i \sigma_i^2(n - r) \\ &= \xi u' \sum_{i=1}^{k} v_i \sum_{s=1}^{\infty} \sigma_i^2(s), \end{aligned}$$

then the theorem follows as before with

$$\sigma^2 = \sum_{i=1}^{k} v_i \sum_{s=1}^{\infty} \sigma_i^2(s). \tag{2.36}$$

Finally, σ^2 in (2.36) vanishes if and only if $\sigma_i^2(s + 1) = 0$ for all $s \geqq 0$ and $1 \leqq i \leqq k$. By (2.34)–(2.36), $\sigma_i^2(s + 1) = 0$ means that for $Z_0 = e_i$

$$Z_1 M^s a' = E\{Z_1 M^s a' \mid Z_0 = e_i\} = \sum_{j=1}^{k} (M^{s+1})_{i,j} a_j \qquad \text{wp } 1. \tag{2.37}$$

But since

$$Z_{r+1} M^s a' = \sum_{i=1}^{k} \sum_{t=1}^{Z_r^i} X_t^i(r) M^s a'$$

and since $X_t^i(r) M^s a'$ has the same distribution as $Z_1 M^s a'$ given $Z_0 = e_i$, it follows from (2.37) that

$$Z_{r+1} M^s a' = \sum_{i=1}^{k} Z_r^i \sum_{j=1}^{k} (M^{s+1})_{i,j} a_j = Z_r M^{s+1} a' \qquad \text{wp } 1.$$

By induction this implies (2.33). On the other hand (2.33) clearly implies $\sigma^2 = 0$.

REMARK 2.2. We briefly investigate when $\sigma^2 = 0$ in Theorem 2.2. First note, that if for some fixed i_0

$$\sigma^2\left(\sum_{\delta=\alpha}^{\beta} Z_1 a'(\delta) \exp(i(s - \gamma)\varphi_\delta) \mid Z_0 = e_{i_0}\right) > 0 \tag{2.38}$$

when $s = s_0$, then (2.38) will hold for all s in a subsequence of positive density. This is an immediate consequence of the fact that for each $\epsilon > 0$

$$\sum_{\delta=\alpha}^{\beta} |\exp i(s - s_0)\varphi_\delta - 1| < \epsilon$$

on a sequence $0 < s_1 < s_2 < \cdots$ of integers of positive density. In fact, by

Theorem 2, p. 421 of [6] one can choose $s_{i+1} - s_i \leqq L$ for some fixed L. Thus, as soon as (2.38) holds for some i_0 and s, σ^2 in (2.32) is strictly positive. Therefore $\sigma^2 = 0$ in (2.32) if and only if for all s and $1 \leqq i \leqq k$

$$(2.39) \quad \sigma^2\left(\textstyle\sum_{\delta=\alpha}^{\beta} Z_1 a'(\delta) \exp i(s - \gamma)\varphi_\delta \mid Z_0 = e_i \right\}$$
$$= E\left| \textstyle\sum_{\delta=\alpha}^{\beta} \sum_{j=1}^{k} (X_t^{i,j}(r) - m_{i,j}) a_j(\delta) \exp i(s - \gamma)\varphi_\delta \right|^2 = 0.$$

Of course (2.39) implies

$$(Z_{r+1} - Z_r M) \textstyle\sum_{\delta=\alpha}^{\beta} (\exp i(n - r - 1 - \gamma)\varphi_\delta) a'(\delta) = 0 \qquad \text{wp } 1$$

because

$$Z_{r+1}^j - (Z_r M)_j = \textstyle\sum_{i=1}^{k} \sum_{t=1}^{Z_r^i} (X_t^{i,j}(r) - m_{i,j}).$$

Hence, if $\sigma^2 = 0$ in (2.32), then the main term in (2.11) will be zero wp 1 for all $n \geqq 0$. This observation suggests that we redefine α, β, γ such that the main term in the right hand side of (2.11) is the highest order term in the expansion of $(Z_n - Z_0 M^n)a'$ which does not vanish wp 1 for all $n \geqq 0$. If no such α, β, γ exist, i.e., if all terms in the expansion of $(Z_n - Z_0 M^n)a'$ are zero wp 1, then $(Z_n - Z_0 M^n)a' \equiv 0$ wp 1. Let us disregard this case. Then with this new definition of α, β, γ we can still prove Theorems 2.1–2.3 provided we replace $Z_n a'$ by $(Z_n - Z_0 M^n)a'$. We shall now be sure that $\sigma^2 \neq 0$ though. More precisely we have

THEOREM 2.4. *Let* (2.11) *hold and assume that the main term of the right hand side of* (2.11) *does not vanish* wp 1 *for all* $n \geqq 0$. *Assume also that* (2.1) *holds and that the integer* t *in* (1.2) *is independent of* i, j.

If $|\rho_\alpha|^2 > \rho$, *then there exist random variables* $X_\alpha, \cdots, X_\beta$ *such that for* $Z_0 = e_i$, $1 \leqq i \leqq k$,

$$\lim_{n\to\infty} [(Z_n - Z_0 M^n)a'/n^\gamma |\rho_\alpha|^n - \textstyle\sum_{\delta=\alpha}^{\beta} \exp(in\varphi_\delta) X_\delta] = 0 \qquad \text{wp } 1.$$

If $|\rho_\alpha|^2 = \rho$, $I = [x_1, x_2]$, $0 < x_1 < x_2$, *then for all* y

$$\lim_{n\to\infty} P\{w \in I, (Z_n - Z_0 M^n)a'/n^{\gamma+\frac{1}{2}}\rho^{n/2} \leqq y \mid Z_0 = e_i\}$$
$$= \textstyle\int_{x_1}^{x_2} d_x P\{w \leqq x \mid Z_0 = e_i\} \Phi(y(x\sigma^2)^{-\frac{1}{2}}),$$

where σ^2 *is given by* (2.31), (2.32) *and* $\sigma^2 > 0$.

If $|\rho_\alpha|^2 < \rho$, $I = [x_1, x_2]$, $0 < x_1 < x_2$, *then for all* y

$$\lim_{n\to\infty} P\{w \in I, (Z_n - Z_0 M^n)a'/\rho^{n/2} \leqq y \mid Z_0 = e_i\}$$
$$= \textstyle\int_{x_1}^{x_2} d_x P\{w \leqq x \mid Z_0 = e_i\} \Phi(y(x\sigma^2)^{-\frac{1}{2}}),$$

where σ^2 *is given by* (2.34)–(2.36). $\sigma^2 > 0$ *unless* (2.33) *holds for all* $n \geqq 0$.

(One can show that $\sigma^2 > 0$ is already implied by the fact that the main term in (2.11) does not vanish wp 1 for all $n \geqq 0$, even without using (2.33).)

This theorem is proved precisely as Theorems 2.1–2.3. In fact, except for arguments concerning the term $Z_0 M^n a'$, everything in those proofs rests on (2.11) only.

EXAMPLE. Consider a 3-dimensional Galton-Watson process with

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_1 & & 0 \\ & \rho_2 & \\ 0 & & \rho_3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$= \left(\frac{1}{3}\right) \begin{pmatrix} \rho_1 + 2\rho_2 & \rho_1 - \rho_2 & \rho_1 - \rho_2 \\ \rho_1 - 2\rho_2 + \rho_3 & \rho_1 + \rho_2 + \rho_3 & \rho_1 + \rho_2 - 2\rho_3 \\ \rho_1 - \rho_3 & \rho_1 - \rho_3 & \rho_1 + 2\rho_3 \end{pmatrix}$$

and $\rho_1 > |\rho_2| > |\rho_3|, \rho_1 \geqq 3|\rho_2|$.

Here $v = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $u = (1, 1, 1)$,

$$BI_1 B^{-1} = \left(\frac{1}{3}\right) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$BI_2 B^{-1} = \left(\frac{1}{3}\right) \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $a = (a_1, a_2, -(a_1 + a_2))$, $a_1 \neq 0$. Then $BI_1B^{-1}a' = 0$ and $(BI_2B^{-1}a')' = (a_1, -a_1, 0) \neq 0$. Thus the highest order non-zero term in (2.7) is $\rho_2^n(BI_2B^{-1})a'$ corresponding to $\alpha = \beta = 2$, $\gamma = 0$. One has $a(2) = (a_1, -a_1, 0)$ and the proper normalization constant for Z_na' is ρ_2^{-n} if $|\rho_2|^2 > \rho_1$, $n^{-\frac{1}{2}}|\rho_2|^{-n} = n^{-\frac{1}{2}}\rho_1^{-n/2}$ if $|\rho_2|^2 = \rho_1$, and $\rho_1^{-n/2}$ if $|\rho_2|^2 < \rho_1$. However, as explained in Remark 2.2 the proper normalization constant for $(Z_n - Z_0M^n)a'$ may be smaller. In fact (2.39) with $\alpha = \beta = 2$ will hold if for each $1 \leqq i \leqq 3$ and $Z_0 = e_i$, $Z_1^1 - Z_1^2 = m_{i,1} - m_{i,2}$ wp 1. This will be the case for instance if the probability distribution of Z_1, given $Z_0 = e_i$ puts all its mass on points of the form $(m_{i,1} + x, m_{i,2} + x, m_{i,3})$. Assume that this is the case. To find the highest term which makes the main term of (2.11) non-zero we compute

$$(BI_3B^{-1}a')' = (0, a_1 + a_2, -a_1 - a_2)$$

and observe that (2.38) with $\alpha = \beta = 3$ will hold if $Z_1^2 - Z_1^3$ is not a constant wp 1 for all $Z_0 = e_i$, $1 \leqq i \leqq 3$. The proper normalization constant for $(Z_n - Z_0M^n)a'$ now becomes $|\rho_3|^{-n}$ if $|\rho_3|^2 > \rho_1$, $n^{-\frac{1}{2}}|\rho_3|^{-n} = n^{-\frac{1}{2}}\rho_1^{-n/2}$ if $|\rho_3|^2 = \rho_1$, and $\rho_1^{-n/2}$ if $|\rho_3|^2 < \rho_1$.

3. "Periodic" branching processes. In this part we shall extend Theorem 1.1 to cover the case when Z_n, $n \geqq 0$, is an indecomposable but not a positively regular branching process. We call such a process periodic since one can find cycles in the behavior over time of the components of Z in just the same way as one can find cycles in the states of a periodic Markov chain. Indeed, one can practically copy the argument on pp. 176–178 of [1] with P replaced by M to conclude the following facts:

There exists an integer $d \geqq 1$ such that the components of Z can be divided into mutually disjoint classes, $\{D_a\}_{1 \leqq a \leqq d}$, and renumbered in such a way that

$$(3.1)\quad m_{i,j} = 0 \quad \text{unless} \quad i \varepsilon D_a \quad \text{and} \quad j \varepsilon D_{a+1} \quad (\text{i.e. in } D_1 \text{ if } a = d) \quad \text{for some } a, \quad 1 \leqq a \leqq d.$$

Moreover, if the components of Z have been renumbered in this way and $i \varepsilon D_a$,

$$(3.2)\qquad (M^n)_{i,j} = 0 \quad \text{unless} \quad j \varepsilon D_b \quad \text{with} \quad b - a = n (\text{mod } d)$$

and for sufficiently large t

$$(3.3)\qquad (M^{td+b-a})_{i,j} > 0 \quad \text{for all} \quad j \varepsilon D_b .$$

We assume throughout this part that the components of Z have been renumbered so that (3.1) holds. Thus M has the form,

$$(3.4)\qquad M = \begin{pmatrix} 0 & M(1,2) & 0 & \cdots & 0 \\ 0 & 0 & M(2,3) & \cdots & 0 \\ \vdots & \vdots & 0 & & \vdots \\ 0 & 0 & 0 & \ddots & M(d-1,d) \\ M(d,1) & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $M(a, a+1)$ denotes the matrix, $(m_{i,j})_{i\varepsilon D_a, j\varepsilon D_{a+1}}$. Easy computations now show that

$$(3.5)\qquad M^{nd} = \begin{pmatrix} N^n(1) & 0 & \cdots & 0 \\ 0 & N^n(2) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N^n(d) \end{pmatrix},$$

where $(N(a))_{i,j} = (M^d)_{i,j} = E\{Z_d^j \mid Z_0 = e_i\}$ for $i, j \varepsilon D_a$. It follows from (3.3) that

$$(3.6)\quad (N^t(a))_{i,j} > 0 \quad \text{for all} \quad i, \ j \varepsilon D_a \quad \text{if } t \text{ is sufficiently large.}$$

Thus, if $Z_0 = e_i$ for some $i \varepsilon D_a$, then the process

$$Z_n(a) = \{Z_{nd}^j : j \varepsilon D_a\}, \qquad n \geqq 0,$$

is a positively regular branching process with moment matrix $N(a)$ and largest positive eigenvalue ρ^d. This fact will be used in the proof of Theorem 3.1 below.

To simplify the statement of the next theorem we will now define several new vectors. Let u and v be respectively a right and left eigenvector of M satisfying (1.4) and (1.5). Then evidently

$$(3.7)\qquad M^d u' = \rho^d u',$$

and

$$vM^d = \rho^d v. \tag{3.8}$$

Thus u and v are also right and left eigenvectors of M^d corresponding to ρ^d. For all $1 \leqq a \leqq d$ we now define[4]

$$u(a) = \{u_i : i \varepsilon D_a\}, \tag{3.9}$$

$$v(a) = \{v_i : i \varepsilon D_a\}, \tag{3.10}$$

$$\bar{u}(a) = (0, \cdots, 0, u(a), 0, \cdots, 0), \tag{3.11}$$

$$\bar{v}(a) = (0, \cdots, 0, v(a), 0, \cdots, 0). \tag{3.12}$$

Then it is easily shown from (1.4) (resp. (1.5)) and (3.1) that for all $1 \leqq a \leqq d$,

$$\bar{v}(a+1) = \rho^{-1}\bar{v}(a)M, \tag{3.13}$$

and

$$\bar{u}'(a-1) = \rho^{-1}M\bar{u}'(a), \tag{3.14}$$

where we interpret $d + 1$ as 1 and 0 as d. Moreover if we choose u and v so that

$$u(1)[v(1)]' = 1, \tag{3.15}$$

then for all $1 \leqq a \leqq d$,

$$\bar{v}(a)\bar{u}'(a) = \rho^{-a+1}\bar{v}(1)M^{a-1}\rho^{-d+a-1}M^{d-(a-1)}\bar{u}'(1) = \bar{v}(1)\bar{u}'(1), \tag{3.16}$$

and, since $\bar{v}(a)\bar{u}'(a) = v(a)u'(a)$,

$$v(a)u'(a) = 1 \quad \text{for all} \quad 1 \leqq a \leqq d. \tag{3.17}$$

Also (3.7) and (3.8) imply

$$N(a)u(a)' = \rho^d u(a)', \qquad 1 \leqq a \leqq d, \tag{3.18}$$

and

$$v(a)N(a) = \rho^d v(a), \qquad 1 \leqq a \leqq d. \tag{3.19}$$

In the remaining parts of the paper we will assume that u and v have been chosen so that (3.15) is satisfied.

THEOREM 3.1. *If* (1.2) *and* (1.3) *hold and if* D_a, $u(a)$ *and* $v(a)$, $1 \leqq a \leqq d$, *are as defined above, then there exists a (one-dimensional) random variable* w *such that for* $Z_0 = e_i$, $i \varepsilon D_a$, *and for any* $j \varepsilon D_b$

$$\lim_{n\to\infty} Z^j_{nd+b-a}/\rho^{nd+b-a} = w\cdot v_j(b) \qquad \text{wp } 1. \tag{3.20}$$

If

---

[4] For simplicity we number the components, $u_i(a)$, by letting i run through D_a rather than by letting i run through the values, $1, 2, \cdots$. Similarly for the entries of $N(a)$. In this way (3.18) and (3.19) have an obvious meaning. Also, the notation of (3.11) means that $\bar{u}_i(a) = 0$ for $i \not\varepsilon D_a$ and $\bar{u}_i(a) = u_i(a)$ for $i \varepsilon D_a$. Similarly for (3.12).

(3.21) $E\{Z_1^{j_0} \log Z_1^{j_0} \mid Z_0 = e_{i_0}\} < \infty$ *for all* $i_0 \varepsilon D_s$ *and* $j_0 \varepsilon D_{s+1}$ *and all* $1 \leqq s \leqq d$,

then

$$E\{w \mid Z_0 = e_i\} = u_i(a). \tag{3.22}$$

If in addition to (3.21) *there exists an* f *and an* $i_1 \varepsilon D_f$ *such that*

(3.23) $\sum_{j \varepsilon D_{f+1}} Z_1^j u_j(f+1)$ *can take at least two values with positive probability, given* $Z_0 = e_{i_1}$,

then for $Z_0 = e_i$, $1 \leqq i \leqq k$, *the distribution of* w *has a jump of magnitude* q_i *at the origin and a continuous density function on the set of positive real numbers. If* (3.23) *does not hold for any pair,* i_1, f, $1 \leqq i_1 \leqq k$, $1 \leqq f \leqq d$, *then the distribution of* w *is concentrated at one point. Finally if* (3.21) *fails to hold for some pair,* i_0, j_0, *then*

$$w = 0 \quad \text{wp } 1. \tag{3.24}$$

PROOF. If $b = a, j \varepsilon D_a$, then (3.20) follows immediately by applying Theorem 1.1 to the process $Z_n(a)$. To obtain (3.20) for general j we use the following lemma which is formulated in greater generality than needed at this moment because of its application in our forthcoming paper [5].

LEMMA 3.1. *If* $X_j(m)$, $m = 1, 2, \cdots; j = 1, 2, \cdots$, *are independent random variables all with the same distribution* $G(\cdot)$, *which has a finite first moment, then for* $\lambda > 1$, $k \geqq 0$, *and* $A > 0$,

$$\lim_{m\to\infty} (m^k\lambda^m)^{-1} \sum_{j=1}^{Am^k\lambda^m} X_j(m) = A \int x\, dG(x) \quad \text{wp } 1. \tag{3.25}$$

PROOF. Let

$$\begin{aligned} Y_j(m) &= X_j(m) && \text{if } |X_j(m)| \leqq m^k\lambda^m \\ &= 0 && \text{if } |X_j(m)| > m^k\lambda^m. \end{aligned}$$

Then

$$P\{\textstyle\sum_{j=1}^{Am^k\lambda^m} X_j(m) \neq \sum_{j=1}^{Am^k\lambda^m} Y_j(m) \text{ infinitely often}\} = 0$$

because

$$\begin{aligned} &\textstyle\sum_{m=1}^{\infty} P\{\sum_{j=1}^{Am^k\lambda^m} X_j(m) \neq \sum_{j=1}^{Am^k\lambda^m} Y_j(m)\} \\ &\qquad \leqq \textstyle\sum_{m=1}^{\infty} Am^k\lambda^m \int_{|x| \geqq m^k\lambda^m} dG(x) \leqq A \int dG(x) \sum_{m^k\lambda^m \leqq |x|} m^k\lambda^m \\ &\qquad = O(\textstyle\int |x|\, dG(x)) < \infty. \end{aligned}$$

Moreover,

$$\lim_{m\to\infty} E\{Y_j(m)\} = E\{X_j(m)\} = \int x\, dG(x) \quad \text{uniformly in } j.$$

The lemma, therefore, follows by standard estimates from

$$\sum_{m=1}^{\infty} \sigma^2((m^k\lambda^m)^{-1} \sum_{j=1}^{Am^k\lambda^m} Y_j(m))$$
$$= A \sum_{m=1}^{\infty}(m^k\lambda^m)^{-1}\sigma^2(Y_j(m)) \leqq A \sum_{m=1}^{\infty} (m^k\lambda^m)^{-1} \int_{|x| \leqq m^k\lambda^m} x^2\, dG(x)$$
$$\leqq A \int x^2\, dG(x) \sum_{m^k\lambda^m \geqq |x|} (m^k\lambda^m)^{-1} = O(\int |x|\, dG(x)) < \infty. \qquad \text{Q.E.D.}$$

COROLLARY. *If $X_j(m) \geqq 0$ wp 1 and if $\{T_n\}_{n\geqq 1}$ is a sequence of non-negative random variables, independent of all the $X_j(m)$'s, such that*

$$\lim_{m\to\infty} T_m/m^k\lambda^m = B \text{ exists} \quad \text{wp } 1, \tag{3.26}$$

then

$$\lim_{m\to\infty} (m^k\lambda^m)^{-1} \sum_{j=1}^{T_m} X_j(m) = B \int_0^\infty x\, dG(x) \qquad \text{wp } 1. \tag{3.27}$$

More generally, even without (3.26)

$$\limsup_{m\to\infty} (m^k\lambda^m)^{-1} \sum_{j=1}^{T_m} X_j(m) \leqq \limsup_{m\to\infty} T_m/m^k\lambda^m \int_0^\infty x\, dG(x) \qquad \text{wp } 1. \tag{3.28}$$

PROOF. (3.25) holds for all rational A simultaneously and since $X_j(m) \geqq 0$, one has on the set $\{\limsup_{m\to\infty} m^{-k}\lambda^{-m}T_m < A\}$

$$\limsup_{m\to\infty} (m^k\lambda^m)^{-1} \sum_{j=1}^{T_m} X_j(m)$$
$$\leqq \lim_{m\to\infty} (m^k\lambda^m)^{-1} \sum_{j=1}^{Am^k\lambda^m} X_j(m) = A \int_0^\infty x\, dG(x),$$

which proves (3.28). (3.27) follows from (3.28) and a similar inequality for lim inf. Q.E.D.

We can now return to the proof of (3.20) for general j. Let $h \varepsilon D_a$ and $j \varepsilon D_b$. For $b \geqq a$ let $V_{nd+b-a}^{h,j}$ denote the number of particles of type j in the $(n d + b - a)$th generation which descend from a particle of type h in the (nd)th generation. (If $b < a$, the same definition holds provided $b - a$ is replaced by $d + b - a$). Then

$$V_{nd+b-a}^{h,j} = \sum_{t=1}^{Z_{nd}^h} X_t(n),$$

where the $X_t(n)$'s are independent random variables that all have the same distribution as Z_{b-a}^j given $Z_0 = e_h$. Hence $E\{X_t(n)\} = (M^{b-a})_{h,j}$ and since we already know that $\rho^{-nd}Z_{nd}^h \to w\cdot v_h(a)$ whenever $Z_0 = e_i$ for some $i \varepsilon D_a$, it follows from (3.27) with $\lambda = \rho^d$, $k = 0$, and $T_n = Z_{nd}^h$ that for this random variable w and for all $h \varepsilon D_a$ *and* $j \varepsilon D_b$ simultaneously

$$\lim_{n\to\infty} V_{nd+b-a}^{h,j}/\rho^{nd} = \lim_{n\to\infty} (Z_{nd}^h/\rho^{nd})(M^{b-a})_{h,j} = wv_h(a)(M^{b-a})_{h,j} \qquad \text{wp } 1.$$

Consequently by (3.13)

$$\lim_{n\to\infty} Z_{nd+b-a}^j/\rho^{nd+b-a} = \lim_{n\to\infty} \rho^{-(b-a)} \sum_{h\varepsilon D_a} V_{nd+b-a}^{h,j}/\rho^{nd}$$
$$= w \sum_{h\varepsilon D_a} v_h(a)((\rho^{-1}M)^{b-a})_{h,j} = w\cdot v_j(b) \qquad \text{wp } 1.$$

This proves (3.20). If (3.21) holds, one easily sees that

$$E\{Z_d^{j_2} \log Z_d^{j_2} \mid Z_0 = e_{i_2}\} < \infty \tag{3.29}$$

for all i_2, $j_2 \varepsilon D_a$. Conversely, if

$$E\{Z_1^{j_0} \log Z_1^{j_0} \mid Z_0 = e_{i_0}\} = \infty$$

for some $i_0 \varepsilon D_b$ and (necessarily) $j_0 \varepsilon D_{b+1}$, then (3.29) must fail for at least one pair i_2, $j_2 \varepsilon D_a$. One merely has to choose i_2, j_2 in such a way that

$$P\{Z_{b-a}^{i_0} > 0 \mid Z_0 = e_{i_2}\} > 0$$

and

$$P\{Z_{a-(b+1)}^{j_2} > 0 \mid Z_0 = j_0\} > 0.$$

Such a choice is possible by (1.2). (Again Z_{b-a} has to be replaced by Z_{d+b-a} if $1 \leqq b \leqq a \leqq d$ and similarly for $Z_{a-(b+1)}$). In view of these observations (3.22) follows from Theorem 1.1 applied to the process $Z_n(a)$, and similarly (3.24) follows if (3.21) is not satisfied. Finally the statements about the distribution of w, if (3.23) holds, are reduced to an application of Theorem 1.1 by observing that

$$\rho^{-n} \sum_{a=1}^{d} \sum_{j \varepsilon D_a} Z_n^{\,j} u_j(a), \qquad n = 0, 1, \cdots,$$

is a martingale. This makes it obvious that (3.23) is a necessary and sufficient condition for the existence of some $i_3 \varepsilon D_a$ such that

$$\sum_{j \varepsilon D_a} Z_d^{\,j} u_j(a) = \sum_{j \varepsilon D_a} Z_1^{\,j}(a) u_j(a)$$

can take at least two values with positive probability when $Z_0 = e_{i_3}$. (See also arguments following (2.32) and (2.36) in [4].) Thus, the conclusions about the distribution of w are contained in Theorem 1.1 and Theorem 3.1 is completely proved. Q.E.D.

REMARK 3.1. Since the processes $Z_n(a)$ are positively regular, we know that every non-zero state is transient for the $Z_n(a)$-process provided it is non-singular (i.e. if not every particle in the $Z_n(a)$-process has exactly one descendant wp 1) (see [3] Theorem II.6.1). Since $Z_n = r$ infinitely often if and only if for each a and $j \varepsilon D_a$, $Z_n^{\,j} = r_j$ infinitely often, we easily conclude that all non-zero states are transient for a non-singular Z_n-process satisfying (1.2).

We also know ([3], Theorem II.7.1) that for $Z_0 = e_i$, $i \varepsilon D_a$, $Z_n(a)$ will die out eventually with probability one if and only if $\rho^d \leqq 1$ and $Z_n(a)$ is non-singular. Since the Z_n-process dies out if and only if all $Z_n(b)$, $1 \leqq b \leqq d$, die out, we conclude, again under (1.2), that the probability of extinction for the Z_n-process is 1 if and only if it is nonsingular and $\rho \leqq 1$.

We shall end this paper by a brief discussion of the analogues of Theorems 2.1–2.4 in the periodic case. First of all, for any vector $c(a) = \{c_j(a) : j \varepsilon D_a\}$ satisfying $v(a)c'(a) = 0$ and $Z_0 = e_i$, $i \varepsilon D_a$, we can immediately find the limit distribution of

$$Z_n(a)c'(a) = \sum_{j \varepsilon D_a} Z_{nd}^{j}\, c_j(a)$$

since Section 2 applies to the positively regular process $Z_n(a)$. Also of course $\sum_{j \varepsilon D_a} Z_n^j c_j(a) = 0$ unless $n \equiv 0(\text{mod } d)$. In the more general case, where one is interested in $\sum_{j \varepsilon D_b} Z_n^j c_j(b)$ for some vector $c(b) = \{c_j(b) : j \varepsilon D_b\}$ and $Z_0 = e_i$, $i \varepsilon D_a$, one has

$$\sum_{j \varepsilon D_b} Z_n^j c_j(b) = 0 \quad \text{unless} \quad n \equiv b - a \,(\text{mod } d),$$

whereas for

$$\sum_{j \varepsilon D_b} Z^j_{nd+b-a} c_j(b) \tag{3.30}$$

one can use the arguments of Section 2. In fact, if one defines $\bar{c} = (0, \cdots, 0, c(b), 0, \cdots, 0)$, then almost all of Section 2 applies to $Z_n \bar{c}'$. Only when one comes to (2.26) does the periodicity of M play a role and this difficulty is easily overcome by restricting R to multiples of d in the proofs of Theorems 2.2 and 2.3. However the computation of σ^2 becomes somewhat messy. Still, for $v(b)c'(b) = 0$ we do obtain a trichotomy as in Section 2 and the limit laws have the same shape as those in Section 2. We shall not give further details here.

Acknowledgment. Shortly after completion of this paper, the authors were informed that the analogues of the results of Section 2 for continuous time multidimensional Markov branching processes had already been obtained by B. Krishnamoorthi in his Stanford Ph.D. thesis. Krishnamoorthi's methods are slightly different from ours and his result will appear in the near future.

REFERENCES

[1] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.
[2] Gantmacher, F. R. (1959). *The Theory of Matrices*. Chelsea, New York.
[3] Harris, T. E. (1963). *The Theory of Branching Processes*. Springer Verlag and Prentice-Hall, New York.
[4] Kesten, H. and B. P. Stigum (1966). A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.* **37** 1211–1223.
[5] Kesten, H. and Stigum, B. P. (1967). Limit theorems for decomposable multidimensional Galton-Watson processes. (To appear in *J. Math. Anal. Appl.*)
[6] van der Corput, J. G. (1931). Diophantische Ungleichungen. I. Zur Gleichverteilung modulo Eins. *Acta Math.* **56** 373–456.