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RENEWAL THEOREMS WHEN THE FIRST OR THE SECOND MOMENT IS INFINITE[1]

BY JOZEF L. TEUGELS

University of Louvain

The classical renewal theorems do not tell much about the renewal function if the mean renewal lifetime is infinite.

To obtain more accurate results we prove a theorem that can be considered as the analogue of Smith's key renewal theorem [8] if $1 - F(t) \sim t^{-\alpha}L(t)$ for $t \to \infty$ where $L(t)$ is slowly varying and $0 < \alpha \leqq 1$.

In Section 3 we consider $1 < \alpha < 2$. An application of the main theorem yields a precise estimate for the renewal function in that case.

1. Regularly varying functions. In this section we collect a number of results that will be applied throughout the entire chapter. For a general discussion, see W. Feller [4].

DEFINITION 1. A function $L(t)$ is called *slowly varying* if $L(t)$ is defined for $t > 0$, positive, and if $\lim_{t\to\infty} L(xt)/L(t) = 1$ for all $x > 0$. We write $L(t)$ is sv.

DEFINITION 2. A distribution function $F(t) \varepsilon V_\alpha$ for $\alpha \geqq 0$ if there exists a slowly varying function $L(t)$ such that

$$(1) \qquad 1 - F(t) \sim t^{-\alpha}L(t) \quad \text{as} \quad t \to \infty.$$

The real number α is the *exponent* of $F(t)$, and $F(t)$ is said to be a *regularly varying distribution* with exponent α. It is easy to show that if (1) holds for some $\alpha \geqq 0$, then this α is unique.

The class V_α is a subclass of the family of regularly varying functions as defined by Feller [4], K. Knopp [6] and others. If $\alpha = 0$ then we assume that $F(t) < 1$ for every $t \geqq 0$. V_0 reduces to a class of slowly varying functions. A paper by S. Aljančić, R. Bojanič and M. Tomić [1] (later on referred to as A.B.T.) contains a number of important results, that will be used later.

LEMMA 1.

(i) *If* $L(t)$ *is* sv *and* $u > 0$ *then* $L(ut)/L(t) \to 1$ *as* $t \to \infty$ *uniformly in every finite interval*; $L(t)t^\gamma \to \infty\ (\to 0)$ *if* $\gamma > 0$ $(\gamma < 0)$;

(ii) *If* $L_1(t)$ *and* $L_2(t)$ are sv, *so are* $L_1(t)L_2(t)$ *and* $L_1(t)/L_2(t)$;

(iii) *If* $L(t)$ *is* sv *for* $t \geqq a$, *so is* $\int_a^t x^{-1}L(x)\,dx$;

The last part is due to S. Parameswaran [7] and W. L. Smith [8].

One of the main properties of sv functions is expressed in the following lemma, which combines an Abelian and Tauberian theorem. An elementary proof is given by Feller in ([4], p. 421).

LEMMA 2. *If* $L(t)$ *is* sv *and* $G(t)$ *is a positive, monotone and right hand continuous*

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function on $[0, \infty]$ *and if* $0 \leqq \alpha < \infty$, *then each of the relations* (*lower case letters stand for* L.S. *Transforms*)

$$g(s) \sim s^{-\alpha}L(s^{-1}), \qquad s \to 0+,$$

and

$$G(t) \sim t^{\alpha}[\Gamma(\alpha + 1)]^{-1}L(t), \qquad t \to \infty,$$

implies each other.

We derive another lemma involving integrals of regularly varying distributions. We assume that if $F \varepsilon V_\alpha$, then $1 - F(t) = t^{-\alpha}L(t)$ for all $t > 0$, and that as $t \to 0+$ $L(t)$ is so defined that $t^{-\alpha}L(t) \to 1$. Moreover if $F \varepsilon V_1$ then we define

$$L^*(t) = \int_0^t [1 - F(x)]\, dx. \tag{2}$$

LEMMA 3.

(i) *Let* $F \varepsilon V_\alpha$, $0 \leqq \alpha < 1$. *Then*

$$\int_0^\infty e^{-st}[1 - F(t)]\, dt \sim s^{\alpha-1}\Gamma(1 - \alpha)L(s^{-1}) \qquad \textit{as} \quad s \to 0+;$$

(ii) *Let* $F \varepsilon V_1$. *Then*

$$\int_0^\infty e^{-st}L^*(t)\, dt \sim s^{-1}L^*(s^{-1}) \qquad \textit{as} \quad s \to 0+;$$

(iii) *Let* $F \varepsilon V_\alpha$. *Then for all* $k < \alpha - 1$

$$\int_t^\infty x^k[1 - F(x)]\, dx \sim [t^{k-\alpha+1}/(-k + \alpha - 1)]L(t) \qquad \textit{as} \quad t \to \infty;$$

(iv) *Let* $F \varepsilon V_\alpha$, $p > 0$ *and* $q > \alpha$. *Then*

$$\int_0^t (t - x)^{p-1}x^{q-1}[1 - F(x)]\, dx \sim t^{p+q-1-\alpha}B(p, q - \alpha)L(t) \qquad \textit{as} \quad t \to \infty.$$

PROOF.

(i) $\int_0^\infty e^{-st}[1 - F(t)]\, dt = s^{\alpha-1} \int_0^\infty e^{-u}u^{-\alpha}L(u/s)\, du.$

Now $\alpha < 1$. Hence A.B.T. Theorem 5 is applicable. There results that the integral on the right is asymptotically equal to $s^{\alpha-1}L(s^{-1}) \int_0^\infty e^{-u}u^{-\alpha}\, du$ as $s \to 0+$. (ii) By Lemma 1 (iii), $L^*(t)$ of (2) is sv and part (i) applies for $\alpha = 0$. Further, (iii) follows directly from A.B.T. Theorem 2, while in (iv) A.B.T. Theorem 1 applies.

2. Renewal theorems for the case where no first moment exists. In this section we prove analogues of renewal theorems that are classical in the case where a finite first moment exists.

An elementary renewal theorem was proved by Feller [5] in connection with fluctuation theory of recurrent events. He considered $1 - F(t) \sim t^{-\alpha}$ where $0 < \alpha < 1$. W. L. Smith proved a result for the two boundary cases $\alpha = 0$, $\alpha = 1$, [9]. See also Feller [4].

Assume from now on that $F \varepsilon V_\alpha$, $0 \leqq \alpha \leqq 1$. Since $h(s) = f(s)/(1 - f(s))$, where $h(s)$ is the L.S.T. of the renewal function $H(t)$, and

$$f(s) = 1 - s\int_0^\infty e^{-st}[1 - F(t)]\, dt; \tag{3}$$

it follows from Lemma 3 that

$$\text{(4)}\quad \text{if}\quad 0 \leqq \alpha < 1 \quad\text{then}\quad h(s) \sim s^{-\alpha}/\Gamma(1-\alpha)L(s^{-1}) \qquad \text{as}\quad s \to 0+$$

$$\text{(5)}\quad \text{and if}\quad \alpha = 1 \quad\text{then}\quad h(s) \sim s^{-1}/L^*(s^{-1}) \qquad \text{as}\quad s \to 0+.$$

THEOREM 1. *If* $F \in V_a$, *then as* $t \to \infty$

$$\begin{aligned} H(t) &\sim (L(t))^{-1} && \textit{if}\quad \alpha = 0 \\ &\sim (t^\alpha/(L(t))\,(\sin \alpha\pi/\alpha\pi) && \textit{if}\quad 0 < \alpha < 1 \\ &\sim t/L^*(t) && \textit{if}\quad \alpha = 1. \end{aligned}$$

PROOF. As is shown in [4], the result for $0 < \alpha < 1$ follows from Lemma 2 and (4). This is also true for $\alpha = 0$. If $\alpha = 1$, then Lemma 2 and (5) yield that

$$\text{(6)}\qquad H(t) \sim t/L^*(t) = t[\textstyle\int_0^t [1 - F(x)]\,dx]^{-1}$$

which finishes the proof.

Formula (6) shows that if the first moment should be finite, then $H(t) \sim t/\mu$ as $t \to \infty$ which is the elementary renewal theorem.

Besides the above theorem, two other renewal theorems are also useful: Blackwell's theorem and Smith's key renewal theorem. Since $F \varepsilon V_\alpha$, where $\alpha \leqq 1$ neither one of them gives more information then a $o(1)$ relation.

From Theorem 1 we always can find a sv function $L_2(t)$ such that

$$\text{(7)}\qquad H(t) \sim t^\alpha L_2(t).$$

To indicate the dependence of $H(t)$ on α, we write $H_\alpha(t) = t^\alpha L_2(t)$ as before.

If $\mu < \infty$ then the key renewal theorem essentially states that a function $Q(t)$, which has the same growth properties as $1 - F(t)$ may be used to obtain a finite limit for the convolution $Q * H(t)$ as $t \to \infty$. If $F \varepsilon V_\alpha$, then an appropriate choice of $Q(t)$ could be $Q(t) \sim t^{-\beta}L_1(t)$ where $0 < \beta < 1$ and $L_1(t)$ slowly varying as $t \to \infty$. Since $L_2(t)$ is of bounded variation over finite intervals [2] there exist two functions $\bar{L}_2(t)$ and $\underline{L}_2(t)$, which are nondecreasing and such that

$$L_2(t) = \bar{L}_2(t) - \underline{L}_2(t).$$

Assume now that

$$\text{(8)}\qquad \lim_{t\to\infty} [(\bar{L}_2(t) + \underline{L}_2(t))/L_2(t)] < \infty.$$

LEMMA 4. *If* (8) *holds then* $\bar{L}_2(t)$ *and* $\underline{L}_2(t)$ *are* sv.

PROOF. The only requirement we have to check is $\lim_{t\to\infty} \bar{L}_2(xt)/\bar{L}_2(t) = 1$ for all $x > 0$. It follows from (8) that $\lim_{t\to\infty} \bar{L}_2(t)/L_2(t) = c < \infty$. Moreover $c \geqq 1$ since $L_2(t) > 0$. Hence

$$\begin{aligned} &\lim_{t\to\infty} \bar{L}_2(xt)/\bar{L}_2(t) \\ &\qquad = \lim_{t\to\infty} [\bar{L}_2(xt)/L_2(xt)]\cdot[L_2(xt)/L_2(t)]\cdot[L_2(t)/\bar{L}_2(t)] = c\cdot 1\cdot c^{-1} = 1. \end{aligned}$$

If $\lim_{t\to\infty} \underline{L}_2(t)/L_2(t) > 0$, then the same argument shows that $\underline{L}_2(t)$ is sv.

We state an analogue of Smith's renewal theorem. The proof is based on a number of lemmas.

THEOREM 2. *Let* $0 < \alpha \leqq 1$, $F \varepsilon V_\alpha$. *Assume that* $L_2(t)$ *satisfies* (8). *For* $0 \leqq \beta < 1$, *let* $Q_\beta(t) = t^{-\beta}L_1(t)$ *where* $L_1(t)$ *is* sv *and* $Q_\beta(t)$ *is nonincreasing. Then as* $t \to \infty$,

$$(9) \quad \int_0^t Q_\beta(t-x)\,dH_\alpha(x) \sim C(\alpha,\beta)\int_0^t Q_\beta(x)\,dx\left(\int_0^t [1-F(x)]\,dx\right)^{-1}$$

where $[C(\alpha,\beta)]^{-1} = (2-\beta)B(\alpha-\beta+1, 2-\alpha)$ *for* $0 < \alpha \leqq 1$.

PROOF. We first prove the theorem for $L_2(t)$ nondecreasing (Part A); then we discuss the case when $L_2(t)$ satisfies condition (8), (Part B).

PART A: $L_2(t)$ nondecreasing. Let ϵ be a fixed positive real number, $0 < \epsilon < \frac{1}{4}$. Then

$$\begin{aligned} I(t) &\equiv \int_0^t Q_\beta(t-x)\,dH_\alpha(x) = \left\{\int_0^{\epsilon t} + \int_{\epsilon t}^{t-\epsilon t} + \int_{t-\epsilon t}^{t}\right\}Q_\beta(t-x)\,dH_\alpha(x) \\ &\equiv I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Since $H_\alpha(x) = x^\alpha L_2(x)$ we can break up $I_2(t)$ and $I_3(t)$ into two parts,

$$\begin{aligned} I_2(t) = \alpha \int_{\epsilon t}^{t-\epsilon t} Q_\beta(t-x)x^{\alpha-1}L_2(x)\,dx \\ + \int_{\epsilon t}^{t-\epsilon t} Q_\beta(t-x)x^\alpha\,dL_2(x) \equiv I_{21}(t) + I_{22}(t). \end{aligned}$$

Similarly, $I_3(t) = I_{31}(t) + I_{32}(t)$.

We show that $I(t)$ is approximated by $I_{21}(t)$ for large values of t. For this reason, we first estimate $I_{21}(t)$.

LEMMA 5. *For* $t \to \infty$

$$\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}L_1[t(1-u)]L_2[ut]\,du \sim L_1(t)L_2(t)\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}\,du.$$

PROOF. Consider (compare Lemma 1 (ii))

$$\begin{aligned} &\left|[L_1(t)L_2(t)]^{-1}\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}L_1[t(1-u)]L_2(ut)\,du - \int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}\,du\right| \\ &\qquad \leqq \int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}\left|(L_1[t(1-u)]L_2(ut)/L_1(t)L_2(t)) - 1\right|\,du. \end{aligned}$$

Since $L_1(t)$ and $L_2(t)$ are sv there exists a constant δ, independent of u (Lemma 1 (i)), such that for $i = 1, 2$, and $t \geqq t_0$

$$|(L_i(ut)/L_i(t)) - 1| \leqq \delta \qquad \text{for all} \quad u \,\varepsilon\, (\epsilon, 1-\epsilon).$$

Hence the above integral is majorized by

$$2\delta(1+\delta)\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}\,du.$$

If $t \to \infty$, then $\delta \to 0$, and hence the lemma follows. For brevity, let us write

$$B(\epsilon) = \int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^{\alpha-1}\,du.$$

Lemma 5 shows that

$$I_{21}(t) \sim \alpha B(\epsilon)t^{\alpha-\beta}L_1(t)L_2(t),$$

or

$$(10) \qquad I_{21}(t) \sim \alpha B(\epsilon)Q_\beta(t)H_\alpha(t).$$

We have to compare the integrals I_1, I_{22}, I_{31} and I_{32} with I_{21}: this is done in the next four lemmas.

LEMMA 6. *$c_1\epsilon^\alpha \leqq \lim_{t\to\infty} I_1(t)/I_{21}(t) \leqq c_2\epsilon^\alpha$, where c_1 and c_2 are constants independent of t and ϵ.*

PROOF. Since $Q_\beta(t)$ is nonincreasing we obtain from the definition of $I_1(t)$ that

$$Q_\beta(t)H_\alpha(\epsilon t) \leqq I_1(t) \leqq Q_\beta[t(1-\epsilon)]H_\alpha(\epsilon t).$$

Now as $t \to \infty$

$$Q_\beta(t)H_\alpha(\epsilon t)/I_{21}(t) \sim \epsilon^\alpha L_2(\epsilon t)/\alpha B(\epsilon)L_2(t) \sim \epsilon^\alpha/\alpha B(\epsilon),$$

and

$$Q_\beta[t(1-\epsilon)]H_\alpha(\epsilon t)/I_{21}(t) \sim (1-\epsilon)^{-\beta}\epsilon^\alpha/\alpha B(\epsilon).$$

But $B(\epsilon) \geqq B(\frac{1}{4})$ since $0 < \epsilon < \frac{1}{4}$. Hence for some constants c_1, c_2 independent of ϵ the lemma will follow.

LEMMA 7. *$I_{22}(t) = o(I_{21}(t))$ as $t \to \infty$.*

PROOF. Clearly,

$$I_{22}(t)/I_{21}(t) \sim [\alpha B(\epsilon)L_1(t)L_2(t)]^{-1}\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^\alpha L_1[t(1-u)]\,dL_2(ut)$$

which by the fact that $L_1(t)$ is sv is majorized by

$$[(1+\delta)/\alpha B(\epsilon)][L_2(t)]^{-1}\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^\alpha\,dL_2(ut)$$

since $L_2(t)$ is nondecreasing.

To estimate the latter integral, we apply a mean value theorem: there exists $c \in [\epsilon, 1-\epsilon]$ such that

$$[L_2(t)]^{-1}\int_\epsilon^{1-\epsilon}(1-u)^{-\beta}u^\alpha\,dL_2(ut)$$
$$= (1-c)^{-\beta}c^\alpha\{L_2[t(1-\epsilon)]/L_2(t) - L_2[\epsilon t]/L_2(t)\}.$$

But $L_2(t)$ is sv. So the expression on the right tends to zero as $t \to \infty$. This proves the lemma.

LEMMA 8. *There exists a constant c_3 independent of ϵ and t such that*

$$0 \leqq \lim_{t\to\infty} I_{31}(t)/I_{21}(t) \leqq c_3\epsilon^{1-\beta}.$$

PROOF. Obviously as $t \to \infty$,

$$I_{31}(t)/I_{21}(t) \sim [\alpha B(\epsilon)L_1(t)L_2(t)]^{-1}\int_{1-\epsilon}^1(1-u)^{-\beta}u^{\alpha-1}L_1[t(1-u)]L_2(ut)\,du$$
$$\leqq (1+\delta)^2(\alpha B(\epsilon))^{-1}\int_{1-\epsilon}^1(1-u)^{-\beta}u^{\alpha-1}\,du$$

where δ was defined similarly as in Lemma 5. The latter integral is majorized by

$$(1+\delta)^2(\alpha B(\epsilon))^{-1}(1-\epsilon)^{\alpha-1}\int_{1-\epsilon}^1(1-u)^{-\beta}\,du \leqq c_3\epsilon^{1-\beta}.$$

This proves the lemma.

Finally:

LEMMA 9. $I_{32}(t) = o(I_{21}(t))$ *as* $t \to \infty$.

PROOF. As before

$$0 \leqq I_{32}(t)/I_{21}(t) \leqq [(1 + \delta)/\alpha B(\epsilon)](L_2(t))^{-1} \int_{1-\epsilon}^{1} (1 - u)^{-\beta} u^{\alpha} \, dL_2(ut)$$
$$\leqq [(1 + \delta)/\alpha B(\epsilon)](L_2(t))^{-1} \int_0^{\epsilon} v^{-\beta} \, dL_2[t(1 - v)].$$

But the last integral is an improper integral. Since $L_2(t)$ is nondecreasing, the existence of this integral is proved as follows: let $0 < \eta < \epsilon$, then

$$\int_0^{\epsilon} v^{-\beta} \, dL_2[t(1 - v)]$$
$$= \lim_{\eta \downarrow 0} \int_{\eta}^{\epsilon} v^{-\beta} \, dL_2[t(1 - v)]$$
$$= \lim_{\eta \downarrow 0} \{\epsilon^{-\beta} L_2[t(1 - \epsilon)] - \eta^{-\beta} L_2[t(1 - \eta)] + \beta \int_{\eta}^{\epsilon} L_2[t(1 - v)] v^{-\beta - 1} \, dv\},$$

and since in (η, ϵ), $L_2[t(1 - v)] \leqq L_2[t(1 - \eta)]$:

$$0 \leqq I_{32}(t)/I_{21}(t) \leqq [(1 + \delta)/\alpha B(\epsilon)] \epsilon^{-\beta} \{(L_2[t(1 - \epsilon)]/L_2[t]) - 1\}$$

which tends to zero as $t \to \infty$. This proves the lemma.

Combining the last five lemmas, we obtain that for every $\epsilon > 0$

$$1 + c_1 \epsilon^{\alpha} \leqq \lim_{t \to \infty} I(t)/I_{21}(t) \leqq 1 + c_2 \epsilon^{\alpha} + c_3 \epsilon^{1-\beta}.$$

Hence for $t \to \infty$ by Lemma 5,

$$I(t) \sim \alpha Q_{\beta}(t) H_{\alpha}(t) \int_0^1 (1 - u)^{-\beta} u^{\alpha - 1} \, du$$

or

$$(11) \qquad I(t) \sim \alpha B(1 - \beta, \alpha) Q_{\beta}(t) H_{\alpha}(t).$$

To finish the proof of Part A, we have to show that (9) and (11) are asymptotically equal. This is proved by using

LEMMA 10. *For* $t \to \infty$,

(i) $\int_0^t [1 - F(x)] \, dx \sim t/H_0(t)$ *if* $\alpha = 0$

$\sim [\sin \alpha\pi/\alpha(1 - \alpha)\pi][t/H_{\alpha}(t)]$ *if* $0 < \alpha < 1$

$\sim t/H_1(t)$ *if* $\alpha = 1$;

(ii) $\int_0^t Q_{\beta}(x) \, dx \sim (t/(1 - \beta)) Q_{\beta}(t)$.

PROOF. Let $\alpha = 0$, then by Lemma 3 (iv) with $p = 1$ and $q = 1 > 0$, and Theorem. 1

$$\int_0^t [1 - F(x)] \, dx \sim tL(t) \sim t/H_0(t).$$

A similar proof using (2) gives the relation for $\alpha = 1$. If $0 < \alpha < 1$, then by putting $p = 1$, $q = 1 > \alpha$ in (iv) of Lemma 3

$$\int_0^t [1 - F(x)] \, dx \sim t^{1-\alpha} B(1, 1 - \alpha) L(t).$$

But by Theorem 1 we also know that $L(t) \sim t^{\alpha} \sin \alpha\pi/\alpha\pi H_{\alpha}(t)$ which proves (i) of the lemma.

Part (ii) is proved similarly.

An elementary computation shows then that

$$\alpha B(1 - \beta, \alpha)Q_{\beta}(t)H_{\alpha}(t) \sim B(1 - \beta, \alpha)(\sin \alpha\pi/\pi)[(1 - \beta)/(1 - \alpha)] \cdot \left(\int_0^t Q_{\beta}(x)\,dx \Big/ \int_0^t [1 - F(x)]\,dx\right) \quad \text{as } t \to \infty$$

which agrees with (9).

This finishes the proof of Part A.

PART B. $L_2(t)$ satisfies condition (8).

Let $L_2(t) = \bar{L}_2(t) - \underline{L}_2(t)$ and put $\bar{J}(t) = \int_0^t Q_{\beta}(t - x)\, d\{x^{\alpha}\bar{L}_2(x)\}$

$$\underline{J}(t) = \int_0^t Q_{\beta}(t - x)\, d\{x^{\alpha}\underline{L}_2(x)\}.$$

Since both $\bar{L}_2(t)$ and $\underline{L}_2(t)$ are nondecreasing and sv by Lemma 4

$$\bar{J}(t) \sim \alpha B(1 - \beta, \alpha)Q_{\beta}(t)t^{\alpha}\bar{L}_2(t), \quad \text{as} \quad t \to \infty, \tag{12}$$

$$\underline{J}(t) \sim \alpha B(1 - \beta, \alpha)Q_{\beta}(t)t^{\alpha}\underline{L}_2(t), \quad \text{as} \quad t \to \infty. \tag{13}$$

If we denote the right hand side of (12) and (13) by $\bar{K}(t)$ and $\underline{K}(t)$ respectively, then there exists a $\delta > 0$ such that for all $t \geqq t_0$,

$$1 - \delta \leqq \bar{J}(t)/\bar{K}(t) \leqq 1 + \delta$$
$$1 - \delta \leqq \underline{J}(t)/\underline{K}(t) \leqq 1 + \delta.$$

Hence

$$I(t)/\alpha B(1 - \beta, \alpha)Q_{\beta}(t)t^{\alpha}L_2(t) = [\bar{J}(t)/\bar{K}(t)]\cdot[\bar{L}_2(t)/L_2(t)] - [\underline{J}(t)/\underline{K}(t)]\cdot[\underline{L}_2(t)/L_2(t)],$$

or for $t \geqq t_0$

$$1 - \delta\{[\bar{L}_2(t) + \underline{L}_2(t)]/L_2(t)\} \leqq I(t)/\alpha B(1 - \beta, \alpha)Q_{\beta}(t)H_{\alpha}(t) \leqq 1 + \delta\{[\bar{L}_2(t) + \underline{L}_2(t)]/L_2(t)\}.$$

By (8) we obtain that for $t \to \infty$,

$$I(t) \sim \alpha B(1 - \beta, \alpha)Q_{\beta}(t)H_{\alpha}(t). \tag{14}$$

However Lemma 10 was not based on the assumption that $L_2(t)$ was nondecreasing. Henceforth it implies the asymptotic equality of (9) and (14).

This finishes the proof of Theorem 2.

We remark that the proof above did not use the fact that $H(t)$ was the renewal function except to put $H(t) \sim t^{\alpha}L_2(t)$ by Theorem 1. Hence the above proof goes through in estimating any integral of the form

$$\int_0^t Q_{\beta}(t - u)\, d\{t^{\alpha}L_2(t)\}$$

where $Q_\beta(t)$ satisfies the condition of Theorem 2, $0 < \alpha \leqq 1$ and $L_2(t)$ any sv function satisfying (S).

In the next theorem we derive an asymptotic result for $E[N^2(t)]$, where $N(t)$ is the number of renewals up to time t. Let $\text{Var}\, N(t) = V(t)$.

THEOREM 3. *If $F \varepsilon V_\alpha$, $0 \leqq \alpha \leqq 1$, then as $t \to \infty$,*

$$\begin{aligned} V(t) &\sim [L^2(t)]^{-1}, \qquad \text{if } \alpha = 0 \\ &\sim (\sin^2 \alpha\pi/\alpha^2\pi^2)\{\pi^{\frac{1}{2}}2^{1-2\alpha}\Gamma(\alpha+1)/\Gamma(\alpha+\tfrac{1}{2}) - 1\}t^{2\alpha}/L^2(t), \qquad \text{if } 0 < \alpha < 1 \\ &\sim o[t^2/L^{*2}(t)], \qquad \text{if } \alpha = 1. \end{aligned}$$

PROOF. It is well-known [8], that

$$E[N^2(t)] = H(t) + 2H * H(t)$$

so that the L.S. Transform of the left hand side equals $h(s) + 2h^2(s)$. The proof for $\alpha = 0$ is obvious by an appeal to Theorem 1 and Lemma 2. Also $\alpha = 1$ follows quickly from the same statements.

If $0 < \alpha < 1$ then we obtain that

$$E[N^2(t)] \sim 2t^{2\alpha}/\Gamma(2\alpha + 1)[\Gamma(1 - \alpha)]^{-2}L^2(t) \quad \text{as} \quad t \to \infty. \tag{15}$$

Combining (15) with Theorem 1 the given result is immediate in view of the identities $\Gamma(\alpha)\Gamma(1 - \alpha) = \pi/\sin \alpha\pi$ and

$$\Gamma(2\alpha + 1) = \pi^{-\frac{1}{2}}2^{2\alpha}\Gamma(\alpha + \tfrac{1}{2})\Gamma(\alpha + 1).$$

This finishes the proof.

The above theorem complements a result of Feller [5] where $F(t)$ is supposed to satisfy the relation $1 - F(t) \sim At^{-\alpha}$ as $t \to \infty$ for $0 < \alpha < 1$.

3. Renewal theorems for the case where only the first moment exists. We assume now that $F \varepsilon V_\alpha$ for $1 < \alpha < 2$, so that $\mu < \infty$ but $\mu_2 \leqq \infty$. Define.

$$F_2(t) = \mu^{-1} \textstyle\int_0^t [1 - F(x)]\, dx; \tag{16}$$

$$G(t) = H(t) - t/\mu + F_2(t); \tag{17}$$

$$H(t) = t\mu^{-1}\bar{L}(t). \tag{18}$$

The importance of the above functions is illustrated in

LEMMA 11.

(i) *$F_2(t)$ is the distribution function of a nonnegative random variable. Its* L.S.T. *$f_2(s)$ is given by $f_2(s) = [\mu s]^{-1}[1 - f(s)]$;*

(ii) *If $F(t)\ \varepsilon\ V_\alpha$ for $1 < \alpha \leqq 2$, then $F_2(t)\ \varepsilon\ V_{\alpha-1}$;*

(iii) *$G(t) = (1 - F_2) * H(t)$;*

(iv) *$\bar{L}(t)$ is slowly varying.*

PROOF. The first part of the lemma is well-known [8], (ii) is a consequence of Lemma 3 (iii) with $k = 0$, and (iv) is trivial since $\mu < \infty$. Let $g(s)$ be the L.S.T.

of $G(t)$ then (17) implies that

$$g(s) = h(s) - (\mu s)^{-1}\{1 - \mu s f_2(s)\} = h(s) - [f_2(s)/(1 - f(s))]f(s)$$

by applying (i) twice. From the last equality (iii) follows immediately.

THEOREM 4. *Let $\bar{L}(t)$ satisfy* (8). *If $F \varepsilon V_\alpha$ for $1 < \alpha < 2$ then as $t \to \infty$,*

$$H(t) - t/\mu \sim [t^{2-\alpha}/\mu^2(\alpha - 1)(2 - \alpha)]L(t).$$

PROOF. From (17) and (iii) of the lemma we obtain

$$\int_0^t [1 - F_2(t - x)]\, dH(x) = H(t) - t/\mu + F_2(t).$$

Let $Q_\beta(x) = 1 - F_2(x)$ with $\beta = \alpha - 1$ and $H(t) = (t/\mu)\bar{L}(t)$; then Theorem 2 yields with $\alpha = 1$ (see the remark there)

$$H(t) - t/\mu + F_2(t) \sim \int_0^t [1 - F_2(x)]\, dx / \int_0^t [1 - F(x)]\, dx. \tag{19}$$

By (i) of the lemma, $F_2(t) = 0(1)$ as $t \to \infty$ and $\int_0^t [1 - F(x)]\, dx \to \mu$ as $t \to \infty$. By Lemma 3 (iv) for $p = q = 1$ and α replaced by $\alpha - 1$ we obtain $\int_0^t [1 - F_2(x)]\, dx \sim (t^{2-\alpha}/\mu(\alpha - 1))B(1, 2 - \alpha)L(t)$.

Using these expressions in (19) the stated result follows. The theorem complements another result of Feller [5]. From Lemma 1 (i) we see that under the given conditions $H(t) - t/\mu$ still tends to infinity as $t \to \infty$.

COROLLARY 2. *Let $\bar{L}(t)$ satisfy* (8). If *$F \varepsilon V_\alpha$ for $1 < \alpha < 2$ then, as $t \to \infty$,*

$$V(t) \sim [2t^{3-\alpha}/\mu^3(3 - \alpha)(2 - \alpha)]L(t).$$

For the case where μ_2 exists we refer the reader to the important paper of Ch. Stone [10]. There precise bounds are given on the renewal function and in Blackwell's theorem, even for the generalized renewal process of the Chung-Pollard type [2].

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REFERENCES

[1] ALJANČIĆ, S. BOJANIČ, R. and TOMIĆ, M. (1954). Sur la valeur asymptotique d'une classe des intégrales définies. *Publ. Inst. Math. Acad.* Serbe **7** 81–94.

[2] ALJANČIĆ, S. and KARAMATA, J. (1956). Fonctions à comportement régulier et l'intégrale de Frullani. *Zbornik Radova Matematičkog Instituta.* **5** 239–248.

[3] CHUNG, K. L. and POLLARD, H. (1952). An extension of renewal theory. *Proc. Amer. Math. Soc.* **3** 303–309.

[4] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2.** Wiley, New York.

[5] FELLER, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67** 98–119.

[6] KNOPP, K. (1943). Über eine Erweiterung des Äquivalenzsatzes der C- und H-Verfahren und eine Klasse regulär wachsender Funktionen. *Math. Z.* **49** 219–255.

[7] PARAMESWARAN, S. (1961). Partition functions whose logarithms are slowly oscillating. *Trans. Amer. Math. Soc.* **100** 217–240.

[8] SMITH, W. L. (1953). Asymptotic renewal theorems. *Proc. Roy. Soc. Edinburgh. Ser. A* **64** 9–48.

[9] SMITH, W. L. (1962). A note on the renewal function when the mean renewal lifetime is infinite. *J. Roy. Statist. Soc. Ser B.* **23** 230–237.

[10] STONE, CH. (1965). On characteristic functions and renewal theory. *Trans. Amer. Math. Soc.* **120** 327–342.