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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Annual meeting, Madison, Wisconsin, August 26–30, 1968. Additional abstracts appeared in earlier issues.*)

25. Optimal and efficient designs of experiments. CORWIN L. ATWOOD, University of Minnesota.

We estimate s out of k regression coefficients. If $s = k$ and the regression functions are of low degree in each variable on a compact finite dimensional space satisfying certain symmetries, then any D-optimal design is supported only on certain points of symmetry. If $s < k$ we give a sufficient condition for D-optimality in terms of convergent sequences of designs, and we give a condition equivalent to D-optimality [modifying Karlin and Studden, *Ann. Math. Statist.* **37** 800–807]. Application of the above to multilinear regression on the simplex characterizes, and in some cases determines, optimal designs. An upper bound is known for the number of points of support necessary for a D-optimal design, $s \leqq k$; we show that the bound is sharp. For $s \leqq k$, we give a lower bound on the D-efficiency of the best design on k or fewer points. If $s = k$ we give a sharp lower bound on the G-efficiency of the best k-point design. Inequalities in both directions are given relating the D- and G-efficiencies of any design, for $s \leqq k$. The inequality in one direction is sharp. Sharp lower bounds are given for the D- and G-efficiencies of a single design used in several models. (Received 24 June 1968.)

6. A controlled transportation queueing process. U. NARAYAN BHAT, Case Western Reserve University.

A transportation queueing process in which taxis arrive in a Poisson process and customers arrive as a renewal process independent of the taxi-arrival process is controlled by calling extra taxis whenever the total number of customers lost to the system reaches a certain pre-determined number. Transient and steady state behavior of this process is studied using renewal theoretic arguments. A method is proposed to determine the best value of the control variable so as to balance the cost to a taxi due to waiting against the cost to a customer. Results given here are extensions of those obtained by Bhat and Erickon [(1966), RM 160, Department of Statistics, Michigan State University] in an inventory system context. (Received 3 July 1968.)

27. Asymptotically optimal sequences in exponential subhouses. M. C. BHATTACHARJEE, University of California, Berkeley.

Let A be a given Borel subset of the set of all probabilities on the real line. Any sequence $\{Y_n : n = 1, 2, \cdots\}$ is called an A-sequence if $\mathfrak{L}(Y_1) \varepsilon A$, $\mathfrak{L}(Y_{n+1} \mid Y_1, \cdots, Y_n) \varepsilon A$ for all n. For $a > 0$, the set of probabilities $\Gamma(a) = \{\gamma : \int \exp(ay)\gamma(dy) \leqq 1\}$ is called an exponential house. For A-sequences, the optimal probability that some partial sum exceeds $s > 0$ is $P_A(s) = \sup P(Y_1 + \cdots + Y_n \geqq s$, some $n)$, the supremum being over all A-sequences. It is well known that $P_{\Gamma(a)}(s) = \exp(-as)$. If the set A of available distributions is smaller, this upper bound for $P_A(s)$ may not be attained. A is an exponential subhouse if A is contained in some $\Gamma(a)$. In this paper we find sufficient conditions under which the smallest such exponential bound is asymptotically attained, using dynamic programming methods. One of the sufficient conditions may be stated as follows: Let $\Gamma^0(a) = \{\gamma : \int \exp(ay)\gamma(dy) = 1\}$. For an exponential subhouse A, let $\Gamma(\alpha_A)$ be the smallest exponential house containing A, $\alpha_A > 0$. If A has a nonempty intersection with the subset of $\Gamma^0(\alpha_A)$ having two jumps, then: s^{-1} [log $P_A(s)$ − log $P_{\Gamma(\alpha_A)}(s)$] $\uparrow 0$ as $s \uparrow \infty$. Finally dynamic programming ideas and methods are used to show that for every sequence $\{X_n: n = 1, 2, \cdots\}$ which is generalized Gaussian, $(S_n/n) \to 0$ a.s., where $S_n = \sum_{i=1}^{n} X_i$, by constructing an exponential bound for $P(|S_n| > n\epsilon \text{ some } n \geqq N)$, $\epsilon > 0$—which is asymptotically sharp. (Received 7 June 1968.)

28. Excessive upper bounds for hitting probabilities in a random walk on the plane. M. C. BHATTACHARJEE, University of California, Berkeley. (By title)

Consider the following random walk problem on the plane. At any (v, s) with $v > 0$, we can choose any random variable X such that $|X| \leqq 1$, $EX = 0$ and move to $(v - EX^2, s - X)$; stop as soon as $v \leqq 0$. Under these rules of play, we study the optimal probability of hitting the set $\{(v, s): v \leqq 0, s \leqq 0\}$ starting from any given point on the plane. After identifying the problem in a dynamic programming framework, it is shown that on the vertical lines through any $v > 0$, the hitting probability of any non-randomized method of play is related to an induced limiting variable. From certain initial states, the goal can be reached in finitely many steps with probability one. 'Excessive' functions, which are functions with range in [0, 1] satisfying simple boundary conditions for $v \leqq 0$ and subharmonic relative to a simple operator are shown to dominate P: the optimal hitting probability. P itself is such a function. We show, each v-section of the minimal failure probability $(1 - P)$ is a distribution function and use this to characterize P as being minimal among an interesting class of functions. 'Excessive' exponential upper bounds for P are constructed. It is also shown that the set of available moves may be extended to include some subfair ones without changing the optimal hitting probability. (Received 7 June 1968.)

29. Approach to degeneracy and the efficiency of some multivariate tests. G. K. BHATACHARYYA and R. A. JOHNSON, University of Wisconsin, Madison.

For the problem of testing p-variate distributions for a shift in location, two important nonparametric competitors of Hotelling's T^2 are the multivariate extensions W of the Wilcoxon test and M of the normal score test. In the univariate case, the normal score test has the commendable property that for all continuous distributions, its ARE with respect to the t-test exceeds 1 and with respect to the Wilcoxon test it exceeds $\pi/6$. This naturally raises the question of whether or not the multivariate extension M inherits this property and if not, what the lower bounds on its ARE with respect to W and T^2 are. It is shown here that the ARE of M with respect to both W and T^2 could be arbitrarily close to zero for some direction and for all $p \geqq 2$. The example consists of a gross error distribution which places most of its mass on a hyperplane and has marginals with high sixth moments. The ARE of W relative to T^2 is shown to be bounded away from zero for the gross error model. However, this ARE could also be arbitrarily close to zero for some other multivariate distribution as is shown by constructing a distribution which places high mass on a line but is not of the gross error type. (Received 12 July 1968.)

30. Binomial sequential design of experiments with general loss and unequal sampling costs. J. D. BORWANKER and H. T. DAVID, University of Minnesota; and University Minnesota and Iowa State University.

We consider how to sequentially select, under a certain general loss structure, one or two available binomial sources of information, differing both in reliability and sampling cost. Let θ_1 and θ_2 be two states of nature, a_1 and a_2 the two terminal decisions, A and B the two sources of information and two signals 0 and 1. The cost of one observation of type A is C_A and $P_r\{1 \mid A, \theta_1\} = P_r\{0 \mid A, \theta_2\} = \pi_A \geqq \frac{1}{2}$. Similarly for B. Let $L_{ij}(y)$ be the terminal loss when θ_i is the true state of nature, a_j is the action taken and y is the sampling cost. Let ξ be the prior probability of θ_1 and $f(\xi, C)$ be the infimum of the Bayes risk over all strategies for which the sampling cost does not exceed C. Let $f(\xi)$ denote the infimum of the Bayes risk over all strategies. It can be shown that if $\lim_{C\to\infty} \min_j \max_i L_{ij}(C)/\min_{i,j,y>C} L_{ij}(y) = 1$ then $\lim_{C\to\infty} f(\xi, C) = f(\xi)$. A characterization of the Bayes strategies for the above situation then follows from this result. Some sufficient conditions for the inadmissibility of one of the two sources of information have also been discussed for the case of a linear loss, i.e., $L_{ij}(y) = y + (1 - \delta_{ij})L$. The case of equal costs have been discussed in Wald (1950) and Chernoff (1959). (Received 8 July 1968.)

31. Nonsequential optimal solutions of sequential decision problems. MIN-TE CHAO, Bell Telephone Laboratories, Holmdel.

Given a general statistical sequential estimation problem specified by a set of standard regularity conditions, it is possible to choose a factor $B(\theta)$ to modify the payoff function in such a way that with respect to the new loss function, the class of all asymptotically Bayes' solutions (with respect to all priors in a certain class) is essentially nonsequential. (Received 7 June 1968.)

32. Factorial examination of alternative fitting equations. CUTHBERT DANIEL, Washington, D. C.

In the empirical fitting of multifactor single-response historical data, we usually start with some preferred full equation containing, say, K constants to be estimated. In trying to shorten this equation, we must consider a maximum of 2^K equations. There may be two possible forms of entry for some independent variables, as well as for the response, e.g. x_1 or $1/x_1$, y or $\log y$. For L such simple alternatives there are 2^L conceivable combinations. There may be more than two, say, S alternatives for some (L') terms, and so $S^{L'}$ factorial combinations of these. After a set of $2^{K+L}S^{L'}$ alternatives—or some balanced fraction of these—has been studied, we may see a set of R data points that are disturbing or distorting some of the best equations. The removal of these points, singly or as a group, should be examined factorially over a reasonable range of the alternatives already studied. (Received 21 June 1968.)

33. The exceedance test for truncation of a supplier's data. J. J. DEELY, D. E. AMOS and G. P. STECK, Sandia Laboratory.

Let Y be an absolutely continuous random variable with distribution F and let X be a random variable truncated at a real number c, i.e. X is a random variable with distribution $G(x) = \min(1, F(x)/F(c))$. Call $\gamma = 1 - F(c)$ the amount of truncation. Let $Y_1, \cdots, Y_k$ be independent observation from F and $X_1, \cdots, X_n$ be independent observations from G, and let K = number of Y_i's $\geqq X_{\max}$. In this paper we show that the exceedance test (i.e. reject if K is too large) of $\gamma = 0$ *vs.* $\gamma > 0$ is the uniformly most powerful rank test, asymptotically uniformly most powerful, and give several typical graphs of the power function from which sampling plans can be obtained. (Received 8 July 1968.)

34. Darmois-Koopman theorems and Cauchy's functional equation on globally and locally arcwise connected spaces. J. L. DENNY, University of Arizona.

Let $(\mathfrak{X}, \mathfrak{I})$ be an arcwise connected space which has a base of arcwise connected sets. Let $(\mathfrak{X}, \mathfrak{A}, \mu_t)$ be a family of measure spaces where $\mu_t \geqq 0$, the μ_t have the same null sets, and where not in general $\mathfrak{J} \subset \mathfrak{A}$. Local solutions of the functional equation in continuous $g: \mathfrak{X} \to R$ and continuous $f: \mathfrak{X}^n \to R$, $\sum g(x_i) = h(f(x_1, \cdots, x_n))$ a.e. μ_t^n, are essentially equivalent to local solutions of the following version of the Darmois-Koopman problem: obtain conditions on a continuous sufficient statistic $f: \mathfrak{X}^n \to R$ which imply that $\{\mu_t\}$ has a representation as a one-parameter exponential family. With a measure-theoretic condition on f analogous to that of Brown [*Ann. Math. Statist.* **35** 1456–1474] and Denny [*Proc. Nat. Acad. Sci. U.S.A.* **57** 1184–1187] we obtain solutions. Each locally convex space is globally and locally arcwise connected and the local solutions are used to characterize some stochastic processes with stationary independent increments with the same null sets for which there is a real-valued sufficient statistic which is continuous. These results extend the results of Brown and the author cited above. (Received 24 June 1968.)

35. A Bayesian test for equality of Bernoulli probabilities. J. M. Dickey and V. K. Murthy, System Development Corporation.

The Bayesian test of the sharp null hypothesis, $H = \{d \equiv p_1 - p_2 = 0\}$, against the diffuse alternative, $\bar{H} = \{p_i \sim \text{beta } (a_i, b_i)$, independent for $i = 1, 2\}$, is based on the ratio of posterior odds to prior odds for H, $L = [P(H \mid \text{Data})/P(\bar{H} \mid \text{Data})]/[P(H)/P(\bar{H})] = P(\text{Data} \mid H)/P(\text{Data} \mid \bar{H}) = P'(d = 0 \mid \text{Data}, \bar{H})/P'(d = 0 \mid \bar{H})$, if $\lim_{d\to 0} P(\text{Data} \mid d, \bar{H}) = P(\text{Data} \mid H)$. The likelihood ratio, L, independent of the stopping rule for the Bernoulli-process statistics N_1, n_1, N_2, n_2, satisfies $L = [B(\sum (a_i + n_i) - 1, \sum (b_i + N_i - n_i) - 1)/B(\sum (a_i) - 1, \sum (b_i) - 1)]/[\prod B(a_i + n_i, b_i + N_i - n_i)/B(a_i, b_i)]$, where B is the complete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. (Received 3 July 1968.)

36. Iterated maximum likelihood estimators (preliminary report). Edward J. Dudewicz, University of Rochester. (By title)

Suppose that θ (a parameter) is in some space Θ and that we have a likelihood function $L(\theta)$ (from Θ to $\mathfrak{R}$). Assume that a unique MLE $\hat{\theta}$ of θ exists, i.e. $\hat{\theta} \varepsilon \Theta$ such that $L(\hat{\theta}) \geqq L(\theta)\ \forall\ \theta \varepsilon \Theta$. Let $g(\cdot)$ be some transformation of Θ, and suppose that $g(\Theta) = \Lambda$. Then if $g(\cdot)$ is $1 - 1$, $g(\hat{\theta})$ is the MLE of θ. If $g(\cdot)$ is not $1 - 1$, Zehna [*Ann. Math. Statist.* **37** (1966) 744] and Berk [*Math. Rev.* **33** (1967) 342–343] both propose to stay with $g(\hat{\theta})$. We discuss their reasons and then consider the likelihood function of the statistic $g(\hat{\theta})$, say L_g. If there is a $\bar{g} \varepsilon \Lambda$ such that $L_g(\bar{g}) \geqq L_g(g')\ \forall\ g' \varepsilon \Lambda$, then $\bar{g}$ is called an *Iterated Maximum Likelihood Estimator* (IMLE) of $g(\theta)$. We note that the IMLE satisfies a criterion of Berk, that one's estimate maximize the likelihood function associated with some random variable. The two estimators of $g(\theta)$ are compared in several cases, e.g. with X_1 normal (μ_1, σ^2), X_2 normal (μ_2, σ^2), X_1 and X_2 independent, μ_1 and μ_2 unknown and $g(\mu_1, \mu_2) = (\mu_{[1]}, \mu_{[2]})$, the ranked parameters. (Received 21 June 1968.)

37. Upper and lower confidence intervals for ranked means (preliminary rereport). Edward J. Dudewicz, University of Rochester. (By title)

Suppose that observations from populations $\pi_1, \cdots, \pi_k$ $(k \geqq 1)$ are normally distributed with unknown means $\mu_1, \cdots, \mu_k$ (respectively) and a common known variance σ^2. Let $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ranked means. Several ranking and selection procedures take n independent observations from each population, denote the sample mean of the n observations from π_i by $\bar{X}_i$ $(i = 1, \cdots, k)$, and utilize the ranked sample means $\bar{X}_{[1]} \leqq \cdots \leqq \bar{X}_{[k]}$. For $i = 1, \cdots, k$ and any γ $(0 < \gamma < 1)$ an upper confidence interval for $\mu_{[i]}$ with minimal probability of coverage γ is $(-\infty, \bar{X}_{[i]} + h_i^*)$ with $h_i^* = (\sigma/n^{\frac{1}{2}})\Phi^{-1}(\gamma^{1/k-i+1})$, where $\Phi(\cdot)$ is the standard normal cdf. A lower confidence interval for $\mu_{[i]}$ with minimal probability of coverage γ is $(\bar{X}_{[i]} - g_i^*, +\infty)$ with $g_i^* = (\sigma/n^{\frac{1}{2}})\Phi^{-1}(\gamma^{1/i})$. For the upper confidence interval on $\mu_{[i]}$ the maximal probability of coverage is $1 - [1 - \gamma^{1/k-i+1}]^i$, while for the lower confidence interval on $\mu_{[i]}$ the maximal probability of coverage is $1 - [1 - \gamma^{1/i}]^{k-i+1}$. Thus the maximal overprotection can always be calculated. The overprotection is tabled for $k = 2, 3$. These results extend to certain translation parameter families. (Received 26 June 1968.)

38. Extensions of the Chernoff and Savage theorems to p-dependent sequences (preliminary report). THOMAS R. FEARS and K. L. MEHRA, Iowa State University.

Let $\{X_i\}$ and $\{Y_i\}$ be two p-dependent stationary sequences of real random variables such that $(X_1, \cdots, X_s)$ is independent of $(Y_r, \cdots)$ if $r - s > p$ and $(Y_1, \cdots, Y_s)$ is independent of $(X_r, \cdots)$ if $r - s > p$. Assuming the marginal distributions of X_i and Y_j to be continuous and that with probability one, no two of the X's or Y's are equal, it is shown that one and two sample rank-order statistics are asymptotically normal under Chernoff-Savage conditions (*Ann. Math. Statist.* **29** (1958) 972–994). An alternative proof under weaker conditions using Govindarajula, LeCam and Raghavachari (*Fifth Berkeley Symp.* 609–638) is also presented. The first proof, however, is a direct extension of the Chernoff-Savage arguments and, in this sense, it is a simpler proof. The results are then used to study the robustness properties of the relative efficiency of certain nonparametric estimates and tests for p-dependent sequences. (Received 5 July 1968.)

39. Distinguishability of probability measures I. LLOYD FISHER and JOHN W. VAN NESS, University of Washington.

The problem is in the general framework considered by Freedman, *Ann. Math. Statist.* **38** 1666–1670. We consider a sequence of iid observations. A countable family Π of probability measure is called distinguishable (written $\Pi \varepsilon D$) if Freedman's (I) holds. The relationship of distinguishability to the Lévy, Lévy-Prokhorov and sup metric (for distribution functions in R^n) is considered for families Π of Borel probability measures on locally compact metrizable groups using a translation invariant metric. For example, if the measures of Π have probability densities with respect to Haar measure which are all Lipschitz with same constant, $\Pi \varepsilon D$ iff each member of Π is isolated in the Lévy-Prokhorov metric. (Received 26 June 1968.)

40. Distinguishability of probability measures II. LLOYD FISHER and JOHN W. VAN NESS, University of Washington.

Independent identically distributed outcomes $X_1, X_2, \cdots$ are obtained sequentially and observed under observational errors $e_1, e_2, \cdots$. Thus we observe $X_1 + e_1, X_2 + e_2, \cdots$ where the $\{e_j\}$ process is independent of the X_j's and stationary. *A priori*, it is known that the common probability distribution, P, of the X_j's is a member of a given (at most countable) family $\Pi = \{P_k\}_{k=1}^{\infty}$. At some time, depending only on the observed data and the tolerable probability of error, one wants to stop and decide which P_k nature has chosen. If this can be done in a suitable manner, Π is said to be distinguishable under the errors, $\{e_j\}$. For example, if the e_j's are independent with distribution Q, then ordinary distinguishability holds if $\{P_k * Q;\ P_k \varepsilon \Pi\}$ satisfies (I) of Freedman, *Ann. Math. Statist.* **38** 1666–1670. Under weak restrictions on Q, we show that this will happen iff each member of Π is isolated from the others in the Lévy-Prokhorov metric. We also study finite distinguishability (see e.g. Hoeffding and Wolfowitz, *Ann. Math. Statist.* **29** 700–718) and more general (e.g. strong mixing and ergodic) error processes. (Received 26 June 1968.)

41. First emptiness of dams with Markovian inputs. J. GANI, University of Sheffield and Stanford University.

In a recent paper, Ali Khan and Gani (*J. Appl. Prob.* **5** (1968) 72–84) have discussed first emptiness probabilities for a dam whose inputs form a Markov chain. Let an infinite dam of initial content $u > 0$ be fed by inputs $X_t = 0, 1, \cdots, r < \infty$ arriving in the consecutive time intervals $(t, t+1)(t = 0, 1, 2, \cdots)$; these are assumed to form a positive irreducible Markov chain with transition probabilities $\{p_{ij}\}_{i,j=0}^{r}$. At the end of each interval the release is unity if the dam content allows it, and zero if not. If $g(T \mid u, 0)$ denotes the first emptiness probability at time $T \geqq u$, given an initial content u, and an input $X_{-1} = 0$, then its pgf $G_0(\theta)$ satisfies the functional equation $G_0(\theta) = \theta\lambda(G_0(\theta))$ subject to $G_0(0) = 0$. Here $\lambda(\theta)$ is the maximum latent root of $P(\theta) = \{p_{ij}\theta^j\}_{i,j=0}^{r}$. It is shown that $\lambda(\theta)$ is not in general a pgf. The probabilities $g(T \mid u, 0)$ are then derived in the form $g(T \mid u, 0) = u\,\lambda_{T-u}^{(T)}/T$ $(T \geqq u > 0)$ where $\lambda_{T-u}^{(T)} = (d/d\theta)^{T-u}\{\lambda^T(\theta)\}/(T-u)!$ (Received 5 July 1968.

42. An elementary method for obtaining lower bounds on the asymptotic power of rank tests. JOSEPH L. GASTWIRTH and STEPHEN S. WOLFF, Johns Hopkins University.

In 1958, Chernoff and Savage (*Ann. Math. Statist.*, p. 972–994) showed that the asymptotic relative efficiency of the normal scores test to the t-test was always greater than or equal to one using variational methods. In this note we give a simpler proof of the result using Jensen's inequality. Our method of proof allows us to generalize Doksum's (*Ann. Math. Statist.* **38** (1967) 1731–1739) lower bound for the asymptotic power of Savage's test for scale change of positive random variables. (Received 3 July 1968.)

43. Sequential confidence interval procedures based on rank tests. JAN C. GEERTSEMA, University of California, Berkeley.

Let $X_1, X_2, \cdots$ be independent random variables with common cdf. $F(x - \theta)$ where $F(x)$ is symmetric about 0 and has density $f(x)$. A confidence interval of prescribed width $2d$ and prescribed coverage probability $1 - 2\alpha$ for θ is desired. Two procedures are proposed: (1) Let $N(1)$ be the first integer $n \geqq n_0 \geqq 2$ such that $X(n, a_n) - X(n, b_n) \leqq 2d$ where $X(n, 1) \leqq X(n, 2) \leqq \cdots \leqq X(n, n)$ are the ordered $X_1, \cdots X_n$; $b_n = \max(1, [n/2 - kn^{\frac{1}{2}}/2])$, $a_n = n - b_n + 1$, k is defined by $\Phi(k) = 1 - \alpha$, Φ is the $N(0, 1)$ cdf and $[x]$ is the largest integer less than or equal to x. Then choose $I(N(1)) = (X(N(1), b_{N(1)}), X(N(1), a_{N(1)}))$ as interval. (2) Let $N(2)$ be the first integer $n \geqq n_0 \geqq 2$ such that $W(n, a_n') - W(n, b_n') \leqq 2d$ where $W(n, 1) \leqq W(n, 2) \leqq \cdots \leqq W(n, n(n+1)/2)$ are the ordered averages $(X_i + X_j)/2$ for $i, j = 1, \cdots, n$ and $i \leqq j$; $b_n' = \max(1, [n(n+1)/4 - k(n(n+1)(2n+1)/24)^{\frac{1}{2}}])$, $a_n' = n(n+1)/2 - b_n' + 1$. Then choose as interval $I(N(2)) = (W(N(2), b'_{N(2)}), W(N(2), a'_{N(2)}))$. Following ideas of Chow and Robbins (*Ann. Math. Statist.* **36** 457–462) we have under regularity conditions, as $d \to 0$: $\lim EN(1)\,d^2 = k^2/4f^2(0)$, $\lim EN(2)\,d^2 = k^2/12(\int f^2(x)\,dx)^2$ and $\lim P(\theta \varepsilon I(N(i))) = 1 - 2\alpha$ for $i = 1, 2$. Defining asymptotic efficiency in terms of asymptotic expected sample size (as d tends to 0), the asymptotic efficiencies of the above procedures and that of Chow and Robbins wrt each other are found to be the same as the Pitman efficiencies between the tests which correspond in a natural way to the various procedures. Monte Carlo studies for the non-asymptotic case are included. (Received 3 July 1968.)

44. On the optimality of the standard blocking procedure for $(p^m)^n$ factorials. DENNIS C. GILLILAND, Michigan State University.

There is a standard procedure for blocking a replicate of an s^n factorial design into s^q blocks where $s = p^m$ and p is a prime number [Kempthorne (1962). *The Design and Analysis* *of Experiments*, John Wiley and Sons, Inc., New York]. Using finite field theory there exists a decomposition of Euclidean s^n-space into $(s^n - 1)/(s - 1)$ mutually orthogonal subspaces each of dimension $s - 1$ and each orthogonal to the vector of all ones. Each of these subspaces can be identified with a basis consisting of $s - 1$ orthogonal factorial effects, each a contrast in blocks of size s^{n-1}. The standard blocking procedure treats each of these subspaces as an atom, either completely confounding it or leaving it orthogonal to the blocks. If a complete set of s^n orthogonal factorial effects (including the sum of treatment effects) is specified, then the standard procedure renders non-estimable s^q factorial effects, the minimum number that can be achieved when blocking into s^q blocks. We prove that the standard blocking procedure is the only one that does not split atoms and at the same time achieves minimal confounding of factorial effects. An example of a nonstandard blocking of a 4^2 into 4 blocks is given which splits atoms and achieves minimal confounding. Such examples are not possible when s is prime for we prove that in this case the standard procedure is the only one that achieves minimal confounding. This result generalizes the previously reported result on the unique optimality of the standard blocking of 2^n factorials into 2^q blocks [*Ann. Math. Statist.* **39** (1968) 1088]. (Received 3 July 1968.)

45. Asymptotic normality and efficiency of two-sample rank order sequential probability ratio test based on Lehmann alternatives. Z. GOVINDARAJULU, University of Kentucky.

Asymptotic normality of a generalized Chernoff-Savage class of statistics is considered. The two-sample rank order sequential probability ratio test statistic based on Lehmann alternatives becomes a special case of the class of statistics considered. The asymptotic relative efficiency of the statistic discussed by Savage and Sethuraman [*Ann. Math. Statist.* **37** (1966) 1154–1160] has been studied and it is inferred that it is asymptotically equivalent to the non-sequential two-sample Savage statistic [*Ann. Math. Statist.* **27** (1956) 590–615]. The asymptotic efficiency relative to its parametric competitor for shift in location and change of scale is studied for normal, exponential and logistic alternatives. (Received 7 June 1968).

46. A central limit theorem for independent summands with infinite variances. Z. GOVINDARAJULU, University of Kentucky.

The Levy-Lindeberg and Lindeberg-Feller forms of the central limit theorem assume finiteness of the second moment of the summands. Gnedenko and Kolmogorov (*Limit Distributions for Sums of Independent Random Variables*, Cambridge: Addison-Wesley Publishing Co., Inc. (1954) 128–132, 171–172) have given some limit theorems when the summands may not have finite second moments. In this paper, using the linear operator's approach employed by Trotter (*Archiv Der Math* **10** (1959) 226–234), sufficient condition for the asymptotic normality of sums of independent random variables which may or may not admit finiteness of second order moments, is obtained. This condition, in the case of identical summands imply the existence of all moments of order less than two. Even if the second moments exist, the basic result of this paper cannot be compared with Lindeberg's form of the central limit theorem because of different standardizations. These results are extended to random sample size case and to multi-variate situations. (Received 12 July 1968.)

47. On weak laws of large numbers. Z. GOVINDARAJULU, University of Kentucky. (By title)

Using the linear operators approach proposed by Trotter (*Archiv Der Math.* **10** (1959) 226–234) a modified version of Feller's weak law of large numbers (*Acta. Scient. Math.*

(Univ. of Szeged) **8** (1936–37) 191–201) for sums of independent random variables (rv's) has been obtained. Also, the following theorem is proved: *Let* $X_1, X_2, \cdots$ *be independent rv's having* $F_1, F_2, \cdots$ *for their distribution functions. Define* $S_n = \sum_1^n (X_k - \alpha_k)/\beta_n$ *where* $\alpha_k = \int_{|x| \leqq \beta_n} x\, dF_k (k = 1, \cdots, n)$ *and* $\{\beta_n\}$ *is a sequence tending to infinity such that* $n/\beta_n \leqq K < \infty$. *Then* S_n *converges to zero in probability if* $B_n(t) \to 0$ *as* $n, t \to \infty$ *where* $B_n(t) = n^{-1}\sum_1^n t[1 - F_k(t) + F_k(-t)]$. *Further,* $S_n \to 0$ *in probability implies that* $B_n(t) \to 0$ *as* $n, t \to \infty$ *whenever* $0 < n/\beta_n < K$ *and* $b/\beta_n \leqq K$ *where* $b = \max(b_1, \cdots, b_n)$ *and for some* $p \equiv 0, b_k \geqq 0$ *are defined by* $F_k(-b_k) \leqq 1 - p$ *and* $F_k(b_k - 0) \geqq p (k = 1, \cdots, n)$. The above theorem, as a special case includes Feller's theorem (*An Introduction to Probability Theory and its Applications*, **2**, New York: John Wiley & Sons, (1966) 231–232) for the identically distributed case. These results are extended to situations where the sample sizes are random, but independent of the underlying random variables, and to multi-variate cases. (Received 12 July 1968.)

48. Orthogonal arrays of strength five (preliminary report). BODH RAJ GULATI and J. J. LUCAS, Eastern Connecticut State College and University of Connecticut.

An orthogonal array of strength t, k factors (or constraints), s symbols (or levels) and N assemblies (or treatment combinations) is a $k \times N$ matrix A if any $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same weight (or index) λ. Such an array is denoted by the parameters (N, k, s, t) where $N = \lambda s^t$. In a technical report RM-189 of Michigan State University, 1967, "Orthogonal Arrays of Odd Index", Esther Seiden showed that for $t \geqq \lambda + 1$, the maximum number of constraints is exactly $t + 1$, from which the construction of arrays (32, 6, 2, 5) and (96, 6, 2, 5) follows immediately. In this paper, methods of construction of arrays with $s = 2$, $t = 5$ are studied for different weights. The orthogonal arrays given below are constructed with the maximum number of constraints; (64, 7, 2, 5), (128, 9, 2, 5), (160, 7, 2, 5), (192, 7, 2, 5) (224, 7, 2, 5), (256, 12, 2, 5) and (288, 8, 2, 5). For some more values of N, arrays are constructed and it is being investigated whether the number of constraints obtained is the best possible. The actual arrays will be given in the paper. (Received 3 June 1968.)

49. Embedding random walks, log likelihood ratios, and other submartingales in Weiner processes with drift. W. J. HALL, University of North Carolina and Stanford University.

Suppose $\{Y_n, \mathfrak{F}_n; n = 1, 2, \cdots\}$ is a submartingale sequence with increments $\{X_n\}$ for which, for some $\theta < 0$, $E(\exp - \theta X_n \mid \mathfrak{F}_{n-1}) \leqq 1$ a.s. for $n = 1, 2, \cdots$, and suppose $\{W_\delta(t); t \geqq 0\}$ is a Weiner process with drift δ. It is proved that there exist successive stopping times $\{T_n\}$ so that the two sequences $(n = 1, 2, \cdots)$ $\{W_\delta(T_n)\}$ and $\{Y_n\}$ are equal in law; moreover, $ET_n = EY_n/\delta$. This is a modification of a theorem of A. V. Skorokhod (1961) (see, e.g., L. Breiman, *Probability*, Addison-Wesley (1968)) on the embedding of martingale sequences $\{Y_n, \mathfrak{F}_n\}$ in Weiner processes without drift. Examples of such submartingale sequences $\{Y_n\}$ include (i) cumulative sums of non-negative random variables, (ii) random walks for which $0 < EX_n \leqq \infty$ and X_n^- has a moment generating function, (iii) cumulative sums of uniformly bounded random variables with positive conditional means bounded away from zero, and (iv) sequences of log likelihood ratios of $(Z_1, \cdots, Z_n)$, say (the 'numerator' distribution of $\{Z_n\}$ being absolutely continuous wrt the 'denominator' distribution). Applications of the embedding theorem include new derivations of the basic theorems of sequential analysis, and some asymptotic properties of nonparametric sequential tests. (Received 8 July 1968.)

50. A note on discrimination in the case of unequal covariance matrices. CHIEN-PAI HAN, Iowa State University.

In assigning an individual $\mathbf{X}_{p\times 1}$ into one of two multivariate normal populations, the likelihood ratio procedure is used. We assume that the populations have different mean vectors and different covariance matrices and that the variates are equally correlated. The covariance matrix of the ith population is of the form $\mathbf{\Sigma}_i = \sigma_i^2[(1 - \rho_i)\mathbf{I} + \rho_i\mathbf{ZZ}']$, $i = 1, 2$, where $\mathbf{I}$ is a $p \times p$ identity matrix and $\mathbf{Z}$ is a $p \times 1$ vector with component unity. The discriminant function derived by the likelihood ratio procedure depends on the size and the shape components of Penrose [*Ann. Eugen.* **13** (1946–47) 228–37] and the component of sum of squares of the variates. (Received 7 June 1968.)

51. Some remarks on selection procedures based on ranks. M. HASSEB RIZVI and GEORGE G. WOODWORTH, Ohio State University and Stanford University; and Stanford University.

Let X_{ij} ($j = 1, \cdots, n; i = 1, \cdots, k$) be independent samples from k populations with cdfs $F(x - \theta_i)$ and let R_{ij} be the rank of X_{ij} and $h(R_{ij})$ a corresponding score. For the problem of selecting the population with the largest θ-value Lehmann (*Math. Annalen* **150** (1963) 268–275) considers the procedure that selects the mth population if $V_m = \max_i V_i$, where $V_i = \sum_j h(R_{ij})$. Puri and Puri (*Ann. Math. Statist.* **37** (1966) 554, Abstract) extend Lehmann's results to the case of selecting t ($\leqq k$) populations with t largest θ-values. These authors obtain the common sample size n such that the probability of a correct selection under a slippage-type configuration of parameters is no smaller than a specified number P^*. That the slippage-type configuration is not the least favorable one is demonstrated in a counter-example provided us by Lehmann. The present paper shows that the slippage-type configuration is, however, least favorable in some asymptotic sense. Asymptotic relative efficiency comparisons of the scores procedure are made with the distribution-free procedure of Sobel (*Ann. Math. Statist.* **38** (1967) 1788–1803). Asymptotic study of some other related selection procedures has also been done. (Received 5 July 1968.)

52. A stochastic model for surpluses and deficits. ROBERT L. HEINY and M. M. SIDDIQUI, Colorado State University.

A sequence $\{X_n\}$ of identically distributed random variables is considered together with an arbitrary constant c. To fix our ideas, we will refer to X_n as the total precipitation at a given location during the nth year. If $X_n > c$, the nth year is called a surplus year and $X_n - c$, the surplus. If $X_n \leqq c$, the nth year is called a deficit year and $c - X_n$ the deficit. A surplus run of length k is a consecutive sequence of k surplus years preceded and succeeded by a deficit year. The surplus sum is the sum of surpluses for this run. The average surplus is the surplus sum divided by the surplus run length. The deficit sum and average deficit are similarly defined. For X_n, $n = 1, 2, \cdots$, independent, the moment generating function, moments, extremes, and approximations to the distribution function of the surplus run length, surplus sum, and average surplus are found. For a first order Markov dependence among the X_n, $n = 1, 2, \cdots$, with a linear condition imposed on the form of the dependence, the moment generating function and moments are determined for the same surplus quantities. A first order autoregressive scheme is presented assuming $X_n = \rho X_{n-1} + \xi n$, $|\rho| < 1$, where $\xi_n \sim N(0, 1)$ is independent for all n. (Received 24 June 1968.)

53. Asymptotic properties of MLE's when the observations are independent but not iid with applications to estimation under variable censoring. ARTHUR BRUCE HOADLEY, Bell Telephone Laboratories, Holmdel.

In this paper, conditions are established under which maximum likelihood estimates are consistent and asymptotically normal in the case where the observations are independent but not identically distributed. Both observations and parameters are vectors. The consistency part is similar to [Wolfowitz, J. (1949). On Wald's proof of the consistency of the maximum likelihood estimate. *Ann. Math. Statist.* **20** 601–602.], and the asymptotic normality part is similar to [Le Cam, L. (1964). Unpublished notes.]. A very interesting motivational example involving estimation under variable censoring is presented. This example invokes the full generality of the theorems with regard to lack of iid and lack of densities wrt Lebesgue or counting measure. (Received 14 June 1968.)

54. On the renewal density matrix of a semi-Markov process. JEFFREY J. HUNTER, University of North Carolina.

This paper is concerned with the derivation of necessary and sufficient conditions for the convergence of the renewal density matrix $\mathbf{h}(x)$ of a semi-Markov process, or more particularly of its associated Markov renewal process, to a limit matrix H as x tends to infinity. The semi-Markov process is assumed to have a finite number of states and an irreducible imbedded Markov chain. The techniques used are generalisations of those used by W. L. Smith in his paper "On the necessary and sufficient conditions for the convergence of the renewal density" [*Trans. Amer. Math. Soc.* **104** (1962) 79–100]. Prior to deriving the minimal conditions, the nature of the limit matrix H is discussed and some renewal theoretic properties of semi-Markov processes presented. Some additional necessary conditions together with a set of simple sufficient conditions that ensure convergence of $\mathbf{h}(x)$ to H as x tends to infinity are also obtained. (Received 3 June 1968.)

55. Sequential analysis of variance with unequal numbers of observations in experimental units. ROGER D. H. JONES and KLAUS HINKELMANN, University of Georgia and Virginia Polytechnic Institute.

For the general linear hypothesis model $y = X\beta + \epsilon$ it is shown that the sequential analysis of variance procedure given by Johnson [*Ann. Math. Statist.* **24** (1953) 614–623] can be extended to the situation where the sampling of observations is done according to a multinomial distribution (with equal or unequal probabilities). This sampling procedure leads to unequal numbers of observations in the experimental units; i.e., to nonorthogonal models. A complication arises then with regard to the estimation of the average sample number (ASN). Numerical studies indicate that for the one-way classification and two-way classification (without and with interaction) model the approximate ASN for the equal subclass number situation as given by Ray [*Biometrika* **43** (1956) 388–403] agrees quite well with the ASN found for the unequal subclass number situation. In addition, a different type of sampling, called nonretentive sampling, is considered which, however, leads to larger ASN experimentwise. (Received 21 June 1968.)

56. A correspondence between Bayesian estimation on Gaussian processes and smoothing by splines. GEORGE KIMELDORF and GRACE WAHBA, University of Wisconsin.

Let $L = \sum_0^m a_j D^j$, $a_m \neq 0$ be a linear differential operator with real constant coefficients and t_i be n distinct real numbers. An L-spline with knots t_i is defined as a function $x \varepsilon C^{2m-2}$

for which $[\sum a_j D^j][\sum a_j(-D)^j]x(t) = 0$ on each open interval $(-\infty, t_1)$, (t_i, t_{i+1}), (t_n, ∞). If $P(\lambda) = \sum_0^m a_j(i\lambda)^j$ has no real zeros and $m > 0$, we can consider the following Bayesian estimation problem. Let the random function $\{x(t), -\infty < t < \infty\}$ have a zero-mean stationary normal prior distribution with spectral density $|2\pi P^2(\lambda)|^{-1}$ and let n observations $y_i = x(t_i) + e_i$ be taken where the vector of measurement errors e_i is independent of $x(t)$ and distributed as $N(0, B^{-1})$, $B^{-1} = [b_{ij}]^{-1}$ known. We show the estimate $\hat{x}(t) = E[x(t) \mid y_1, \cdots, y_n]$, the posterior mean, is an L-spline with knots t_i and is the unique function x having absolutely continuous $(m - 1) - st$ derivative which minimizes $\sum\sum [x(t_i) - y_i]b_{ij}[x(t_j) - y_j] + \int (Lx)^2$. The usual conditional expectation formula yields a convenient expression for $\hat{x}(t)$ as a linear combination of the n functions $K_i(t) =$ prior covariance of $x(t)$ and $x(t_i)$. The reproducing kernel Hilbert space associated with the process $\{x(t)\}$ is shown to be useful for solving a large class of interpolation and smoothing problems involving L-spines. (Received 5 July 1968.)

57. A class of ADF tests for subhypothesis in the multiple linear regression.

HIRA LAL KOUL, Michigan State University.

In the regression model $Y = \beta_1 x_1 + \beta_2 x_2 + Z$, under suitable conditions, a class of asymptotically distribution free (ADF) tests, for testing $H_0 : \beta_1 = 0$ when β_2 is unknown, is given. This class does not include test corresponding to the score function Φ^{-1}. It turns out that the Wilcoxon type tests are not suitable for the above problem. But the tests of Freund-Ausari type, Mood-type, among others, are in the class. (Received 3 July 1968.)

58. Simultaneous tests and multiple decision procedures for multi-response growth curves.

P. R. KRISHNAIAH and KANTA JAYACHANDRAN, Aerospace Research Laboratories, Wright-Patterson Air Force Base. (By title)

Let $\mathbf{x}_{it}' = (x_{it1}, \cdots, x_{itp})$ for $i = 1, 2, \cdots, k$ and $t = 1, 2, \cdots, n_i$ where x_{itj} denotes the observed value on ith sample unit at tth time point on jth variate. In addition, let $(\mathbf{x}_{i1}', \cdots, \mathbf{x}_{in_i}')$ be distributed as a multivariate normal with means $E(\mathbf{x}_{it}) = \beta_{i0} + \beta_{i1}t + \cdots + \beta_{iq}t^q$ and covariance matrix $\Omega_i \times \Sigma$ where Ω_i and Σ are unknown and $\Omega_i = (\omega_{irs}) : n_i \times n_i$, $\omega_{irs} = \rho_i^{|r-s|}$. Also, let $n_i = 2m_i + 1$ and $H_j : c_{u1}\beta_{1j} + \cdots + c_{uk}\beta_{kj} = 0$. Krishnaiah ("Simultaneous Tests for Multiple Comparisons of Growth Curves," ARL Tech Report) proposed procedures for testing the hypotheses H_{ju} simultaneously, when $\rho_i = \rho$, by considering the conditional distributions of $x_{i,2a,b+1}$ holding $\mathbf{x}_{i,2a-1}$, $\mathbf{x}_{i,2a+1}$, $\mathbf{x}_{i,2a,1}$, $\cdots$ $\mathbf{x}_{i,2a,b}$ fixed. In this paper, alternative test procedures based on correlated Hotelling T^2 statistics are proposed for testing the hypotheses H_{ju} simultaneously when $\rho_i = \rho$ by considering the conditional distributions of $\mathbf{x}_{i,2a}$ holding $\mathbf{x}_{i,2a-1}$ and $\mathbf{x}_{i,2a+1}$ fixed. Test procedures are also proposed for multiple comparisons among $\rho_1, \cdots, \rho_k$. Procedures for selecting "better" populations are proposed by using different criteria. Procedures are also proposed for ranking populations according to their importance. Finally, the relationships of some simultaneous test procedures with some of the ranking and selection procedures are discussed. (Received 12 July 1968.)

59. Minimum discrimination information estimation of cell probabilities in a contingency table—applications.

H. H. KU and S. KULLBACK, National Bureau of Standards and George Washington University.

Under a null hypothesis of interest, the cell probabilities of a contingency table are either completely specified, or are estimated from observed or given marginal frequencies. A method of estimation based on minimizing the discrimination information, $I(p:\pi) =$

$\sum p \ln p\pi^{-1}$, has been proposed where p and π with appropriate indices are the observed (hypothesized) and the hypothesized (observed) cell probabilities respectively. It has been shown that these estimates, p^*, (A) are RBAN, (B) can be computed by a convergent iterative scheme, and (C) can be expressed in a logarithmic linear additive form of parameters. In addition (D) $2nI(p^*:\pi)$ is asymptotically distributed as χ^2. Application of this estimation procedure and the resulting test statistic includes tests of second- and higher-order interactions, and estimation of paramters. Results obtained by maximum likelihood (Grizzle), and minimum logit χ^2 (Berkson), and minimum discrimination information estimation for five examples are compared. (Received 2 July 1968.)

60. Estimation in a waiting time discrimination under Markovian dependence. RICHARD A. LAITNEN and J. S. RUSTAGI, Ohio State University.

The probability distribution of a number of Bernoulli trials required to obtain a preassigned number of successes under Markovian dependence arises in the study of certain weapon systems. If the sequences of observations are completely known, the maximum likelihood estimates can be easily obtained for the parameters in the distribution. However, when only partial information is available on the observed sequences, other methods of estimation such as the method of moments are employed. Certain properties of the moment estimates are studied. (Received 5 July 1968.)

61. Finite discrete probability distributions as polynomials. ANDRE G. LAURENT, Wayne State University.

A real discrete finite random variable X can be considered as an element of a finite associative algebra A, with modulus u, over the field of real numbers so that X be specified by (1) its minimal function $\pi(x)$ (with real roots x_i, $i = 1$ to k), (2) a linear functional $E[\ ]$ on A, subject to $E[u] = 1$, $E[1_i] = p_i > 0$ (read $P(X = x_i)$, all i's, where the l_i's are idempotent, constitute a basis for the subalgebra B generated by $(u, X, \cdots, X^{k-1})$, and $l_i = l_i(X)$, where the $l_i(x)$'s are fundamental interpolatory polynomials associated with $\pi(x)$. $l_i(X)$ is the indicator of $(X = x_i)$; $g(X) = \sum g(x_i)l_i(X)$, where $g(t)$ is a polynomial. Let $l'(X) = (l_1(X), \cdots, l_k(X))$, $p' = (p_1, \cdots, p_k)$; let $\varphi' = (\varphi_1, \cdots, \varphi_k)$ be a basis for B. $E[\ ]$ can be defined by $E[\varphi] = m_\varphi$ (generalized moments); then $p = W_\varphi^{-1}m_\varphi$, where W_φ' is a generalized Vandermonde matrix. If φ and ψ are bases for B, then $p(x) = E[l'(X)]\, l(x) = m_\varphi' R^{-1}\psi(x)$, where $R = V_\psi W_\varphi'$ is a generalized Hankel matrix. Different bases are studied, in particular when X is defined on $(0, 1, \cdots, n)$. New derivations and expressions for the Poincare-Jordan-Frechet formulae are given. Uses of the mapping $(x_1, \cdots, x_k) \rightarrow (0, 1, \cdots, k - 1)$ are suggested. Basic statistical problems concerning discrete distributions are considered within this unifying frame of reference. (Received 5 July 1968.)

62. Ellipsoidal distributions I: The elliptical case *i* (preliminary report). ANDRE G. LAURENT, Wayne State University. (By title)

Let X be a random vector. X has an ellipsoidal distribution (ed) will mean that the pdf of X, $f(X) = p(Q)$, depends only on $Q = X'\Sigma^{-1}X$ (Σ positive definite). Basic properties (pertaining to characteristic functions, projections, etc.) are studied. In applied work it is of interest to generate ed by setting a model (1) for the pdf of $\xi'\xi$, where ξ is spherical, then ellipsoidizing through $X = T\xi$, (2) for the pdf of the projection P_AX of X on a linear subspace A. Many problems concerning ed involve the pdf of some random variable defined on the unit sphere with uniformly distributed probability mass. Elliptical distributions are studied as special cases of ed. Obtaining the pdf of $X'X = R^2$ or R (radial error) can be reduced to a classical problem of potential theory. In case the pdf of $\xi'\xi = \rho^2$ or ρ is an entire function, expansions involving Legendre polynomials in the "coefficient of non-circularity" Θ of Σ are obtained for the pdf of R^2 or R. Bivariate generalizations of Laplace, Student, Cauchy and other univariate distributions are proposed and studied. (Received 5 July 1968.)

63. A comparison of two procedures for deciding among several hypotheses. JAMES A. LECHNER, Research Analysis Corporation.

For a particular test case, the Wald-Sobel technique for deciding between three alternative hypotheses is evaluated by Monte Carlo, and compared with a procedure obtained by direct generalization of the SPRT. The latter is a special case of the tests (called GSPRT's) studied by Barr (Sequential decision rules for a multiple-choice problem, NASA CR-437, Colorado State Univ., April 1966); see also abstract by this author, in *Ann. Math. Statist.*, **37** (1966) 1428. The Monte Carlo technique was used to evaluate the OC curves and average sample time for both tests. The test case is the choice among three values for the mean (drift parameter) of a Gaussian stochastic process with known variance. Although the OC curves for the Wald-Sobel test are obtainable from Wald's formulas, they were also calculated, as a check on the simulation. (Received 8 July 1968.)

64. Moments of random variables related to zero crossings. RAOUL D. LE PAGE, Columbia University.

Let X indicate the sample path of a real, sample-continuous random process over an interval I of the real line. Supposing X to have at most a finite number of zeros, with probability one, and letting T be generic for zero crossings by X on I, conditions are given so that for real functions g and integers $k > 0$, $E[\sum_T g(T)]^k$ may be expressed as a limit of k-space integrals involving the joint distributions of $X(t_1), \cdots, X(t_k)$ over I^k. The approach used differs from that employed by others in previous work connected with the moments of the number of zero crossings in that nowhere in the present work is a linear interpolation of X employed. (Received 2 July 1968.)

65. Asymptotic results and equilibrium conditions for branching Poisson processes. PETER A. W. LEWIS, IBM Research Center. (By title)

A branching Poisson process is a superposition of the events in a *main* Poisson process with parameter α, and the events in the sequence of *subsidiary* processes generated by the main events. Each subsidiary process contains S events, where S has generating function $\phi_S(z)$, the events being separated by independent identically distributed random variables Y_i. The Y_i have survivor function $R_Y(x) = \text{prob}\ (Y > x)$. Generalizations of the asymptotic results given in Lewis [*J. Roy Statist. Soc. Ser. B* (1964)] are obtained, in particular on the moments of the number of events in the process, and on the forward recurrence-time distribution. It is shown that the number of events in an interval $(0, x]$, suitably normalized, is asymptotically normal provided that $R_Y(x)$ is $o(x^{-\gamma})$, where $\gamma > \frac{1}{2}$. The process is generalized to include a random number of subsidiary processes running at the start of the process. Unique conditions on the number and structure of these initial subsidiary processes are given to make the generalized process an equilibrium branching Poisson process. (Received 12 June 1968.)

66. A system of Markov chains with random lifetimes. FREDERICK W. LEYSIEFFER, Florida State University.

Let $\{X(n), n \geqq 0\}$ be an irreducible Markov chain with stationary n-step probabilities $P_n(x, y)$. Assume $\{Y(n), n \geqq 0\}$ independent of $X(n)$, $Y(n) = 0$ or 1, $Y(n) = 1 \Rightarrow Y(n +$

$1) = 1$ and let $p(n) = P[Y(n) = 0]$, $\pi = \lim_{n\to\infty} p(n)$. Let $\{Z(n) \equiv (X(n), Y(n)), n \geqq 0\}$ be a two dimensional process with state space Λ. Limiting properties of a system of such random processes are investigated. At each integer time $n \geqq 0$, M_n particles enter Λ at $(0, 0)$. The M_n are iid Poisson λ, and independent of $X(k)$ and $Y(k)$. Also at time $n \geqq 1$, particles which entered the system at time k, $k = 0, 1, \cdots, n-1$, are allowed to move according to the probability law of $Z(n-k)$, each independently of the others. Let $B = \{(x, y) \mid x \neq 0, y = 0\} \subset \Lambda$ be finite, $C = \{x \mid (x, 0) \varepsilon B\}$, $A_n(B)$ be the number of particles in B at time n, $S_n(B) = \sum_{k=1}^n A_n(B)$ and $S(B) = \lim_{n\to\infty} S_n(B)$. THEOREM 1. *If either* (a) $\{X(n), n \geqq 0\}$ *is transient or* (b) $\{X(n), n \geqq 0\}$ *is persistent and* $p(k) = O(k^{-(1+\delta)})$ *for some* $\delta > 0$, then $P[\lim_{n\to\infty} S_n(B)/n = \lambda ES(B)] = 1$. THEOREM 2. $P[\lim_{n\to\infty} \{\sum_{m=1}^n A_m(B)/m\}/n = \lambda\pi u(C)] = 1$ *where* $u(x) = \lim_{n\to\infty} P_n(x, x)$, $u(C) = \sum_{x \varepsilon C} u(x)$. THEOREM 3. *If for some* $\delta > 0$, $k^{1+\delta} = O(EN_k^2(C))$ *for sufficiently large* k, *where* $N_k(C)$ *is the occupation time of* $X(n)$ *in* C *to time* k, *then* $\{S_n(B) - ES_n(B)\}/\text{Var } S_n(B)$ *is asymptotically* $\mathfrak{N}(0, 1)$. Analogous results for a different system were obtained by Port, *Ann. Math. Statist.* **37** (1966) 406–411. (Received 3 July 1968.)

67. Bayesian analysis of time truncated samples (preliminary report). ROBERT H. LOCHNER, and A. P. BASU, University of Wisconsin and IBM.

Suppose a sample of n units from a given life distribution, with failure time cdf $F(x)$, are put on test and the ith unit $(i = 1, \cdots, n)$ is removed from testing at min (X_i, t_i) where X_i is the failure time of the ith unit and t_i is a fixed truncation time. Then X_i is known only if $X_i < t_i$. Assume $F(0) = 0$. Setting $t^* = \max_i t_i$ we partition $(0, t^*)$ into m subintervals $\{(\tau_{i-1}, \tau_i)\}$, $(i = 1, \cdots, m)$, where $0 = \tau_0 < \tau_1 < \cdots < \tau_m = t^*$ and each t_i is equal to some τ_j. We set $p_i = F(\tau_i) - F(\tau_{i-1})$, $(i = 1, \cdots, m+1)$, $\tau_{m+1} = \infty$, and assign a Dirichlet prior to $\mathbf{p} = (p_1, \cdots, p_m)$. By suitably grouping the sample data, sufficient statistics for $\mathbf{p}$ are obtained. The joint posterior distribution of $\mathbf{p}$ and the marginal posterior distribution of $F(\tau_i)$, $(1 \leqq i \leqq m)$, are obtained. Methods for selecting m, $(\tau_1, \cdots, \tau_{m-1})$ and the parameters of the prior distribution are discussed. The robustness of the posterior distribution of $F(\tau_i)$ under variation in the prior parameters is studied. (Received 24 June 1968.)

68. The asymptotic behavior of a certain Markov chain with non-stationary transition probabilities. BERNARD MC CABE, Bellcomm Inc.

The process studied is a model for a certain kind of growth process. It is defined recursively as follows: $S_1 = 1$, $S_{n+1} = S_n + X_n(n + 1 - S_n)$, where $\{X_n\}$ is a sequence of independent random variables concentrated on the interval $[0, 1]$ each having the same mean m, $m(1-m) \neq 0$. This represents a model for a growth process attracted to a "saturation level" which it cannot exceed, and the amount of attraction is governed by the distribution of the X_n. The basic properties of this process have been analyzed and the following two results are fundamental: THEOREM A. S_n/n *converges to* 1, a.e.; and THEOREM B. *The sequence* $\{\text{Var } S_n\}$ *is bounded; moreover if* $E(X_n^2) \equiv m_2$, *then* $\lim \text{Var } S_n$ *exists and is equal to* $(m_2 - m^2)/m^2(2m - m_2)$. Theorem A follows readily from Theorem B, and the latter is proven by analyzing a difference equation satisfied by the sequence Var S_n. (Received 17 June 1968.)

69. Investigations on general incomplete multiresponse designs. L. L. MC DONALD, Colorado State University. (By title)

A general incomplete multiresponse (GIM) design [see Srivastava, *Ann. Inst. Statist. Math.* (1967)] has u sets $(S_1, \cdots, S_u)$ of experimental units where the subset $R_i = \{V_{l_{i1}}, \cdots, V_{l_{ip_i}}\}$ of the p responses $\{V_1, \cdots, V_p\}$ is measured on the units of S_i. Let $U_r =$

$\{S_i : V_r$ is measured on S_i, $i = 1, \cdots, u\}$. If in a GIM design there exists a permutation of $(1, \cdots, p)$, say $(r_1, \cdots, r_p)$, such that $U_{r_1} \supseteq \cdots \supseteq U_{r_p}$, the design is called hierarchical multiresponse (HM). Here the orthogonality and regularity [see Srivastava, *Ann. Math. Statist.* (1968)] of certain HM and GIM designs is considered. Suppose the sets S_i $(i = 1, \cdots, u)$ form BIB designs with parameters $(v, b_i, r_i, k_i, \lambda_i)$. For several cases, where there is "balance" wrt the responses, the incomplete multiresponse (IM) design is shown, in this paper, to be regular if each response combination, R_i, included is measured on two sets, S_i and $S_{i'}$ (say), where $\lambda_i/k_i = \lambda_{i'}/k_{i'}$. Assuming the sets S_i form randomized block designs with $2b_i$ blocks and v treatments, the IM design is shown to be regular irrespective of the pattern of incomplete responses. When the sets S_i form connected block-treatment designs with $v(\geqq 2)$ treatments the IM design is not orthogonal ($\perp$). The desirability of $\perp$ designs is considered and examples given where, wrt the "trace criterion", $\perp$ designs are undesirable. Hierarchical multiresponse response surface (HMRS) designs are defined and for $p = 2$, n. and s. c. are given such that the HMRS design is $\perp$. Finally, a procedure for construction of $\perp$ HMRS designs (with p arbitrary) is given. (Received 16 July 1968.)

70. On the probability of trivariate Student's *t* distribution. S. A. Patil and J. L. Kovner, Rocky Mountain Forest and Range Experiment Station.

The trivariate t density considered is given by

$$f(u_1, u_2, u_3) = \Gamma((n+3)/2)/[\Gamma(n/2)(n\pi)^{3/2}(1-\rho^2)^{1/2}]^{-1}[1 + n^{-1}((u_1^2 - 2\rho u_1 + u_2^2)(1-\rho^2)^{-1}) + u_3^2]^{-(n+3)/2}, \qquad -\infty < u_i < \infty \quad i = 1, 2, 3.$$

The density is expressed in the powers of n^{-1} by Taylor series. The coefficients of the terms of the series are of the form $\exp\,(-\frac{1}{2}[(u_1^2 - 2\rho u_1 u_2 + u_2^2)\ (1-\rho^2)^{-1}) + u_3^2]\}u_3^k((u_1^2 - 2\rho u_1 u_2 + u_2^2)\ (1-\rho^2)^{-1})^l$, $k, l \geqq 0$. Each variable of the terms of the series is integrated from $-d$ to d. The integrals are expressed in terms of the probabilities of chi-square distribution. The integrated terms can be added to construct the tables of trivariate t for selected values of ρ, d and various values of n. The convergence is slow for small n, say, $n < 10$, powers up to n^{-5} or more may be required for desirable tabular precision. The method can also be used for finding probabilities of $-d_i \leqq U_i \leqq d_i$ $i = 1, 2, 3$. The expressions for probabilities are considerably simpler if one is interested in getting probabilities of $-\infty < U_i \leqq -d$, $i = 1, 2, 3$. One application of the tables is to find the linear segment confidence intervals for the difference line of two simple linear regressions with a common independent variable. (Received 3 July 1968.)

71. A robust point estimator in a generalized regression model. P. V. Rao and J. I. Thornby, University of Florida.

In this paper an Hodges-Lehman type estimator is given for the parameter θ in the model $Y_j = \tau + g_j(\theta) + Z_j$, $j = 1, 2, \cdots, N$, where τ and θ are unknown parameters, $g_1, g_2, \cdots, g_N$ are real valued functions of real variables satisfying suitable conditions and $Z_1, Z_2, \cdots, Z_N$ are independent identically distributed random variables having a distribution function belonging to a specified class. Some special cases of the above model having practical significance are obtained by taking (i) $g_j(\theta) = \theta x_j$, (ii) $g_j(\theta) = K|x_j - \theta|$, (iii) $g_j(\theta) = K \log(1 + \theta + x_j)$, and (iv) $g_j(\theta) = K\theta^{x_j}$, $0 < \theta < 1$, where $x_1, x_2, \cdots, x_N$ and K are assumed to be known. Some small sample (e.g. median unbiasedness) and large sample (e.g. asymptotic normality) properties of the proposed estimator are also given in the paper. (Received 5 July 1968.)

72. A distributional analysis of linear programs under risk. J. K. SENGUPTA and B. C. SANYAL, Iowa State University.

The sensitivity of static linear programming solutions is analyzed here empirically through a set of nonlinear programs which are so constructed as to reflect the distributional characteristics of net prices in the objective function. The implications of substituting the two extremes rather than the average of sample values and also the second best and the third best optimal solutions besides the first best one are discussed in this connection in order to evolve an approach towards range analysis for the optimal solutions of a static linear program arising in a problem of resource allocation for an optimal crop-mix. This paper deals essentially with applied aspects of probabilistic programming and extends the ideas contained in the papers "Sensitivity Analysis Methods for a Crop-Mix Problem" (to be published in the *J. Amer. Statist. Assoc.*) and "A Fractile Programming Approach with Extreme Sample Estimates of Parameters" (to be presented in the winter meetings of the Econometric Society to be held in Chicago) by the same authors. (Received 15 July 1968.)

73. Asymptotic normality of linear combinations of functions of order statistics. GALEN SHORACK, University of Washington.

Let $X_1, \cdots, X_N$ be iid rv's, which by the probability integral transformation may be assumed to have a uniform $(0, 1)$ distribution. Let $T_N = N^{-1}\sum_{i=1}^{N} c_{Ni}g(X_{Ni})$ where $X_{N1} \leqq \cdots \leqq X_{NN}$ are the ordered $X_1, \cdots, X_N$, g is a specified function and c_{Ni}'s are specified constants. Let $L_N(t) = N^{1/2}[g(F_N^{-1}(t)) - g(t)]$ for $0 < t < 1$ where F_N is the empirical df. Theorems yielding asymptotic normality of $T_N^* = N^{1/2}(T_N - \mu_N)$, for appropriate location constants μ_N, are derived. Conditions are given under which the L_N-processes may be replaced by equivalent $\tilde{L}_N$-processes which converge to a limiting L_0-process in the sense that

$$(*) \qquad \|\tilde{L}_N - L_0\|_\nu \to_p 0 \text{ as } N \to \infty$$

where $\|f\|_\nu = \int_0^1 |f| \, d|\nu|$ and $|\nu|$ is the total variation measure of an appropriate signed measure ν. It is then shown that $T_N^* = \Phi(L_N) + \theta_N$ where Φ is an appropriate integral functional and $\theta_N \to_p 0$ as $N \to \infty$; and this is used in connection with (*) to show that T_N^* is asymptotically distributed as $\Phi(L_0)$. $\Phi(L_0)$ is then recognized as having a particular normal distribution. (Received 3 July 1968.)

74. Some contributions to the theory of construction of strata. RAVINDRA SINGH and B. V. SUKHATME, Punjab University and Iowa State University.

In most of the theoretical investigations of the problem of optimum stratification, both the estimation variable and the stratification variable are taken to be the same. Since the distribution of the estimation variable Y is generally not known, it is desirable to stratify on the basis of some auxiliary variable X. This paper considers the problem of optimum stratification on the basis of some suitably chosen auxiliary variable X having a finite range when the form of the regression of Y on X and the conditional variance $V(Y \mid X = x)$ are known. Under this setup, minimal equations giving optimum points of stratification have been obtained using various types of estimates such as simple mean estimate, ratio estimate and regression estimate in the case of simple random sampling. Since these equations cannot be solved exactly, various approximations to the exact solutions have been obtained.

The asymptotic equivalence of the approximate and the exact solutions has been established. Suitable modifications for the case when the variable X has infinite range have been suggested. The results have also been extended to the case of sampling with varying probabilities. Finally, the paper gives certain numerical illustrations in the case of simple random sampling. (Received 22 June 1968.)

75. Some density functions of a counting variable in the simple random walk. JOHN SLIVKA, National Cancer Institute.

Let $\langle X_i ; i = 1, 2, \cdots \rangle$ be a sequence of completely independent and identically distributed random variables such that $X_i = +1, -1$ with probabilities p $(0 < p < 1)$ and $q = 1 - p$ respectively and let $\langle S_n ; n = 1, 2, \cdots \rangle$ be the corresponding sequence of partial sums, i.e. $S_n = \sum_1^n X_i$. Let the indicator variables $Y_n = Y_n(p, \lambda)$, $n = 1, 2, \cdots$, be defined by $Y_n = 1$ if $S_n > (\mu + \lambda)n$ and $Y_n = 0$ otherwise, where $\mu = 2p - 1$ is the mean of X_i and λ is a positive number less than 2. The counting variable of interest, $N = N(p, \lambda)$, is defined by $N = \sum_1^\infty Y_n$. By the strong law of large numbers, $P\{N < \infty\} = 1$. THEOREM. *If* $\mu + \lambda = k/(k + 2)$, $k = 1, 2, \cdots$, *then* $P\{N = 0\} = 1 - x$ *and* $P\{N = j\} = (1 - x)j^{-1}\sum_{i=0}^{s} (j - [k + 2]i)\binom{j}{i}p^{j-i}q^i, j = 1, 2, \cdots$, *where* x *is the unique root of* $qy^{k+2} - y + p$ *such that* $0 < x < 1$ *and* s *is the greatest integer not exceeding* $j/(k + 2)$. The proof employs time-honored developments by E. S. Andersen in fluctuation theory, A. Dvoretzky and Th. Motzkin in ballot problems, and W. Feller in the theory of recurrent events. (Received 5 July 1968.)

76. The factorial subassembly association scheme and the construction of multidimensional partially balanced designs. J. N. SRIVASTAVA and D. A. ANDERSON, Colorado State University and University of Wyoming.

For $k = 0, 1, \cdots, m$, define $S_{mk} = \{\omega \mid \omega = a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_k}^{j_k} ; 1 \leqq i_1 < i_2 < \cdots < i_k \leqq m; j_u = 0 \text{ or } 1; u = 1, 2, \cdots, k\}$, and the set of unordered pairs $S_{mk}^* = \{(\omega, \bar{\omega}) \mid \omega \varepsilon S_{mk}$, and $\bar{\omega}$ is obtained from ω by interchanging 0's and 1's in the superscripts$\}$. The elements of S_{mk} are called subassemblies of order k from a 2^m factorial; the a_u $(u = 1, 2, \cdots, m)$ being the factors, and j_u denoting the levels. An element $\rho \varepsilon S_{mi}$ is said to be an (α_1, α_2) associate of $\omega \varepsilon S_{mk}$ if ω and ρ have exactly α_1 factors in common, and of these α_1 factors exactly α_2 have common levels. It is obvious that $\bar{\rho}$ is then an $(\alpha_1, \alpha_1 - \alpha_2)$ associate of ω. An element $(\rho, \bar{\rho})$ is said to be an (α_1, α_2^*) associate of $\omega \varepsilon S_{mk}$, if ρ and $\bar{\rho}$ have α_1 factors in common with ω, and among these α_1 factors, either ρ or $\bar{\rho}$ has $\alpha_2^* = \max[\alpha_2, \alpha_1 - \alpha_2]$ common levels. Finally $(\rho, \bar{\rho})$ is said to be an (α_1, α_2^*) associate of $(\omega, \bar{\omega}) \varepsilon S_{mk}^*$ if it is an (α_1, α_2^*) associate of $\omega \varepsilon S_{mk}$. In this paper, the above scheme is shown to be a multidimensional partially balanced (MDPB) association scheme, (Bose and Srivastava, *Sankhyā*, (1964)), and all parameters of the scheme are calculated. Methods of construction of MDPB designs using this scheme are discussed. For example, let Σ_1, Σ_2, and Σ_3, be any three sets, distinct or not, on and between which a MDPB scheme is defined. Let $T(\alpha, \beta, \gamma)$ be the set of all assemblies (x_1, x_2, x_3), where $x_i \varepsilon \Sigma_i$ $(i = 1, 2, 3)$, and (x_1, x_2), (x_1, x_3) and (x_2, x_3) are respectively αth, βth and γth associates. Then it is known that T is a MDPB design. (Received 5 July 1968.)

77. Estimation of probability density function based on random number of observations with applications. R. C. SRIVASTAVA, Ohio State University and Stanford University.

In this paper we deal with the problem of estimating the probability density function of a random variable based on a random number of observations. The asymptotic properties of the estimate, of the type suggested by Parzen (*Ann. Math. Statist.* **33** (1962), 1065–1076.) are studied. Two cases are considered separately. First we discuss the case when the sample size N is independent of observations $X_1, \cdots, X_N$ and then consider the second case when the sample size may depend on the observations. These results are applied to estimate (i), the probability density function of the velocity distribution of vehicles on roads, (ii), the reliability and the hazard rate and, (iii) the probability density function of the inter-arrival time and the service time in some queuing models. (Received 5 July 1968.)

78. The use of random allocation for the control of selection bias. STEPHEN M. STIGLER, University of Wisconsin.

In the problem of comparing two treatments, it may be true that, as in many clinical trials, the suitable subjects arrive sequentially and must be treated at once. In such situations, if the experiment calls for fixed treatment numbers, the experimenter can, using his knowledge of the number of treatments left to be assigned, bias the experiment by his selection of subjects. If we consider the method of assigning treatments as an experimental design, Blackwell and Hodges (*Ann. Math. Statist.* **28** 449–460) have shown that the minimax design is the truncated binomial. In this paper we show that random allocation is a restricted Bayes design within the class of Markov designs, and is in many senses preferable to the minimax design. In particular, it is possible for the random allocation design to eliminate the bias asymptotically when the minimax design does not, and in no case will random allocation have a much worse performance than the minimax. (Received 5 July 1968.)

79. A general Skorohod space and its applications to the weak convergence of multidimensional stochastic processes (preliminary report). MIRON L. STRAF, University of Chicago.

Stochastic processes with jump discontinuities, e.g., an empirical distribution function, may be handled by treating them as random elements of the space $D[0, 1]$ of all functions on the unit interval with discontinuities of at most the first kind. However, the required measurability of important empirical processes with respect to the Borel sets of $D[0, 1]$ precludes us from using the uniform norm on this space. An ingenious topology invented by Skorohod satisfies our measurability requirements and makes $D[0, 1]$ a topologically complete separable metric space so that Prohorov's theorem characterizing the relative weak compactness of a family of probability measures on such spaces may be applied. (Cf. Billingsley, *Weak Convergence of Probability Measures*, J. Wiley 1968.) Motivated by the Skorohod topology, we present a space $D(T)$ of functions with possible jumps on an arbitrary topological space T and a natural collection of metrics for $D(T)$. Exploring the relation between these metrics, we determine which ones make the function space separable and complete, and characterize relative weak compactness of probability measures on $D(T)$. When $T = [0, 1]$, our work reduces to results known for $D[0, 1]$; when $T \subset R^k$, this study provides a method for the analysis of multidimensional stochastic processes. (Received 3 July 1968.)

80. Generating functions on idempotent semigroups with applications to combinatorial analysis. MELVIN TAINITER, Brookhaven National Laboratory.

It has been shown in earlier work that many combinatorial problems can be treated analytically by considering the algebraic structure of the set on which one is counting. The main tool in this analysis is the "incidence algebra" over a partially ordered set and two of its elements, the zeta function and its inverse the Möbius function. When the partially ordered set is a commutative semigroup of idempotents (semilattice) the zeta function is shown to be multiplicative. We exploit the multiplicativity of the zeta function to define a generating function which converts semigroup convolutions to ordinary multiplication. Examples and applications of the approach are given. (Received 15 July 1968.)

81. On the class of uniformly minimum variance unbiased estimators when the class of probability distributions has a finite rank. KEI TAKEUCHI, Courant Institute of Mathematical Sciences of New York University and University of Tokyo.

R. R. Bahadur (*Sankhyā* (1957)) proved that the class of all UMV estimators is equal to the class of all finite variance functions measurable with respect to some subfield S_0 of the field S reduced by the minimum sufficient statistic. We shall discuss the case when the class of probability distributions has only p linearly independent elements. Then S_0 is finite, and has at most p disjoint sets. For any $A \varepsilon S$, $P(A)$ be defined by the number of linearly independent conditional distributions given A, then it is proved that $A \varepsilon S_0$ if and only if $P(A) + P(\bar{A}) = p$. If the sample space consists of finite points, explicit criterion for the partition S_0 is given. It is applied to the case of multinomial distribution, and the class S_0 is explicitly given for several situations. (Received 3 July 1968.)

82. Asymptotic efficiency of ranking procedures. YUNG LIANG TONG, University of Nebraska. (By title)

The idea of "good" statistic for ranking and selection problems in terms of its asymptotic efficiency is discussed. Let $\mathfrak{F}$ denote a class of statistics and let t, t' be elements of $\mathfrak{F}$, n_1, n_2 be the sample sizes needed to achieve the probability requirement when the decision is made based on t or t' in a ranking and selection problem. The asymptotic relative efficiency of t wrt t' is defined by eff $(t, t') = \lim_{P^* \to 1} n_2/n_1$ where P^* is the probability of correction selection. A statistic $t \varepsilon \mathfrak{F}$ is said to be asymptotically relatively efficient (ARE) in $\mathfrak{F}$ if eff $(t, t') \geqq 1$ for every $t' \varepsilon \mathfrak{F}$. It is shown that the "good" ranking statistic in a large class $\mathfrak{F}$ which contains statistics with asymptotically normal distributions can be found. Under certain regularity conditions, an asymptotically efficient estimator for the one-sample estimation problem is ARE in $\mathfrak{F}$ for the k-sample ranking and selection problem. In particular, for location parameter families ranking procedures based on the maximum likelihood estimator (if it exists) or the "best" linear combination of order statistics given by Chernoff, Gastwirth and Johns [*Ann. Math. Statist.* **38** 52–72] are ARE in $\mathfrak{F}$. The means procedure (based on the sample means) considered by Lehmann [*Math. Annalen* **150** 268–275] may not be ARE in $\mathfrak{F}$. Examples are given. (Received 24 June 1968.)

83. An equivalence between decision procedures. B. J. TRAWINSKI, University of Alabama in Birmingham.

The problem of selecting a subset (whose size is a random variable, say $\mathfrak{n}$) from k populations, so that it contains the superior one with probability not less than a preassigned positive number P^*, has been discussed in many settings without reference to the distribution of $\mathfrak{n}$. It is the purpose of this work to deduce properties of this distribution from those of the *multiple decision procedure* for selecting a subset, and to investigate the relationship of the procedure to that of hypothesis testing (at admissible configurations of the population parameters). In terms respectively of the expected value of $\mathfrak{n}$ and the power of the test, the multiple decision procedure and hypothesis testing (of the parameters involved) are under certain conditions found to be asymptotically (in the sample size n) equivalent. This equivalence is discussed in particular in relation to tests whose size $\alpha_n \to 0$ as $n \to \infty$. (Received 5 July 1968.)

84. Inference with tested priors. V. B. WAIKAR and S. K. KATTI, Florida State University.

This paper considers the problem of estimating the unknown mean μ of a distribution when we have some prior knowledge about the mean in the form of an initial estimate μ_0. The proposed two-stage estimation scheme is as follows: Obtain a sample of size n_1 first and using μ_0, construct a region R in the space of $\bar{X}_1$. If $\bar{X}_1 \varepsilon R$, take $\hat{\mu} = \bar{X}_1$, but if $\bar{X}_1 \varepsilon \bar{R}$, obtain another sample of size n_2 and take $\hat{\mu} = (n_1\bar{X}_1 + n_2\bar{X}_2)/(n_1 + n_2)$. For a multivariate population with known Σ, it was shown that the optimal R which minimizes the generalized mean squared error of $\hat{\mu}$ when μ_0 is the true mean (determinant of $E[(\hat{\mu} - \mu_0)(\hat{\mu} - \mu_0)']$) is an ellipsoid in the space of $\bar{X}_1$. For the multivariate normal case a comparison of $\hat{\mu}$ with the fixed sample mean $\bar{X}$ based on an equivalent sample size shows that $\hat{\mu}$ is better when μ_0 is the true mean. Next for the univariate populations, the consequences of the departure of μ_0 from the true mean are explored and to represent these consequences in a concise way the weighted mean squared error of $\hat{\mu}$ is considered. Further a comparison of $\hat{\mu}$ with the Bayes' estimate based on an equivalent sample size shows that under certain conditions $\hat{\mu}$ is better. (Received 21 June 1968.)

85. The first passage time distribution of Brownian motion with positive drift. M. T. WASAN and L. K. ROY, Queen's University, Kingston, Ontario.

Some results concerning the sampling distribution are obtained. Then an admissible minimax estimate of the parameter $1/\lambda$ is discussed, and compared to mean unbiased and maximum likelihood estimates of the same parameter. An empirical Bayes estimate for $E(\lambda \mid x)$ is also presented. Next we consider results concerning the acceptance region of the uniformly most powerful test of the hypothesis $1/\lambda = 1/\lambda_0$, and lastly an example is given to demonstrate how to detect a sample outlier. (Received 14 June 1968.)

86. Confidence limits for the distribution function based on the first two order statistics. JOHN S. WHITE, University of Minnesota.

We consider the analysis of a life testing experiment in which n similar units are run until two have failed at which time the experiment is censored. We assume that the underlying distribution of failure times is of the form Prob (failure time $< x$) $= G(x) = F((x - m)/s)$ where the functional form of F is known and m and s are unknown parameters. We wish to obtain an upper confidence limit for the distribution function G at a specific time t. If x and y are the values taken on by the first two order statistics X and Y, then there exists a real number b such that $(1 - b)x + by = t$ and we may consider t as the value taken on by the random variable $T = (1 - b)X + bY$. A $100\,c\%$ confidence limit for $G(t)$ is $G(t) < p$, where prob $(G(T) < p) = c$. It is shown that, for b positive and $p = F(z)$, p is the solution of

$$(*) \qquad c = 1 - (1 - p)^n - n\textstyle\int_{-\infty}^{z} (1 - F((z - (1 - b)x)/b)^{n-1}\, dF(x).$$

For $F(x) = 1 - \exp(-\exp(x))$ a graphical technique for obtaining confidence intervals has been obtained earlier (J. S. White, Transactions of the 1967 technical conference of the American Society for Quality Control). (Received 24 June 1968.)

87. A consistent estimator for the identification of finite mixtures. SIDNEY J. YAKOWITZ, University of Arizona.

H. Teicher has initiated a valuable study on the identifiability of finite mixtures; he and others have proven that many of the popular families of distribution functions do generate identifiable mixtures. (Yakowitz and Spragins, On the identifiability of finite mixtures, *Ann. Math. Statist.* **39** 209–214, gives a review of the status of the theory and a bibliography.) The methods thus far used to prove that various families generate identifiable finite mixtures are not constructive. The purpose of the present paper is to reveal a method for constructing an estimator for the unknown mixing distribution function. The estimator is proven to be consistent for all one-dimensional families listed in the literature as generating identifiable finite mixtures. The analysis reveals some facts about the mathematical structure of the identifiability problem; the results may have relevance to certain problems in the domain of empirical Bayes methods. (Received 14 June 1968.)

88. Ranking and rank correlation (preliminary report). T. YANAGIMOTO and M. OKAMOTO. The Institute of Statistical Mathematics, Tokyo and Iowa State University.

For fixed n let Π be the set of all permutations $\pi = (i_1, \cdots, i_n)$ of $(1, \cdots, n)$. A partial order can be defined in Π as follows. Given two elements $\pi = (i_1, \cdots, i_n)$ and $\pi' = (j_1, \cdots, j_n)$ of Π we write $\pi \rightarrow \pi'$ iff there exists h $(1 \leqq h < n)$ such that $i_h = j_{h+1} < i_{h+1} = j_h$ and $i_k = j_k$ for $k \neq h, h+1$. Then we define $\pi \geqq \pi'$ iff there exists a chain $\pi = \pi_0 \rightarrow \pi_1 \rightarrow \cdots \rightarrow \pi_m = \pi'$. THEOREM 1. *$\Pi$ is a lattice. Now a real-valued function ψ defined over Π is called non-decreasing* iff *$\pi \geqq \pi'$ implies $\psi(\pi) \geqq \psi(\pi')$. For any integer k and real numbers $x_1, \cdots, x_n, y_1, \cdots, y_n$ we denote by i_k the rank of x_h among x_i's, assuming that the rank of y_h among y_i's is k.* THEOREM 2. *Let $(x_1, y_1), \cdots, (x_n, y_n)$ be a random sample from bivariate normal distribution with correlation coefficient ρ. If ψ is a non-decreasing function over Π, then the statistic $R = \psi(i_1, \cdots, i_n)$ is stochastically non-decreasing in ρ.* Both rank correlations, Kendall's τ and Spearman's ρ, are examples of Theorem 2. (Received 24 June 1968.)

89. Bayes invariant estimators of the common mean of two normal distributions based on small samples of equal size. S. ZACKS, University of New Mexico.

Let $X_1, \cdots, X_n$ be iid random variables having a $\mathfrak{N}(\mu, \sigma_1^2)$ distribution; $-\infty < \mu < \infty$, $0 < \sigma_1^2 < \infty$. Let $Y_1, \cdots, Y_n$ be iid random variables having a $\mathfrak{N}(\mu, \sigma_2^2)$ distribution; $0 < \sigma_2^2 < \infty$. The X's and the Y's are mutually independent, and the ratio σ_2^2/σ_1^2 is unknown. We consider the problem of estimating the common mean μ. Let $(\bar{X}, \bar{Y}, S_1, S_2)$ be the minimal sufficient statistic, where $\bar{X}$ and $\bar{Y}$ are the sample means and S_1, S_2 are the sample ssd. Let $Z_i = S_i/(\bar{Y} - \bar{X})^2$, $i = 1, 2$. Any translation invariant and scale preserving estimator of μ, based on $(\bar{X}, \bar{Y}, S_1, S_2)$ is of the form: $\hat{\mu}_\psi = \bar{X} + (\bar{Y} - \bar{X})\psi(Z_1, Z_2)$, where $\psi(\cdot, \cdot)$ is an appropriately measurable function. We consider the problem of choosing $\psi(Z_1, Z_2)$ in a Bayesian decision framework, for square error loss functions. The optimal choice of $\psi(Z_1, Z_2)$ for a given prior distribution of $\rho = \sigma_2^2/\sigma_1^2$, satisfying certain conditions, is generally expressed as a ratio of two integrals. Special cases are worked out numerically for small values of n and the efficiency of the corresponding Bayes invariant estimators is investigated. We also show that whenever the samples size n is odd, $\hat{\mu}_\psi$ is unbiased for all integrable ψ. (Received 12 July 1968.)

(Abstracts of papers presented at the European meeting, Amsterdam, Netherlands, September 2–7, 1968. Additional abstracts appeared in earlier issues.)

10. Some results on the sampling distribution of the mean difference. GIOVANNI GIRONE, Università de Baria.

Let $X_1, \cdots, X_n$ be a random sample from a continuous distribution function $F(x)$. Let $X_{(1)}, \cdots, X_{(n)}$ denote the order statistics of the sample. Alternative formulas for

Gini's mean difference Δ are $\Delta = 2\sum_1^{n-1} i(n-i)[X_{(i+1)} - X_{(i)}]/n(n-1) = 2\sum_1^n (2i - n - 1)X_{(i)}/n(n-1) = 2\sum_1^{n/2} (n - 2i + 1)[X_{(n-i+1)} - X_{(i)}]/n(n-1)$. (A) Firstly, let the sample come from the exponential distribution, then the random variables $Y_i = 2[X_{(i+1)} - X_{(i)}]/n(n-1)$ are independent exponentially distributed random variables and Δ is their convolution. Particularizing a problem of Feller on the sampling distribution of the convolution of n independent exponentially distributed random variables I derived the distribution function of Δ:

$$G(\Delta) = 1 - \sum_1^{n-1} i^{n-2}(-1)^{n-i-1}[(i-1)!(n-i-1)!]^{-1} \exp\{-\alpha n(n-1)\Delta(2i)^{-1}\}, \quad \Delta > 0.$$

By the same way I derived easily the moments of Δ. (B) Let now the sample come from the rectangular distribution. By a direct method, but with very cumbersome calculus I derived the sampling distribution of Δ given by the third formula for $n = 2, 3, \cdots, 10$. (C) Finally, using recent results by Jung, Govindarajulu, Chernoff and others, Moore, concerning the asymptotic normality of linear functions of order statistics when the weight function has four bounded derivatives, I have shown that the asymptotic distribution of Δ is normal with mean and variance depending on the form of the continuous distribution function $F(x)$ of X_i's. (Received 12 August 1968.)

11. Bivariate Laplace I laws of errors, applications to bombing problems.

ANDRE G. LAURENT, Wayne State University. (By title)

Univariate df $f(\xi)$ are generalized to bivariate df $f(x)$, $x' = (x_1, x_2)$, by the conditions that $f(x)$ be "elliptic," i.e., $f(x) = p(x'\Sigma^{-1}x)$, Σ positive definite, and that (1) the df of the projections (U, X), or (2) the conditional df of X_2 given x_2/x_1 (signed radial error given the azimuth) belong to an (admissible) hypothetized class C. In this paper the case when C is the class of Laplace I distributions is studied. Several expansions are given for the df of $X'X$ (squared radial error). Applications to bombing problems are stressed and discussed. The case when C is a subclass of unfolded incomplete gamma distributions is also considered. (Received 12 July 1968.)

12. On a new family of tests for multivariate analysis.

J. N. SRIVASTAVA, Colorado State University.

For many general classes of problems in multivariate analysis, the test procedure is based on the roots $c_1 \geqq c_2 \geqq \cdots \geqq c_p(\geqq 0)$ of a certain matrix T. For example, in multivariate analysis of variance, $T = S_h S_e^{-1}$, where as usual S_h and S_e are respectively the sum of products matrices for the hypothesis and error. Similarly, in the problem of independence between two sets of variates, $T = S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where S_{ij} $(i, j = 1, 2)$ is the matrix of covariances between the ith and jth set of variates; and so on. Various tests based on the c's are well known. In this paper, the following new family of tests is introduced. Let μ's be real numbers satisfying $\infty \geqq \mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_p \geqq 0$. Then the test region is $(c_1 \leqq \mu_1,\ c_1 + c_2 \leqq \mu_1 + \mu_2, \cdots, c_1 + c_2 + \cdots + c_i \leqq \mu_1 + \cdots + \mu_i, \cdots, c_1 + c_2 + \cdots + c_p \leqq \mu_1 + \cdots + \mu_p)$. The whole family is obtained by considering different sets of values of the μ's subject to the conditions indicated. Under certain very mild conditions on the μ's, tests belonging to this family are shown to correspond *uniquely* to tests based on the set of all symmetric gauge functions of $c_1^{\frac{1}{2}}, \cdots, c_p^{\frac{1}{2}}$. Simultaneous confidence bounds arising from these are discussed. The monotonicity property (wrt each individual noncentrality parameter) of such tests, and also their admissibility (with appropriate restrictions) is proved. Many other important features are also pointed out. (Received 10 July 1968.)

13. On the cost-wise optimality of hierarchical multiresponse designs in the class of general incomplete multiresponse designs. L. L. Mc Donald and J. N. Srivastava, Colorado State University. (By title)

A general incomplete multiresponse (GIM) design has u sets $(S_1, \cdots, S_u)$ of experimental units where the subset $(V_{l_{i1}}, \cdots, V_{l_{ip_i}})$ of the p responses $(V_1, \cdots, V_p)$ is measured on the units of S_i [see, e.g., Srivastava, *Ann. Inst. Statist. Math.* (1967)]. Let $U_r = \{S_i(i = 1, \cdots, u): V_r \text{ is measured on } S_i\}$. If in a GIM design there exists a permutation of $(1, \cdots, p)$, say $(r_1, \cdots, r_p)$, such that $U_{r_1} \supseteq \cdots \supseteq U_{r_p}$, the design is called hierarchical multiresponse (HM). Assume each combination of the responses is measured on a (possibly null) set S_i $(i = 1, \cdots, (2^p - 1))$, where S_i possesses a randomized block design with b_i (say) blocks and v treatments. Let Δ denote this class of GIM designs (containing Δ^*, its subclass of HM designs). Given $D \varepsilon \Delta$, let Σ_d denote the dispersion matrix of the piecewise estimators [see the above paper] of a set of normalized orthogonal contrasts of the treatment parameters. Let Δ_c^* and $(\Delta - \Delta^*)_c$ denote respectively the designs of Δ^* and $(\Delta - \Delta^*)$ which satisfy the "cost" restriction, $c \geqq c_0 \sum_{i=1}^{2^p-1} b_i + \sum_{r=1}^{p} (c_r \sum_{i \varepsilon U_r} b_i)$, where c is the total capital available, c_0 the initial cost for one block, and c_r the cost of measuring V_r on all units in one block. In this paper, it is shown that, given $D \varepsilon (\Delta - \Delta^*)_c$, there exists a HM design, $D^* \varepsilon \Delta_c^*$, such that trace $(\Sigma_d) \geqq$ trace (Σ_{d^*}). Formulae (for $p = 2, 3$) (involving the ratios of the variances of the responses) are given for the b_i's of the optimum HM design in Δ_c^*, and estimation procedures (for these ratios) are proposed assuming that data from a "pilot" experiment is available. For $p = 2$, similar results are established, if the units in S_i $(i = 1, 2, 3)$ are arranged in certain cyclic PBIB designs (instead of randomized block designs). (Received 26 July 1968.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. Density estimation by orthogonal series. Geoffrey S. Watson, Johns Hopkins University.

Given a random sample $x_1 \cdots x_n$ from density $f(x) = \sum \alpha_m \varphi_m(x)$ where $\{\varphi_m(x)\}$ is an orthogonal basis, the estimator $f_n^*(x) = \sum \lambda_m(n) a_m \varphi_m(x)$ where $a_m = n^{-1} \sum_{k=1}^{n} \varphi_m(x_k)$ is suggested. $f_n^*(x)$ will be a minimum integrated mean square error estimator in its class if $\lambda_m(n) = \alpha_m^2 \{\alpha_m^2 + \text{var}(\varphi_m(x)) n^{-1}\}^{-1}$. These estimators are related to the kernel estimators discussed by Watson and Leadbetter (*Ann. Math. Statist.* **34** (1963) 480–491). (Received 7 June 1968.)

2. On an extended compound decision problem. Dennis C. Gilliland and James F. Hannan, Michigan State University.

Consider a decision problem based on $(x_1, \cdots, x_k) \equiv \mathbf{x}^k$ with distribution $\times_1^k P_i$, a loss function L depending on $\mathbf{P}^k$ only through P_k, decision functions ϕ, risk functions $R_k(\mathbf{P}^k, \phi) = \int L(P_k, \phi(\mathbf{x}^k))\, d(\times_1^k P_i)$, prior distributions G^k on $\mathbf{P}^k$ and Bayes envelope $R_k(G^k) = \inf_\phi \int R_k(\mathbf{P}^k, \phi)\, dG^k$. For $n \geqq k$, let G_n^k be the empirical distribution of the k-tuples $(P_1, \cdots, P_k), (P_2, \cdots, P_{k+1}), \cdots, (P_{n-k+1}, \cdots, P_n)$ and let ψ_n^k be Bayes wrt G_n^k. The following theorems extend some results of Swain [Stanford Tech. Report No. 81 (1965)] for squared error loss estimation components and Johns [*Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** (1967) 463–478] for two action components. Theorem 1. $(n - k)R_{k+1}(G_n^{k+1}) \leqq (n - k + 1)R_k(G_n^k)$. Theorem 2. $\sum_k^n R_k((P_{i-k+1}, \cdots, P_i), \psi_i^k) \leqq (n - k + 1)R_k(G_n^k) \leqq \sum_k^n R_k((P_{i-k+1}, \cdots, P_i), \psi_{i-1}^k)$. Theorem 3. *If G^∞ is strictly stationary with marginal G^k on P^k and ϕ_i is $\mathbf{x}^i$ measurable for each i and such that*

$n^{-1}\sum_1^n R_i(\mathbf{P}^i, \phi_i) \leqq R_k(G_n{}^k) + o(1)$ *uniformly in* $\mathbf{P}^n$, *then* $\limsup n^{-1}\sum_1^n \int R_i(\mathbf{P}^i, \phi_i)\, dG^\infty \leqq R_k(G^k)$. *If, in addition,* $\mathbf{P}^{i-k}$ *is independent of* $(P_{i-k+1}, \cdots, P_i)$ *for all* $i > k$, *the above limit exists and equality holds.* (Received 3 July 1968.)

3. A note on asymptotic efficiency of Chernoff-Savage class of tests. Z. GOVINDARAJULU, University of Kentucky.

Sufficient conditions which enable one to interchange the operations of limit and integration while computing Pitman efficiency for the Chernoff-Savage class of tests, are obtained. These conditions are less restrictive than those given by Puri [*Ann. Math. Statist.* **35** (1964) 102–121] for the Chernoff-Savage class, which in turn are extensions of the sufficient conditions of Hodges and Lehmann [*Proc. Fourth Berkeley Symp. Math. Statist. and Prob.* **1** (1961) 307–317] for the normal scores test. It is of particular interest to examine the Lehmann type of alternatives and translation and change of scale alternatives for the Chernoff-Savage class in the light of these sufficient conditions. Furthermore, a table is provided which tells us for what distribution, for what alternatives and for what rank tests, one can pass the limit underneath the integral sign. In particular, it is shown that, for Savage's test [*Ann. Math. Statist.* **27** (1956) 590–615] with exponential translation alternatives, the sufficient conditions enables us to interchange the operations of limit and integration. (Received 7 June 1968.)

4. A Bayesian approach to efficiency of experimental designs (preliminary report). IRWIN GUTTMAN, University of Wisconsin.

We suppose that a random variable y has distribution $f(y \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a $g \times 1$ vector-parameter, and prior information on $\boldsymbol{\theta}$ may be summarized by the "honest" prior $\rho(\boldsymbol{\theta})$. Frequently, interest lies in $\phi = \mathbf{l}_0'\boldsymbol{\theta}$ or in $\boldsymbol{\tau} = (\mathbf{l}_1'\boldsymbol{\theta}, \cdots, \mathbf{l}_k'\boldsymbol{\theta})$, where the vectors $\mathbf{l}_i$ are $(q \times 1)$ and specified by the interest of the experimenter. Suppose now that the experimenter has a choice of using a design D_1 or a design D_2. The question is which should he use. For this situation, and as advocated in Guttman [*J. Roy. Statist. Soc. Ser. B* **29** (1967) 83], we believe that a preposterior analysis should be utilized, and, accordingly, we make use of the following definition. DEFINITION. A design D_1 is preferred to D_2 (written $D_1 \succ D_2$), with respect to $\phi = \mathbf{l}_0'\boldsymbol{\theta}$, if in the presence of the same prior information on $\boldsymbol{\theta}$, the use of D_1 will lead to a posterior for $\mathbf{l}_0'\boldsymbol{\theta}$ with strictly smaller expected variance than will the use of D_2, where the expectation is taken over the future distribution of $\mathbf{y}$, the observations to be taken by either design. In obvious notation, then, $D_1 \succ D_2 \Leftrightarrow E_{\mathbf{y}}[V(\mathbf{l}_0'\boldsymbol{\theta} \mid \mathbf{y}, D_1)] < E_{\mathbf{y}}[V(\mathbf{l}_0'\boldsymbol{\theta} \mid \mathbf{y}, D_2)]$. (If D_1 is not preferred to D_2 we say that D_2 is indifferent to D_1 *or* D_2 is preferred to D_1 and we write $D_1 \preceq D_2$.) Further, D_1 is preferred to D_2, with respect to $\boldsymbol{\tau} = (\mathbf{l}_1'\boldsymbol{\theta}, \cdots, \mathbf{l}_k'\boldsymbol{\theta})'$, if we have the following ordering of the determinants of the expected variance covariance matrix of $\boldsymbol{\tau}$ under D_1 or D_2, namely $D_1 \succ D_2 \Leftrightarrow \det E_{\mathbf{y}}[V(\boldsymbol{\tau} \mid \mathbf{y}, D_1)] < \det E_{\mathbf{y}}[V(\boldsymbol{\tau} \mid \mathbf{y}, D_2)]$. Many interesting applications can be made, ranging from sample survey design comparisons to the usual regession situations (factorial designs, rotatable, etc.) If $D_1 \succ D_2$ we define the relative efficiency (RE) of D_2 with respect to D_1 as either RE $(D_2 \mid D_1) = E_{\mathbf{y}}[V(\mathbf{l}_0'\boldsymbol{\theta} \mid \mathbf{y}, D_1)]/E_{\mathbf{y}}[V(\mathbf{l}_0'\boldsymbol{\theta} \mid \mathbf{y}, D_2)]$ if interest is in $\mathbf{l}_0'\boldsymbol{\theta}$, or RE $(D_2 \mid D_1) = \det E_{\mathbf{y}}[V(\boldsymbol{\tau} \mid \mathbf{y}, D_1)]/\det E_{\mathbf{y}}[V(\boldsymbol{\tau} \mid \mathbf{y}, D_2)]$, if interest is in $\boldsymbol{\tau}$. The extension to sequential designs will be attempted. (Received 3 June 1968.)

5. Optimal learning with finite memory. MARTIN E. HELLMAN and THOMAS M. COVER, Stanford University.

This paper develops the theory of the design and performance of optimal finite-memory systems for the two-hypothesis testing problem. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables drawn according to a probability measure P. Consider the standard two-hypothesis testing problem with probability of error loss criterion in which $P = P_0$, with probability π_0; and $P = P_1$ with probability π_1. Let the data be summarized after each new observation by a statistic $T \varepsilon \{1, 2, \cdots, m\}$ which is updated according to the rule $T_n = f(T_{n-1}, X_n)$, where f may be a random function, independent of n and the data. (The time-varying rule $T_{n+1} = f(T_{n-1}, X_n, n)$ has been investigated in another paper—Cover, T., "Hypothesis testing with finite statistics", submitted to *Ann. Math. Statist.*) Let $\delta:\{1, 2, \cdots, m\} \rightarrow \{0, 1\}$ be a fixed decision function taking action $\delta(T_n)$ at time n, and let $P_e(f, \delta)$ be the asymptotic probability of error of the scheme (f, δ) as the number of trials $n \rightarrow \infty$. Define $P_e^* = \inf_{f,\delta} P_e(f, \delta)$. Let $\bar{l} = \sup (P_0(A)/P_1(A))$ and $\underline{l} = \inf (P_0(A)/P_1(A))$ where the supremum and infinum are taken over all measurable sets A for which $P_0(A) + P_1(A) > 0$. Define $\gamma = \bar{l}/\underline{l}$. In this paper it is shown that $P_e^* = (2(\pi_0\pi_1\gamma^{m-1})^{\frac{1}{2}} - 1)/(\gamma^{m-1} - 1)$, under the nondegeneracy condition $P_e^* \leqq \min \{\pi_0, \pi_1\}$. Also a simple family of ϵ-optimal (f, δ)'s is exhibited. Thus γ summarizes the resolving power of an m-valued statistic, and it may be concluded that the asymptotic probability of error of the optimal m-state machine tends to zero exponentially in the number of states. (Received 26 June 1968.)

6. A matrix representation of a class of denumerable homogeneous Markov chains. John A. Higginson, Queen's University, Kingston, Ontario.

Let k be an integer >0. Then a *block-spreading chain of block size* k is a Markov chain having the non-negative integers as states and whose transition matrix P consists of k by k submatrices $P_{[i,j]}$ such that $P_{[i,i+1]}$ is invertible for all i and $P_{[i,j]} = 0$ if $j > i + 1$. Let S be the denumerable matrix defined by $S_{i,i+1} = 1$ and all other entries $= 0$. Then any block-spreading chain P of block size k can be obtained form S^k by a similarity transformation, i.e. there are computable lower block triangular matrices R and Q such that $R_{[0,0]} = Q_{[0,0]} = I$, $RQ = QR = I$, and $P = QS^kR$. These properties determine R and Q. Hence if all states are transient, $N = \sum_{m=0}^{\infty} Q(S^{mk})R$. A transient block-spreading chain has at most k boundary points (*à la* R. S. Martin) and this is best possible. The ladder chain of a block-spreading chain is also block-spreading, and its transition and representation matrices can be easily computed. (Received 3 July 1968.)

7. Diffusion approximations in collective risk theory. Donald L. Iglehart, Stanford University.

Let $X(t)$ be the total assets of an insurance company at time t. The company has an initial capital u and policy-holders pay a gross risk premium of a per unit time. At the jumps of a renewal process, $\{N(t):t \geqq 0\}$, a sequence of claim $\{X_i\}$ are made against the company. The $\{X_i\}$ are iid random variables and we let $S_0 = 0$ and $S_i = X_1 + \cdots + X_i$. Then $X(t) = u + at - S_{N(t)}$. Consider now a sequence of such processes indexed by n and defined as $X_n(t) = u_n + a_n \cdot nt - S_{N(nt)}^{(n)}$, where the claims $\{X_i^{(n)}\}$ for the nth process are iid with mean μ_n and variance σ_n^2. Then X_n is a random function in $D[0, 1]$ and if $u_n = un^{\frac{1}{2}} + o(n^{\frac{1}{2}})$, $a_n = an^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$, $\mu_n = \mu n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$, $\sigma_n^2 \rightarrow \sigma^2$, and $E[(X_i^{(n)})^{2+\epsilon}]$ is bounded in n for some $\epsilon > 0$, then the process $n^{-\frac{1}{2}}X_n$ converges weakly to the process $u + (a - \mu\lambda)t + \sigma\lambda^{\frac{1}{2}}W(t)$, where $W(t)$ is Brownian motion and λ^{-1} is the expected time between jumps of $N(t)$. A limit theorem for the time to ruin is also obtained and the density of the limiting distribution given explicitly. (Received 15 July 1968.)

8. Asymptotic global proximity probabilities for a bivariate n-sample. Roger E. Miles, Australian National University.

The results of Miles [*Ann. Math. Statist.* **39** (1968) 1071] extend as follows. Consider an independent n-sample from the bivariate distribution with (smooth) pdf $f(\mathbf{x})$ in E^2, and define $p\{i, \theta\} = 1 - q\{i, \theta\} = e^{-\theta}\{1 + \theta + \cdots + (\theta^i/i!)\}$. (i) If $f(\mathbf{x}) \leqq M < \infty$ in E^2, then $\sup_{R>0} |P$ (every disc of radius R in E^2 contains *at most* j sample points) $-$ exp $[-n(j + 1)\int q\{j - 1, \pi n R^2 f(\mathbf{x})\} f(\mathbf{x})\, d\mathbf{x}]| \to 0$ as $n \to \infty$ $(j = 1, 2, \cdots)$. This generalizes Efron [*Ann. Math. Statist.* **38** (1967) 298] when his $d = 2$. (ii) If $f(x) \geqq m > 0$ in a (sufficiently regular) set A and is 0 elsewhere, if $A_R = \{\mathbf{x}$: disc (centre $\mathbf{x}$, radius R) $\subset A\}$ and $I = \sup \{R: A_R \neq \varnothing\}$, then $\sup_{0<R<I} |P$ (every disc of radius R contained in A contains *at least* k sample points) $-$exp $[-nk\int_{A_R} p\{k, \pi n R^2 f(\mathbf{x})\} f(\mathbf{x})\, d\mathbf{x}]| \to 0$ as $n \to \infty$ $(k = 1, 2, \cdots)$. In the uniform case with $k = 1$, this agrees well with Gilbert's [*Biometrika* **52** (1965) 330] simulation in which A is a disc of radius $4R$. (i), (ii) give rise to statistical tests, in which discs of radius R are constructed in E^2 about each sample point as centre. They generalize to all dimensions. (Received 24 June 1968.)

9. Some studies on intersection tests in multivariate analysis of variance. J. N. SRIVASTAVA, Colorado State University.

Consider a null hypothesis H_0, expressible as an intersection of component subhypotheses H_{0j}. Let D_{0j} be a 'natural' acceptance region for testing H_{0j} (against not $-H_{0j}$), so that $D = \bigcap_j D_{0j}$, is the acceptance region of S. N. Roy's union-intersection test for H_0. A major part of this paper is concerned with the problem of the appropriate sizes for the D_{0j} (given a fixed size for D), so that the power of the test based on D is maximised. Here, the H_{0j} are univariate linear hypotheses and F_j (the F-test for H_{0j}) has the same degrees of freedom for all j. Three situations are considered according as the F_j are (a) completely independent, (b) quasi-independent (i.e. have the same denominator, but independent numerators), (c) dependent, with correlated numerators and a common denominator. For (a), (b) and a special case in (c), it is shown that the larger the noncentrality parameter for H_{0j}, the smaller should be the size of D_{0j}. Finally, multivariate linear hypotheses H_0 are considered wrt their responsewise and contrastwise decompositions, and the use and importance of the above results in this context is pointed out. (Received 24 June 1968.)

10. Order statistics from a class of non-normal distributions. K. SUBRAHMANIAM, University of Manitoba.

In this paper we have considered the distribution of the order statistics from a population with the probability density function (pdf): $f(x) = [1 + (\lambda_3/6)H_3(x) + (\lambda_4/24)H_4(x) + (\lambda_3^2/72)H_6(x)]\phi(x)$, with $\phi(x)$ representing the standard normal pdf and the $H_j(x)$ are Hermite polynomials of degree j. Expressions for the moments of the rth order statistic are obtained in terms of the corresponding normal moments. The results due to Ruben, [*Biometrika* (1954)], are used for this purpose in the case of extreme order statistics. Tables of the constants involved will be compiled. Asymptotic expressions for the first four moments of the median are obtained using a method proposed by David and Johnson [*Biometrika*, (1954)]. The joint distribution of two order statistics is studied, in particular that of the range. Following the development of Ruben, some low moments of the range are given. (Received 26 July 1968.)