The Annals of Mathematical Statistics
1969, Vol. 40, No. 3, 1116–1119

A NOTE ON CHERNOFF-SAVAGE THEOREMS[1]

BY RONALD PYKE AND GALEN R. SHORACK

The University of Washington

Let $X_1, \cdots, X_m$; $Y_1 \cdots, Y_n$ be independent random samples from continuous df's F and G respectively; and let F_m and G_n be the corresponding empirical df's. Let $N = m + n$ and $\lambda_N = m/N$. Set

$$T_N = m^{-1} \sum_{i=1}^{N} c_{Ni}^* Z_{Ni} \tag{1}$$

where $\{c_{Ni}^* : 1 \leqq i \leqq N, N \geqq 1\}$ is a set of given constants and Z_{Ni} equals 1 (or 0) if the ith largest from $\{X_1, \cdots, X_m, Y_1, \cdots, Y_n\}$ is an X (or a Y).

The asymptotic normality of the class of statistics of the form (1) was studied first by Chernoff and Savage [1]. Since then several other approaches to this problem have been considered. In one of these approaches, [2], the authors presented some results (cf. Proposition 5.1, Corollary 5.1 and the related discussion in [2]) to indicate in what sense the results of [1] follow from those of [2]. The purpose of this note is to strengthen greatly these results by showing that a different decomposition of T_N makes Theorem 5.1 (a) of [2] more directly applicable and enables condition (i) of Theorem 5.1 (b) of [2] to be replaced by more easily verifiable conditions.

All notations undefined below are to be given their meaning according to Pyke and Shorack [2]. We recall only the following. For $N \geqq 1$ the L_N-process on [0, 1] is given by $L_N(t) = N^{\frac{1}{2}}[F_m \circ H_N^{-1}(t) - F \circ H^{-1}(t)]$ with $H_N = \lambda_N F_m + (1 - \lambda_n)G_n$ and $H = \lambda_N F + (1 - \lambda_N)G$; and the L_0-process is the natural limit of these processes. The signed measure ν and the related right continuous function J, which is of bounded variation on $[\epsilon, 1 - \epsilon]$ for all $\epsilon > 0$, satisfy

$$-\nu((a, b]) = J(b) - J(a) \qquad \text{for all} \quad 0 < a < b < 1.$$

Finally $\mathbf{Q}$ is the class of functions defined in [2]; an example of $q \varepsilon \mathbf{Q}$ is $q(t) = [t(1 - t)]^{\frac{1}{2}-\delta}$ for $\delta > 0$. Let

$$\tilde{T}_N = N^{\frac{1}{2}}[T_N - \mu] \tag{2}$$

where

$$\mu = \int_0^1 J d(F \circ H^{-1}). \tag{3}$$

Then

$$\tilde{T}_N = \tilde{S}_N + \theta_N + \epsilon_N \tag{4}$$

Received 16 October 1968.
[1] This research was supported in part by National Science Foundation Grant GP-8882.

where

$$
\begin{aligned}
\tilde{S}_N &= N^{\frac{1}{2}}[S_N - \mu_N], \\
\theta_N &= N^{\frac{1}{2}}[T_N - S_N - J(1 - 1/N)], \qquad (5)\\
\epsilon_N &= N^{\frac{1}{2}}[\mu_N - \mu + J(1 - 1/N)], \\
S_N &= \int_{I_N} F_m \circ H_N^{-1}\, d\nu, \qquad \mu_N = \int_{I_N} F \circ H^{-1}\, d\nu,
\end{aligned}
$$

and $I_N = (1/N, 1 - 1/N]$. Thus

$$(6) \qquad \tilde{S}_N = \int_{I_N} L_N\, d\nu.$$

Let $\alpha \wedge \beta = \min(\alpha, \beta)$. The following conditions are used in the theorem below.

(C1) There exists $q \varepsilon \mathbf{Q}$ such that $\int_0^1 q\, d\,|\nu| < M < \infty$.

(C2) $N^{-\frac{1}{2}} \sum_{i=1}^N |c_{Ni}^* - J(i/N \wedge (1 - 1/N))| \leqq \delta_N$ where $\delta_N = o(1)$.

(C3) $N^{\frac{1}{2}}|\lambda_N - \lambda_0| \leqq M_N$ where $M_N = O(1)$ and $F \circ H_0^{-1}$ is differentiable a.e. $- |\nu|$.

(C4) F, G (which may depend on N) and $\{\lambda_N\}$ are such that the functions $F \circ H^{-1}$ have derivatives a_N which form a uniformly equicontinuous family and a_N converges uniformly to a_0 as $N \to \infty$. I.e. given $\epsilon > 0$ there exists δ_ϵ, N_ϵ such that $|a_N(s) - a_N(t)| < \epsilon$ for all $|s - t| < \delta_\epsilon$ and for all N, and $N > N_\epsilon$ implies $|a_N(t) - a_0(t)| < \epsilon$ for all t.

(C2′) (a) $\sum_{i=1}^{N-1} [c_{Ni}^* - J(i/N)]Z_{Ni} = o_p(N^{-\frac{1}{2}})$ (b) $c_{NN}^* = o(N^{\frac{1}{2}})$ (c) $J(1 - 1/N) = o(N^{\frac{1}{2}})$.

(C3′) $F \circ H^{-1}$ is differentiable a.e. $- |\nu|$ for all N exceeding some N_0.

(C4′) Condition (C4) without the requirement that a_N converges to a_0.

THEOREM 1. (i) *Under* (C1) *and* (C3)

$$\tilde{S}_N \to_p \int_0^1 L_0\, d\nu \qquad as \quad N \to \infty,$$

where the limiting rv *is normal* (*see* [2] *for the variance*).

(ii) *Under* (C1) *and* (C2′)

$$\theta_N \to_p 0 \qquad and \qquad \epsilon_N \to 0 \qquad as \quad N \to \infty.$$

(iii) *Thus under* (C1), (C2′) *and* (C3)

$$\tilde{T}_N \to_p \int_0^1 L_0\, d\nu \qquad as \quad N \to \infty.$$

(iv) *Under* (C1), (C2), (C3) *and* (C4)

$$\tilde{T}_N \to_p \int_0^1 L_0\, d\nu \qquad as \quad N \to \infty$$

and the convergence is uniform in the set of all F, G, λ_N, c_{Ni}^**'s and* J *such that the conditions hold for fixed* q, M, δ_N*'s*, M_N*'s*, δ_ϵ*'s and* N_ϵ*'s.*

(v) *Parts* (i), (iii) *and* (iv) *remain true if* (C3′) *replaces* (C3) *and* "$\cdots - \int_0^1 L_{0N}\, d\nu \to_p 0$" *replaces* "$\cdots \to_p \int_0^1 L_0\, d\nu$" *everywhere.*

REMARK. According to Theorem 1(v), (C1), (C2′) and (C3′) imply

$$\tilde{T}_N - \int_0^1 L_{0N}\, d\nu \to_p 0 \quad \text{as} \quad N \to \infty.$$

These conditions are weaker than those of Chernoff and Savage's Theorem 1. Conditions (C2′) (a) and (b) are conditions (2) and (3) of their Theorem 1, and (C2′) (c) is implied by the bound on J of their condition (4); these conditions tell how well their J_N must approximate J. Condition (C2) implies (C2′), but is much easier to verify. Condition (C1) is the basic growth condition on the function J. Notice that it is implied by their condition (4), that J', J'' and the absolute continuity of J are not mentioned, and that the growth condition on J is weaker than that of Chernoff and Savage. For example if $c_{Ni}^* = J(i/N \wedge (1 - 1/N)$ with $J(t) = [t^{\frac{1}{2}} \log^4 t]^{-1}$ on $(0, \frac{1}{2}]$ and $J(t) = J(\frac{1}{2})$ for t in $(\frac{1}{2}, 1)$, then Theorem 1 above holds with $q(t) = -t^{\frac{1}{2}} \log t$ but the Chernoff-Savage hypotheses fail. Conditions (C3) and (C3′) are essentially weak smoothness conditions on J.

PROOF. (i) By Theorem 5.1 (a) of [2] and (6)

$$|\tilde{S}_N - \textstyle\int_0^1 L_0\, d\nu| \to_p 0 \qquad \text{as} \quad N \to \infty.$$

(ii) For $2 \leqq i \leqq N - 1$, let

$$\tilde{c}_{Ni} = \nu(((i-1)/N, i/N]) = -[J(i/N) - J((i-1)/N)]$$

and let $\tilde{c}_{N1} = \tilde{c}_{NN} = 0$; let $\tilde{c}_{Ni}^* = \tilde{c}_{Ni} + \cdots + \tilde{c}_{NN}$ for $1 \leqq i < N$ and $N \geqq 1$. Then $mS_N = \sum_{i=1}^N \tilde{c}_{Ni}^* Z_{Ni}$ by summation by parts so that

$$\begin{aligned} m\theta_N &= N^{\frac{1}{2}} \textstyle\sum_{i=1}^N [c_{Ni}^* - \tilde{c}_{Ni}^* - J(1 - 1/N)] Z_{Ni} \\ &= N^{\frac{1}{2}} \textstyle\sum_{i=1}^N [c_{Ni}^* - J(i/N)] Z_{Ni} \\ &\quad + N^{\frac{1}{2}} \textstyle\sum_{i=2}^{N-1} [J(i/N) - J((i-1)/N)] Z_{Ni} \\ &\quad + N^{\frac{1}{2}} [c_{NN}^* - J(1 - 1/N)] Z_{NN} \\ &\equiv (\theta_{1N} + \theta_{2N} + \theta_{3N}) m. \end{aligned}$$

Now (C2′) (a) implies $\theta_{1N} \to_p 0$ and (C2′)(b) and (c) imply $\theta_{3N} \to_p 0$. Also $|\theta_{2N}| \leqq N^{-\frac{1}{2}} \lambda_*^{-1} |\nu| ([1/N, 1 - 1/N]) \to 0$ by (C1) since

$$\infty > M > \textstyle\int_0^1 q\, d|\nu| \geqq [N^{\frac{1}{2}} \min(q(1/N), q(1 - 1/N))] \cdot [|\nu|([1/N, 1 - 1/N])/N^{\frac{1}{2}}]$$

implies $|\nu| ([1/N, 1 - 1/N]) N^{-\frac{1}{2}} \to 0$ as $N \to \infty$ since $q(u) u^{-\frac{1}{2}} \to \infty$ as $u \to 0$ for every $q \in \mathbf{Q}$. Note also that under (C1) and (C2) $|\theta_N| \to_p 0$ as $N \to \infty$ uniformly as stated in (iv).

Let $E = \{(t_1, t_2): a < t_i \leqq b \text{ for } i = 1, 2 \text{ and } t_2 < t_1\}$ and $E' = \{(t_1, t_2): a < t_i < 1 \text{ for } i = 1, 2 \text{ and } t_2 \geqq t_1\}$. Apply Fubini's theorem to the indicator function of E (of E') using the measure $-\nu$ and the measure associated with $K = F \circ H^{-1}$ (with $1 - K$) to obtain

$$\textstyle\int_{(a,b)} K\, d\nu + K(a+)[J(b) - J(a)] = \int_{(a,b)} J\, dK - J(b)[K(b+) - K(a+)]$$

(we set $K(a+) = 0$ when $a = 0$) and

$$\textstyle\int_{(a,1)} (1 - K)\, d\nu = \int_{(a,1)} J\, d(1 - K) + J(a)[1 - K(a+)].$$

Now

$$\epsilon_N = N^{\frac{1}{2}}[\int_{I_N} K\, d\nu - \int_{(0,1/N]} J\, dK - \int_{I_N} J\, dK + \int_{(1-1/N,1)} J\, d(1-K) + J(1-1/N)]$$

which with these integration-by-parts formulae reduces to

$$\epsilon_N = N^{\frac{1}{2}}[\int_{(1-1/N,1)} (1-K)\, d\nu - \int_{(0,1/N]} K\, d\nu]$$

so that $\epsilon_N = o(1)$ by (C1); and this is uniform as stated in (iv).

(iii) This follows immediately from (i) and (ii).

(iv) From the uniformity shown in (ii) we need only show that $\tilde{S}_N \to_p \int_0^1 L_0\, d\nu$ uniformly as stated. From the discussion at the beginning of Section 4 of [2] we need show only that $\rho(A_N, a_0) \to_p 0$; but the proof of Lemma 4.2 of [2] establishes this.

(v) Section 5 of [2] yields (i) and (iii) and Section 4 with Lemma 4.2 establishes $\rho(A_N, a_N) \to_p 0$ to give (iv). □

REFERENCES

[1] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

[2] Pyke, R. and Shorack, G. (1968). Weak convergence of a two-sample empirical process and a new approach to Chernoff-Savage theorems. *Ann. Math. Statist.* **39** 755–771