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MARTINGALES WITH INDEPENDENT INCREMENTS[1]

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0. Summary. In section one, L_1 bounds are obtained for transforms of discrete parameter martingales having independent increments. Section two develops the theory of quadratic variation for continuous parameter martingales having independent increments. An application of the results of these two sections to the theory of stochastic integrals appears in section three.

1. Discrete parameter martingales with independent increments. Let $(\Omega, \mathfrak{A}, P)$ be a probability space, and $\mathfrak{A}_n$, $n = 0, 1, 2, \cdots$, an increasing family of sub sigma fields of $\mathfrak{A}$. Let $f = \{f_n, \mathfrak{A}_n, n = 1, 2, \cdots\}$ be a martingale, and $d_1 = f_1$, $d_2 = f_2 - f_1, \cdots$. Let $v = \{v_n, n = 1, 2, \cdots\}$ be a sequence of real random variables where v_n is $\mathfrak{A}_{n-1}$ measurable; then the sequence $g = \{g_n, n = 1, 2, \cdots\}$ will be called the transform of f under v provided that $g_n = \sum_{k=1}^{n} v_k d_k$. If the increments $\{d_k\}$ of f form an independent sequence, if $\mathfrak{A}_{n-1}, d_n, d_{n+1}, \cdots$ are independent for each $n \geqq 1$, and if $\sup_n |v_n| \leqq 1$, then we show in this section that

$$\text{(a)} \quad E|g_n| \leqq 2E|f_n|, \qquad \text{and} \qquad \text{(b)} \quad Eg_n^* \leqq 2Ef_n^*.$$

Here $g_n^* = \sup_{1 \leqq k \leqq n} |g_k|$. Note that the condition concerning the sigma fields $\mathfrak{A}_n$ will always be satisfied if we take $\mathfrak{A}_n$ to be the smallest sigma field generated by $\{d_k, 1 \leqq k \leqq n\}$, and $\mathfrak{A}_0$ to be the trivial sigma field.

If f does not have independent increments, then inequality (a) is false, in general, no matter what constant one inserts in place of '2.' If, however, f satisfies only $E|f_n|^p < \infty, n = 1, 2, \cdots$, for some $p > 1$, then it follows from Burkholder's theory of martingale transforms ([2], page 1502) and from Doob's martingale inequalities ([6], page 317) that there are constants A_p and B_p (depending on p only) such that analogously:

$$\text{(a}'\text{)} \quad E|g_n|^p \leqq A_p E|f_n|^p \qquad \text{and} \qquad \text{(b}'\text{)} \quad E(g_n^*)^p \leqq B_p E(f_n^*)^p.$$

Finally, we point out that if f has independent increments and v is a sequence of *constants* with $\sup_n |v_n| \leqq 1$, then inequalities (a$'$) and (b$'$) follow with $p = 1$ from results of Marcinkiewicz and Zygmund ([11], Theorems 5, 6).

An analogue of inequality (a) for certain continuous parameter martingales is developed in Section 3.

LEMMA 1.1. *Let φ be a real valued function on $[-1, 1] \times R$ (where R is the real line) with the properties that φ is bounded from below, $\varphi(-1, x) = \varphi(1, -x)$ and $\varphi(\cdot, x)$ is convex for each x. If X is a symmetrically distributed random variable, and if b is a real number satisfying $-1 \leqq b \leqq 1$, then $E\varphi(b, X) \leqq E\varphi(1, X)$.*

PROOF. By passing to an appropriate product space if necessary, we may suppose that there exists a random variable B, independent of X, such that

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$P(B = -1) = (1 - b)/2$, $P(B = 1) = (1 + b)/2$. Then $E\varphi(b, X) = E\varphi(E(B \mid X), X) \leqq E\{E[\varphi(B, X) \mid X]\} = E\varphi(B, X) = E\varphi(1, X)I(B = 1) + E\varphi(-1, X)I(B = -1) = E\varphi(1, X)P(B = 1) + E\varphi(1, -X)P(B = -1) = E\varphi(1, X)P(B = 1) + E\varphi(1, X)P(B = -1) = E\varphi(1, X)$. Here we have used $I(A)$ to denote the indicator of the set A.

LEMMA 1.2. *Let f be a martingale with symmetrically distributed, independent increments, and with $\mathfrak{A}_{n-1}$, d_n, d_{n+1}, $\cdots$ independent for $n \geqq 1$. Let g be a transform of f under v, with* $\sup_n |v_n| \leqq 1$. *Then $E\,|g_n| \leqq E\,|f_n|$ and $Eg_n^* \leqq Ef_n^*$.*

PROOF. We prove the second inequality first. Let n and k be positive integers with $k \leqq n$. Let $\varphi_k(b, x; a_1, \cdots, a_{n-1}) = \max_{1 \leqq j \leqq n} R_{jk}(b, x; a_1, \cdots, a_{n-1})$, where $R_{jk} = |a_1 + \cdots + a_j|$ if $1 \leqq j < k$, and $R_{jk} = |a_1 + \cdots + a_{j-1} + bx|$ if $k \leqq j \leqq n$. Then, for each k, $\varphi_k(\cdot, \cdot; a_1, \cdots, a_{n-1})$ satisfies the condition of Lemma 1. Therefore,

$$
\begin{aligned}
Eg_n^* &= E\varphi_n(v_n, d_n; v_1 d_1, \cdots, v_{n-1} d_{n-1}) \\
&= E[E\{\varphi_n(v_n, d_n; v_1 d_1, \cdots, v_{n-1} d_{n-1}) \mid \mathfrak{A}_{n-1}\}] \\
&\leqq E[E\{\varphi_n(1, d_n; v_1 d_1, \cdots, v_{n-1} d_{n-1}) \mid \mathfrak{A}_{n-1}\}] \\
&= E\varphi_{n-1}(v_{n-1}, d_{n-1}; v_1 d_1, \cdots, v_{n-2} d_{n-2}, d_n) \\
&= E[E\{\varphi_{n-1}(v_{n-1}, d_{n-1}; v_1 d_1, \cdots, v_{n-2} d_{n-2}, d_n) \mid \mathfrak{A}_{n-2} \vee d_n\}] \\
&\leqq E\varphi_{n-1}(1, d_{n-1}; v_1 d_1, \cdots, v_{n-2} d_{n-2}, d_n) \\
&= E\varphi_{n-2}(v_{n-2}, d_{n-2}; v_1 d_1, \cdots, v_{n-3} d_{n-3}, d_{n-1}, d_n) \\
&\leqq \cdots \leqq E\varphi_1(1, d_1; d_2, \cdots, d_n) = Ef_n^*.
\end{aligned}
$$

Using a similar argument with $\varphi_k(b, x; a_1, \cdots, a_{n-1}) = |a_1 + \cdots + a_{n-1} + bx|$, for all k one obtains the first inequality of the lemma.

THEOREM 1.1. *Let f be a martingale with independent increments, and with $\mathfrak{A}_{n-1}$, d_n, d_{n+1}, $\cdots$ independent for $n \geqq 1$. Let g be a transform of f under v with $\sup_n |v_n| \leqq 1$. Then $E\,|g_n| \leqq 2E\,|f_n|$ and $Eg_n^* \leqq 2Ef_n^*$.*

PROOF. Let $\mathfrak{A}_\infty$, d_1', d_2', $\cdots$ be independent, where $\mathfrak{A}_\infty = \bigvee_{n=1}^{\infty} \mathfrak{A}_n$, and for each $n \geqq 1$, d_n' is a random variable having the same distribution as d_n. Let $\mathfrak{A}_n'$ be the smallest sigma field generated by $d_1', \cdots, d_n'$. Let $\mathfrak{A}_0'' = \mathfrak{A}_0$, and for $n \geqq 1$, $\mathfrak{A}_n'' = \mathfrak{A}_n \vee \mathfrak{A}_n'$. Then v_n is $\mathfrak{A}_{n-1}''$ measurable, and $E(d_n - d_n' \mid \mathfrak{A}_{n-1}'') = E(d_n \mid \mathfrak{A}_{n-1}) - E(d_n' \mid \mathfrak{A}_{n-1}') = 0$, $n > 1$. Therefore the sequence $\{f_n - f_n'\}$ is a martingale, where $f_n' = \sum_{k=1}^{n} d_k'$, and $g_n - g_n' = \sum_{k=1}^{n} v_k(d_k - d_k')$ is a transform of $\{f_n - f_n'\}$. Using Lemma 2, one sees that

$$
\begin{aligned}
2Ef_n^* &\geqq E \sup_{1 \leqq k \leqq n} |f_k - f_k'| \\
&\geqq E \sup_{1 \leqq k \leqq n} |g_k - g_k'| \\
&= E[E\{\sup_{1 \leqq k \leqq n} |g_k - g_k'| \mid \mathfrak{A}_\infty\}] \\
&\geqq E \sup_{1 \leqq k \leqq n} |E\{g_k - g_k' \mid \mathfrak{A}_\infty\}| \\
&= E \sup_{1 \leqq k \leqq n} |g_k - E\, d_1 E v_1| \\
&\geqq Eg_n^* - E\, d_1 E v_1 .
\end{aligned}
$$

If $E\,d_1 = 0$, then the theorem is proved. If $E\,d_1 \neq 0$, let r be a random variable independent of f, with $P(r = \pm 1) = \frac{1}{2}$. Let $F = \{F_1, F_2, \cdots\}$ be the martingale (relative to $\{\mathcal{A}_n \vee \sigma(r), n \geqq 1\}$ obtained by setting $F_k = rf_k$. Then $EF_1 = Er\,d_1 = 0$, and so if $G = \{G_n, n \geqq 1\}$ is the transform of F under v, $EG_n^* \leqq 2EF_n^*$. Since $G_n^* = g_n^*$ and $F_n^* = f_n^*$, it follows that $Eg_n^* \leqq 2Ef_n^*$.

The inequality $E\,|g_n| \leqq 2E\,|f_n|$ is obtained in a similar manner.

2. Quadratic variation of martingales with independent increments. Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space, $\{\mathcal{A}(t), 0 \leqq t \leqq T\}$ an increasing sequence of sub sigma fields of $\mathcal{A}$, and $X = \{X(t), \mathcal{A}(t), 0 \leqq t \leqq T\}$ a martingale having independent increments. For $p \geqq 1$, we will write $\|X\|_p^p = \sup_{0 \leqq t \leqq T} E\,|X(t)|^p$. Let $\pi = \{\pi_n\}$, $n = 1, 2, \cdots$, be a sequence of partitions of $[0, T]$, with $\pi_n : 0 \leqq t_{n1} < t_{n2} < \cdots \leqq T$. We will suppose that for each n, π_{n+1} is a refinement of π_n. Let $Q_n(X, \pi, T)$ be the non-negative square root of $[X(t_{n1})]^2 + \sum_{k \geqq 1} [X(t_{n,k+1}) - X(t_{n,k})]^2$. We will often abbreviate this $Q_n(X)$ or $Q_n(T)$, depending on context.

The behavior of $Q_n(X, \pi, T)$ as n increases has frequently been studied under various assumptions on X and π. In 1940, Lévy showed ([9]) that $Q_n^2(X)$ converges a.e. and in L_2 norm, when X is standard Brownian motion and the π_n become dense in $[0, T]$. (Independently, Cameron and Martin ([3]) obtained the same result, assuming that t_{nk} is of the form $k2^{-n}$). Another proof of Lévy's result has been given by Doob ([6], page 395) who made the important observation that, when X is Brownian motion, $Q_n^2(X)$ is a reversed martingale. There are several ways one may try to extend Lévy's result: (i) one may consider, instead of Brownian motion, more general martingales; (ii) one may consider, instead of Brownian motion, more general processes with independent increments. Of course, there are other interesting ways to extend Lévy's result (see, for example, Baxter ([1]), who considers more general Gaussian processes), but the two ways indicated are the most relevant for this paper.

A generalization of the second type has been given by Cogburn and Tucker ([4]): If X is a separable, continuous, infinitely divisible process with law (α_T, Ψ_T), where α_T is a function of bounded variation on $[0, T]$, then $\lim_n Q_n(X)$ exists a.e. Here, (α_T, Ψ_T) denotes the characteristic function of $X(t)$, which under the given hypotheses must have the form $\exp \psi_t$, where

$$\psi_t(u) = iu\alpha_t + \int [\exp(iux) - 1 - (iux)(1 + x^2)^{-1}](1 + x^2)x^{-2}\, d\Psi_t(x).$$

The assumptions of Cogburn and Tucker serve to restrict the path irregularities of X. In particular, under those assumptions, almost all paths are bounded, have right and left limits at every point, and all discontinuities are jumps; there are no fixed discontinuities.

Very recently, several attempts have been made to generalize Lévy's result to martingales other than Brownian motion. Here the emphasis has been on norm convergence and convergence in probability, rather than a.e. convergence. We indicate briefly this development. In 1966, Fisk ([7]) showed that $Q_n(X)$ converges in L_2 norm if X is a square integrable martingale having almost all paths continuous. Kunita and Watanabe ([8]) independently obtained the same result, but used a specially chosen sequence of stochastic partitions π_n (the t_{nk}'s were taken to be specially chosen stopping times). Meyer ([12]) obtained the convergence of $Q_n(X)$ in L_2, assuming that X was a right continuous, square integrable martingale which was also quasi left continuous. Independently of the preceding, Millar ([13]) obtained two results: (a) if almost all paths of X are continuous and if $\|X\|_p < \infty$ for some $p > 1$, then $Q_n(X)$ converges in L_p norm; (b) if X has right continuous paths, then $Q_n(X)$ converges in $L_{p'}$ norm, $p > p' \geqq 1$, provided $\|X\|_p < \infty$, and converges in probability otherwise. To prove (b), the partitions were assumed to be stochastic partitions of a certain type. Finally, Doléans ([5]) proved that if X has right continuous paths, then $Q_n(X)$ converges in L_p norm if $\|X\|_p < \infty$, $(p > 1)$, and converges in probability if $\|X\|_1 < \infty$.

In this section, we develop the theory of quadratic variation for continuous parameter martingales X on $[0, T]$, having independent increments. In contrast to the lines of research described above, we will need no assumptions about path regularity, nor will we require the separability of the process. We will prove first (Theorem 2.1) that there exists a universal constant K such that for every $\lambda \geqq 0$, $\lambda P(\sup_n Q_n(X) > \lambda) \leqq KE|X(T)|$. A maximal inequality of this type is new to the theory of quadratic variation. The proof of this result will yield the consequence that $E \sup_n Q_n(X) < \infty$, whenever $\|X\|_p < \infty$ for some $p > 1$. Next we prove (Theorem 2.2) that if $\|X\|_1 < \infty$, then $Q_n(X)$ converges in L_1 norm. This result should be compared with that of Doléans for right continuous martingales satisfying the same norm condition. It is known, incidentally, that $Q_n(X)$ need not converge in L_1 norm if X is a (right continuous) martingale not having independent increments. Theorem 2.2 is of value to the theory of stochastic integrals (see Section 3). Finally, we prove (Theorem 2.3) that if $E \sup_n Q_n(X) < \infty$ (a condition satisfied, for example, if $\|X\|_p < \infty$ for some $p > 1$) then $Q_n(X)$ converges a.e. Almost everywhere convergence of $Q_n(X)$ has not been established for general martingales, and this is a first attempt in that direction. The result is both more and less general than that of Cogburn and Tucker for infinitely divisible processes. It is more general in the sense that we do not make any assumptions on path regularity or about infinite divisibility; it is less general in that we have a norm condition on the process (viz. $\|X\|_p < \infty$) which Cogburn and Tucker do not need.

In what follows, we will make frequent use of an inequality of Marcinkiewicz and Zygmund ([11], Theorem 5) which we state for the convenience of the reader. Let $\{e_k, k = 1, 2, \cdots\}$ be a sequence of independent random variables having mean 0. Let $h_n = \sum_{k=1}^n e_k$, and $S_n = [\sum_{k=1}^n e_k^2]^{\frac{1}{2}}$. Then there are positive constants A_p and B_p, depending on p only, such that $A_p E S_n^p \leqq E|h_n|^p \leqq B_p E S_n^p$, whenever $p \geqq 1$.

LEMMA 2.1. (a) *Let X be a martingale with symmetrically distributed, independent increments. Suppose that $\|X\|_p < \infty$. Then $\{Q_n^p(X), n = 1, 2, \cdots\}$ is a reversed supermartingale if $1 \leqq p \leqq 2$, and a reversed submartingale if $p \geqq 2$.*

(b) *Let X and X' be two independent martingales, each with symmetric, independent increments. Then $P_n(X, X') = \sum_k d_{nk} d'_{nk}$ is a reversed martingale, where $d_{nk} = X(t_{n,k+1}) - X(t_{nk})$.*

PROOF. (a) Let $n_0 < n_1 < \cdots < n_r$ be positive integers. We will show that

$$E[Q_{n_o}^p \mid Q_{n_1}^p, \cdots, Q_{n_r}^p] \leqq Q_{n_1}^p \quad \text{if} \quad 1 \leqq p \leqq 2$$
$$\geqq Q_{n_1}^p \quad \text{if} \quad 2 \leqq p < \infty.$$

Let $\rho = (\rho_1, \rho_2, \cdots)$ be the Rademacher sequence on the Lebesgue unit interval: $\rho_k(s) = 1$ if $s \,\varepsilon\, [2j/2^k, (2j+1)/2^k)$ for some $j = 0, 1, \cdots, 2^{k-1} - 1$, and $\rho_k(s) = -1$ otherwise. Then ρ is a sequence of identically distributed, independent random variables on $[0, 1]$, with $\mu[\rho_k = \pm 1] = \frac{1}{2}$, where μ is Lebesgue measure. For each $s \,\varepsilon\, [0, 1]$ define a martingale $Y(s)\cdot = Y(s, \cdot)$ on $(\Omega, \mathcal{A}, P)$ with index $t \,\varepsilon\, [0, T]$ as follows:

$$Y(s, t) = \rho_1(s)X(t), \qquad \text{if} \quad 0 \leqq t \leqq t_{n_1,1},$$
$$= \rho_1(s)X(t_{n_1,1}) + \rho_2(s)[X(t) - X(t_{n_1,1})], \qquad \text{if} \quad t_{n_1,1} < t \leqq t_{n_1,2},$$

and so forth. Note that, for each s, $Y(s) \sim X$ (i.e., $Y(s)$ and X have the same distribution) and that if t' and t'' both belong to the same interval in the partition π_{n_1}, then

$$|Y(s, t'') - Y(s, t')| = |X(t'') - X(t')|.$$

Therefore, $Q_{n_j}(Y(s)) = Q_{n_j}(X)$, $j = 1, 2, \cdots, r$. Let f be a bounded Borel measurable function on R^r, and let $g(X) = f(Q_{n_1}(X)^p, \cdots, Q_{n_r}(X)^p)$. Then $g(Y(s)) = g(X)$ and we have

$$\int_\Omega Q_{n_0}(X)^p g(X)\, dP = \int_0^1 \int_\Omega Q_{n_0}(Y(s))^p g(Y(s))\, dP\, ds \qquad (\text{since } Y(s) \sim X)$$
$$= \int_\Omega g(X) \int_0^1 Q_{n_0}(Y(s))^p\, ds\, dP$$
$$\geqq \int_\Omega g(X) Q_{n_1}(X)^p\, dP \quad \text{if} \quad 2 \leqq p < \infty$$
$$\leqq \int_\Omega g(X) Q_{n_1}(X)^p\, dP \quad \text{if} \quad 1 \leqq p \leqq 2,$$

since
$$\int_0^1 Q_{n_0}(Y(s))^p\, ds \leqq [\int_0^1 Q_{n_0}(Y(s))^2\, ds]^{p/2}$$
$$= Q_{n_1}(X)\, Q_{n_1}(X)^p \quad \text{if} \quad 1 \leqq p \leqq 2$$

and
$$\int_0^1 Q_{n_0}(Y(s))^p\, ds \geqq [\int_0^1 Q_{n_0}(Y(s))^2\, ds]^{p/2}$$
$$= Q_{n_1}(X)^p \quad \text{if} \quad 2 \leqq p < \infty.$$

(b) Since it is readily verified that $P_n(X, X')$ is integrable, we need only show that $E[P_{n_0} \mid P_{n_1}, \cdots, P_{n_r}] = P_{n_1}$, where $n_0 < n_1 < \cdots < n_r$. If $\rho = (\rho_1, \rho_2, \cdots)$ is again the Rademacher sequence, define for each $s \,\varepsilon\, [0, 1]$ martingales $Y(s)$ and $Y'(s)$ corresponding to X and X' as described in (a). Then $P_{n_j}(Y(s), Y'(s)) = P_{n_j}(X, X')$, $j = 1, 2, \cdots, r$. Let f be a bounded Borel measurable function on R^r, and let $g(X, X') = f(P_{n_1}(X, X'), \cdots, P_{n_r}(X, X'))$.

Then $g(Y(s), Y'(s)) = g(X, X')$ and we have

$$\begin{aligned}\int_\Omega g(X, X')P_{n_0}(X, X')\,dP &= \int_0^1 \int_\Omega g(Y(s), Y'(s))P_{n_0}(Y(s), Y'(s))\,dP\,ds\\ &= \int_\Omega g(X, X') \int_0^1 P_{n_0}(Y(s), Y'(s))\,ds\,dP\\ &= \int_\Omega g(X, X') \int_0^1 P_{n_1}(Y(s), Y'(s))\,ds\,dP\\ &= \int_\Omega g(X, X')P_{n_1}(X, X')\,dP.\end{aligned}$$

I am indebted to D. L. Burkholder for suggesting the device of the Rademacher functions used above; the resulting proof is simpler than my original one.

THEOREM 2.1. *Let X be a martingale with independent increments. There exists a universal constant K such that for all $\lambda \geqq 0$:*

$$\lambda P\{\sup_n Q_n(X) > \lambda\} \leqq KE\,|X(T)|.$$

PROOF. Suppose first that X has symmetric, independent increments. Then $Q_n(X)$ is a reversed supermartingale, and, by an inequality of Doob ([6], page 314), $\lambda P\{\sup_n Q_n(X) > \lambda\} \leqq \sup_n EQ_n(X)$. But, by the inequality of Marcinkiewicz and Zygmund, $EQ_n(X) \leqq KE\,|X(T)|$, so the theorem is true in the special case.

If X does not have symmetric increments, let X' be a martingale with the same distribution as X, but independent of X. Then the martingale $X - X'$ has independent, symmetric increments, so that

$$2KE\,|X(T)| \geqq KE\,|X(T) - X'(T)| \geqq \lambda P\{\sup_n Q_n(X - X') > \lambda\}.$$

Since $Q_n(X - X') \geqq |Q_n(X) - Q_n(X')|$, we have

$$\begin{aligned}(1)\qquad 2KE\,|X(T)| &\geqq \lambda P\{\sup_n |Q_n(X) - Q_n(X')| > \lambda\}\\ &\geqq (\lambda/2)P\{\sup_n |Q_n(X) - \mu_n| > \lambda\}\end{aligned}$$

by the symmetrization inequalities (Loève, [10], page 247). Here μ_n is the median of Q_n. If U is a non-negative, integrable random variable with median m, then m minimizes $E\,|U - b|$ as a function of b, so that $m = E\,|m - U + U| \leqq E\,|m - U| + EU \leqq 2EU$. Thus, if $W = 2K \sup_n E\,|X(T)|$, then $\mu_n \leqq W$ and

$$(2)\qquad P\{\sup_n |Q_n(X) - \mu_n| > \lambda\} \geqq P\{\sup_n Q_n(X) > \lambda + W\}.$$

The final inequality now follows from (1) and (2).

COROLLARY. *If X is a martingale with independent increments and $\|X\|_p < \infty$ for some $p > 1$, then $E \sup_n Q_n(X) < \infty$.*

PROOF. Apply the proof of Theorem 2.1 to $Q_n^p(X)$, and obtain

$$\lambda P\{\sup Q_n^p(X) > \lambda\} \leqq K\,\|X\|_p^p.$$

The corollary follows from this.

LEMMA 2.2. *Let X be a martingale with independent increments. Then $Q_n(X)$ converges in probability.*

PROOF. If X has symmetric increments, then $Q_n(X)$ is a reversed supermartingale, and so converges a.e. *A fortiori*, $Q_n(X)$ converges in probability.

If X does not have independent increments, then consider X' and $Q_n(X - X')$ as in Theorem 2.1. Then

$$Q_n^2(X - X') = Q_n^2(X) + Q_n^2(X') - 2 \textstyle\sum_k d_{nk} d'_{nk} \tag{3}$$

converges a.e. We show next that $\sum_k d_{nk} d'_{nk}$ converges in probability. Let X'' and X''' be independent martingales, each with the same distribution as X, but independent of X and X'. By Lemma 2.1, $P_n = \sum_k (d_{nk} - d''_{nk})(d'_{nk} - d'''_{nk})$ is an L_1 bounded, reversed martingale, and so converges a.e. and also in L_1 (since it is reversed). Hence $E(P_n \mid X, X')$ must converge in L_1. Since X, X', X'', X''' are all independent, the preceding statement implies that $\sum_k d_{nk} d'_{nk}$ converges in L_1, by an elementary calculation. It follows that $Q_n^2(X) + Q_n^2(X')$ converges in probability. But,

$$\begin{aligned}
&[\{Q_n^2(X) - Q_m^2(X) > \epsilon\} \cap \{Q_n^2(X') - Q_m^2(X') > \epsilon\}] \cup \\
&\qquad [\{Q_m^2(X) - Q_n^2(X) > \epsilon\} \cap \{Q_m^2(X') - Q_n^2(X') > \epsilon\}] \\
&\qquad\qquad \subset \{|Q_n^2(X) + Q_n^2(X') - Q_m^2(X) - Q_m^2(X')| > 2\epsilon\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&P^2\{Q_n^2(X) - Q_m^2(X) > \epsilon\} + P^2\{Q_m^2(X) - Q_n^2(X) > \epsilon\} \\
&\qquad \leqq P\{|Q_n^2(X) + Q_n^2(X') - Q_m^2(X) - Q_m^2(X')| > 2\epsilon\},
\end{aligned}$$

since $X \sim X'$. Therefore, $P\{|Q_n^2(X) - Q_m^2(X)| > \epsilon\} \to 0$ as $m, n \to \infty$, proving the lemma.

THEOREM 2.2. *Let X be a martingale with independent increments. Then $Q_n(X)$ converges in L_1 norm.*

PROOF. By Lemma 2.2, it is enough to show that the family $\{Q_n(X)\}$ is uniformly integrable. If X' is chosen as in Theorem 2.1, then $Q_n(X - X')$ is a reversed, L_1 bounded supermartingale, by Lemma 2.1. Therefore, the family $\{Q_n(X - X')\}$ is uniformly integrable. Since $Q_n(X - X') \geqq |Q_n(X) - Q_n(X')|$, the family $\{Q_n(X) - Q_n(X')\}$ is likewise uniformly integrable. By Lemma 2.2, $Q_n(X) - Q_n(X')$ converges in probability and so converges in L_1 norm. Hence, $E\{Q_n(X) - Q_n(X') \mid X\} = Q_n(X) - EQ_n(X')$ converges in L_1, implying that the family $\{Q_n(X) - EQ_n(X)\}$ is uniformly integrable. Since the constants $EQ_n(X)$ are uniformly bounded by the result of Marcinkiewicz and Zygmund used before, we conclude that $\{Q_n(X)\}$ is uniformly integrable.

THEOREM 2.3. *Let X be a martingale with independent increments such that $E \sup_n Q_n(X) < \infty$. Then $Q_n(X)$ converges almost everywhere.*

Note that the hypothesis on $\sup_n Q_n(X)$ will be satisfied if, for example, $\|X\|_p < \infty$ for some $p > 1$, by the corollary to Theorem 2.1.

PROOF. As in the proof of Lemma 2.2, $Q_n^2(X - X') = Q_n^2(X) + Q_n^2(X')$

$- 2 \sum_k d_{nk} d'_{nk}$ converges a.e. To show that $\sum_k d_{nk} d'_{nk}$ converges a.e., note that, if $\{P_n\}$ is the sequence defined in the proof of Lemma 2.2,

$$\begin{aligned}|P_n| &\leqq [\textstyle\sum_k (d_{nk} - d''_{nk})^2]^{\frac{1}{2}}[\sum_k (d'_{nk} - d'''_{nk})^2]^{\frac{1}{2}} \\ &\leqq 2[Q_n(X) + Q_n(X'')][Q_n(X') + Q_n(X''')],\end{aligned}$$

implying that

$$\begin{aligned}E \sup_n |P_n| &\leqq 2E(\sup_n Q_n(X) + \sup_n Q_n(X''))(\sup_n Q_n(X') + \sup_n Q_n(X''')) \\ &= 8[E \sup_n Q_n(X)]^2 < \infty.\end{aligned}$$

Since P_n converges a.e., it follows from the dominated convergence theorem for conditional expectations that $E(P_n \mid X, X')$ converges a.e. Hence, $\sum_k d_{nk} d'_{nk}$ converges a.e. and therefore so does $Q_n^2(X) + Q_n^2(X')$. If we can show that $Q_n^2(X) - Q_n^2(X')$ converges a.e., then the theorem will follow upon adding the two. To do this, let M be a large positive number, and let

$$A_1 = \{\omega: \sup_n Q_n(X) \leqq M\}, \qquad A_2 = \{\omega: \sup_n Q_n(X') \leqq M\}$$

and I_i = indicator of A_i. By the dominated convergence theorem for conditional expectations,

$$E[I_1 I_2\{Q_n^2(X) + Q_n^2(X')\} \mid X] = Q_n^2(X) I_1 P(A_2) + I_1 E\{Q_n^2(X') I_2\}$$

converges a.e., and, by conditioning on X' instead of X, we find also that $Q_n^2(X') I_2 P(A_2) + I_2 E\{Q_n^2(X) I_1\} = Q_n^2(X') I_2 P(A_2) + I_2 E\{Q_n^2(X') I_2\}$ (since $X \sim X'$) converges a.e. Hence,

$$[Q_n^2(X) I_1 - Q_n^2(X') I_2] P(A_2) + (I_1 - I_2) E\{Q_n^2(X') I_2\},$$

which we obtain by subtracting the two expressions in the preceding sentence, converges a.e. Therefore, $Q_n^2(X) - Q_n^2(X')$ converges a.e. on $A_1 \cap A_2$. Let $M \uparrow \infty$ so that $A_1 \cap A_2 \uparrow \Omega$ to obtain the result.

We conclude this section with a conjecture: if X is a martingale on $[0, T]$ having independent increments, then $E \sup_n Q_n(X) < \infty$.

3. An application to the theory of stochastic integrals. In this section we apply Theorems 1.1 and 2.2 to the theory of stochastic integrals. We refer the reader to [13] for more detail on the concepts and terminology used below. Suppose now that $X = \{X(t), \mathfrak{A}(t), t \geqq 0\}$ is a right continuous, L_1 bounded martingale having independent increments. Let $\pi = \{\pi_n\}$ be a sequence of partitions of $[0, t]$ that become dense in the interval. For each interval $[0, t]$ we may compute (by Theorem 2.2) $\lim_{n\to\infty} Q_n(X, t) = Q(t)$ as a limit in L_1 norm. The increasing process $Q^2(t)$ so obtained may be taken to have right continuous paths. Let v be a left continuous step function (see [13]), and define the norm $n_1(v) = E^{\frac{1}{2}} \int v^2(t)\, dQ^2(t)$. Since $Q(t)$ exists as a limit in L_1 norm, the discussion of ([13], section 8) yields the following result: If $\{v_n\}$ is a sequence of step functions converging to v in the norm n_1, then the step function integrals $\int v_n(t)\, dX(t)$ converge in probability to a limit denoted by $\int v(t)\, dX(t)$. This extends the definition of stochastic integral to all processes v which belong to the completion of the step functions under the norm n_1. Moreover, by Theorem 1.1, we have, whenever v_n is a step function: $E\,|\int_0^T v_n(t)\,dX(t)| \leqq 2E\,|X(T)|$. An application of Fatou's lemma then yields, for all processes v in the n_1 completion of the step functions, the inequality

$$E\,|\textstyle\int_0^T v(t)\,dX(t)| \leqq 2E\,|X(T)|.$$

This generalizes Theorem 1.1 to the continuous parameter case.

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REFERENCES

[1] Baxter, G. (1956). A strong limit theorem for Gaussian processes. *Proc. Amer. Math. Soc.* **7** 522–527.

[2] Burkholder, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[3] Cameron, R. H. and Martin, W. T. (1947). The behavior of measure and measurability under a change of scale in Wiener space. *Bull. Amer. Math. Soc.* **53** 130–137.

[4] Cogburn, R. and Tucker, H. G. (1961). A limit theorem for a function of the increments of a decompsable process. *Trans. Amer. Math. Soc.* **99** 278–284.

[5] Doléans, C. (1968). Variation quadratique des martingales continues à droite. *Ann. Math. Statist.* **40** 284–289.

[6] Doob, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[7] Fisk, D. L. (1966). Sample quadratic variation of sample continuous, second order martingales. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **6** 273–278.

[8] Kunita, H. and Watanabe, S. (1967). On square integrable martingales. *Nagoya Math. J.* **30** 209–245.

[9] Lévy, P. (1940). Le mouvement Brownien plan. *Amer. J. Math.* **62** 487–550.

[10] Loève, M. (1963). *Probability Theory.* Van Nostrand, New York.

[11] Marcinkiewicz, J. and Zygmund, A. (1938). Quelques théorèmes sur les fonctions indépendantes. *Studia Math.* **7** 104–120.

[12] Meyer, P. A. (1967). Intégrales stochastiques, I. *Séminaire de Probabilités I, Université de Strasbourg*, Springer-Verlag, Heidelberg. (Lecture Notes in Mathematics, vol. 39).

[13] Millar, P. W. (1968). Martingale integrals. *Trans. Amer. Math. Soc.* **133** 145–166.