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ON THE PROBABILITY OF LARGE DEVIATIONS AND EXACT SLOPES[1]

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1. Summary. The purpose of this paper is to investigate a certain probability of a large deviation for a sequence of random variables $\{W_n\}$ which have moment-generating functions. We will assume that the mean of W_n is given by $n\mu_n$ and the variance by $n\sigma_n^2$, where $\{\mu_n\}$ and $\{\sigma_n^2\}$ are covergent sequences. We seek the limit, as $n \to \infty$, of the expression

$$n^{-1} \ln P[W_n > na_n],$$

where $\{a_n\}$ is a convergent sequence with $\lim a_n > \lim \mu_n$. It is shown that, if the moment-generating function of W_n satisfies certain limiting conditions, the above expression has a limit which depends on certain limits of this moment-generating function and its derivative. This result can be used in the computation of exact slopes for test statistics whose moment-generating function is known under the null hypothesis. Some applications are given.

2. Introduction. Let $W_1, W_2, \cdots$ be a sequence of nondegenerate random variables. We denote the cdf of W_n by $H_n(w) = P[W_n \leqq w]$, the moment-generating function of W_n by $m_n(t) = \int_{-\infty}^{\infty} e^{tw}\, dH_n(w)$, and let $\Psi_n(t) = \ln m_n(t)$. We assume the following conditions:

(i) $m_n(t) < \infty$ for some interval of t values, $-A < t < B$, $A, B > 0$.

(ii) For $t \varepsilon [0, B)$, $n^{-1}\Psi_n(t)$ has a finite limit as $n \to \infty$, which we shall denote by

$$c_0(t) = \lim_{n\to\infty} n^{-1}\Psi_n(t).$$

(iii) For $t \varepsilon [0, B)$, $n^{-1}\Psi_n'(t)$ has a finite limit as $n \to \infty$, which we shall denote by

$$c_1(t) = \lim_{n\to\infty} n^{-1}\Psi_n'(t),$$

and moreover, for fixed t,

$$|n^{-1}\Psi_n'(t) - c_1(t)| = O(n^{-1}).$$

(iv) For $t \varepsilon [0, B)$, $n^{-1}\Psi_n''(t)$ has a finite limit as $n \to \infty$, which we shall denote by

$$c_2(t) = \lim_{n\to\infty} n^{-1}\Psi_n''(t)$$

and

$$c_2(t) > 0.$$

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(v) For $t \varepsilon [0, B)$ and $\{t_n\}$ an arbitrary sequence of numbers in $(-A, B)$ converging to t, we have

$$n^{-1}\Psi_n'''(t_n) = O(1) \quad \text{as} \quad n \to \infty.$$

For $0 < h < B$, we define a cdf which is a function of h and $H_n(w)$ by the equation

$$\bar{H}_n(w) = (1/m_n(h)) \int_{(-\infty,w]} e^{hy}\, dH_n(y).$$

We shall denote a random variable having the cdf $\bar{H}_n(w)$ by $\bar{W}_n$. The moment-generating function of $\bar{W}_n$ is

$$\bar{m}_n(t) = m_n(t+h)/m_n(h)$$

and it is clear from condition (i) that $\bar{m}_n(t) < \infty$ for $-A - h < t < B - h$. Let $\bar{\Psi}_n(t) = \ln \bar{m}_n(t)$.

The importance of $\bar{H}_n(w)$ lies in the relation

$$H_n(w) = m_n(h) \int_{(-\infty,w]} e^{-hy}\, d\bar{H}_n(y), \tag{1}$$

which enables us to express the cdf of W_n in terms of the cdf of $\bar{W}_n$.

THEOREM 1. *Assume that the sequence* $\{W_n\}$ *satisfies conditions* (i)–(v). *Assume that* $a \varepsilon I = \{c_1(t): 0 < t < B\}$ *and that* $\{a_n\}$ *is a sequence such that*

$$a_n = a + (\epsilon_n n^{-\frac{1}{2}}),$$

where $\lim_n \epsilon_n = \epsilon$, $-\infty < \epsilon < \infty$. *Then*

$$\lim_n \{-n^{-1} \ln P[W_n > na_n]\} = ha - c_0(h), \tag{2}$$

where $h \varepsilon (0, B)$ *is the unique solution to the equation* $a = c_1(h)$.

The result in (2) also holds for the same expression with $>$ replaced by $\geqq$.

The following special case has been treated in [6], but is included here for completeness. Suppose $W_n = X_1 + \cdots + X_n$, where $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with common moment-generating function $m(t) < \infty$ for $-A < t < B$, $A, B > 0$. Further, suppose $a \varepsilon \{m'(t)/m(t): 0 < t < B\}$ and $\{a_n\}$ is a sequence such that $a_n = a + (\epsilon_n n^{-\frac{1}{2}})$, where $\lim_n \epsilon_n = \epsilon$, $-\infty < \epsilon < \infty$. Then

$$\lim_n \{(-n^{-1}) \ln P[W_n > na_n]\} = ha - \ln m(h), \tag{3}$$

where $h \varepsilon (0, B)$ is the unique solution to the equation $a = m'(h)/m(h)$. The result (3) follows from (2) since $n^{-1}\Psi_n(t) = \ln m(t)$ is independent of n and, as a consequence, conditions (i)–(v) are satisfied with $c_0(t) = \ln m(t)$ and $c_1(t) = m'(t)(m(t))^{-1}$.

As another useful special case, suppose $W_n = K(X_1) + \cdots + K(X_n)$, where $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with common exponential density function

$$f(x; \theta) = \exp[\theta K(x) + S(x) + q(\theta)]; \qquad \alpha < x < \beta, \qquad \gamma < \theta < \delta.$$

Assume $q'''(\theta)$ exists, $\gamma < \theta < \delta$. Assume that $a \varepsilon \{-q'(\theta + t): 0 < t < \delta - \theta\}$ and that $\{a_n\}$ is a sequence such that $a_n = a + (\epsilon_n n^{-\frac{1}{2}})$, where $\lim_n \epsilon_n = \epsilon$, $-\infty < \epsilon < \infty$.
Then

$$(4) \qquad \lim_n \{-n^{-1} \ln P[W_n > na_n]\} = ha - q(\theta) + q(\theta + h)$$

where $h \varepsilon (0, \delta - \theta)$ is the unique solution to the equation $a = -q'(\theta + h)$. The result (4) follows from (2) since $(1/n)\Psi_n(t) = q(\theta) - q(\theta + t)$ is independent of n and, as a consequence, conditions (i)–(v) are satisfied with $c_0(t) = q(\theta) - q(\theta + t)$ and $c_1(t) = -q'(\theta + t)$.

3. Proof of Theorem 1. We begin with some asymptotic properties of $\bar{W}_n$ and shall then use equation (1) to establish equation (2).

LEMMA 1. *Under the assumptions of the theorem,*

$$(5) \qquad \lim_n n^{-1}E[\bar{W}_n] = a$$

and

$$(6) \qquad |n^{-1}E[\bar{W}_n] - a| = O(n^{-1}).$$

PROOF. Since $E[\bar{W}_n] = \bar{\Psi}_n'(0) = \Psi_n'(h)$ and $c_1(h) = a$, the lemma is immediate from condition (iii).

LEMMA 2. *Under the assumptions of the theorem,*

$$(7) \qquad \text{Var}\,(\bar{W}_n) = nc_n,$$

where $c_n > 0$ for all n and

$$(8) \qquad \lim_n c_n = c_2(h) > 0.$$

PROOF. We have $\text{Var}\,(\bar{W}_n) = \bar{\Psi}_n''(0) = \Psi_n''(h) = nc_n$, where

$$(9) \qquad c_n = n^{-1}\Psi_n''(h).$$

Since nc_n is the variance of a nondegenerate random variable, $c_n > 0$, and equation (8) follows from condition (iv).

LEMMA 3. *Under the assumptions of the theorem,*

$$U_n = (\bar{W}_n - na)n^{-\frac{1}{2}}$$

has a limiting normal distribution with mean 0 *and variance* $c_2(h)$.

PROOF. The moment-generating function of U_n is

$$E[e^{tU_n}] = e^{-tn^{\frac{1}{2}}a} m_n(h + tn^{-\frac{1}{2}}) m_n^{-1}(h)$$

and $\ln E[e^{tU_n}] = -tn^{\frac{1}{2}}a + \Psi_n(h + tn^{-\frac{1}{2}}) - \Psi_n(h)$. Using Taylor's expansion of the function $\Psi_n(s)$ about the point h, we have for $s = h + tn^{-\frac{1}{2}}$,

$$(10) \quad \ln E[e^{tU_n}] = -tn^{\frac{1}{2}}a + tn^{-\frac{1}{2}}\Psi_n'(h) + (t^2/2n)\Psi_n''(h) + [t^3/(3!\,n^{\frac{3}{2}})]\Psi_n'''(\xi_n)$$
$$= tn^{\frac{1}{2}}[n^{-1}\Psi_n'(h) - a] + (t^2/2n)\Psi_n''(h) + [t^3/(3!\,n^{\frac{3}{2}})]\Psi_n'''(\xi_n),$$

where ξ_n is between h and $h + tn^{-\frac{1}{2}}$. From equation (6), since $\Psi_n'(h) = E[\bar{W}_n]$, the first term of equation (10) is $O(n^{-\frac{1}{2}})$. From equation (9), we have $t^2 c_n/2$ for the second term of equation (10) and for the last term we observe that $\lim_n \xi_n = h$ and by condition (v), $(1/n)\Psi_n'''(\xi_n)$ is $O(1)$ as $n \to \infty$. Hence the last term of equation (10) is $O(n^{-\frac{1}{2}})$. Then

$$\ln E[e^{tU_n}] = t^2 c_n/2 + O(n^{-\frac{1}{2}}),$$

and using equation (8), the limit of this expression as $n \to \infty$ is $t^2 c_2(h)/2$. The lemma then follows from a continuity theorem for moment-generating functions ([7], page 432).

Proof of Theorem 1. Using equation (1), we can write

$$P[W_n > na_n] = m_n(h) \int_{(na_n,\infty)} e^{-hw}\, d\bar{H}_n(w).$$

If we let $\sigma_n^2 = \mathrm{Var}(\bar{W}_n) = nc_n$ and change the variable of integration to $z = (w - na)/\sigma_n$, we have

$$P[W_n > na_n] = m_n(h)e^{-hna} \int_{(\bar{a}_n,\infty)} e^{-h\sigma_n z}\, d\bar{H}_n(\sigma_n z + na)$$

where $\bar{a}_n = \epsilon_n c_n^{-\frac{1}{2}}$. Thus

$$-n^{-1} \ln P[W_n > na_n] = ha - n^{-1}\Psi_n(h) - n^{-1} \ln f(n) \tag{11}$$

where we have let

$$f(n) = \int_{(\bar{a}_n,\infty)} e^{-h\sigma_n z}\, d\bar{H}_n(\sigma_n z + na).$$

Since the lower limit of integration, $\bar{a}_n$, has a finite limit as $n \to \infty$, there exists finite N, a' and a'' such that for all $n > N$, $a' < \bar{a}_n < a''$. Then for $n > N$,

$$f(n) \leqq \int_{(a',\infty)} e^{-h\sigma_n z}\, d\bar{H}_n(\sigma_n z + na) \leqq e^{-h\sigma_n a'} P[a' < V_n] = e^{-h\sigma_n a'} d_n,$$

where $V_n = (\bar{W}_n - na)\sigma_n^{-1}$ and $d_n = P[a' < V_n]$,
and further,

$$f(n) \geqq \int_{(a'',a''+1)} e^{-h\sigma_n z}\, d\bar{H}_n(\sigma_n z + na)$$
$$\geqq e^{-h\sigma_n (a''+1)} P[a'' < V_n < a'' + 1] = e^{-h\sigma_n (a''+1)} d_n^*,$$

where $d_n^* = P[a'' < V_n < a'' + 1]$.
Using Lemma 3, we have

$$\lim_n d_n = 1 - \Phi(a') > 0 \tag{12}$$

and

$$\lim_n d_n^* = \Phi(a'' + 1) - \Phi(a'') > 0, \tag{13}$$

where $\Phi(x)$ is the standard normal cdf. There exists finite $N' > N$ such that for all $n > N'$, $d_n > 0$ and $d_n^* > 0$. Then for $n > N'$,

$$-h(a'' + 1)\sigma_n + \ln d_n^* \leqq \ln f(n) \leqq -ha'\sigma_n + \ln d_n$$
$$ha' c_n^{\frac{1}{2}} n^{-\frac{1}{2}} - n^{-1} \ln d_n \leqq -n^{-1} \ln f(n) \leqq h(a'' + 1)c_n^{\frac{1}{2}} n^{-\frac{1}{2}} - n^{-1} \ln d_n^*.$$

From equations (8), (12) and (13), the extremes of the above inequality are $O(n^{-\frac{1}{2}})$. Then $-n^{-1} \ln f(n)$ is $O(n^{-\frac{1}{2}})$ and hence $o(1)$ as $n \to \infty$. Applying this to equation (11) we have

$$\lim_n \{-n^{-1} \ln \Pr [W_n > na_n]\} = \text{ha} - \lim_n n^{-1}\Psi_n(h) = ha - c_0(h).$$

Finally, to show that the equation $a = c_1(h)$ has a unique solution h for given a, we shall show that $c_1(t)$ is a strictly increasing function of t, $0 \leqq t < B$. For arbitrary $t_1, t_2 \varepsilon [0, B)$, $t_1 < t_2$, we have,

$$\Psi_n'(t_2) - \Psi_n'(t_1) = \int_{t_1}^{t_2} \Psi_n''(t)\, dt$$

for each n. If we multiply by n^{-1}, pass to the limit as $n \to \infty$ and apply Fatou's lemma, we have

$$c_1(t_2) - c_1(t_1) = \lim_n \int_{t_1}^{t_2} n^{-1}\Psi_n''(t)\, dt \geqq \int_{t_1}^{t_2} c_2(t)\, dt > 0.$$

Hence, $c_1(t)$ is a strictly increasing function.

4. Rate of convergence to 0 of the significance level of a test. For each n, let W_n be a statistic with a distribution determined by a real valued parameter θ taking values in an interval $\Omega = [\theta_0, \infty)$, $-\infty < \theta_0 < \infty$. Let $H_n(w; \theta) = P_\theta[W_n \leqq w]$ denote the cdf of W_n. When θ_0 obtains, let $m_n(t) = \int_{-\infty}^{\infty} e^{tw}\, dH_n(w; \theta_0)$ denote the moment-generating function of W_n, and let $\Psi_n(t) = \ln m_n(t)$.

Assume that $m_n(t)$ and $\Psi_n(t)$ satisfy conditions (i)–(v). Further, assume that the sequence $\{W_n\}$ satisfies the following conditions when other θ values obtain:

(vi) $$E_\theta(W_n) = n\mu(\theta) \qquad \text{for } \theta \varepsilon \Omega,$$

and $\mu(\theta)$ is a strictly increasing function of θ which does not depend on n;

(vii) $$\mathrm{Var}_\theta (W_n) = nb_n(\theta) \qquad \text{for } \theta \varepsilon \Omega,$$

and $\lim_n b_n(\theta) = b(\theta)$, $0 < b < \infty$; and

(viii) $$(W_n - n\mu(\theta))/(nb_n(\theta))^{\frac{1}{2}}$$

has a limiting normal distribution with mean 0 and variance 1 for all $\theta \varepsilon \Omega$.

Suppose we wish to test the hypothesis $H_0{:}\theta = \theta_0$ against $H_1{:}\theta > \theta_0$. For each n, consider the test which rejects H_0 if W_n is observed to be greater than some critical value, say w_n. Let the power functions of these tests be denoted $P_n(\theta) = P_\theta[W_n > w_n]$.

Fix an alternative $\theta > \theta_0$ and let $\{w_n\}$ be any sequence of critical values satisfying

$$w_n > n\mu(\theta), \qquad (w_n - n\mu(\theta))n^{-\frac{1}{2}} = \epsilon_n \to \epsilon, \tag{14}$$

as $n \to \infty$, where $0 < \epsilon < \infty$. For such a sequence of critical values, $P_n(\theta)$ is asymptotically bounded away from 0 or 1 since $\lim_n P_n(\theta) = 1 - \Phi(\epsilon/(b(\theta))^{\frac{1}{2}})$, and the rate of convergence to 0 of the significance levels can be specified. To see this, note that from (14) we can write $w_n = n[\mu(\theta) + \epsilon_n n^{-\frac{1}{2}}]$. If $\mu(\theta) \varepsilon I = \{c_1(t){:}0 < t < B\}$, then the sequence satisfies the hypothesis of the theorem and as a result we have

(15) $\lim_n \{-n^{-1} \ln P_{\theta_0}[W_n > w_n]\} = h(\theta)\mu(\theta) - c_0(h(\theta)) = e(\theta)$ say,

where $h(\theta) \varepsilon (0, B)$ is the solution to the equation $c_1(h(\theta)) = \mu(\theta)$.

In typical examples, $I = \{\mu(\theta):\theta \varepsilon \Omega\}$.

The function $e(\theta)$ defined in equation (15) provides a measure of the rate of convergence to 0 of the significance levels of the tests. Bahadur refers to $2e(\theta)$ as the exact slope of $\{W_n\}$.

Let $\alpha_n(\theta) = P_{\theta_0}[W_n > w_n]$ and for $\delta \varepsilon (0, 1)$, define $N(\delta, \theta)$ to be the least integer m such that

$$\alpha_m(\theta) \leqq \delta. \tag{16}$$

Then

$$\lim_{\delta\to 0} (-\ln \delta)/N(\delta, \theta) = e(\theta). \tag{17}$$

The result (17) is essentially the same as a result in [5], Section 5; it follows from (15) and the inequalities

$$\alpha_{N(\delta,\theta)}(\theta) \leqq \delta < \alpha_{N(\delta,\theta)-1}(\theta).$$

Suppose that two sequences of test statistics, $\{W_n^{(i)}\}$, $i = 1, 2$, have been proposed to test the hypothesis H_0 against H_1. Let $e^{(i)}(\theta)$, $i = 1, 2$, denote their respective limits in (15). Then Bahadur [3] has defined the asymptotic relative efficiency (A.R.E.) of $\{W_n^{(1)}\}$ to $\{W_n^{(2)}\}$, when $\theta > \theta_0$ obtains, as

$$e_{1,2}(\theta) = e_1(\theta)/e_2(\theta).$$

If for each sequence $\{W_n^{(i)}\}$ and $\delta \varepsilon (0, 1)$ we consider the integer $N^{(i)}(\delta, \theta)$ as defined in (16), $i = 1, 2$, then

$$\lim_{\delta\to 0} N^{(2)}(\delta, \theta)/N^{(1)}(\delta, \theta) = e_{1,2}(\theta).$$

This is immediate from (17).

5. Applications.

EXAMPLE 1. *Sign Test.* Let m denote the median of a probability distribution with a continuous cdf $F(x)$. Let $X_1, X_2, \cdots$ denote a sequence of independent random variables which have common cdf $F(x)$. Consider testing $H_0:m = m_0$ against $H_1:m > m_0$ for some number m_0. If we let $\theta = 1 - F(m_0)$, the above is equivalent to the test of $H_0:\theta = \frac{1}{2}$ against $H_1:\theta > \frac{1}{2}$. Let $\Omega = [\frac{1}{2}, 1)$ denote the parameter space.

For each n, let $W_n = \#X_i > m_0:1 \leqq i \leqq n$; where "#" reads "the number of." It is clear that W_n has a binomial (n, θ) distribution and if $\theta = \frac{1}{2}$, the moment-generating function of W_n is

$$m_n(t) = [\tfrac{1}{2}(1 + e^t)]^n.$$

Conditions (i)–(viii) are satisfied with $\mu(\theta) = \theta$, $b(\theta) = \theta(1 - \theta)$, $c_0(t) = \ln [\frac{1}{2}(1 + e^t)]$ and $c_1(t) = e^t/(1 + e^t)$. Note $\{c_1(t):t \geqq 0\} = \Omega$.

Using (15), a simple calculation gives

$$e(\theta) = \theta \ln (2\theta) + (1 - \theta) \ln [2(1 - \theta)],$$

a result which was obtained earlier by Bahadur [2] by other means.

EXAMPLE 2. *Sample Median Test.* Let $F(x)$ be a cdf which admits a continuous density $f(x)$ such that $f(x)$ is symmetric about 0 and $f(0) > 0$. Let $X_1, X_2, \cdots$ denote a sequence of independent random variables which have common cdf $F(x - \theta)$. For each n, let $W_n = nU_n$, where $U_n = \text{median } \{X_1, \cdots, X_n\}$. (To avoid trivial complications, assume in what follows that n takes odd integer values.)

To test the hypothesis $H_0{:}\theta = \theta_0$ against $H_1{:}\theta > \theta_0$, $-\infty < \theta_0 < \infty$, consider for each n, the test based on W_n; large values of which are significant. The sequence $\{W_n\}$ satisfies conditions (vi)–(viii) with $\mu(\theta) = \theta$ and $b(\theta) = \frac{1}{4}[f(0)]^{-2}$. However, the moment-generating function of W_n is not available and the theorem is not directly applicable to determine (15). Instead, we relate the distribution of U_n to the binomial distribution.

For $a > \theta_0$ such that $F(a - \theta_0) < 1$, consider

$$P_{\theta_0}[U_n > a] = P_{\theta_0}[Z_n \geqq \tfrac{1}{2}(n + 1)],$$

where $Z_n = \# X_i > a{:}1 \leqq i \leqq n$. When θ_0 obtains, Z_n has a binomial (n, p) distribution with $p = p(a) = 1 - F(a - \theta_0)$. Note, $0 < p < \frac{1}{2}$. Further, the moment-generating function of Z_n is given by $m_n(t) = [1 - p + pe^t]^n$ and conditions (i)–(v) are satisfied with

$$c_0(t) = \ln [1 - p + pe^t] \quad \text{and} \quad c_1(t) = pe^t[1 - p + pe^t]^{-1}.$$

Since $\frac{1}{2} \varepsilon \{c_1(t){:}0 < t < \infty\} = (p, 1)$, we can apply the theorem to give

$$\begin{aligned} \lim_n \{-n^{-1} \ln P_{\theta_0}[U_n > a]\} &= \lim_n \{-n^{-1} \ln P_{\theta_0}[Z_n \geqq \tfrac{1}{2}(n + 1)]\} \\ &= \tfrac{1}{2}h - c_0(h) \\ &= e(a) \quad \text{say,} \end{aligned}$$

where $h = h(a)$ is the solution to $\frac{1}{2} = c_1(h)$. A calculation shows

$$e(a) = -\tfrac{1}{2} \ln [4p(1 - p)].$$

Consider the tests based on $\{W_n\}$. As in Section 4, for a fixed alternative $\theta > \theta_0$, let $\{w_n\}$ be a sequence of critical values satisfying (14). Then the significance levels of the tests are $P_{\theta_0}[W_n > w_n] = P_{\theta_0}[U_n > \theta + \epsilon_n n^{-\frac{1}{2}}]$. As $n \to \infty$, the significance levels converge to 0 at the same rate as $P_{\theta_0}[U_n > \theta]$. (The significance levels are asymptotically bounded above and below by sequences of the form $P_{\theta_0}[U_n > a]$ for suitable a and $e(a)$ is a continuous function of a.) Hence, for the sample median test, the limit in (15) is given by

$$e_U(\theta) = e(\theta) = -\tfrac{1}{2} \ln [4p(1 - p)]$$

where $p = p(\theta) = 1 - F(\theta - \theta_0)$.

It is interesting to compare the sign test to the median test in this example.

For the sign test we have (from Example 1 with a change of notation)

$$e_S(\theta) = q \ln (2q) + (1 - q) \ln [2(1 - q)]$$

where $q = q(\theta) = 1 - F(\theta_0 - \theta)$. Note $q(\theta) = 1 - p(\theta)$. An algebraic argument shows that $e_S(\theta) < e_U(\theta)$ for all $\theta > \theta_0$. Hence, the A.R.E. of the sign test to the median test, $e_{S,U}(\theta) = e_S(\theta)/e_U(\theta)$, is strictly less than unity for all $\theta > \theta_0$. Also, $e_{S,U}(\theta) \to 1$ as $\theta \to \theta_0$ (which agrees with the Pitman efficiency) and $e_{S,U}(\theta) \to 0$ as $\theta \to \infty$.

EXAMPLE 3. *Sample Variance Test.* Let $X_1, X_2, \cdots$ denote a sequence of independent random variables which have a common normal distribution with mean μ and variance θ. Consider testing the hypothesis $H_0:\theta = \theta_0$ against $H_1:\theta > \theta_0$, for some number $\theta_0 > 0$.

Let $\Omega = [\theta_0, \infty)$ denote the parameter space and fòr each n, let

$$W_n = (n/(n - 1)) \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

It is clear that W_n is of the form $n\theta Y/(n - 1)$, where Y has a chi-square distribution with $n - 1$ degrees of freedom and if $\theta = \theta_0$, the moment-generating function of W_n is given by

$$m_n(t) = \{1 - (2n\theta_0 t/(n - 1))\}^{-\frac{1}{2}(n-1)}; \qquad t < (n - 1)/(2n\theta_0).$$

Conditions (i)–(viii) are satisfied with $\mu(\theta) = \theta$, $b(\theta) = 2\theta^2$, $c_0(t) = -\frac{1}{2} \ln [1 - 2\theta_0 t]$ and $c_1(t) = \theta_0/(1 - 2\theta_0 t)$ for $t < 1/(2\theta_0)$. Note that

$$\{c_1(t):t \geqq 0\} = \Omega.$$

From (15), with a simple calculation, we have

$$e(\theta) = \tfrac{1}{2}[(\theta/\theta_0) - 1 + \ln (\theta_0/\theta)].$$

EXAMPLE 4. *One- and two-sample tests of location for normal populations.* Consider the problem of testing the hypothesis $H_0:\theta = 0$ against $H_1:\theta > 0$ for a random variable X having a normal distribution with mean θ and variance σ^2 and the related two-sample problem of testing the same hypothesis for a pair of independent, normally distributed random variables, X and Y, with respective means μ and $\mu + \theta$ and equal variances σ^2. Without loss of generality, assume $\sigma^2 = 1$.

Consider the tests based on the sample mean $\bar{X}$, the one-sample t statistic $T^{(1)}$ and the one-sample Wilcoxon W for the one-sample problem and the tests based on the difference of the sample means $\bar{Y} - \bar{X}$, the two-sample t statistic $T^{(2)}$ and the Mann-Whitney M for the two-sample problem. (Assume equal sample sizes.)

For the one-sample tests, we have

$$e_{\bar{X}}(\theta) = \tfrac{1}{2}\theta^2,$$

$$e_{T^{(1)}}(\theta) = \tfrac{1}{2} \ln [1 + \theta^2], \quad \text{and}$$

$$e_W(\theta) = 2h'p_\theta' - \ln [\cosh (h')],$$

where $p_\theta' = P_\theta[X_1 + X_2 > 0] - \frac{1}{2}$, and $h' = h'(\theta)$ satisfies the equation

$$\int_0^1 x \tanh(h'x)\, dx = p_\theta'.$$

Further, we have the following relations between the one- and two-sample tests:

$$e_{\bar{X}}(\tfrac{1}{2}\theta) = \tfrac{1}{2}e_{\bar{Y}-\bar{X}}(\theta), \tag{18}$$

$$e_{T^{(1)}}(\tfrac{1}{2}\theta) = \tfrac{1}{2}e_{T^{(2)}}(\theta), \tag{19}$$

and

$$e_W(\tfrac{1}{2}\theta) = \tfrac{1}{2}e_M(\theta). \tag{20}$$

(In the computation of (15) for the two-sample statistics, the combined sample size is $2n$.)

From (18), (19) and (20), it readily follows that

$$e_{W,\bar{X}}(\tfrac{1}{2}\theta) = e_{M,\bar{Y}-\bar{X}}(\theta), \quad \text{and} \quad e_{W,T^{(1)}}(\tfrac{1}{2}\theta) = e_{M,T^{(2)}}(\theta).$$

That is to say, for corresponding pairs of tests, the efficiency at θ in the two-sample case is the same as the efficiency at $\frac{1}{2}\theta$ in the one-sample case. (This relation was suggested by Hoadley in [8], Section 8.)

To verify these relations, we must establish (18), (19) and (20).

The formulas for $e_{\bar{X}}$ and $e_{T^{(1)}}$ have been given in [2] and e_W is from [10]. Moreover, $e_{T^{(2)}}(\theta) = \ln[1 + \frac{1}{4}\theta^2]$ (see [11] or [8], line (8.6)) and $e_{\bar{Y}-\bar{X}}(\theta) = \frac{1}{4}\theta^2$ (see [11] or [1], lemma 3). Hence (18) and (19) are clear.

For the Mann-Whitney M, we have from [11]

$$e_M(\theta) = 2hp_\theta + \ln(4) - 2\ln[e^h + 1]$$

where $p_\theta = P_\theta[X < Y]$ and $h = h(\theta)$ is the solution to

$$\int_1^2 xe^{hx}/(e^{hx} - 1)\, dx - \int_0^1 xe^{hx}/(e^{hx} - 1)\, dx = p_\theta.$$

This is a new form of the formula given in [8].

One can check that $p_\theta = \Phi(2^{-\frac{1}{2}}\theta)$ and $p_\theta' = \Phi(2^{\frac{1}{2}}\theta) - \frac{1}{2}$, where $\Phi(x)$ is the standard normal cdf. Hence $p_{\frac{1}{2}\theta}' = p_\theta - \frac{1}{2}$.

It can also be verified that if $h' = h'(\frac{1}{2}\theta)$ is the solution to the equation

$$\int_0^1 x \tanh(h'x)\, dx = p_{\frac{1}{2}\theta}',$$

then $2h' = h$. To see this, write

$$\begin{aligned} p_\theta - \tfrac{1}{2} = p_{\frac{1}{2}\theta}' &= \int_0^1 x(e^{h'x} - e^{-h'x})/(e^{h'x} + e^{-h'x})\, dx \\ &= \int_0^1 x(e^{2h'x} - 1)/(e^{2h'x} + 1)\, dx \\ &= \int_0^1 x(e^{4h'x} - 2e^{2h'x} + 1)/(e^{4h'x} - 1)\, dx \\ &= \int_0^1 2x(e^{4h'x} - e^{2h'x})/(e^{4h'x} - 1)\, dx - \int_0^1 x\, dx, \end{aligned}$$

or

$$\begin{aligned} p_\theta &= \int_0^1 2x(e^{4h'x} - e^{2h'x})/(e^{4h'x} - 1)\, dx \\ &= \int_0^1 \{[2x(2e^{4h'x})/(e^{4h'x} - 1)] - [2x(e^{4h'x} + e^{2h'x})/(e^{4h'x} - 1)]\}\, dx \\ &= \int_0^2 xe^{2h'x}/(e^{2h'x} - 1)\, dx - 2\int_0^1 xe^{2h'x}/(e^{2h'x} - 1)\, dx \\ &= \int_1^2 xe^{2h'x}/(e^{2h'x} - 1)\, dx - \int_0^1 xe^{2h'x}/(e^{2h'x} - 1)\, dx. \end{aligned}$$

Hence $2h' = h$. Then

$$\begin{aligned} e_W(\tfrac{1}{2}\theta) &= 2h'p'_{\frac{1}{2}\theta} - \ln[\tfrac{1}{2}(e^{h'} + e^{-h'})] \\ &= 2h'p_\theta + \ln(2) - \ln[e^{2h'} + 1] \\ &= \tfrac{1}{2}[2hp_\theta + \ln(4) - 2\ln(e^h + 1)] \\ &= \tfrac{1}{2}e_M(\theta), \text{ which is (20).} \end{aligned}$$

EXAMPLE 5. *A nonparametric test of independence.* Let $Z_1, Z_2, \cdots$ denote a sequence of independent random variables $Z_n = (X_n, Y_n)$ which have a common bivariate distribution with continuous cdf $F(x, y)$ and continuous marginal cdf's $G(x)$ and $H(y)$. Suppose we wish to test the hypothesis $H_0{:}F(x, y) = G(x)H(y)$.

For each n, if the ranks of $Y_1, \cdots, Y_n$ are arranged in the natural order 1, 2, $\cdots$, n, then the ranks of the corresponding X's will be a permutation of 1, 2, $\cdots$, n and one way to measure the disarray of the ranks of the X's from the natural order is by counting the number of inversions of order among the ranks of the X's, say Q_n. If we let $V_{ij} = \operatorname{sgn}(X_i - X_j)\operatorname{sgn}(Y_i - Y_j)$, where $\operatorname{sgn}(a) = +1(-1)$ if $a > 0 (a < 0)$, then

$$Q_n = \sum_{1 \leq i < j \leq n} \tfrac{1}{2}(1 - V_{ij}).$$

Under H_0, the statistic

$$T_n = 1 - [4Q_n/n(n-1)]$$

is symmetrically distributed on $[-1, 1]$ and hence has expectation 0. Under H_0, $\operatorname{Var}(T_n) = 2(2n + 5)/9n(n - 1)$. In general,

$$E(T_n) = E[V_{12}] = \tau \text{ say,}$$

and

$$\operatorname{Var}(T_n) = (2/n(n-1))[\operatorname{Var}(V_{12}) + 2(n-2)\operatorname{Cov}(V_{12}, V_{13})].$$

Kendall [9] has discussed using T_n as a nonparametric test of the hypothesis H_0.

Consider $W_n = nT_n$. Under H_0, since the moment-generating function of Q_n is given by

$$(1/n!)\prod_{i=1}^n [(e^{it} - 1)/(e^t - 1)],$$

the moment-generating function of W_n is

$$m_n(t) = (e^{nt}/n!) \prod_{i=1}^{n} [(e^{-4it/(n-1)} - 1)/(e^{-4t/(n-1)} - 1)].$$

Then

$$\begin{aligned}\Psi_n(t) &= \ln m_n(t)\\ &= nt - \ln (n!) + \sum_{i=1}^{n} \ln [1 - e^{-4it/(n-1)}] - n \ln [1 - e^{-4t/(n-1)}]\\ &= nt - \sum_{i=1}^{n} \ln (i/n) + \sum_{i=1}^{n} \ln [1 - e^{-4it/(n-1)}] - n \ln [n(1 - e^{-4t/(n-1)})].\end{aligned}$$

Thus we have

$$\begin{aligned}c_0(t) &= \lim_{n\to\infty} n^{-1}\Psi_n(t)\\ &= t - \int_0^1 \ln (x)\, dx + \int_0^1 \ln [1 - e^{-4tx}]\, dx - \ln (4t)\\ &= (t + 1) - \ln (4t) + \int_0^1 \ln [1 - e^{-4tx}]\, dx.\end{aligned}$$

Also

$$\Psi_n'(t) = n + \sum_{i=1}^{n} (4i/(n - 1))/(e^{4it/(n-1)} - 1) - (4n/(n - 1))/(e^{4t/(n-1)} - 1)$$

and

$$c_1(t) = \lim_{n\to\infty} n^{-1}\Psi_n'(t) = 1 - t^{-1} + \int_0^1 4x/(e^{4tx} - 1)\, dx.$$

Note that $\{c_1(t) : t > 0\} = (0, 1)$.

In a similar manner, it can be checked that $\Psi_n''(t)$ and $\Psi_n'''(t)$ satisfy conditions (iv) and (v). Conditions (vi) and (vii) hold with $E[W_n] = n\tau$ and

$$\text{Var}\,(W_n) = (2n/(n - 1))[\text{Var}\,(V_{12}) + 2(n - 2)\,\text{Cov}\,(V_{12}, V_{13})].$$

Condition (viii) is verified in [9]. Thus from (15), we have

$$e_W = h\tau - c_0(h)$$

where $h = h(\tau)$ is the solution to $\tau = c_1(h)$.

EXAMPLE 6. *Tests of location for double exponential distributions.* Let $X_1, X_2, \cdots$ denote a sequence of independent random variables which have a common double exponential distribution with density function

$$f(x, \theta) = \tfrac{1}{2}e^{-|x-\theta|}; \quad -\infty < x < \infty,\ \theta \geqq \theta_0.$$

Suppose we wish to test the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$. Let $\Delta = \theta - \theta_0$.

For each n, consider the sign test based on $W_n^{(1)} = \#X_i > \theta_0 : 1 \leqq i \leqq n$. Then from Example 1 we have

$$e_1(\theta) = q \ln (2q) + (1 - q) \ln [2(1 - q)],$$

where $q = 1 - \frac{1}{2}e^{-\Delta}$. This can be written as

$$e_1(\theta) = [1 - \tfrac{1}{2}e^{-\Delta}] \ln [2e^{\Delta} - 1] - \Delta.$$

For each n, consider the test based on the sample median $W_n^{(2)}$. Then from Example 2 we have

$$e_2(\theta) = -\tfrac{1}{2} \ln [4p(1 - p)],$$

where $p = \frac{1}{2}e^{-\Delta}$. This can be written as

$$e_2(\theta) = \tfrac{1}{2}\Delta - \tfrac{1}{2} \ln [2 - e^{-\Delta}].$$

For each n, consider the test based on $W_n^{(3)} = X_1 + \cdots + X_n$. Under the null hypothesis, the moment-generating function of $W_n^{(3)}$ is

$$m_n(t) = [e^{\theta_0 t}/(1 - t^2)]^n; \qquad |t| < 1.$$

Conditions (i)–(viii) are satisfied with $\mu(\theta) = \theta$, $b(\theta) = 2$, $c_0(t) = \theta_0 t - \ln(1 - t^2)$ and $c_1(t) = \theta_0 + [2t/(1 - t^2)]$. Further, $\{c_1(t): 0 < t < 1\} = (\theta_0, \infty)$ and using (15) we have

$$e_3(\theta) = (\theta - \theta_0)h + \ln (1 - h^2)$$

where h satisfies the equation $\theta_0 + [2h/(1 - h^2)] = \theta$. This can be written as

$$e_3(\theta) = -1 + (1 + \Delta^2)^{\frac{1}{2}} + \ln [2\{-1 + (1 + \Delta^2)^{\frac{1}{2}}\}/\Delta^2].$$

Bahadur [4], [5] has shown, under certain general conditions, that the likelihood ratio test of Neyman and Pearson has an optimal exact slope given by $2J(\theta)$

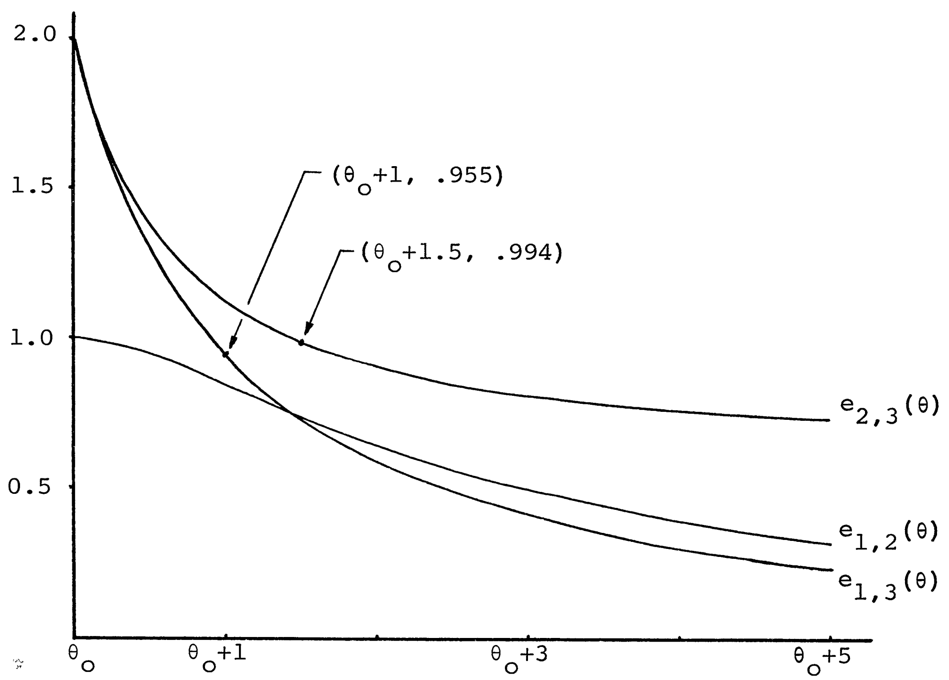

FIG. 1. $e_{1,2}(\theta)$, $e_{1,3}(\theta)$ and $e_{2,3}(\theta)$

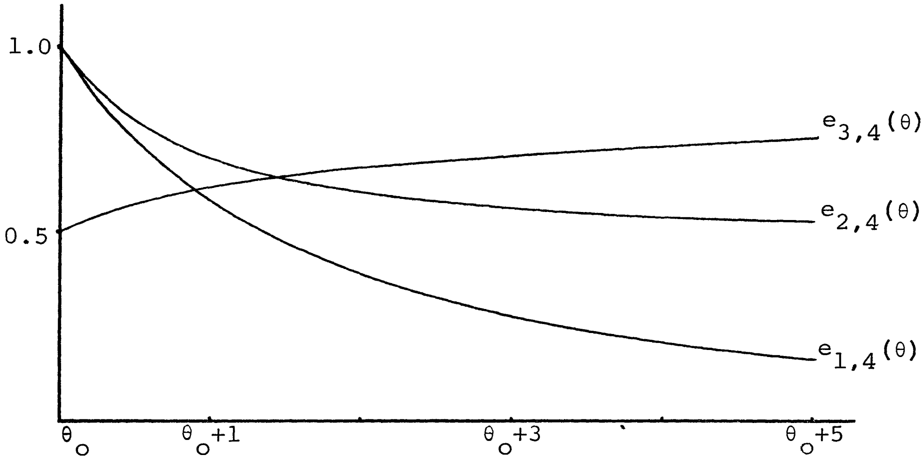

FIG. 2. $e_{1,4}(\theta)$, $e_{2,4}(\theta)$ and $e_{3,4}(\theta)$

(see [5], page 310). In the double exponential case of this example the required conditions (given in [4]) are met and

$$J(\theta) = E_\theta[\log \{f(x, \theta)/f(x, \theta_0)\}] = \Delta - 1 + e^{-\Delta}.$$

Hence, for the sequence of likelihood ratio statistics, the limit in (15) is $J(\theta) = e_4(\theta)$ say.

Certain values of the A.R.E. curves $e_{i,j}(\theta) = e_i(\theta)/e_j(\theta)$, $1 \leqq i < j \leqq 4$, were determined on a computer and the results are plotted in Figures 1 and 2. The limits as $\theta \to \theta_0$ were checked analytically and agree with the Pitman efficiencies for these cases. Also $e_{1,4}(\infty) = 0$, $e_{2,4}(\infty) = \frac{1}{2}$, $e_{3,4}(\infty) = 1$, $e_{1,2}(\infty) = 0$, $e_{1,3}(\infty) = 0$ and $e_{2,3}(\infty) = \frac{1}{2}$.

From Figure 1, we can conclude that $W_n^{(1)}$ and $W_n^{(2)}$ are locally more efficient than $W_n^{(3)}$, but the reverse is true for alternatives farther away from the null hypothesis. As indicated in example 2, $W_n^{(1)}$ is less efficient than $W_n^{(2)}$. From Figure 2, we can conclude that the likelihood ratio test is uniformly (in θ) more efficient than the other tests. It is interesting to note that $W_n^{(1)}$ and $W_n^{(2)}$ become fully efficient as $\theta \to \theta_0$ and that $W_n^{(3)}$ becomes fully efficient as $\theta \to \infty$.

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