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EXTENSION OF A RESULT OF SENETA FOR THE SUPER-CRITICAL GALTON–WATSON PROCESS

BY C. C. HEYDE

Australian National University

1. Introduction. Let $Z_0 = 1, Z_1, Z_2 \cdots$ denote a super-critical Galton–Watson process whose non-degenerate offspring distribution has probability generating function $F(s) = \sum_{j=0}^{\infty} s^j \Pr(Z_1 = j)$, $0 \leqq s \leqq 1$, where $1 < m = EZ_1 < \infty$. The Galton–Watson process evolves in such a way that the generating function $F_n(s)$ of Z_n is the nth functional iterate of $F(s)$ and, for the super-critical case in question, the probability of extinction of the process, q, is well known to be the unique real number in $[0, 1)$ satisfying $F(q) = q$. It is the main purpose of this paper to establish the following theorem which gives an ultimate form of the limit result for the case in question.

THEOREM 1. *There exists a sequence of positive constants* $\{c_n, n \geqq 1\}$ *with* $c_n \to \infty$ *and* $c_n^{-1}c_{n+1} \to m$ *as* $n \to \infty$ *such that the random variables* $c_n^{-1}Z_n$ *converge almost surely to a non-degenerate random variable* W *for which* $\Pr(W = 0) = q$ *and which has a continuous distribution on the set of positive real numbers. Let* s_0 *be any fixed number in* $(0, -\log q)$. *Then,* c_n *can be taken as* $[h_n(s_0)]^{-1}$ *where* $h_n(s)$ *is the inverse function of* $k_n(s) = -\log E\{\exp(-sZ_n)\}$.

This result constitutes an extension of the main result of Seneta [6] where convergence in distribution was established. It should be remarked that, when $EZ_1 = \infty$, it is not possible to find a sequence of positive constants $\{c_n\}$ for which $c_n^{-1}Z_n$ converges in distribution to a non-degenerate limit law ([7] Theorem 4.4).

By way of comparison with Theorem 1, we note that:

THEOREM A. (Stigum [8], Kesten and Stigum [3]). *As* $n \to \infty$, $m^{-n}Z_n$ *converges almost surely to a random variable* W_1 *for which* $\Pr(W_1 = 0) = q$ *or* 1 *and which, if* $\Pr(W_1 = 0) < 1$, *has a continuous density on the set of positive real numbers. Moreover, the following two conditions are equivalent:*

(i) $E(Z_1 \log Z_1) < \infty$.
(ii) $\Pr(W_1 = 0) = q$.

Thus, when $E(Z_1 \log Z_1) = \infty$, the norming by m^n is not appropriate and a more subtle norming is required to obtain a non-degenerate limit law. Almost sure convergence in Theorem A is based on the fact (due to Doob) that the process $\{m^{-n}Z_n\}$ is a martingale. The process $\{h_n(s_0)Z_n\}$ is, as was noted in [6], a submartingale but the submartingale convergence theorem is only applicable when $E(Z_1 \log Z_1) < \infty$.

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2. Proof of Theorem 1. Firstly we note the following results of [6]. $k_n(s) = -\log E\{\exp(-sZ_n)\}, s \geqq 0$, is the nth functional iterate of

$$k(s) = -\log E\{\exp(-sZ_1)\}.$$

$k_n(s)$ is continuous, strictly monotone, and strictly concave for $s \geqq 0$ and its inverse function $h_n(s)$ (the nth functional iterate of $h(s) = k^{-1}(s)$) exists for $0 \leqq s < -\log q$ and has properties which are dual to those of $k_n(s)$. Let s_0 be any fixed number in $(0, -\log q)$.

Now, for $n \geqq 1$ let $\mathscr{F}_n$ be the σ-field generated by $Z_1, \cdots, Z_n$ and consider the process $\{\exp(-h_n(s_0)Z_n)\}$. Then,

$$\begin{aligned} E[\exp(-h_{n+1}(s_0)Z_{n+1}) \mid \mathscr{F}_n] &= [E[\exp(-h_{n+1}(s_0)Z_1)]]^{Z_n} \\ &= \exp(-Z_n k(h_{n+1}(s_0))) \\ &= \exp(-h_n(s_0)Z_n), \end{aligned}$$

so that $\{\exp(-h_n(s_0)Z_n), \mathscr{F}_n\}$ is a martingale. Furthermore, $0 \leqq \exp\{-h_n(s_0)Z_n\} \leqq 1$, so the martingale convergence theorem gives the almost sure convergence of $\{\exp(-h_n(s_0)Z_n)\}$ to a finite limit. It has already been demonstrated in [6] that $h_n(s_0)Z_n$ converges in distribution to a non-degenerate limit so almost sure convergence to a non-degenerate random variable W is established.

It is not shown explicitly in [6] that $h_n(s_0)[h_{n+1}(s_0)]^{-1} \to m$ as $n \to \infty$ but it follows readily from the results given therein since

$$h_n(s_0)[h_{n+1}(s_0)]^{-1} = h_n(s_0)[h(h_n(s_0))]^{-1} \to m$$

as $n \to \infty$. Furthermore, Seneta has not shown that the limit distribution function is continuous on the set of positive real numbers. It follows simply, however, from Equation 3.1 of [6], that the characteristic function $\phi(t)$ of W satisfies the functional equation

$$\phi(mt) = F(\phi(t)) \tag{1}$$

which is just that studied by Stigum [8]. Then, following [8] and noting that $\Pr(W = 0) = q$, we define a characteristic function

$$\Psi(t) = [\phi((1-q)t) - q]/(1-q),$$

and a probability generating function

$$h(s) = [F((1-q)s+q) - q]/(1-q),$$

so that, using (1),

$$\Psi(mt) = h(\Psi(t)).$$

It can then be deduced from Lemma 2 of [8] that $\lim_{|t|\to\infty} |\Psi(t)| = 0$. This ensures that the distribution function corresponding to Ψ is continuous ([5], 27), and hence that W has a continuous distribution on the set of positive real numbers. This completes the proof of the theorem.

3. A Wald type identity. Let T be a stopping rule on the sequence $\{Z_n\}$. That is, T is an integer-valued random variable such that the event $\{T \leqq n\} \in \mathscr{F}_n$ for every $n \geqq 1$ and $P(T < \infty) = 1$. We shall establish the following theorem.

THEOREM 2. *For any* s *in* $[0, -\log q)$, *we have* $e^s E\{\exp(-h_T(s)Z_T)\} = 1$.

PROOF. We have seen in the proof of Theorem 1 that, for fixed s in $(0, -\log q)$, $\{\exp(-h_n(s)Z_n), \mathscr{F}_n\}$ is a martingale. Also, the family $\{\exp(-h_n(s)Z_n)\}$ is trivially seen to be uniformly integrable so we may apply Theorem 2.2, Chapter 7, of Doob [1] and obtain

$$E\{\exp(-h_T(s)Z_T)\} = E\{\exp(-h(s)Z_1)\}$$
$$= \exp\{-k(h(s))\} = \exp\{-s\},$$

as required.

Theorem 2 is included in this paper as it follows so simply from the proof of Theorem 1. The result will be explored elsewhere.

4. An application of Theorem 1. In this section we shall establish the consistency in a certain sense of the estimator $\sum_{j=1}^{n} Z_j / \sum_{j=0}^{n-1} Z_j$ for m. This estimator has been discussed by Harris [2] who has shown that it is a maximum likelihood estimator for m and that, if $EZ_1^{\,2} < \infty$, it is consistent in the sense that

$$\lim_{n\to\infty} \Pr\left(\left|\left(\sum_{j=1}^{n} Z_j / \sum_{j=0}^{n-1} Z_j\right) - m\right| \geqq \varepsilon \,\middle|\, Z_n > 0\right) = 0$$

for every $\varepsilon > 0$. We shall strengthen this result and remove the restriction that $EZ_1^{\,2} < \infty$.

Firstly, we need the following theorem which is of some independent interest.

THEOREM 3. *If* $c_n^{-1} Z_n \to_{\text{a.s.}} W$ *where* $c_n \to \infty, c_n^{-1} c_{n+1} \to m$ *as* $n \to \infty$, *then* $c_n^{-1} \sum_{j=0}^{n} Z_j \to_{\text{a.s.}} mW/(m-1)$ *as* $n \to \infty$. ("a.s." denotes almost sure convergence).

PROOF. Take $c_0 = 1$ for convenience. Since $c_n^{-1} Z_n - W \to_{\text{a.s.}} 0$ as $n \to \infty$ we have, using the Toeplitz Lemma (e.g. Loève [4] 238),

$$\left\{\sum_{k=0}^{n} c_k [c_k^{-1} Z_k - W] / \sum_{k=0}^{n} c_k\right\} \to_{\text{a.s.}} 0,$$

which yields

$$\left(\sum_{k=0}^{n} c_k\right)^{-1} \sum_{k=0}^{n} Z_k \to_{\text{a.s.}} W. \tag{2}$$

Also, since $c_n^{-1} c_{n+1} \to m$ as $n \to \infty$, a further application of the Toeplitz Lemma gives

$$\left\{\sum_{k=0}^{n} c_k [c_k^{-1} c_{k+1} - m] / \sum_{k=0}^{n} c_k\right\} \to 0, \qquad \text{that is,}$$
$$\left\{1 + (c_{n+1} - 1) / \sum_{k=0}^{n} c_k\right\} \to m$$

as $n \to \infty$. This yields

$$\sum_{k=0}^{n} c_k \sim m c_n / (m-1) \tag{3}$$

and the desired result follows immediately from (2) and (3).

THEOREM 4. *Let $\mathscr{E}$ denote the event $\{Z_k > 0, k = 1, 2, 3, \cdots\}$. Then, for arbitrary $\varepsilon > 0$,*

$$\lim_{n\to\infty} \Pr(\max_{k\geqq n} |(\sum_{j=1}^{k} Z_j / \sum_{j=0}^{k-1} Z_j) - m| \geqq \varepsilon \mid \mathscr{E}) = 0.$$

PROOF. Define the random variables U_n, $n = 1, 2, 3, \cdots$, W^* as follows:

$$\begin{aligned} U_n &= h_n(s_0) \textstyle\sum_{j=0}^{n} Z_j && \text{if } Z_n > 0, \\ &= 1 && \text{if } Z_n = 0; \\ W^* &= W && \text{if } W > 0, \\ &= 1 && \text{if } W = 0. \end{aligned}$$

Then, it is clear from Theorem 1 and Theorem 3 that U_n converges almost surely to $mW^*/(m-1)$ as $n \to \infty$, the random variable W^* having a distribution function which is continuous at zero. We therefore have, since $\Pr(\mathscr{E}) = 1-q$,

$$\begin{aligned} &\Pr(\max_{k\geqq n} |(\textstyle\sum_{j=1}^{k} Z_j / \sum_{j=0}^{k-1} Z_j) - m| \geqq \varepsilon \mid \mathscr{E}) \\ &\quad = (1-q)^{-1} \Pr(\max_{k\geqq n} U_{k-1}^{-1} |h_{k-1}(s_0)\{[h_k(s_0)]^{-1} U_k - 1\} - mU_{k-1}| \geqq \varepsilon; \mathscr{E}) \\ &\quad \leqq (1-q)^{-1} \Pr(\max_{k\geqq n} U_{k-1}^{-1} |h_{k-1}(s_0)\{[h_k(s_0)]^{-1} U_k - 1\} - mU_{k-1}| \geqq \varepsilon). \end{aligned}$$

The result of the theorem then follows readily because $\Pr(W^* = 0) = 0$ and

$$\begin{aligned} &h_{k-1}(s_0)\{[h_k(s_0)]^{-1} U_k - 1\} - mU_{k-1} \\ &\quad = (h_{k-1}(s_0)[h_k(s_0)]^{-1} - m)U_k - h_{k-1}(s_0) + m(U_k - U_{k-1}) \to_{\text{a.s.}} 0 \end{aligned}$$

as $n \to \infty$ since $h_{k-1}(s_0)[h_k(s_0)]^{-1} \to m$, $h_{k-1}(s_0) \to 0$ and U_k converges almost surely.

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