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MINIMUM VARIANCE ORDER WHEN ESTIMATING THE LOCATION OF AN IRREGULARITY IN THE DENSITY

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1. Introduction and summary. Let $f(y)$ be a probability density on the real line, with

$$f(y) = {}^{+}R(y) \quad (y > 0), \qquad f(y) = {}^{-}R(|y|) \quad (y < 0) \tag{1}$$

where ${}^{+}R$ and ${}^{-}R$ are normalized slowly varying functions as $y \to 0$ (cf. [6] chapter 8, sect. 8). Let θ be a location parameter. Denote by X a sample $(x_1, \cdots, x_n)$ of n independent observations from the distribution defined by $f(x-\theta)$. By $t = t(X)$ we denote an unbiased estimator of θ. In this paper we study lower bounds for the variances $V_\theta(t)$, with special reference to their order, in n.

Considering only densities with regular variation at θ, (1) includes all cases where θ is the location either of a cusp or of a discontinuity with finite and positive values of $f(0-)$ and $f(0+)$.

Under some conditions on ${}^{+}R$ and ${}^{-}R$, we calculate a function $\psi(h)$ such that

$$V_\theta(t) \geqq K(\psi^{-1}(n^{-1}))^2 \qquad (\text{all } t) \quad (0 < K < \infty). \tag{2}$$

It is surmised that this lower variance bound is of the best possible order of n. The bound is sometimes $o(n^{-1})$; hyperefficient estimators should then be possible.

It is found that $\psi(h)$ depends heavily on the function ${}^{+}\varepsilon(s)$ defined by ${}^{+}R(y) = {}^{+}A \exp\{-\int_y^1 {}^{+}\varepsilon(s)/s\, ds\}$, and on the corresponding function ${}^{-}\varepsilon(s)$. In view of [6] chapter 8, sect. 9 (or (10) below), this form of ${}^{+}R(y)$ is not a constraint on $f(y)$.

The estimators t_0 constructed by Daniels [5] and Prakasa Rao [10] (for particular ${}^{+}R$ and ${}^{-}R$) have variances of the order of (2). This order is thus optimal with the densities considered: we have $V_\theta(t_0) \geqq \inf V_\theta(t) \geqq K(\psi^{-1}(n^{-1}))^2$. The Prakasa Rao estimators are hyperefficient.

The calculations are based, partly, on ideas from the author's paper [9].

Since a cusp may be a mode, the results of this paper contribute to the discussion on the estimation of the mode (see [4], [11] and references therein).

The generalization of (1) to regularly varying $f(y)$ as $y \downarrow 0$ and $y \uparrow 0$ will be treated elsewhere ([8]).—The generalization of (2) to biased estimators t (or mean square error) is straightforward, but some conditions on the bias function will be necessary.

Notation. K and K' denote positive, finite constants.

If there exist K, K' such that $K < a(x)/b(x) < K'$ for all x, $|x| < x_0$, we shall write $a(x) = \Omega(b(x))$ $(x \to 0)$; sometimes we omit $(x \to 0)$.

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2. The theorem. Two examples.

THEOREM. *Let* $t = (X)$ *be an unbiased estimator of* θ. *If*

(i) *the set* $\mathfrak{X} = \{x \,|\, f(x-\theta) > 0$ *if* $x \neq \theta, f(0) \geqq 0\}$ *does not depend on* θ

(ii) *there is an* h_0 *such that* $\mathscr{H} = \{h \,|\, |h| < h_0\} \subset \{h \,|\, \int (f(y-h))^2/f(y)\,dy$ is defined$\} \cap \{h \,|\, \theta + h$ *is a possible parameter value*$\}$

(iii) $f(y) = {}^{+}R(y) = {}^{+}A \exp\{-\int_y^1 {}^{+}\varepsilon(s)/s\,ds\}$ $\qquad y > 0$

$\qquad = {}^{-}R|y| = {}^{-}A \exp\{-\int_{|y|}^1 {}^{-}\varepsilon(s)/s\,ds\}$ $\qquad y < 0$

where ${}^{+}R$ *and* ${}^{-}R$ *are differentiable for* $|y| < y_0$,

$\lim {}^{+}\varepsilon(y) = \lim {}^{-}\varepsilon(y) = 0\ (y \to 0)$,

${}^{+}\varepsilon(y)$ *or* $-{}^{+}\varepsilon(y)$, *and* ${}^{-}\varepsilon(y)$ *or* $-{}^{-}\varepsilon(y)$ *vary regularly as* $y \to 0$, *and* $\pm({}^{+}\varepsilon(y) - {}^{-}\varepsilon(y))$ *varies regularly if both* $\pm{}^{+}\varepsilon(y)$ *and* $\pm{}^{-}\varepsilon(y)$ *vary slowly*

(iv) $\lim {}^{+}R(y)/{}^{-}R(y) = C$ *exists (finite or infinite) as* $y \to 0$

(v) $0 < \lim h^{-2} \int_{|y|>\eta} \{f(y-h)/f(y) - 1\}^2 f(y)\,dy < \infty$ $(h \to 0, \eta > 0$ *is fixed*$)$

then

$$\begin{aligned}
\int_{-\infty}^{+\infty} (f(y-h))^2/f(y)\,dy &= 1 + \psi(h) \\
&= 1 + \Omega(|h|{}^{+}R(|h|)({}^{+}\varepsilon(|h|))^2 + |h|{}^{-}R(|h|)({}^{-}\varepsilon(|h|))^2 \\
&\quad + h^2 \int_{\lambda|h|}^{\eta} y^{-2}({}^{+}\varepsilon(y))^2\, {}^{+}R(y)\,dy + h^2 \int_{\lambda|h|}^{\eta} y^{-2}({}^{-}\varepsilon(y))^2\, {}^{-}R(y)\,dy \\
&\quad + h^2 + I_0(h))
\end{aligned} \tag{3}$$

(*where* $I_0(h)$ *is found in Lemma* 2 *below*), *and*

$$V_\theta(t) \geqq K(\psi^{-1}(n^{-1}))^2. \tag{4}$$

For the proof we need two lemmas.

LEMMA 1. *Let* $H = |h|$,

$$\begin{aligned}
{}^{+}g_1(y,h) &= \{{}^{+}R(y-h)/{}^{+}R(y) - 1\}^2\, {}^{+}R(y) && (h > 0) \\
{}^{+}g_2(y,h) &= \{{}^{+}R(y)/{}^{+}R(y-H) - 1\}^2\, {}^{+}R(y-H) && (h < 0).
\end{aligned}$$

For $i = 1, 2$, *under the conditions of the theorem,*

$$I_{1i} = \int_H^{\lambda H} {}^{+}g_i(y,h)\,dy = \Omega(H\,{}^{+}R(H)({}^{+}\varepsilon(H))^2) \tag{5}$$

$$I_{2i} = \int_{\lambda H}^{\eta} {}^{+}g_i(y,h)\,dy = \Omega(H^2 \int_{\lambda H}^{\eta} y^{-2}({}^{+}\varepsilon(y))^2\, {}^{+}R(y)\,dy). \tag{6}$$

(5) *and* (6) *remain true if we change* ${}^{+}R$ *and* ${}^{+}\varepsilon$ *into* ${}^{-}R$ *and* ${}^{-}\varepsilon$ *throughout.*

$$\int_{|y|>\eta} (f(y-h)/f(y) - 1)^2 f(y)\,dy = \Omega(h^2). \tag{7}$$

The proof is deferred to the next section.

Next, put $H = |h|$, define C as in (iv) of the theorem and

$$I_0(h) = \int_0^h \{{}^-R(h-y)/{}^+R(y)-1\}^2 \; {}^+R(y)\,dy \qquad h > 0 \tag{8}$$
$$= \int_0^H \{{}^+R(H-y)/{}^-R(y)-1\}^2 \; {}^-R(y)\,dy \qquad h < 0.$$

LEMMA 2.

Case	*Order of I_0 if $h > 0$*	*Order of I_0 if $h < 0$*
1. $0 < C < \infty$, $C \neq 1$	$h\,{}^+R(h)$	$H\,{}^-R(H)$
2. $C = 0$	$h\,{}^+R(h)$	$H({}^+R(H))^2({}^-R(H))^{-1}$
3. $C = \infty$	$h({}^-R(h))^2({}^+R(h))^{-1}$	$H\,{}^-R(H)$
4. $C = 1$	$h\,{}^+R(h)\,{}^+Z(h)$	$H\,{}^-R(H)\,{}^-Z(H)$

Here, ${}^+Z(h)$ is defined by

$${}^+Z(h) = ({}^-\varepsilon(h))^2 + \{\textstyle\int_0^h ({}^-\varepsilon(s) - {}^+\varepsilon(s))s^{-1}ds\}^2 \text{ if both } \pm{}^+\varepsilon \text{ and } \pm{}^-\varepsilon \text{ vary slowly,}$$
$$= \{\textstyle\int_0^h {}^-\varepsilon(s)/s\,ds\}^2 + \{\int_0^h {}^+\varepsilon(s)/s\,ds\}^2 \quad \text{otherwise.}$$

${}^-Z$ is obtained from ${}^+Z$ by interchanging ${}^+\varepsilon$ and ${}^-\varepsilon$.
The proof of Lemma 2 is also deferred to the next section.

PROOF OF THE THEOREM. From the results of Chapman and Robbins [2], it follows that

$$V_\theta(t) \geqq h^2\{(1+\psi(h))^n - 1\}^{-1} \qquad (\text{any } h \in \mathscr{H}). \tag{9}$$

Since $(f(y-h))^2 = -(f(y))^2 + 2f(y-h)f(y) + (f(y-h)-f(y))^2$, we have

$$1+\psi(h) = -1+2+\int_{-\infty}^{+\infty}\{f(y-h)/f(y)-1\}^2 f(y)\,dy.$$

Split this integral at the points $-\eta$, $-\lambda|h|$, $-|h|$, 0, $|h|$, $\lambda|h|$, η and use (1). The lemmas then apply directly, giving (3). If we choose $h = \psi^{-1}(n^{-1})$, the denominator of (9) remains bounded, and (4) follows. The theorem is proved.

We notice that when I_0 has different orders for $h > 0$ and $h < 0$, the best bound (9) is obtained when I_0 tends most quickly to zero as $|h| \to 0$ (provided that the order of I_0 determines the order of $\psi(h)$). On the other hand, it is, of course, the term of (3) that tends to zero most slowly that decides the order of $\psi(h)$.

We give two examples to illustrate the theorem.

(a) Let ${}^+\varepsilon(s) = {}^-\varepsilon(s) = -vs^v$ and ${}^+A = {}^-A$. Then, for $v > 0, f(y) = K\exp(-|y|^v)$ has a cusp at $y = 0$. From (3) we then obtain

$$1+\psi(h) = 1+\Omega(h^2) \qquad v > \tfrac{1}{2}$$
$$= 1+\Omega(-h^2\log h) \qquad v = \tfrac{1}{2}$$
$$= 1+\Omega(h^{2v+1}) \qquad 0 < v < \tfrac{1}{2}.$$

Inverting the functions h^2, $-h^2\log h$ and h^{2v+1}, we find that $V(t) \geqq Kn^{-1}$ $(v > \frac{1}{2})$ $V(t) \geqq K(n\log n)^{-1}$ $(v = \frac{1}{2})$, $V(t) \geqq Kn^{-2/(2v+1)}$ $(0 < v < \frac{1}{2})$—the "steepness" of the cusp decides the order of the variance bound.

Prakasa Rao [10] (for $0 < v < \frac{1}{2}$) and Daniels [5] (for $v > \frac{1}{2}$) have constructed estimators in this case, which have variances of the orders just found. As indicated in the introduction, this is then the order of the minimum variance attainable. The case $v = \frac{1}{2}$ is left open by both these authors.

(b) Let ${}^{+}\varepsilon(s) = {}^{-}\varepsilon(s) = 0$. Then $f(y)$ is constant, say $f(y) = \beta > 0$ when $y < 0$ and $f(y) = \gamma > 0$ when $y > 0$. We may define $f(x-\theta)$ over the interval (0, 1)—this is the situation of [3]. Since C of (iv) and Lemma 2 equals β/γ, we get

$$\psi(h) = \Omega(h) \quad \text{if} \quad \beta \neq \gamma, \qquad \psi(h) = \Omega(h^2) \quad \text{if} \quad \beta = \gamma$$

and so $V(t) \geqq Kn^{-2}$ $(\beta \neq \gamma)$, in accordance with the result of [3], but $V(t) \geqq Kn^{-1}(\beta = \gamma)$—there is no discontinuity when $\beta = \gamma$!

3. Proofs of the lemmas. For reference, we recall that any slowly varying function at zero can be written as (cf. [6] chapter 8, sect. 9)

$$S(y) = a(y) \exp\{-\textstyle\int_y^1 \varepsilon(s)/s\, ds\} \tag{10}$$

where $a(y) \to A \neq 0$, and $\varepsilon(y) \to 0$ as $y \to 0$. If $a(y) \equiv A$, S is normalized. From (10), it is easy to show (i) that for each $\kappa > 0$ there is a $y_0 > 0$ such that

$$y^{\kappa} < S(y) < y^{-\kappa} \qquad (\text{all } y < y_0), \tag{11}$$

and (ii) that for each $\kappa > 0$, there is a $K > 0$ such that

$$(u/v)^{\kappa} < S(hu)/S(hv) < (u/v)^{-\kappa} \qquad (0 < u < v < K, \quad \text{all} \quad h \in \mathscr{H}). \tag{12}$$

Further, for $p > -1$, we have

$$\textstyle\int_0^x y^p S(y)\, dy \sim (p+1)^{-1} x^{p+1} S(x) \qquad (x \to 0). \tag{13}$$

PROOF OF LEMMA 1. In I_{11}, we put $(1-h/y) = t$ and obtain (writing R for ${}^{+}R$ and, later, ε for ${}^{+}\varepsilon$)

$$I_{11} = \textstyle\int_0^{1-\lambda^{-1}} \{R(ht/(1-t))/R(h/(1-t)) - 1\}^2 R(h/(1-t))(1-t)^{-2} h\, dt.$$

We extract $(1-t)^{-2}R(h/(1-t))$ by aid of the mean value theorem of integral calculus, and write the integrand in the form of (iii). We then get

$$I_{11} = \Omega(hR(h)) \textstyle\int_0^{1-\lambda^{-1}} \{\exp\{\int_J \varepsilon(s)/s\, ds\} - 1\}^2\, dt \tag{14}$$

where $J = (ht/(1-t),\ h/(1-t))$. In [9] we showed that $I_{11} = O(hR(h))$; here we need the exact order, which is always $o(hR(h))$. Because of (iii), put $\varepsilon(s) = \pm s^w W(s)$ with $w \geqq 0$ and slowly varying W. If $w > 0$, we have from (13)

$$\begin{aligned} \left|\pm \textstyle\int_J s^{w-1} W(s)\, ds\right| &\sim [w^{-1} s^w W(s)]_J \\ &= w^{-1}(1-t)^{-w} h^w W(h/(1-t))[1 - t^w W(ht/(1-t))\{W(h/(1-t))\}^{-1}]. \end{aligned}$$

From (11) and (12), it then follows that the exponent of (14) tends to zero as $h \to 0$. We then obtain

$$\begin{aligned} I_{11} &= \Omega(hR(h) \textstyle\int_0^{1-\lambda^{-1}} \{w^{-1}(1-t)^{-w} h^w W(h/(1-t))\}^2 (1-t^{w\pm\kappa})^2\, dt) \\ &= \Omega(hR(h) h^{2w} (W(h))^2) \qquad (w > 0) \end{aligned}$$

since the integral in t converges.

If $w = 0$, $\varepsilon(s) = \pm W(s)$, and the exponent in (14) may not tend to zero as $h \to 0$. The integration interval of I_{11} is divided into $(0, \tau)$ and $(\tau, 1-\lambda^{-1})$, where $\tau = \tau(h) = (W(h))^2$. For the interval $(0, \tau)$, we write the integral of (14) as

$$\int_0^{\tau(h)} \{R(ht/(1-t))/R(h/(1-t))-1\}^2 \, dt.$$

Using (13), we find that this is

$$O(\tau(h)\{(R(h\tau))^2(R(h))^{-2}+1\}) \tag{15}$$

Now, $|\log R(h\tau)/R(h)| = \int_{h\tau}^{h} W(s)/s \, ds \leqq \sup_{h\tau<s<h} W(s) \log \tau^{-1}$. But $W(s)$ varies slowly: $W(s) = a(s) \exp\{-\int_s^1 v(z)/z \, dz\}$. Then, $(W(h))^{-1} \sup W(s) = a(x)/a(h) \exp\{\int_h^x v(z)/z \, dz\} = a(x)/a(h) \exp\{v(y) \log x/h\} \leqq (1+K)\tau^{-|v(y)|}$, where $h\tau \leqq x \leqq h$, $x \leqq y \leqq h$. Since $v(s) \to 0 (s \to 0)$, we obtain

$$|\log R(h\tau)/R(h)| \leqq W(h)(1+K)\tau^{-\kappa} \log \tau^{-1}. \tag{16}$$

Since $\kappa > 0$ is arbitrary, and $\tau = (W(h))^2$, the logarithm tends to zero when $h \to 0$. Thus, from (15) and (16), the integral over $(0, \tau)$ is $O(\tau) = O(W^2) = O(\{\varepsilon(h)\}^2)$.

From (16), we know that $R(ht/(1-t))/R(h/(1-t)) \to 1$ for all t in $(\tau, 1-\lambda^{-1})$. The integrand of (14) is then $\Omega((\int_J W(s)/s \, ds)^2)$ in that interval. Applying the reasoning leading to (16), we find that this is $\Omega((W(h))^2)$. We have then proved (5) with $i = 1$. The proof for $i = 2$ follows in the same way if we first extract ${}^+R(y)/{}^+R(y-H)$ from the integrand of (5) and make use of (12).

Now consider (6). It is clear, that for every y in $(\lambda h, \eta)$,

$$R(y-h)/R(y) \to 1 \qquad (h \to 0). \tag{17}$$

Using (10), we then have $I_{21} = \Omega(\int_{\lambda h}^{\eta} \{\int_{y-h}^{y} \varepsilon(s)/s \, ds\}^2 R(y) \, dy)$. With $\varepsilon(y) = \pm y^w W(y)$, we have

$$\begin{aligned} \left|\int_{y-h}^{y} \varepsilon(s)/s \, ds\right| &\leqq \sup_{y-h<s<y} |\varepsilon(s)| \log y/(y-h) \\ &= \sup s^w W(s)\{y^w W(y)\}^{-1} |\varepsilon(y)| \, \Omega(h/y) \qquad (h/y \to 0) \\ &= \Omega(1) |\varepsilon(y)| \, \Omega(h/y) \end{aligned} \tag{18}$$

where the $\Omega(1)$ factor is obtained as in the reasoning leading to (16), if η is small enough. To obtain $\Omega(h/y)$, we must have λ large. A lower bound for I_{21}, of the same form as (18), can be found in the same manner, and so

$$I_{21} = \Omega(\int_{\lambda h}^{\eta} (\varepsilon(y))^2 h^2 y^{-2} R(y) \, dy).$$

The proof for I_{22} follows if we extract ${}^+R(y)/{}^+R(y-H)$ from the integrand of (6) and recall (17).

Finally, (7) follows from condition (v). Lemma 1 is proved.

Proof of Lemma 2. Take $h > 0$. The results for $h < 0$ will not be proved separately; they follow at once from the definition (8). Divide the integration interval into $(0, h/2)$ and $(h/2, h)$; call the integrals over these I_{01} and I_{02}. In I_{01}, put $y/h = u$, in I_{02}, put $h/y-1 = t$. Then,

$$I_{01} = \int_0^{\frac{1}{2}} \{{}^-R(h(1-u))/{}^+R(hu)-1\}^2 \; {}^+R(hu)h\,du$$

$$I_{02} = \int_0^1 \{{}^-R(ht/(1+t))/{}^+R(h/(1+t))-1\}^2 \; {}^+R(h/(1+t))(1+t)^{-2}h\,dt$$

In the integrand of I_{01}, we write

$$\text{(19)} \quad {}^-R(h(1-u))/{}^+R(hu) = \{{}^-R(h(1-u))/{}^-R(hu)\}\{{}^-R(hu)/{}^+R(hu)\}$$
$$= \exp\{\textstyle\int_{hu}^{h(1-u)} {}^-\varepsilon(s)/s\,ds\}(C+o(1)) \qquad (\text{as } hu \to 0).$$

If $\pm{}^-\varepsilon(s)$ varies regularly with positive exponent (that is, ${}^-\varepsilon(s) = \pm s^{w_1}W_1(s)$ with $w_1 > 0$), the exponent of (19) tends to zero as $h \to 0$, and so $I_{01} = \int_0^{\frac{1}{2}}\{(1+o(1))(C+o(1))-1\}^2\{{}^+R(hu)/{}^+R(h)\}{}^+R(h)h\,du$. When $C < \infty$, $C \neq 1$, the squared factor is $\Omega(1)$. From (12), we estimate ${}^+R(hu)/{}^+R(h)$ by $u^{\pm\kappa}$. The integral in u then converges, and we obtain $I_{01} = \Omega(h\,{}^+R(h))$—for the Cases 1, 2 with $h > 0$ and 3 with $h < 0$. I_{02} can be treated in the same manner; the result is $I_{02} = \Omega(h\,{}^+R(h))$ in the same cases.

If $\pm{}^-\varepsilon(s)$ varies slowly ($w_1 = 0$), we proceed as in the corresponding part of the proof for I_{11}, dividing the interval $(0, \frac{1}{2})$ into $(0, v(h))$ and $(v(h), \frac{1}{2})$ with suitably chosen v. We then again obtain $I_{01} = \Omega(h\,{}^+R(h)) = I_{02}$.

For Case 2 with $h < 0$ and Case 3 with $h > 0$, we estimate (cf. (12)) $(1-u)^{-\kappa_1}u^{\kappa_1} < {}^-R(h(1-u))/{}^-R(hu) < (1-u)^{\kappa_1}u^{-\kappa_1}$; $u^{\kappa_2} < {}^-R(hu)/{}^-R(h) < u^{-\kappa_2}$; $u^{\kappa_3} < {}^+R(h)/{}^+R(hu) < u^{-\kappa_3}$. Then I_{01} takes values in the interval with endpoints (upper signs to be used together)

$$\int_0^{\frac{1}{2}}\{(1-u)^{\mp\kappa_1}u^{\pm(\kappa_1+\kappa_2+\kappa_3)}({}^-R(h)/{}^+R(h))-1\}^2u^{\pm\kappa_3} \; {}^+R(h)h\,du.$$

We can always choose κ_i so that the integral in u converges. If ${}^-R/{}^+R \to \infty$, this is then $\Omega(\{{}^-R(h)/{}^+R(h)\}^2 \, {}^+R(h)h)$. I_{02} is treated analogously.

Case 4 remains. Using conditions (iii) and (iv) with $C = 1$, we obtain

$${}^-R(y)/{}^+R(y) = \exp\{-\textstyle\int_y^1 {}^-\varepsilon(s)/s\,ds + \int_y^1 {}^+\varepsilon(s)/s\,ds + \log{}^-A/{}^+A\}$$
$$= \exp\{\textstyle\int_0^y({}^-\varepsilon(s)-{}^+\varepsilon(s))s^{-1}\,ds\}.$$

Since $C = 1$, the exponent must tend to zero, as $y \to 0$. For the squared factor in the integrand of I_{01}, we then obtain

$$\{{}^-R(h(1-u))/{}^+R(hu)-1\}^2 = \Omega(\{\textstyle\int_0^{h(1-u)} {}^-\varepsilon(s)/s\,ds - \int_0^{hu} {}^+\varepsilon(s)/s\,ds\}^2).$$

From condition (iii), it follows that this expression cannot vanish identically unless ${}^+\varepsilon = {}^-\varepsilon \equiv 0$.—If at least one of ${}^+\varepsilon$ and ${}^-\varepsilon$ varies regularly with positive exponent ($w > 0$ or $w_1 > 0$), it follows that each integral tends to zero as $h \to 0$. As before in this proof, we then extract integrable functions of u to obtain the integration interval $(0, h)$ of the lemma.

If both $\pm{}^+\varepsilon$ and $\pm{}^-\varepsilon$ vary slowly, however, we must again divide the interval $(0, \frac{1}{2})$ into $(0, v(h))$ and $(v(h), \frac{1}{2})$. As before, $v(h)$ can be chosen such that the integral over the second of these intervals decides the order, while at the same time $\int_{hu}^{h(1-u)} s^{-1} W_1(s)\, ds$ tends to zero as $h \to 0$, for all u in $(v(h), \frac{1}{2})$. We need then only calculate the order of

$$\int_{v(h)}^{\frac{1}{2}} \{\int_{hu}^{h(1-u)} s^{-1} W_1(s)\, ds + \int_0^{hu} ({}^{-}\varepsilon(s) - {}^{+}\varepsilon(s))s^{-1}\, ds\}^2 h^{+} R(h) u^{\pm\kappa}\, du$$

where $|{}^{-}\varepsilon - {}^{+}\varepsilon|$ varies regularly, according to (iii). Using the same techniques as above, we arrive at the order of the lemma. The proof for I_{02} follows the same pattern. Lemma 2 is proved.

4. Comments. When $\mathfrak{X}$ of condition (i) of the theorem is of the form $(\theta + a, \theta + b)$, the methods of this paper still apply if the modification described by Blischke *et al.* [1] pages 51–52, is carried out. Depending on the behavior of $f(y)$ at the endpoints, the theory of [9] may have to be applied.

The integrals appearing in I_0 will sometimes be of the form $J(h) = \int_0^h S(s)/s\, ds$ with slowly varying S, and it is known that $J \to 0$ as $h \to 0$. In [7], rules are given that determine a slowly varying function $T(h)$ $(\to K \leqq \infty)$ such that $J(h) = \Omega(S(h)T(h))$ $(h \to 0)$.

The integrals in (3) can be calculated by the aid of [7] if $w = \frac{1}{2}$ or $w_1 = \frac{1}{2}$.

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