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A CLASS OF ADF TESTS FOR SUBHYPOTHESIS IN THE MULTIPLE LINEAR REGRESSION

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Summary. In the regression model $Y = \beta_1 x_1 + \beta_2 x_2 + Z$ a class of asymptotically distribution free (ADF) tests for testing $H_0: \beta_1 = 0$ when β_2 is unknown is given. It turns out that if one uses Wilcoxon type tests then there is no reasonable distribution of Z under which the test would be ADF unless x_1 and x_2 are orthogonal to each other when centered. On the other hand if one is sampling from double exponential, then the class of tests is reasonably large. The tests of the Freund–Ausari type, Mood-type, among others, are in the class.

Section 1 consists of introduction, notation and assumptions.

In Section 2, we prove a uniform continuity Theorem 2.2 for rank statistics. Theorem 2.1 and Lemma 2.3 are proved before Theorem 2.2. These latter two results are based on the work done in [4]. Theorem 2.4 gives the desired result. Finally generalization to the situation where one has multiple linear regression model and has more than one parameter under H_0 with more than one unknown is mentioned.

1. Introduction, notation and assumptions. Suppose we are observing random observations $\{Y_{in} 1 \leqq i \leqq n\} n \geqq 1$ independently such that

$$\Pr[Y_{in} \leqq y] = F(y - \beta_1 x_{in}(1) - \beta_2 x_{in}(2)) \tag{1.1}$$

where β_1, β_2 are the parameters of interest and $\{x_{in}(\alpha) i = 1, \cdots, n\} n \geqq 1$ $\alpha = 1, 2$ are sequences of known regression scores.

Consider the hypothesis $H_0: \beta_1 = 0$, when nothing is known about β_2.

Here we give a class of rank tests which are shown to be asymptotically distribution free for H_0 under some suitable conditions on the regression scores, on the underlying F and on an estimator $\hat{\beta}_n$ of β_2. One essentially constructs a test based on the ranks of $Y_{in} - \hat{\beta}_n x_{in}(2)$ and shows that asymptotically it is equivalent to the one based on the ranks of $Y_{in} - \beta_0 x_{in}(1)$, where β_0 is the true value of the parameter β_2, in probability. In the case of two sample problems, when testing for scale with unknown locations, a similar thing has been done by Gross [2] for a class of statistics.

The following assumptions and notations will be used in the sequel.

We shall suppress the dependence of $\{x_{in}(j)\}$ on n and write $\{x_{i\alpha}\}$ instead. Assume

$$\lim \max_{1 \leqq i \leqq n} x_{i\alpha}^2 / \sum x_{i\alpha}^2 = 0 \qquad \alpha = 1, 2 \tag{1.2}$$

$$0 < \lim n^{-1} \sum x_{i\alpha}^2 \leqq k_\alpha < \infty \qquad \alpha = 1, 2$$

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and

$$\lim n^{-1} \sum x_{i1} x_{i2} \leqq k < \infty.$$

Here and in the sequel lim will always be taken as $n \to \infty$.

Let $M = \{\varphi: [0, 1) \to R;$

(i) φ absolutely continuous and is such that for every $0 \leqq u \leqq 1, \varphi(u) = \int_{\frac{1}{2}}^{u} \varphi'(v)\, dv$ where φ' is the derivative of φ which exists almost everywhere;

(ii) $\|\varphi\| = \int_0^1 |\varphi'(u)|\, du < \infty\}$.

Note that $\varphi \in M$ implies φ is bounded and hence square integrable.

The distribution F is not known but is a member of $\mathscr{F}_0 = \{F;\, F$ is an absolutely continuous cdf and has absolutely continuous density with $\sup_x f(x) \leqq \lambda < \infty$. F strictly increasing$\}$.

Let R_{in} be the rank of $Y_{in} - tx_{i2}$ among $\{Y_{in} - tx_{i2}\, 1 \leqq i \leqq n\}$ for some t. Clearly R_{in} is a function of t.

Define for a score function φ the test statistic

$$S_n(t) = n^{-1} \sum_{i=1}^{n} x_{i1}\, \varphi\left(\frac{R_{in}}{n+1}\right). \tag{1.3}$$

The test is to reject H_0 in favor of positive values of β_1 if $S_n(\hat{\beta}_n)$ is large.

Finally let us represent S_n as a Chernoff–Savage type statistic. This representation will be very helpful in the sequel.

For any two numbers t and x, define

$$v_n(t, x) = n^{-1} \textstyle\sum_{i=1}^{n} I(Y_{in} - tx_{i2} \leqq x), \qquad q_n(t, x) = Ev_n(t, x), \tag{1.4}$$

$$\mu_n(t, x) = n^{-1} \textstyle\sum_{i=1}^{n} x_{i1}\, I(Y_{in} - tx_{i2} \leqq x), \qquad m_n(t, x) = E\mu_n(t, x), \tag{1.5}$$

$$L_n(t, x) = n^{\frac{1}{2}}[\mu_n(t, x) - m_n(t, x)], \tag{1.6}$$

where $I(A)$ is the indicator of the set A and E is the expectation taken under H_0, when the true parameter value of β_2 is 0. With these definitions one can write

$$S_n(t) = \textstyle\int_{-\infty}^{\infty} \varphi[(n/n+1)v_n(t, x)]\, d\mu_n(t, x) \tag{1.7}$$

where x is the integrating variable.

Define

$$A_n(t) = \textstyle\int_{-\infty}^{\infty} \varphi[q_n(t, x)(n/n+1)]\, dm_n(t, x). \tag{1.8}$$

Finally for any rv $X \mathscr{L}_\beta(X)$ is probability law of X when θ is the true parameter value.

2. A uniform continuity theorem and main theorem. Without loss of generality we may assume that the true value of the parameter β_2 is zero. Let P_n denote the probability measure generated by $\{Y_{in}\, 1 \leqq i \leqq n\}$ under the above assumption. The following results have been proved in [4] and will be restated here without proof for the sake of completeness.

LEMMA 2.1. *If conditions* (1.2) *are satisfied by the regression scores and* $F \in \mathscr{F}_0$ *then for every* $\varepsilon > 0$, *and any* $0 < a < \infty$, *there is an* $n(\varepsilon, a)$ *and* $b(\varepsilon)$ *such that* $n \geqq n(\varepsilon, a)$ *implies*

(i) $P_n[\sup\{|v_n(tn^{-\frac{1}{2}}, x) - q_n(tn^{-\frac{1}{2}}, x)|; |t| \leqq a, x \in [-\infty, +\infty]\} > \varepsilon] < \varepsilon$.
(ii) $P_n[\sup\{n^{\frac{1}{2}}|v_n(tn^{-\frac{1}{2}}, x) - q_n(tn^{-\frac{1}{2}}, x)|; |t| \leqq a, x \in [-\infty, \infty]\} \geqq b(\varepsilon)] \leqq \varepsilon$.

LEMMA 2.2. *Under the conditions of Lemma* 2.1

$$\sup\{L_n(tn^{-\frac{1}{2}}, x); x \in [-\infty, +\infty], |t| \leqq a\}$$

has a limit determined by a Gaussian process with continuous sample paths in P_n *probability and consequently if* $G_n(t, \xi)$ *and* $M_n(t, \xi)$, $\xi \in [0, 1]$, *are two real-valued processes such that*

$$\sup\{|G_n(tn^{-\frac{1}{2}}, \xi) - M_n(tn^{-\frac{1}{2}}, \xi)|; |t| \leqq a, \xi \in [0, 1]\} \tag{2.1}$$

tends to zero in probability, then

$$\lim_{n\to\infty} P_n[\sup\{|L_n(tn^{-\frac{1}{2}}, G_n(tn^{-\frac{1}{2}}, \xi)) - L_n(tn^{-\frac{1}{2}}, M_n(tn^{-\frac{1}{2}}, \xi))|; \\ |t| \leqq a, 0 \leqq \xi \leqq 1\} \geqq \varepsilon] = 0 \tag{2.2}$$

for any $0 < a < \infty$. *Also*

$$\lim_{n\to\infty} P_n[\sup\{|L_n(tn^{-\frac{1}{2}}, x) - L_n(0, x)|; |t| \leqq a, x \in [-\infty, +\infty]\} \geqq \varepsilon] = 0 \tag{2.3}$$

for any $0 < a < \infty$.

Now let us look at the function $q_n(t, x) = n^{-1}\sum F(x + tx_{i2})$. Clearly for every t, $q_n(t, \cdot)$ is a distribution function and has the same properties as does F. Also note that $v_n(t, \cdot)$ is nothing but an empirical cdf and as such is nondecreasing. Define for $0 \leqq \xi \leqq 1$

$$G_n(t, \xi) = \inf\{x; v_n(t, x) \geqq \xi\} \tag{2.4}$$
$$M_n(t, \xi) = \inf\{x; q_n(t, x) \geqq \xi\}.$$

From (i) of Lemma 2.1 it then follows that (2.1) is satisfied for the above G_n and M_n and consequently (2.2) also holds. We will need this fact and therefore shall state it as a

COROLLARY 2.1. *With* G_n *and* M_n *defined by* (2.4), (2.2) *holds.*

For the purpose of proving the following theorem, since F is continuous and strictly increasing, we shall without loss of any generality assume $F(y) \equiv y\ 0 \leqq y \leqq 1$.

THEOREM 2.1. *Let* $\{Y_{in}\ 1 \leqq i \leqq n\}\ n \geqq 1$ *be a sequence of independent* rv's *satisfying* (1.1). *Let* $\{x_{i\alpha}, 1 \leqq i \leqq n, \alpha = 1, 2\}\ n \geqq 1$ *satisfy* (1.2) *and* $F \in \mathscr{F}_0$. *Then for every* $\varepsilon > 0$, *any* $0 < a < \infty$, *there is* $\mathscr{N}(\varepsilon, a)$ *which may depend on* F *and* $\{x_{i\alpha}\}$, *such that* $n \geqq \mathscr{N}(\varepsilon, a)$ *implies*

$$P_n[\sup(n^{\frac{1}{2}}|\{S_n(tn^{-\frac{1}{2}}) - A_n(tn^{-\frac{1}{2}})\} - \{S_n(0) - A_n(0)\}|; |t| \leqq a) > \varepsilon] < \varepsilon$$

for all $\varphi \in M'$ *where* $M' \subset M$ *is such that* $\sup\{||\varphi||; \varphi \in M'\} < \infty$.

PROOF. Throughout the proof we shall assume that for any $\varepsilon > 0$ there is an $n(\varepsilon)$ and a $b(\varepsilon)$ independent of n. The statements then will be made for $n \geqq n(\varepsilon, a)$. Because of the boundedness of the score function φ; we can replace the argument of φ in the definition of S_n by $v_n(t, x)$ above. Consequently we shall prove the theorem for the statistic $S_n(t) = \int \varphi(v_n(t, x))\, d\mu_n(t, x)$. Consider the decomposition

$$\begin{aligned} n^{\frac{1}{2}}\{S_n(t) - A_n(t)\} &= \int \varphi(q_n)\, dL_n + n^{\frac{1}{2}} \int (v_n - q_n)\varphi(q_n)\, dm_n \\ &\quad + \int \{\varphi(v_n) - \varphi(q_n)\}\, dL_n + n^{\frac{1}{2}} \int \{\varphi(v_n) - \varphi(q_n) - (v_n - q_n)\varphi(q_n)\}\, dm_n \\ &= B_{n1}(t) + B_{n2}(t) + B_{n3}(t) + R_n(t) \quad \text{say.} \end{aligned}$$

We set to prove

$$P_n[\sup\{|R_n(tn^{-\frac{1}{2}})|;\ |t| < a\} > \varepsilon] < \varepsilon. \tag{2.5}$$

Now note that by Lemma 2.1 (ii) there exists a $b(\varepsilon)$ such that for every

$$|t| \leqq an^{-\frac{1}{2}}, \qquad -\infty \leqq x \leqq +\infty,$$

$$v_n(t, x) = q_n(t, x) + un^{-\frac{1}{2}}, \qquad |u| \leqq b(\varepsilon) \tag{2.6}$$

with probability at least $1 - \varepsilon$.

For $F \in \mathscr{F}_0$, and by (1.2) there are constants λ_0 and K_1 such that for every $|t| \leqq an^{-\frac{1}{2}}$

$$|dm_n(t, M_n(t, y))| \leqq K_1 \lambda\, dy \qquad 0 \leqq y \leqq 1. \tag{2.7}$$

Using (2.6), except for probability at most ε, one can write for all $|t| \leqq an^{-\frac{1}{2}}$,

$$R_n(t) = n^{\frac{1}{2}} \int \{\varphi(q_n(t, x) + un^{-\frac{1}{2}}) - \varphi(q_n(t, x)) - un^{-\frac{1}{2}}\varphi'(q_n(t, x))\}\, dm_n(t, x).$$

Now making a change of variable from $q_n(t, x)$ to y (note that y may depend on t, but that presents no difficulty for integration purposes) and using (2.7), we have except for probability at most ε, $\sup_{|t| \leqq a} |R_n(tn^{-\frac{1}{2}})| \leqq K_1 \lambda \int_0^1 \rho_n(y)\, dy$ where

$$\rho_n(y) = \sup_{|u| \leqq b(\varepsilon)} n^{\frac{1}{2}} |\varphi(y + un^{-\frac{1}{2}}) - \varphi(y) - un^{-\frac{1}{2}}\varphi'(y)|$$

which tends to zero for almost all y and hence $\int \rho_{0n}^{1}(y)\, dy$ tends to zero. Therefore (2.5) is proved.

Next consider $B_{n1}(t) = \int \varphi(q_n(t, x)\, dL_n(t, x)$. Noting that

$$P_n[\sup\{\varphi(q_n(t, \pm\infty)) \pm L_n(t, \pm\infty);\ |t| \leqq an^{-\frac{1}{2}} = 0] = 1$$

for all n, we have $B_{n1}(t) = -\int L_n(t, x)\, d\varphi(q_n(t, x))$ for every $|t| \leqq a$ with probability 1. But by substitution $B_{n1}(t) = -\int_0^1 L_n(t, M_n(t, \xi))\varphi'(\xi)\, d\xi$. Hence

$$\begin{aligned} &\sup\{|B_{n1}(tn^{-\frac{1}{2}}) - B_{n1}(0)|;\ |t| \leqq a\} \\ &\quad \leqq \sup\{|L_n(tn^{-\frac{1}{2}}, M_n(tn^{-\frac{1}{2}}, \xi)) - L_n(0, M_n(0, \xi))|;\ |t| \leqq a, 0 \leqq \xi \leqq 1\} \|\varphi\|. \end{aligned} \tag{2.8}$$

But since by Assumption (1.2) and $F \in \mathscr{F}_0$

$$\begin{aligned} &\sup\{|q_n(tn^{-\frac{1}{2}}, x) - q_n(0, x)|;\ |t| \leqq a,\ -\infty < x < \infty\} \\ &\quad \leqq n^{-1} \sum_i \sup\{|F(x + tx_{i2}n^{-\frac{1}{2}}) - F(x)|;\ |t| \leqq a,\ -\infty < x < \infty\} \to 0 \end{aligned}$$

which implies $\sup\{|M_n(tn^{-\frac{1}{2}}, \xi)-M_n(0, \xi)|;\ |t| \leqq a, 0 \leqq \xi \leqq 1\} \to 0$ and hence by (2.2)

$$P_n[\sup\{|L_n(tn^{-\frac{1}{2}}, M_n(tn^{-\frac{1}{2}}, \xi))-L_n(0, M_n(0, \xi))|;\ |t| \leqq a, 0 \leqq \xi \leqq 1\} > \varepsilon] \leqq \varepsilon$$

for any $0 < a < \infty$.

Consequently since $\varphi \in M$, we have from (2.8) that

$$P_n[\sup\{|B_{n1}(tn^{-\frac{1}{2}})-B_{n1}(0)|;\ |t| \leqq a\} > \varepsilon] < \varepsilon$$

for any $0 < a < \infty$.

Term B_{n2} is dealt with in similar fashion to B_{n1}. The term B_{n3} is dealt with in fashion similar to R_n. However here we integrate by parts first which is justified as both functions $L_n(t, \cdot)$ and $v_n(t, \cdot)$ are the functions of bounded variation and φ is continuous, then use of Corollary 2.1 is made. It remains to prove that convergence is uniform with respect to φ. This will follow if we show that the terms $R_n(t)$, $B_{ni}(t)-B_{ni}(0) i = 1, 2$, and $B_{n3}(t)$ satisfy, as a function of φ, the Lipschitz condition (LC) in the norm $\|\varphi\| = \int_0^1 |\varphi'|$ for large n with large probability, uniformly in $|t| \leqq an^{-\frac{1}{2}}$. By looking at (2.8) it is clear that $B_{n1}(t)-B_{n1}(0)$ satisfy LC condition. Similarly one can easily see that this is so for $B_{n2}(t)-B_{n2}(0)$ and $B_{n3}(t)$. We verify this condition for $R_n(t)$ term here. Let

$$R_{n1}(t) = n^{\frac{1}{2}} \int [\varphi(v_n(t, x))-\varphi(q_n(t, x))]\, dm_n(t, x)$$

$$R_{n2}(t) = n^{\frac{1}{2}} \int [v_n(t, x)-q_n(t, x)]\varphi'(q_n(t, x))\, dm_n(t, x).$$

Clearly, in view of Lemma 2.1 (ii) and (2.7) $\sup_{|t| \leqq a} |R_{n2}(tn^{-\frac{1}{2}})| \leqq K_1 \lambda b(\varepsilon)\|\varphi\|$ with probability at least $1-\varepsilon$. Integrating by parts and making change of variables from $v_n(t, x)$ to y, $0 \leqq y \leqq 1$ and q_n to $0 \leqq y \leqq 1$, we get $R_{n1}(t) = -n^{\frac{1}{2}} \int_0^1 [m_n(t, G_n(t, y))- m_n(t, M_n(t, y))]\varphi'(y)\, dy$ so that

$$\sup_{|t| \leqq an^{-1/2}} |R_{n2}(t)| \leqq \sup_{0 \leqq y \leqq 1} \sup_{|t| \leqq an^{-1/2}} n^{\frac{1}{2}} |m_n(t, G_n(t, y)) - m_n(t, M_n(t, y))|\, \|\varphi\|.$$

It may be readily verified that for $F \in \mathscr{F}_0$, the first term is bounded in probability. Thus the proof is complete after noticing that $R_n(t) = R_{n1}(t)-R_{n2}(t)$.

We next prove a series of lemmas and theorems which will lead us to ADF tests.

LEMMA 2.3. *Assume that* $\varphi \in M$ *is such that*

$$\int_0^1 |\varphi'|^2 < \infty, \qquad \sup_{0 \leqq u \leqq 1} |\varphi''(u)| \leqq K_1 \tag{2.9}$$

and $F \in \mathscr{F}_0$ *is such that*

$$I(f) = \int_{-\infty}^{\infty} (f'/f)^2\, dF < \infty. \tag{2.10}$$

About $\{x_{ij}\} 1 \leqq i \leqq n, j = 1, 2$, *we assume that* (1.2) *holds. Then*

$$n^{\frac{1}{2}}\|A_n(tn^{-\frac{1}{2}})-A_n(0)-tn^{-\frac{1}{2}}\dot{A}_n(0)\| \to 0 \tag{2.11}$$

where $\quad \dot{A}_n(0) = n^{-2} \sum_{i=1}^n x_{i1} \sum_{j=1}^n (x_{j2}-x_{i2}) \int_{-\infty}^{\infty} f(x)\varphi'(F(x))\, dF(x)$

and $\|\cdot\|$ *stands for sup norm taken over* $|t| \leqq a$, *for any* $0 < a < \infty$.

PROOF. Recalling the definition of A_n function from (1.8) one notes that $A_n(t)$ can be expressed by $A_n(t) = n^{-1}\sum x_{i1}\int_{-\infty}^{\infty}\varphi[q_n(t, x-tx_{i2})]\,dF(x)$, by mere substitution process. Also observe that $(n/n+1)$ in the argument of φ is dropped which can be done. Using Taylor's expansion of φ around $q_n(0, x) =_{\mathrm{df}} F(x)$, and putting $h_{ni}(t, x) = q_n(t, x-tx_{i2}) - q_n(0, x)$, $\delta_{ij} = (x_{j2}-x_{i2})$, $b_{ni} = n^{-1}\sum_j \delta_{ij}$, one gets

$$
\begin{aligned}
(2.12)\qquad & n^{\frac{1}{2}}\|A_n(tn^{-\frac{1}{2}}) - A_n(0) - tn^{-\frac{1}{2}}\dot{A}_n(0)\| \\
&\leqq \|n^{-\frac{1}{2}}\textstyle\sum_{i=1}^n x_{i1}\int_{-\infty}^{\infty}\{h_{ni}(tn^{-\frac{1}{2}}, x) - tn^{-\frac{1}{2}}b_{ni}f(x)\}\varphi'(F(x))\,dF(x)\| \\
&\qquad + K_1\|n^{-\frac{1}{2}}\textstyle\sum_i x_{i1}\int_{-\infty}^{\infty} h_{ni}^2(tn^{-\frac{1}{2}}, x)\,dF(x)\|.
\end{aligned}
$$

But

$$
\begin{aligned}
(2.13)\qquad h_{ni}(t, x) &= n^{-1}\textstyle\sum_j [F(x+t\delta_{ij}) - F(x)] \\
&= tn^{-1}\textstyle\sum_j \delta_{ij} f(x+\xi\delta_{ij})
\end{aligned}
$$

where $|\xi| \leqq |t|$ and it may depend on various quantities involved. Also note that

$$
\begin{aligned}
(2.14)\qquad & \textstyle\int_{-\infty}^{\infty}[n^{-1}\sum \delta_{ij} f(x+\xi\delta_{ij})]^2\,dF(x) \\
&\leqq (n^{-1}\textstyle\sum_j \delta_{ij}^2)(n^{-1}\sum_{j=1}^n \int_{-\infty}^{\infty} f^2(x+\xi\delta_{ij})\,dF(x)) \\
&= (n^{-1}\textstyle\sum_j \delta_{ij}^2)(n^{-1}\sum_{j=1}^n \int_{-\infty}^{\infty} f^2(x)\,dF(x-\xi\delta_{ij})) \\
&\leqq (n^{-1}\textstyle\sum_j \delta_{ij}^2)\lambda^2
\end{aligned}
$$

where λ comes from the fact that $\sup_x f(x) \leqq \lambda < \infty$. Using (2.13) and (2.14) one gets $\|n^{-\frac{1}{2}}\sum_i x_{i1}\int_{-\infty}^{\infty} h_{ni}^2(tn^{-\frac{1}{2}}, x)\,dF(x)\| \leqq \lambda^2 a^2 n^{-\frac{1}{2}}|n^{-2}\sum_i x_{i1}\sum_j \delta_{ij}^2|$ which $\to 0$ in view of (1.2) and the fact that $n^{-\frac{1}{2}} \to 0$ as $n \to \infty$. Furthermore

$$
(2.15)\qquad 2!\{h_{ni}(tn^{-\frac{1}{2}}, x) - tn^{-\frac{1}{2}}b_{ni}f(x)\} = t^2 n^{-1}n^{-1}\textstyle\sum_j \delta_{ij}^2 f'(x+\xi\delta_{ij}n^{-\frac{1}{2}}).
$$

Also for all $1 \leqq i, j \leqq n$,

$$
\begin{aligned}
& |\textstyle\int_{-\infty}^{\infty} f'(x+\xi\delta_{ij}n^{-\frac{1}{2}})\varphi'(F(x))\,dF(x)| \\
(2.16)\qquad &\leqq \{(\textstyle\int_{-\infty}^{\infty} f'^2(x)\,dF(x-\xi\delta_{ij}n^{-\frac{1}{2}}))(\int_0^1 \varphi'^2(u)\,du)\}^{\frac{1}{2}} \\
&\leqq \lambda[I(f)\cdot\textstyle\int_0^1 \varphi'^2(u)\,du]^{\frac{1}{2}} < \infty \qquad \text{by (2.9) and (2.10).}
\end{aligned}
$$

Hence combining (2.16) and (2.15) we observe that $\|n^{-\frac{1}{2}}\sum_i x_{i1}\int_{-\infty}^{\infty}\{h_{ni}(tn^{-\frac{1}{2}}, x) - tn^{-\frac{1}{2}}b_{ni}f(x)\}\varphi'(F(x))\,dF(x)\| \leqq (a/2)n^{-\frac{1}{2}}(n^{-2}\sum_i|x_{i1}|\sum_j \delta_{ij}^2)\lambda[I(f)\int_0^1\varphi'^2(u)\,du] \to 0$ as $n \to \infty$ for by (1.2) $\lim_{n\to\infty}\{n^{-2}\sum|x_{i1}|\sum_j \delta_{ij}^2\} < \infty$. Hence one concludes (2.14). This terminates the proof.

After integrating $\dot{A}_n(0)$ by parts it is easy to see that

$$
\begin{aligned}
\dot{A}_n(0) &= n^{-1}\textstyle\sum_i (x_{i1}-\bar{x}_1)(x_{i2}-\bar{x}_2)\int_0^1 \varphi(u, f)\varphi(u)\,du \\
&= b_n(\varphi, f) \quad \text{say}
\end{aligned}
$$

where $\varphi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u))$. Hence one has

COROLLARY 2.2. $n^{\frac{1}{2}}\|A_n(tn^{-\frac{1}{2}})-A_n(0)-tn^{-\frac{1}{2}}b_n(\varphi,f)\| \to 0$.
Combining this with Theorem 2.1 one gets

THEOREM 2.2. *Under the conditions of Theorems* 2.1 *and Lemma* 2.3 *on the underlying quantities one has for every* $\varepsilon > 0$

$$P_n[n^{\frac{1}{2}}\|S_n(tn^{-\frac{1}{2}})-S_n(0)-tn^{-\frac{1}{2}}b_n(\varphi,f)\| \geqq \varepsilon] \to 0 \tag{2.17}$$

where $\|\cdot\|$ *is* sup norm *taken with respect to all* $|t| \leqq a$, $0 < a < \infty$.
An immediate consequence of the above is the following

THEOREM 2.3. *Assume* $\{Y_{in}\, 1 \leqq i < n\}$, $n \geqq 1$ *as in model* 1.1. *Let the conditions of Theorem* 2.2 *be satisfied. Furthermore suppose* $\hat{\beta}_n$ *is an estimator of* β_2 *such that for every* $\varepsilon > 0$ *there exists* $n(\varepsilon)$ *and* $b_1(\varepsilon)$, *depending only on* ε, *such that* $n \geqq n(\varepsilon)$ *implies*

$$\Pr[n^{\frac{1}{2}}|\hat{\beta}_n-\beta_2| \leqq b_1(\varepsilon)] \geqq 1-\varepsilon. \tag{2.18}$$

Then for every $\varepsilon > 0$ *there is an* $n(\varepsilon)$ *such that for all* $n \geqq n(\varepsilon)$

$$\Pr[n^{\frac{1}{2}}|S_n(\hat{\beta}_n)-S_n(\beta_2)+(\hat{\beta}_n-\beta_2)b_n(\varphi,f)| \geqq \varepsilon] \leqq \varepsilon \tag{2.19}$$

for every β_2, *where probability in* (2.19) *and* (2.18) *is computed when* β_2 *is the true parameter point.*

Before stating and proving our final result, we note that from (2.19) it follows that the test based on $S_n(\hat{\beta}_n)$ would be asymptotically distribution free if $b_n(\varphi,f) = 0$.

THEOREM 2.4. *Let* $\{Y_{in}, 1 \leqq i \leqq n\}$ *and* $\{x_{i\alpha}\}$ *be as in* (1.1). *Assume* $F \in \mathscr{F}_0$ *and satisfies* (2.10), $\{x_{i\alpha}\}$ *satisfy* (1.2) *and* $\varphi \in M$ *and satisfies* (2.9). *Furthermore suppose there is an estimator* $\hat{\beta}_n$ *of* β_2 *satisfying* (2.18). *Then*

$$b_n(\varphi,f) = 0 \tag{2.20}$$

entails $\mathscr{L}_{\beta_2}(n^{\frac{1}{2}}S_n(\hat{\beta}_n)) \to N(a, \sigma^2)$, *for some* a, σ^2 *independent of* F. *Consequently test* $n^{\frac{1}{2}}S_n(\hat{\beta}_n)$ *of* $H_0: \beta_1 = 0$ *when* β_2 *is not known is asymptotically distribution free.*

PROOF. The proof follows from Theorem 2.3 and the fact that $\mathscr{L}_{\beta_2}(n^{\frac{1}{2}}S_n(\beta_2)) \to N(a, \sigma^2)$, the proof of which may be found in Hájek [2].

The following lemma gives a sufficient condition for (2.20) to hold.

LEMMA 2.4. *If either*

$$n^{-1}\sum(x_{i1}-\bar{x}_1)(x_{i2}-\bar{x}_2) = 0 \qquad \textit{or} \tag{2.21}$$

$$\varphi(u) = \varphi(1-u) \qquad 0 \leqq u \leqq 1 \quad \textit{and} \tag{2.22}$$

$$F(x) = 1-F(-x) \qquad -\infty < x < +\infty$$

then $b_n(\varphi,f) = 0$.

PROOF. Recall that $b_n(\varphi,f) = n^{-1}\sum_i(x_{i1}-\bar{x}_1)(x_{i2}-\bar{x}_2)b(\varphi,f)$ with $b(\varphi,f) = -\int_{-\infty}^{\infty}[f'(x)/f(x)]\varphi(F(x))\,dF(x)$. That $(2.21) \Rightarrow b_n(\varphi,f) = 0$ is clear. We will show $(2.22) \Rightarrow b(\varphi,f) = 0$ so that $b_n(\varphi,f) = 0$. Now note that

$$-b(\varphi,f) = \int_{-\infty}^{\infty} f'(x)\varphi(F(x))\,dx.$$

But

$$\begin{aligned} f'(-x)\varphi(F(-x)) &= f'(x)\varphi(1-F(x)) && \text{by (2.22)} \\ &= -f'(x)\varphi(F(x)) && \text{by (2.22)} \end{aligned}$$

which implies $-b(\varphi,f) = 0$.

REMARKS. Symmetry of φ seems to be quite reasonable in some cases and not so reasonable in other cases. If $\varphi(u) = u-\frac{1}{2}$, then there is no nontrivial $F \in \mathscr{F}_0$, satisfying (2.22) and (2.10) such that $b(\varphi,f) = 0$. Next observe that if $\varphi(u) = \varphi(u,f)$, then $b(\varphi,f) = 0$ if and only if f is uniform density. So if one would expect asymptotically most powerful type tests to be ADF, one must have orthogonal regression score.

Next suppose $F(x) = 1/(1+e^{-x})$, $-\infty < x < +\infty$, then $\varphi(u) = (2u-1)^{\alpha}$, $0 \leqq u \leqq 1$, α an even integer, is a suitable class of functions for ADF tests.

If $F(x) = (\frac{1}{2})e^{-|x|} - \infty < x < +\infty$, any symmetric function φ, symmetric about $(\frac{1}{2})$, and satisfying condition of Theorem 2.4, is reasonable. But there are non-symmetric functions φ for which $b(\varphi,f) = 0$.

To see that both these comments are valid we first note that for the above F,

$$\begin{aligned} b(\varphi,f) &= \int_0^1 \operatorname{sgn}(F^{-1}(u))\varphi(u)\,du \\ &= -\int_0^{\frac{1}{2}} \varphi(u)\,du + \int_{\frac{1}{2}}^1 \varphi(u)\,du \end{aligned}$$

so that $b(\varphi,f) = 0 \Leftrightarrow \int_0^{\frac{1}{2}}\varphi(u)\,du = \int_{\frac{1}{2}}^{1}\varphi(u)\,du$ which is trivially true for any φ symmetric about $(\frac{1}{2})$. But for φ defined by

$$\begin{aligned} \varphi(u) &= u(\tfrac{1}{2}-u) && 0 \leqq u \leqq \tfrac{1}{2} \\ &= u/6 - 1/12 && \tfrac{1}{2} \leqq u \leqq 1 \end{aligned}$$

we also have $\int_0^{\frac{1}{2}}\varphi(u)\,du = \int_{\frac{1}{2}}^1\varphi(u)\,du = 1/48$. Besides this φ satisfies all of our conditions. Thus if F is symmetric, symmetry of φ is not necessary for a rank test to be ADF test.

Finally, for any F symmetric about zero, $\varphi(u) = [u-\frac{1}{2}]^2$ is another example where our methods are valid.

In the conclusion it might be remarked that the above results remain valid for the general multiple linear regression model where one may have p parameters and would like to test the hypothesis about any $k < p$ parameters when the remaining $p-k$ are unknown. The test statistics is taken to be the suitable quadratic combination of corresponding linear rank statistics, which has an asymptotic χ^2 distribution with k degrees of freedom. Here the estimator vector for the unknown could be either a least squares estimate or the one defined in [4], besides any other satisfying the analogue of (2.18).

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