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PARAMETER ESTIMATION FOR AN *R*-DIMENSIONAL PLANE WAVE OBSERVED WITH ADDITIVE INDEPENDENT GAUSSIAN ERRORS[1]

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Let $\varepsilon(t, \mathbf{x})$ denote a stationary Gaussian process with $E\varepsilon(t, \mathbf{x}) = 0$, $E\varepsilon^2(t, \mathbf{x}) = \sigma^2 \forall t \in \mathscr{T}, \mathbf{x} \in \mathscr{X}$, and $E\varepsilon(t_1, \mathbf{x}_1)\varepsilon(t_2, \mathbf{x}_2) = 0 \forall t_1 \neq t_2$ or $\mathbf{x}_1 \neq \mathbf{x}_2$. Let $\mathscr{T}$ be the set of integers and $\mathscr{X}$ a subset of the r-dimensional Euclidean space R^r. Given a coordinate system in R^r and a time origin, observe $y(t, \mathbf{x}) = s(t, \mathbf{x}) + \varepsilon(t, \mathbf{x})$, where $s(t, \mathbf{x}) = \sum_{j=0}^{T-1} A(\omega_j) \exp\{i[\omega_j t - \boldsymbol{\kappa}(\omega_j)'\mathbf{x}]\}$, $\omega_j = 2\pi j/T$, $j = 0, 1, \cdots, T-1$, and $\boldsymbol{\kappa}(\omega_j)$ is a vector of parameters in R^r. If $\boldsymbol{\kappa}(\omega) = (\omega/v)\mathbf{e}$, where $\mathbf{e}'\mathbf{e} = 1$, $s(t, \mathbf{x})$ is the r-dimensional generalization of a (discrete-time) plane wave which is propagating with phase velocity v in a direction parallel to $\mathbf{e}$. For a finite time let the process $y(t, \mathbf{x})$ be simultaneously observed at each $\mathbf{x} \in \mathscr{X} = S_1 \times S_2 \times \cdots \times S_r$, $S_j = \{1, 2, \cdots, n\}$. The maximum likelihood estimators $\hat{A}(\omega_j)$ and $\hat{\boldsymbol{\kappa}}(\omega_j)$ of $A(\omega_j)$ and $\boldsymbol{\kappa}(\omega_j)$, respectively, have a joint limiting normal distribution in which appropriately normalized estimators of the r components of $\boldsymbol{\kappa}(\omega_j)$ are mutually independent, for each $j = 1, \cdots, T-1$. The distributions of the estimators for different ω_j's are mutually independent. The analysis is generalized to the case where $s(t, \mathbf{x})$ is a sum of plane waves with separation between the phase velocities.

1. Introduction. Let $\varepsilon(t, \mathbf{x})$, $t \in \mathscr{T}$ and $\mathbf{x} \in \mathscr{X}$, denote a stationary Gaussian random field, where

$$\begin{aligned} E\varepsilon(t, \mathbf{x}) &= 0\,, & t \in \mathscr{T}\,, \quad \mathbf{x} \in \mathscr{X}\,, \\ E\varepsilon^2(t, \mathbf{x}) &= \sigma^2\,, & t \in \mathscr{T}\,, \quad \mathbf{x} \in \mathscr{X}\,, \\ E\varepsilon(t_1, \mathbf{x}_1)\varepsilon(t_2, \mathbf{x}_2) &= 0\,, & t_1 \neq t_2 \quad \text{or} \quad \mathbf{x}_1 \neq \mathbf{x}_2\,. \end{aligned} \tag{1.1}$$

Let $\mathscr{T}$ be the set of integers and $\mathscr{X}$ a subset of the r-dimensional Euclidean space R^r. Given a coordinate system in R^r and a time origin, suppose that we observe the real-valued random process

$$y(t, \mathbf{x}) = \sum_{m=1}^{M} s_m(t, \mathbf{x}) + \varepsilon(t, \mathbf{x})\,, \tag{1.2}$$

where for each m, $s_m(t, \mathbf{x})$ is the following periodic function in t (of period T) for each $\mathbf{x} \in \mathscr{X}$:

$$s_m(t, \mathbf{x}) = \sum_{j=0}^{T-1} A_m(\omega_j) \exp\{i[\omega_j t - \boldsymbol{\kappa}_m(\omega_j)'\mathbf{x}]\}\,, \tag{1.3}$$

where $\omega_j = 2\pi j/T$, $j = 0, 1, \cdots, T-1$, and for each ω_j and m $\boldsymbol{\kappa}_m(\omega_j) \in R^r$ is a vector of parameters contained in a closed set in the interior of one of the orthants of $K = [-\pi, \pi]^r$. That is, each component of each $\boldsymbol{\kappa}_m(\omega_j)$ is bounded

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away from $-\pi$, 0, and π. The coefficients $A_m(\omega_j)$ are the Fourier coefficients of $s_m(t, \mathbf{0})$, i.e.,

$$(1.4) \qquad \begin{aligned} A_m(\omega_j) &= \frac{1}{T} \textstyle\sum_{t=0}^{T-1} e^{-i\omega_j t} s_m(t, \mathbf{0}) , \\ \bar{A}_m(\omega_j) &= A_m(\omega_{T-j}) , \qquad j = 0, 1, \cdots, T-1 . \end{aligned}$$

From now on for simplicity we assume $A_m(0) = 0$, $m = 1, \cdots, M$. If $A_m(0) \neq 0$ it is necessary to calculate and remove the series mean before proceeding with the analysis in this paper. Details when $r = 1$ are in Walker [19].

Suppose that the index t has units of time associated with it, $r = 3$, the coordinates of the vector index $\mathbf{x} \in R^3$ have units of length and $\boldsymbol{\kappa}_m(\omega) = \omega v_m^{-1}(\omega)\mathbf{e}$, where $\mathbf{e}$ is a fixed unit length vector in R^3 and $\{v_m(\omega)\}$ is a set of real functions of $\omega > 0$. Then (1.3) defines a discrete-time *plane wave* which is propagating in a dispersive medium with phase velocity $v_m(\omega)$ (Courant and Hilbert [5], Chapter III, Section 3). The direction of propagation of the wave is parallel to $\mathbf{e}$.

In order to simplify the exposition let us first discuss the case where $M = 1$ with the subscript dropped from the expressions involved in $s_1(t, \mathbf{x})$. The analysis will be generalized in Section 5.

Although in general for propagating waves the phase velocity $v(\omega)$ depends on ω, in most applications $v(\omega)$ is a constant for the frequencies ω of interest. If $v(\omega_j) = v$ for $j = 1, \cdots, T-1$, the plane wave is called *non-dispersive* and $s(t, \mathbf{x})$ depends on t and $\mathbf{x}$ only as a function of $t - \boldsymbol{\alpha}'\mathbf{x}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)' = v^{-1}\mathbf{e}$, i.e.,

$$s(t, \mathbf{x}) = s(t - \boldsymbol{\alpha}'\mathbf{x}) = \textstyle\sum_{j=1}^{T-1} A(\omega_j) \exp\{i\omega_j(t - \boldsymbol{\alpha}'\mathbf{x})\} .$$

The parameters $\alpha_1, \alpha_2, \alpha_3$ are called the *wave slowness* components of the propagating wave (their units are time/distance), and

$$\alpha_k = \frac{1}{v} e_k = \frac{1}{\omega} \kappa_k(\omega) , \qquad k = 1, 2, 3 ,$$

does not depend on ω, with $\boldsymbol{\kappa}(\omega) = (\kappa_1(\omega), \kappa_2(\omega), \kappa_3(\omega))'$.

Let us now review the relevant literature dealing with the estimation of the parameters of a propagating non-dispersive plane wave. Suppose that for $t = 1, \cdots, T$ the process $y(t, \mathbf{x}) = s(t, \mathbf{x}) + \varepsilon(t, \mathbf{x})$ is simultaneously observed at a finite set of positions $\{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ where $\mathbf{x}_p = (x_{p1}, x_{p2}, x_{p3})'$. From the set of observations of $y(t, \mathbf{x})$ we wish to estimate the wave slowness components $\alpha_1, \alpha_2, \alpha_3$ and the Fourier coefficients $A(\omega_j)$, $j = 1, \cdots, \frac{1}{2}(T-1)$.[2]

The classical applied problem involves waves which are measured by a collection of sensors located on a *plane* (called a *planar array*). Examples are

[2] From now on for exposition we assume T is odd.

seismic waves measured by an array of seismometers on the earth's surface (Capon, Greenfield, and Kolker [2], Laster and Linville [10], Smith [17]), acoustic and gravity waves measured by arrays of microphones and barographs (sensitive barometers) (Clay and Hinich [4], Green, Kelly, and Levin [7]), underwater acoustic waves measured by hydrophones on the ocean floor (Clay [3] and Horton [9]), and electromagnetic waves measured by an array of radio receivers (Barber [1]). If we choose the coordinate system of R^3 such that the third axis is perpendicular to the array plane, the wave slowness components are

$$\alpha_1 = \frac{\sin \gamma \cos \theta}{v}, \qquad \alpha_2 = \frac{\sin \gamma \sin \theta}{v}, \qquad \alpha_3 = \frac{\cos \gamma}{v},$$

where γ is the angle of propagation with respect to the normal to the array plane, θ is the azimuth angle of propagation in the plane, and v is the phase velocity; see Figure 1. The coordinates of the pth sensor in the planar array are just $\mathbf{x}_p = (x_{p1}, x_{p2}, 0)'$.

The method which is generally used to estimate the parameters is called "delay-and-sum beam forming." (See [1]–[3], [7], [10]–[11], [13], [16]–[17].) Let $y(t, \mathbf{x}_p)$ denote the output of the pth sensor, and given delays α_1 and α_2 let $v(t, \boldsymbol{\alpha})$ be the sum

$$y(t, \boldsymbol{\alpha}) = \sum_{p=1}^{N} y(t + \boldsymbol{\alpha}'\mathbf{x}_p)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0)'$. In many applications the analyst has strong a priori

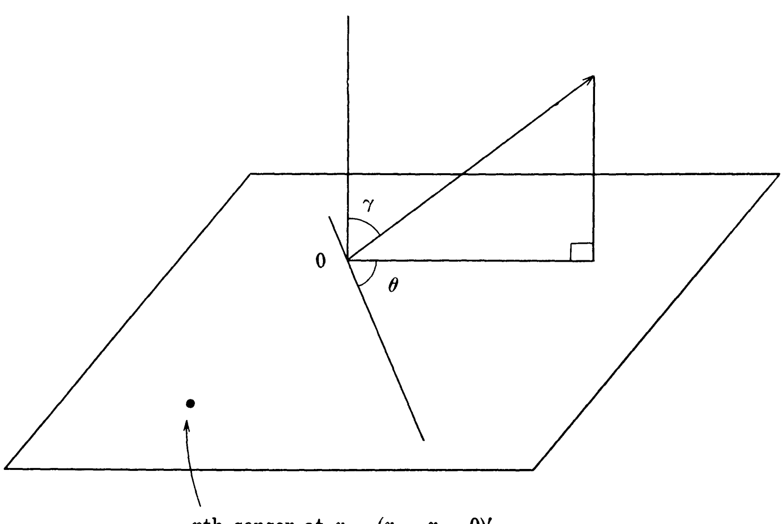

FIG. 1. The diagram shows the array plane, direction of propagation (arrow), the elevation angle of propagation γ, and the azimuth angle of propagation in the plane θ.

notions about the shape of the waveform and he searches over α_1 and α_2 to find delays such that he visually finds a pattern in $y(t, \boldsymbol{\alpha})$ similar to the waveform (Embree, Burg, and Backus [6] and Robinson [16]). In the many schemes which have been employed to detect and analyze waves, there has been no formal analysis of the parameter estimation problem. The work of Levin [11] indicates that if $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are a pair of delays which maximize $\sum_{t=0}^{T} y^2(t, \boldsymbol{\alpha})$, then $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are maximum likelihood estimators of the wave slowness components α_1 and α_2, provided that $\boldsymbol{\alpha}$ is restricted to lie in a closed bounded subset in R^3 and $s(t) = 0$ for $t > T - \max_p (\boldsymbol{\alpha}'\mathbf{x}_p)$ and $t < -\min_p (\boldsymbol{\alpha}'\mathbf{x}_p)$ in order to avoid end-effects problems (Hinich [8]). The maximum likelihood estimator of the waveform is $y(t, \hat{\boldsymbol{\alpha}})$. The large-sample properties of these estimators, however, have not been determined, except for a special linear array problem (MacDonald and Schultheiss [13]). In this paper we treat the general parameter estimation problem given an equally-spaced array of sensors which is observing a possibly dispersive wave embedded in a spatially-incoherent Gaussian noise field. However our analysis of the problem concentrates on the statistical aspects and avoids the specific physical nature of waves.

2. The statistical model and preliminaries. We now formally state the model which is the basis of the statistical problem considered in this paper. We continue to restrict attention to a single r-dimensional plane wave [$M = 1$ in (1.2)]. The case $M > 1$ will be considered in Section 5.

Let $\mathscr{X}$ be the set of $N = n^r$ lattice vectors given by $\mathscr{X} = S_1 \times S_2 \times \cdots \times S_r$, where $S_j = \{1, \cdots, n\}$, i.e., the elements of $\mathscr{X}$ are of the form $\mathbf{x} = (x_1, \cdots, x_r)'$ for $x_j = 1, \cdots, n, j = 1, \cdots, r$. Assume for $t = 0, \cdots, T-1$ and $\mathbf{x} \in \mathscr{X}$ we observe

$$y(t, \mathbf{x}) = \sum_{j=1}^{T-1} A(\omega_j) \exp\{i[\omega_j t - \boldsymbol{\kappa}(\omega_j)'\mathbf{x}]\} + \varepsilon(t, \mathbf{x}) , \tag{2.1}$$

where $\omega_j = 2\pi j/T$ and $\boldsymbol{\kappa}(\omega_j) = (\kappa_1(\omega_j), \cdots, \kappa_r(\omega_j))'$. For each $\mathbf{x} \in \mathscr{X}$ and ω_j, $j = 1, \cdots, \frac{1}{2}(T-1)$, define

$$\begin{aligned} z(\omega_j, \mathbf{x}) &= \frac{1}{T} \sum_{t=0}^{T-1} e^{-i\omega_j t} y(t, \mathbf{x}) \\ &= A(\omega_j) \exp[-i\boldsymbol{\kappa}(\omega_j)'\mathbf{x}] + u(\omega_j, \mathbf{x}) , \\ &= z_1(\omega_j, \mathbf{x}) - i z_2(\omega_j, \mathbf{x}) , \qquad j = 1, \cdots, \tfrac{1}{2}(T-1) , \end{aligned} \tag{2.2}$$

where for each ω_j and $\mathbf{x}$

$$u(\omega_j, \mathbf{x}) = \frac{1}{T} \sum_{t=1}^{T} e^{-i\omega_j t} \varepsilon(t, \mathbf{x}) = u_1(\omega_j, \mathbf{x}) - i u_2(\omega_j, \mathbf{x}) \tag{2.3}$$

is a complex normal random variable and

$$
\begin{aligned}
&Eu_g(\omega_j, \mathbf{x}) = 0\,, && \forall\,\omega_j, \mathbf{x} \in \mathscr{X}\,, \quad g = 1, 2\,, \\
(2.4)\qquad &Eu_g^2(\omega_j, \mathbf{x}) = \frac{\sigma^2}{2T}\,, && \forall\,\omega_j, \mathbf{x} \in \mathscr{X}\,, \quad g = 1, 2\,, \\
&Eu_1(\omega_j, \mathbf{x})u_2(\omega_j, \mathbf{x}) = 0\,, && \forall\,\omega_j, \mathbf{x} \in \mathscr{X}\,, \\
&Eu_g(\omega_j, \mathbf{x}_1)u_h(\omega_k, \mathbf{x}_2) = 0\,, && j \neq k \quad \text{or} \quad \mathbf{x}_1 \neq \mathbf{x}_2\,, \quad g, h = 1, 2\,.
\end{aligned}
$$

We wish to estimate the $\kappa_k(\omega_j)$ for $k = 1, \cdots, r$ for each ω_j. These parameters are called *wavenumbers*.

Since the $u(\omega_j, \mathbf{x})$ variables are independent for different values of j it will be convenient to fix j and write simply

$$z(\mathbf{x}) = Ae^{-i\boldsymbol{\kappa}'\mathbf{x}} + u(\mathbf{x})\,, \tag{2.5}$$

where $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_r)'$, $A = \frac{1}{2}(a + ib)$, $z(\mathbf{x}) = z_1(\mathbf{x}) - iz_2(\mathbf{x})$, and $u(\mathbf{x}) = u_1(\mathbf{x}) - iu_2(\mathbf{x})$. The $u(\mathbf{x})$'s are independent normal variables, and $u_1(\mathbf{x})$ and $u_2(\mathbf{x})$ are independent, each having mean 0 and variance $\frac{1}{2}\sigma^2/T$. The estimators of the κ_j's which we shall consider are the maximum likelihood estimators, which are obtained by minimizing the expression

$$
\begin{aligned}
(2.6)\qquad U_n(a, b, \boldsymbol{\kappa}) &= \textstyle\sum_{x_1,\cdots,x_r=1}^{n} |z(\mathbf{x}) - Ae^{-i\boldsymbol{\kappa}'\mathbf{x}}|^2 \\
&= \textstyle\sum_{x_1,\cdots,x_r=1}^{n} |z(\mathbf{x})|^2 - 2\mathscr{R}A \sum_{x_1,\cdots,x_r=1}^{n} \bar{z}(\mathbf{x})e^{-i\boldsymbol{\kappa}'\mathbf{x}} + n^r|A|^2
\end{aligned}
$$

which is $-\sigma^2/T$ times the exponent in the likelihood of the $z(\mathbf{x})$'s, $\mathbf{x} \in \mathscr{X}$. Denote

$$
\begin{aligned}
(2.7)\qquad D_n(\boldsymbol{\kappa}) &= D_n(\kappa_1, \cdots, \kappa_r) = \textstyle\prod_{j=1}^{r} D_n(\kappa_j) \\
&= \textstyle\sum_{x_1,\cdots,x_r=1}^{n} e^{i\boldsymbol{\kappa}'\mathbf{x}} = \prod_{j=1}^{r} \dfrac{\sin \frac{1}{2}\kappa_j n}{\sin \frac{1}{2}\kappa_j} \exp\left[i\tfrac{1}{2}\kappa_j(n + 1)\right].
\end{aligned}
$$

We assume $\boldsymbol{\kappa}$ is contained in a closed set in the interior of one of the orthants of $K = [-\pi, \pi]^r$ (i.e., each component of $\boldsymbol{\kappa}$ is bounded away from $-\pi$, 0, and π). This assumption is included to avoid identifiability problems in the model (2.2). Minimization of (2.6) with respect to a, b, and $\boldsymbol{\kappa}$ gives

$$\hat{A} = \tfrac{1}{2}(\hat{a} + i\hat{b}) = \frac{1}{n^r} \textstyle\sum_{x_1,\cdots,x_r=1}^{n} z(\mathbf{x})e^{i\hat{\boldsymbol{\kappa}}'\mathbf{x}} \tag{2.8}$$

and $\hat{\boldsymbol{\kappa}} = (\hat{\kappa}_1, \cdots, \hat{\kappa}_r)'$ such that

$$I_n(\hat{\kappa}_1, \cdots, \hat{\kappa}_r) = \max_{\boldsymbol{\kappa} \in K} I_n(\kappa_1, \cdots, \kappa_r)\,, \tag{2.9}$$

where

$$
\begin{aligned}
(2.10)\qquad I_n(\boldsymbol{\kappa}) &= \frac{1}{(2\pi n)^r} \left|\textstyle\sum_{x_1,\cdots,x_r=1}^{n} z(\mathbf{x})e^{i\boldsymbol{\kappa}'\mathbf{x}}\right|^2 \\
&= \frac{1}{(2\pi)^r} \textstyle\sum_{m_1,\cdots,m_r=-(n-1)}^{n-1} e^{-i\boldsymbol{\kappa}'\mathbf{m}} C(m_1, \cdots, m_r)
\end{aligned}
$$

is an r-dimensional periodogram and $C(m_1, \cdots, m_r)$ is the sample autocovariance function of $z(\mathbf{x})$ defined by

$$C(m_1, \cdots, m_r) = \frac{1}{n^r} \sum_{x_1 \in X_{m_1}} \cdots \sum_{x_r \in X_{m_r}} z(x_1, \cdots, x_r)\bar{z}(x_1 + m_1, \cdots, x_r + m_r) ,$$

where $X_m = \{1, \cdots, n - m\}$ for $m \geqq 0$ and $X_m = \{1 - m, \cdots, n\}$ for $m < 0$. (See Priestly [14] for the case $r = 2$.) We shall use $I_n{}^u(\boldsymbol{\kappa})$ to denote the periodogram with $z(\mathbf{x})$ replaced by $u(\mathbf{x})$.

We shall have occasion to consider some partial derivatives of $D_n(\boldsymbol{\kappa})$. In particular,

$$\left|\frac{\partial D_n(\boldsymbol{\kappa})}{\partial \kappa_j}\right| = \left|\sum_{x_1,\cdots,x_r=1}^{n} x_j e^{i\boldsymbol{\kappa}'\mathbf{x}}\right| \leqq \tfrac{1}{2}n^{r+1} + O(n^r) , \qquad j = 1, \cdots, r , \tag{2.11}$$

$$\begin{aligned} \left|\frac{\partial^2 D_n(\boldsymbol{\kappa})}{\partial \kappa_j \, \partial \kappa_k}\right| &= \left|\sum_{x_1,\cdots,x_r=1}^{n} x_j x_k e^{i\boldsymbol{\kappa}'\mathbf{x}}\right| \\ &\leqq \tfrac{1}{3}n^{r+2} + O(n^{r+1}) , \qquad j = k , \\ &\leqq \tfrac{1}{4}n^{r+2} + O(n^{r+1}) , \qquad j \neq k , \quad j, k = 1, \cdots, r , \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \left|\frac{\partial^3 D_n(\boldsymbol{\kappa})}{\partial \kappa_j \, \partial \kappa_k \, \partial \kappa_l}\right| &= \left|\sum_{x_1,\cdots,x_r=1}^{n} x_j x_k x_l e^{i\boldsymbol{\kappa}'\mathbf{x}}\right| \\ &\leqq \tfrac{1}{4}n^{r+3} + O(n^{r+2}) , \qquad j = k = l , \\ &\leqq \tfrac{1}{6}n^{r+3} + O(n^{r+2}) , \qquad j = k \neq l , \\ &\leqq \tfrac{1}{8}n^{r+3} + O(n^{r+2}) , \\ &\qquad j \neq k , \quad j \neq l , \quad k \neq l , \quad j, k, l = 1, \cdots, r . \end{aligned} \tag{2.13}$$

Much tighter bounds are possible in (2.11)–(2.13) if each component of $\boldsymbol{\kappa}$ is bounded away from 0 and every integer multiple of 2π. Then (2.11) is $O(n)$, (2.12) is $O(n^2)$, and (2.13) is $O(n^3)$.

The following lemma will be used to establish the consistency of $\hat{\boldsymbol{\kappa}}$. It is a straightforward generalization of a result in Walker [18] for $r = 1$.

LEMMA 2.1.

$$E \max_{\boldsymbol{\kappa} \in K} I_n{}^u(\kappa_1, \cdots, \kappa_r) = O(n^{\frac{1}{2}r}) .$$

The next two lemmas give further properties of the periodogram of the $u(\mathbf{x})$ process.

LEMMA 2.2.

$$E \max_{\boldsymbol{\kappa} \in K} \frac{1}{(2\pi n)^r} \left|\sum_{x_1,\cdots,x_r=1}^{n} e^{i\boldsymbol{\kappa}'\mathbf{x}} x_j u(\mathbf{x})\right|^2 = O(n^{\frac{1}{2}r+2}) , \qquad j = 1, \cdots, r .$$

LEMMA 2.3.

$$E \max_{\boldsymbol{\kappa} \in K} \frac{1}{(2\pi n)^r} \left| \sum_{x_1, \cdots, x_r = 1}^{n} e^{i\boldsymbol{\kappa}'\mathbf{x}} x_j x_k u(\mathbf{x}) \right|^2 = O(n^{\frac{1}{2}r+4}), \qquad j, k = 1, \cdots, r.$$

In Section 3 we shall show that the maximum likelihood estimators $\hat{A}$ and $\hat{\boldsymbol{\kappa}}$ are consistent and have a joint limiting normal distribution when appropriately normalized. The example when $r = 3$ mentioned in Section 1 is further discussed in Section 4. In Section 5 we treat $M > 1$ [see (1.2)].

3. Properties of the maximum likelihood estimators. We follow techniques employed by Walker [19] for the case $r = 1$. In the following discussion A_0 and $\boldsymbol{\kappa}_0 = (\kappa_{10}, \cdots, \kappa_{r0})'$, when used, will distinguish the true values of A and $\boldsymbol{\kappa}$, respectively. (When there is no ambiguity the subscript 0 will not be used.) Each component of $\boldsymbol{\kappa}_0$ is assumed to be bounded away from $-\pi$, 0, and π, and furthermore $\boldsymbol{\kappa}_0 \in K = [-\pi, \pi]^r$. We shall denote this set of admissible values of $\boldsymbol{\kappa}_0$ by K^I.

THEOREM 3.1. *Let $\hat{\boldsymbol{\kappa}} = (\hat{\kappa}_1, \cdots, \hat{\kappa}_r)'$ be defined by* (2.9). *Then*

$$\hat{\kappa}_j - \kappa_{j0} = o_p(n^{-1}), \qquad j = 1, \cdots, r, \tag{3.1}$$

as $n \to \infty$.

PROOF. By (2.10)

$$\begin{aligned}(2\pi n)^r I_n(\boldsymbol{\kappa}) &= \left| \sum_{x_1, \cdots, x_r = 1}^{n} e^{i\boldsymbol{\kappa}'\mathbf{x}} (A_0 e^{-i\boldsymbol{\kappa}_0'\mathbf{x}} + u(\mathbf{x})) \right|^2 \\ &= (2\pi n)^r I_n^u(\boldsymbol{\kappa}) + 2\mathscr{R}\{A_0 D_n(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0) \textstyle\sum_{x_1, \cdots, x_r = 1}^{n} e^{-i\boldsymbol{\kappa}'\mathbf{x}} \bar{u}(\mathbf{x})\} \\ &\quad + |A_0|^2 |D_n(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)|^2 .\end{aligned} \tag{3.2}$$

When $\boldsymbol{\kappa} = \boldsymbol{\kappa}_0$ the real and imaginary parts of $\sum_{x_1, \cdots, x_r = 1}^{n} e^{i\boldsymbol{\kappa}_0'\mathbf{x}} u(\mathbf{x})$ each have variance $\sigma^2 n^r/(2T)$, and (3.2) is therefore $|A_0|^2 n^{2r} + O_p(n^{\frac{3}{2}r})$. That is,

$$\left(\frac{2\pi}{n}\right)^r I_n(\boldsymbol{\kappa}_0) = |A_0|^2 + O_p(n^{-\frac{1}{2}r}). \tag{3.3}$$

From (3.2) and Lemma 2.1 we have

$$\begin{aligned}\max_{\boldsymbol{\kappa} \in K} &|(2\pi)^r I_n(\boldsymbol{\kappa}) - n^{-r} |A_0|^2 |D_n(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)|^2| \\ &= O_p(n^{\frac{1}{2}r}) + O_p(n^{\frac{3}{4}r}).\end{aligned} \tag{3.4}$$

If δ is small and n is sufficiently large (depending on δ)

$$\max_{|\kappa_j - \kappa_{j0}| \geqq n^{-1}\delta, j=1, \cdots, r, \boldsymbol{\kappa} \in K} |D_n(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)|^2 = \left(\frac{\sin \frac{1}{2}\delta}{\sin \frac{1}{2}n^{-1}\delta}\right)^{2r}$$

(see Walker [19]), and by (3.4)

$$\begin{aligned}\left(\frac{2\pi}{n}\right)^r &\max_{|\kappa_j - \kappa_{j0}| \geqq n^{-1}\delta, j=1, \cdots, r, \boldsymbol{\kappa} \in K} I_n(\boldsymbol{\kappa}) \\ &\leqq n^{-2r} |A_0|^2 \left(\frac{\sin \frac{1}{2}\delta}{\sin \frac{1}{2}n^{-1}\delta}\right)^{2r} + o_p(1) \\ &= |A_0|^2 \left(\frac{\sin \frac{1}{2}\delta}{\frac{1}{2}\delta}\right)^{2r} \left(\frac{\frac{1}{2}n^{-1}\delta}{\sin \frac{1}{2}n^{-1}\delta}\right)^{2r} + o_p(1).\end{aligned}$$

Then

$$p \lim_{n\to\infty} \left(\frac{2\pi}{n}\right)^r \max_{|\kappa_j - \kappa_{j0}| \geq n^{-1}\delta, j=1,\cdots,r, \boldsymbol{\kappa}\in K} I_n(\boldsymbol{\kappa})$$

$$= |A_0|^2 \left(\frac{\sin \frac{1}{2}\delta}{\frac{1}{2}\delta}\right)^{2r} < |A_0|^2 = p \lim_{n\to\infty} \left(\frac{2\pi}{n}\right)^r I_n(\boldsymbol{\kappa}_0) ,$$

and (3.1) follows because δ can be chosen arbitrarily small.

Consistency of the estimator $\hat{A}$ given by (2.8) is a consequence of Theorem 3.1.

THEOREM 3.2.

$$p \lim_{n\to\infty} \hat{A} = A_0 .$$

PROOF. By (2.8)

$$|\hat{A} - A_0| \leq \frac{|A_0|}{n^r} |D_n(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}_0) - n^r| + \frac{1}{n^r} |\textstyle\sum_{x_1,\cdots,x_r=1}^n e^{i\hat{\boldsymbol{\kappa}}'\mathbf{x}} u(\mathbf{x})| .$$

By the law of the mean

$$D_n(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}_0) - n^r = \textstyle\sum_{j=1}^r (\hat{\kappa}_j - \kappa_{j0}) \left.\dfrac{\partial D_n(\boldsymbol{\kappa})}{\partial \kappa_j}\right|_{\boldsymbol{\kappa}^*} \tag{3.5}$$

where $\boldsymbol{\kappa}^* = (\kappa^*_1, \cdots, \kappa^*_r)' = \theta(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}_0)$ and $0 < \theta < 1$. By (2.11) and Theorem 3.1 n^{-r} times (3.5) is $o_p(1)$. Finally, the result follows from Lemma 2.1.

The main result of this section is that the estimators (2.8)–(2.9) have a joint limiting normal distribution when appropriately normalized. By the law of the mean, with U_n defined by (2.6),

$$\begin{bmatrix} \dfrac{\partial U_n}{\partial a} \\ \dfrac{\partial U_n}{\partial b} \\ \dfrac{\partial U_n}{\partial \boldsymbol{\kappa}} \end{bmatrix}_{a_0,b_0,\boldsymbol{\kappa}_0} = \begin{bmatrix} \dfrac{\partial^2 U_n}{\partial a^2} & \dfrac{\partial^2 U_n}{\partial a\, \partial b} & \dfrac{\partial^2 U_n}{\partial a\, \partial \boldsymbol{\kappa}} \\ \dfrac{\partial^2 U_n}{\partial a\, \partial b} & \dfrac{\partial^2 U_n}{\partial b^2} & \dfrac{\partial^2 U_n}{\partial b\, \partial \boldsymbol{\kappa}} \\ \dfrac{\partial^2 U_n}{\partial a\, \partial \boldsymbol{\kappa}} & \dfrac{\partial^2 U_n}{\partial b\, \partial \boldsymbol{\kappa}} & \dfrac{\partial^2 U_n}{\partial \boldsymbol{\kappa}^2} \end{bmatrix}_{a^*,b^*,\boldsymbol{\kappa}^*} \begin{bmatrix} a_0 - \hat{a} \\ b_0 - \hat{b} \\ \boldsymbol{\kappa}_0 - \hat{\boldsymbol{\kappa}} \end{bmatrix} , \tag{3.6}$$

where a^*, b^*, $\boldsymbol{\kappa}^*$ denotes a point on the line segment between the vectors $(a_0, b_0, \boldsymbol{\kappa}_0')'$ and $(\hat{a}, \hat{b}, \hat{\boldsymbol{\kappa}}')'$. In fact, different values of a^*, b^*, $\boldsymbol{\kappa}^*$ are possible for each row of the matrix in (3.6). Here $\partial U_n/\partial \boldsymbol{\kappa}$ denotes the r-component vector with entries $\partial U_n/\partial \kappa_j$, $\partial^2 U_n/\partial \boldsymbol{\kappa}^2$ the $r \times r$ matrix with entries $\partial^2 U_n/\partial \kappa_i\, \partial \kappa_j$, etc.

First we note the joint limiting distribution of the left-hand side of (3.6), after appropriate normalization. From (2.5) and (2.6)

$$\left.\frac{\partial U_n}{\partial a}\right|_{a_0,b_0,\boldsymbol{\kappa}_0} = - \textstyle\sum_{x_1,\cdots,x_r=1}^n [\cos \boldsymbol{\kappa}_0'\mathbf{x}\, u_1(\mathbf{x}) + \sin \boldsymbol{\kappa}_0'\mathbf{x}\, u_2(\mathbf{x})] . \tag{3.7}$$

The other derivatives are evaluated similarly. The covariance matrix of the left-hand side of (3.6) is to terms of greatest order

$$(3.8)\qquad \frac{\sigma^2 n^r}{2T}\begin{bmatrix} 1 & 0 & \frac{1}{2}b_0 n\mathbf{1}_r' \\ 0 & 1 & -\frac{1}{2}a_0 n\mathbf{1}_r' \\ \frac{1}{2}b_0 n\mathbf{1}_r & -\frac{1}{2}a_0 n\mathbf{1}_r & 4|A_0|^2 n^2\mathbf{C}_r \end{bmatrix},$$

where $\mathbf{1}_r = (1, \cdots, 1)'$ is $r \times 1$ and $\mathbf{C}_r$ is $r \times r$ with $\frac{1}{3}$'s along the main diagonal and $\frac{1}{4}$'s elsewhere. By the Lindeberg condition the limiting distribution of $n^{-\frac{1}{2}r}$ times (3.7) is $N(0, \frac{1}{2}\sigma^2/T)$, and the same result holds for $\partial U_n/\partial b|_{a_0,b_0,\kappa_0}$. The limiting distribution as $n \to \infty$ of $n^{-\frac{1}{2}r-1}$ times each variable $\partial U_n/\partial\kappa_j|_{a_0,b_0,\kappa_0}$ is $N(0, \frac{2}{3}\sigma^2|A_0|^2/T)$.

By the Lindeberg condition (Loève, Section 21.2) applied to an arbitrary linear combination the joint limiting distribution of

$$(3.9)\qquad \left(n^{-\frac{1}{2}r}\frac{\partial U_n}{\partial a}, n^{-\frac{1}{2}r}\frac{\partial U_n}{\partial b}, n^{-\frac{1}{2}r-1}\frac{\partial U_n}{\partial \boldsymbol{\kappa}}\right)'_{a_0,b_0,\kappa_0}$$

is normal with mean vector $\mathbf{0}$ and covariance matrix $\frac{1}{2}\sigma^2/T$ times

$$(3.10)\qquad \boldsymbol{\Sigma}_0 = \begin{bmatrix} 1 & 0 & \frac{1}{2}b_0\mathbf{1}_r' \\ 0 & 1 & -\frac{1}{2}a_0\mathbf{1}_r' \\ \frac{1}{2}b_0\mathbf{1}_r & -\frac{1}{2}a_0\mathbf{1}_r & 4|A_0|^2\mathbf{C}_r \end{bmatrix}.$$

Now we analyze the second-order partial derivatives appearing in (3.6). First we note

$$(3.11)\qquad \frac{\partial^2 U_n}{\partial a^2} = \frac{\partial^2 U_n}{\partial b^2} = \tfrac{1}{2}n^r, \qquad \frac{\partial^2 U_n}{\partial a\,\partial b} = 0.$$

Next

$$(3.12)\qquad \left.\frac{\partial^2 U_n}{\partial a\,\partial\kappa_j}\right|_{a^*,b^*,\kappa^*} = \sum_{x_1,\cdots,x_r=1}^n x_j[\tfrac{1}{2}a_0\sin(\boldsymbol{\kappa}^* - \boldsymbol{\kappa}_0)'\mathbf{x} + \tfrac{1}{2}b_0\cos(\boldsymbol{\kappa}^* - \boldsymbol{\kappa}_0)'\mathbf{x} + \sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x}) - \cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(\mathbf{x})].$$

By the law of the mean

$$(3.13)\qquad i\sum_{x_1,\cdots,x_r=1}^n x_j\exp[i(\boldsymbol{\kappa}^* - \boldsymbol{\kappa}_0)'\mathbf{x}] = \left.\frac{\partial D_n(\boldsymbol{\kappa})}{\partial\kappa_j}\right|_0 + \sum_{k=1}^r(\kappa_k^* - \kappa_{k0})\left.\frac{\partial^2 D_n(\boldsymbol{\kappa})}{\partial\kappa_j\,\partial\kappa_k}\right|_{\kappa^{**}}, \qquad j = 1, \cdots, r,$$

with $\boldsymbol{\kappa}^{**} = \theta(\boldsymbol{\kappa}^* - \boldsymbol{\kappa}_0)$ and $0 < \theta < 1$. By Theorem 3.1 $\kappa_k^* - \kappa_{k0} = o_p(n^{-1})$. Therefore (3.13) is $i\frac{1}{2}n^r(n+1) + o_p(n^{r+1})$, where we have used (2.12). Thus (3.12) is

$$\left.\frac{\partial^2 U_n}{\partial a\,\partial\kappa_j}\right|_{a^*,b^*,\kappa^*} = \tfrac{1}{4}b_0 n^{r+1} + o_p(n^{r+1}) + \sum_{x_1,\cdots,x_r=1}^n x_j[\sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x}) - \cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(\mathbf{x})], \qquad j = 1, \cdots, r.$$

Therefore, by Lemma 2.2

$$(3.14)\qquad p\lim_{n\to\infty} n^{-r-1}\left.\frac{\partial^2 U_n}{\partial a\,\partial\kappa_j}\right|_{a^*,b^*,\kappa^*} = \tfrac{1}{4}b_0, \qquad j = 1, \cdots, r.$$

A similar analysis yields

$$(3.15)\qquad p\lim_{n\to\infty} n^{-r-1} \frac{\partial^2 U_n}{\partial b\,\partial\kappa_j}\bigg|_{a^*,b^*,\kappa^*} = -\tfrac{1}{4}a_0\,, \qquad j=1,\cdots,r\,.$$

Other second-order partial derivatives in (3.6) include

$$\begin{aligned}(3.16)\qquad \frac{\partial^2 U_n}{\partial\kappa_j^2}\bigg|_{a^*,b^*,\kappa^*} &= \textstyle\sum_{x_1,\cdots,x_r=1}^{n} x_j^2[\tfrac{1}{2}(a^*a_0+b^*b_0)\cos(\boldsymbol{\kappa}^*-\boldsymbol{\kappa}_0)'\mathbf{x} \\ &\quad -\tfrac{1}{2}(a^*b_0-a_0b^*)\sin(\boldsymbol{\kappa}^*-\boldsymbol{\kappa}_0)'\mathbf{x} \\ &\quad + a^*(\cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x})+\sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(\mathbf{x})) \\ &\quad + b^*(\sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x})-\cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(x))]\,, \quad j=1,\cdots,r\,.\end{aligned}$$

The law of the mean yields

$$\begin{aligned} i^2 \textstyle\sum_{x_1,\cdots,x_r=1}^{n} x_j^2 \exp[i(\boldsymbol{\kappa}^*-\boldsymbol{\kappa}_0)'\mathbf{x}] &= \frac{\partial^2 D_n(\boldsymbol{\kappa})}{\partial\kappa_j^2}\bigg|_{\mathbf{0}} + \textstyle\sum_{k=1}^{r}(\kappa_k^*-\kappa_{k0})\frac{\partial^3 D_n(\boldsymbol{\kappa})}{\partial\kappa_j^2\,\partial\kappa_k}\bigg|_{\boldsymbol{\kappa}^{**}} \\ &= -\tfrac{1}{6}n^r(n+1)(2n+1)+o_p(n^{r+2})\,, \qquad j=1,\cdots,r\,,\end{aligned}$$

where $\boldsymbol{\kappa}^{**}=\theta(\boldsymbol{\kappa}^*-\boldsymbol{\kappa}_0)$, $0<\theta<1$, by Theorem 3.1 and (2.13). Therefore (3.16) is

$$\begin{aligned}\frac{\partial^2 U_n}{\partial\kappa_j^2}\bigg|_{a^*,b^*,\kappa^*} &= \tfrac{1}{6}(a^*a_0+b^*b_0)n^{r+2}+o_p(n^{r+2}) \\ &\quad + \textstyle\sum_{x_1,\cdots,x_r=1}^{n} x_j^2[a^*(\cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x})+\sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(\mathbf{x})) \\ &\quad + b^*(\sin\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_1(\mathbf{x})-\cos\boldsymbol{\kappa}^{*\prime}\mathbf{x}\,u_2(\mathbf{x}))]\,, \qquad j=1,\cdots,r\,.\end{aligned}$$

By Lemma 2.3 and Theorem 3.2

$$(3.17)\qquad p\lim_{n\to\infty} n^{-r-2}\frac{\partial^2 U_n}{\partial\kappa_j^2}\bigg|_{a^*,b^*,\kappa^*} = \tfrac{2}{3}|A_0|^2\,, \qquad j=1,\cdots,r.$$

The same steps as above show

$$(3.18)\qquad p\lim_{n\to\infty} n^{-r-2}\frac{\partial^2 U_n}{\partial\kappa_j\,\partial\kappa_k}\bigg|_{a^*,b^*,\kappa^*} = \tfrac{1}{2}|A_0|^2\,, \quad j\neq k\,,\quad j,k=1,\cdots,r\,.$$

THEOREM 3.3. *Let $z(\mathbf{x})=A\exp(i\boldsymbol{\kappa}'\mathbf{x})+u(\mathbf{x})$, where $\boldsymbol{\kappa}\in K^I$, $A=\frac{1}{2}(a+ib)$, and the variables $u(\mathbf{x})=u_1(\mathbf{x})-iu_2(\mathbf{x})$ are independently and normally distributed, with $u_1(\mathbf{x})$ and $u_2(\mathbf{x})$ being independent and each having mean 0 and variance $\frac{1}{2}\sigma^2/T$. If $\hat a$, $\hat b$, and $\hat{\boldsymbol{\kappa}}$ are defined by (2.8) and (2.9), then as $n\to\infty$ $(n^{\frac{1}{2}r}(\hat a-a), n^{\frac{1}{2}r}(\hat b-b), n^{\frac{1}{2}r+1}(\hat{\boldsymbol{\kappa}}-\boldsymbol{\kappa})')'$ has a limiting normal distribution with mean vector $\mathbf{0}$ and covariance matrix*

$$(3.19)\qquad \frac{2\sigma^2}{T}\boldsymbol{\Sigma}^{-1} = \frac{\sigma^2}{2T|A|^2}\begin{bmatrix} 4|A|^2+3rb^2 & -3rab & -6b\mathbf{1}_r' \\ -3rab & 4|A|^2+3ra^2 & 6a\mathbf{1}_r' \\ -6b\mathbf{1}_r & 6a\mathbf{1}_r & 12\mathbf{I}_r \end{bmatrix},$$

where $\mathbf{1}_r = (1, \cdots, 1)'$ *is* $r \times 1$ *and* $\mathbf{I}_r$ *is the identity matrix of order* r.

PROOF. We drop the subscript 0 from a_0, b_0, $\boldsymbol{\kappa}_0$, and $\boldsymbol{\Sigma}_0$ as in the statement of the theorem. Denote the matrix appearing in (3.6) by

$$\begin{bmatrix} \mathbf{U}_{11}^{n*} & \mathbf{U}_{12}^{n*} \\ \mathbf{U}_{21}^{n*} & \mathbf{U}_{22}^{n*} \end{bmatrix},$$

where $\mathbf{U}_{11}^{n*}$ is 2×2, $\mathbf{U}_{12}^{n*} = (\mathbf{U}_{21}^{n*})'$ is $2 \times r$, and $\mathbf{U}_{22}^{n*}$ is $r \times r$. By (3.11), (3.14), (3.15), (3.17), and (3.18)

$$p \lim_{n\to\infty} \begin{bmatrix} n^{-r}\mathbf{U}_{11}^{n*} & n^{-r-1}\mathbf{U}_{12}^{n*} \\ n^{-r-1}\mathbf{U}_{21}^{n*} & n^{-r-2}\mathbf{U}_{22}^{n*} \end{bmatrix} = \tfrac{1}{2}\boldsymbol{\Sigma}, \tag{3.20}$$

defined by the right-hand side of (3.10). From (3.6) we obtain

$$\begin{bmatrix} n^{\frac{1}{2}r}(\hat{a} - a) \\ n^{\frac{1}{2}r}(\hat{b} - b) \\ n^{\frac{1}{2}r+1}(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}) \end{bmatrix} = - \begin{bmatrix} n^{-r}\mathbf{U}_{11}^{n*} & n^{-r-1}\mathbf{U}_{12}^{n*} \\ n^{-r-1}\mathbf{U}_{21}^{n*} & n^{-r-2}\mathbf{U}_{22}^{n*} \end{bmatrix}^{-1} \begin{bmatrix} n^{-\frac{1}{2}r}\dfrac{\partial U_n}{\partial a} \\ n^{-\frac{1}{2}r}\dfrac{\partial U_n}{\partial b} \\ n^{-\frac{1}{2}r-1}\dfrac{\partial U_n}{\partial \boldsymbol{\kappa}} \end{bmatrix}_{a_0, b_0, \boldsymbol{\kappa}_0}.$$

The desired result follows from (3.20) and the fact that (3.9) has limiting distribution $N(0, (\frac{1}{2}\sigma^2/T)\boldsymbol{\Sigma})$ as $n \to \infty$. Theorem 3.3 with $r = 1$ was originally stated by Whittle [20] and was rigorously proved by Walker [19]. See also Rao [15].

THEOREM 3.4. *Let* $z(\omega_j, \mathbf{x}) = A(\omega_j) \exp\{-i\boldsymbol{\kappa}(\omega_j)'\mathbf{x}\} + u(\omega_j, \mathbf{x})$, *where* $\boldsymbol{\kappa} \in K^I$, $A(\omega_j) = \frac{1}{2}[a(\omega_j) + ib(\omega_j)]$, *and the variables* $u(\omega_j, \mathbf{x})$ *are normally distributed and satisfy* (2.3) *and* (2.4), $j = 1, \cdots, \frac{1}{2}(T-1)$. *Let* $\hat{a}(\omega_j)$, $\hat{b}(\omega_j)$, *and* $\hat{\boldsymbol{\kappa}}(\omega_j)$ *be defined by* (2.8) *and* (2.9) *with* $z(\mathbf{x})$ *replaced by* $z(\omega_j, \mathbf{x})$ *and* $\boldsymbol{\kappa}$ *replaced by* $\boldsymbol{\kappa}(\omega_j)$ *in the corresponding expressions. Then the distributions of* $(\hat{a}(\omega_j), \hat{b}(\omega_j), \hat{\boldsymbol{\kappa}}(\omega_j)')'$ *are independent*, $j = 1, \cdots, \frac{1}{2}(T-1)$, *and as* $n \to \infty$ $(n^{\frac{1}{2}r}[\hat{a}(\omega_j) - a(\omega_j)], n^{\frac{1}{2}r}[\hat{b}(\omega_j) - b(\omega_j)], n^{\frac{1}{2}r+1}[\hat{\boldsymbol{\kappa}}(\omega_j) - \boldsymbol{\kappa}(\omega_j)]')'$ *has a limiting normal distribution with mean vector* $\mathbf{0}$ *and covariance matrix* $(2\sigma^2/T)\boldsymbol{\Sigma}_j^{-1}$, *where the latter is given by* (3.19) *with* a, b, *and* A *replaced by* $a(\omega_j)$, $b(\omega_j)$, *and* $A(\omega_j)$, *respectively.*

4. Planar array example. Recall the definition of a planar array of sensors given in the Introduction. Conforming to the sampling design used in this paper, the array consists of a square of n^2 sensors located at the grid points $\mathbf{x} = (x_1, x_2, 0)'$ for $x_1, x_2 = 1, \cdots, n$. The process $y(t, \mathbf{x})$ is simultaneously observed for $t = 1, \cdots, T$ at each sensor position. Suppose that the plane wave is non-dispersive, $M = 1$ and the Fourier coefficients $A(\omega_j)$ are zero for every ω_j except for one frequency which we denote simply ω, i.e., the wave is

$$s(t, \mathbf{x}) = A \exp\{i\omega(t - \alpha_1 x_1 - \alpha_2 x_2)\},$$

where $\alpha_1 = (\sin\gamma\cos\theta)/v$, $\alpha_2 = (\sin\gamma\sin\theta)/v$, $0 < \gamma < \pi(\gamma \neq \frac{1}{2}\pi)$, and $0 < \theta < 2\pi(\theta \neq \frac{1}{2}\pi, \pi, \frac{3}{2}\pi)$, and $A(\omega) = A$. Such a wave is called *monochromatic* and is characterized by its *wavelength* $\lambda = 2\pi v/\omega$. In order to avoid spatial aliasing it is assumed that the distance between adjacent sensors along the axes of the planar array is at most $\frac{1}{2}\lambda$.

Let $\hat{\kappa}_1$, $\hat{\kappa}_2$ denote the maximum likelihood estimators of the wavenumbers κ_1, κ_2, given by (2.9), with $r = 2$ as the appropriate dimension for the planar array. Since $\boldsymbol{\kappa} = \omega\boldsymbol{\alpha}$, it follows that if $\hat{\kappa}_2 > 0$

$$\hat{\theta} = \arc\tan(\hat{\kappa}_2/\hat{\kappa}_1) \tag{4.1}$$

is the maximum likelihood estimator of the azimuth θ. If $\hat{\kappa}_2 < 0$, $\hat{\theta} = \arc\tan(\hat{\kappa}_2/\hat{\kappa}_1) + \pi$. By the delta method and Theorem 3.3 the large sample mean square error of $\hat{\theta}$ is

$$\begin{aligned} E(\hat{\theta} - \theta)^2 &\sim (1 + \kappa_2^2\kappa_1^{-2})^{-2}(\kappa_2^2\kappa_1^{-4} + \kappa_1^{-2})E(\hat{\kappa}_1 - \kappa_1)^2 \\ &= (\kappa_1^2 + \kappa_2^2)^{-1}E(\hat{\kappa}_1 - \kappa_1)^2 \\ &\sim \frac{1}{n^4}\frac{3\lambda^2\sigma^2}{2\pi^2T|A|^2\sin^2\gamma}, \end{aligned} \tag{4.2}$$

since $E(\hat{\kappa}_1 - \kappa_1)^2 = E(\hat{\kappa}_2 - \kappa_2)^2$.

If the wave is not monochromatic, then $\hat{\theta}$ is computed for each ω_j where $A(\omega_j) \neq 0$ and an estimator of θ is a weighted average of these $\hat{\theta}(\omega_j)$'s.

Let us now discuss estimation of v when the angle γ is known. The maximum likelihood estimator of v is

$$\hat{v} = \frac{\omega\sin\gamma}{(\hat{\kappa}_1^2 + \hat{\kappa}_2^2)^{\frac{1}{2}}}. \tag{4.3}$$

By the delta method and Theorem 3.3 the large sample mean square error of $\hat{v}$ is

$$\begin{aligned} E(\hat{v} - v)^2 &\sim \frac{v^4}{\omega^2}\sin^{-2}\gamma E(\hat{\kappa}_1 - \kappa_1)^2 \\ &\sim \frac{1}{n^4}\frac{3\lambda^2\sigma^2v^2}{2\pi^2T|A|^2\sin^2\gamma}. \end{aligned} \tag{4.4}$$

A planar array design using n^2 sensors can be prohibitively expensive. A commonly used design consists of sensors located on the arms of a cross (Barber [1]). To be more explicit, suppose that the array consists of $2n$ sensors, where n sensors are located at the points $\mathbf{x} = (x_1, 0, 0)'$ for $x_1 = \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}n$, and the remaining sensors are at the points $\mathbf{x} = (0, x_2, 0)'$ for $x_2 = \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}n$ (assuming n even). See Figure 2. The arm along the x_1-axis is a *linear array* in a one-dimensional Euclidean space, and similarly for the arm along the x_2-axis. The wave in the x_1 space is $A\exp[i\omega(t - \alpha_1x_1)]$ and the wave in the x_2

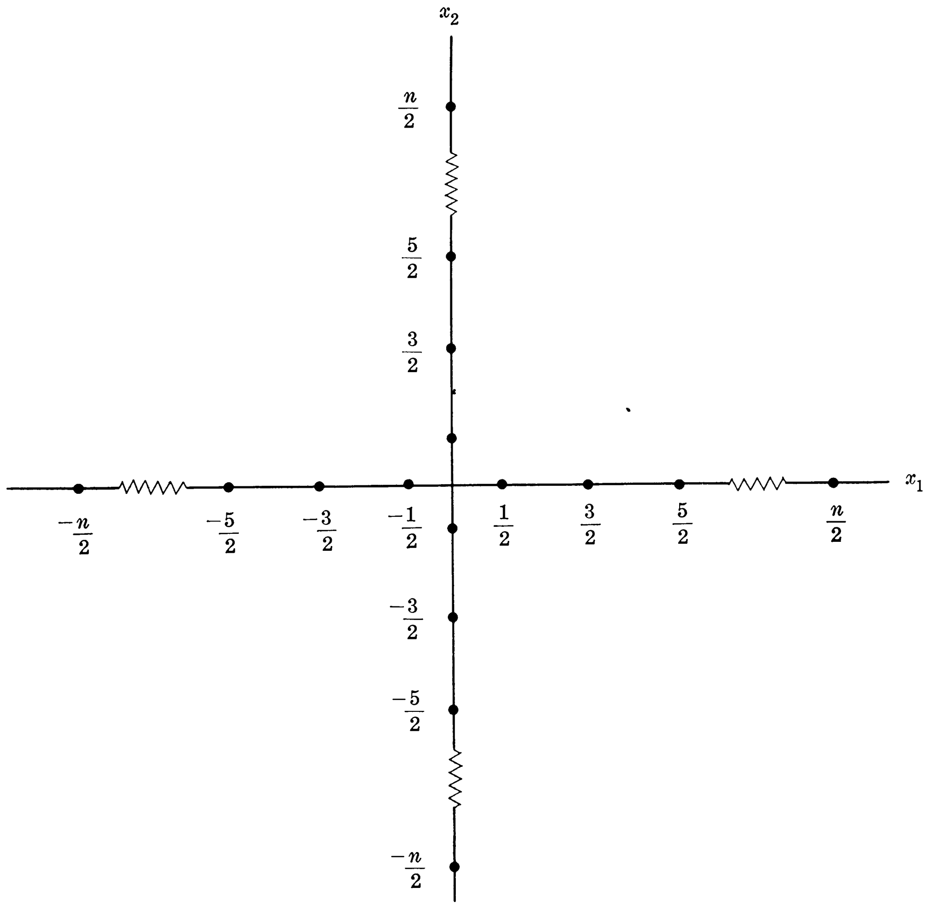

FIG. 2. Cross array with sensors on coordinate axes.

space is $A \exp [i\omega(t - \alpha_2 x_2)]$. The errors $(t = 1, \cdots, T)$ for each sensor are independent and normal, with mean 0 and variance σ^2, and errors for different sensors are independent.

Let $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ be the maximum likelihood estimators of $\kappa_1 = \omega\alpha_1$ and $\kappa_2 = \omega\alpha_2$, respectively. Since the two linear arrays have no common sensor, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are independent. From Theorem 3.3, as $n \to \infty$ $n^{\frac{3}{2}}(\tilde{\kappa}_1 - \kappa_1)$ has limiting distribution $N(0, 6\sigma^2/[T|A|^2])$, and similarly for $\tilde{\kappa}_2$.

The maximum likelihood estimator of θ is $\tilde{\theta} = \arctan(\tilde{\kappa}_2/\tilde{\kappa}_1)$ if $\tilde{\kappa}_2 > 0$, and its large sample mean square error is, from (4.2) and Theorem 3.3,

$$E(\tilde{\theta} - \theta)^2 \sim nE(\hat{\theta} - \theta)^2 \sim \frac{1}{n^3} \frac{3\lambda^2\sigma^2}{2\pi^2 T|A|^2 \sin^2 \gamma} .$$

Clay and Hinich [4] used an *ad hoc* estimator for θ which had the same properties as $\tilde{\theta}$, given a cross array. Similarly the maximum likelihood estimator of v is $\tilde{v} = \omega \sin \gamma / (\hat{\kappa}_1^2 + \hat{\kappa}_2^2)^{\frac{1}{2}}$, and its large sample mean square error is $\mathrm{Var}(\tilde{v}) \sim n \mathrm{Var}(\hat{v})$. Thus the number of sensors in a cross array is a factor of $2/n$ times that of a square array, but the large sample variances are n times as large.

5. Estimation for a sum of plane waves. In this section we treat the model (1.2) when $M > 1$, generalizing the treatment by Walker [19] of $r = 1$. The plane waves are of the form (1.3) and $A_m(0) = 0$, $m = 1, \cdots, M$, is assumed to hold. Then

$$\begin{aligned}(5.1)\qquad z(\omega_j, \mathbf{x}) &= \frac{1}{T} \textstyle\sum_{t=0}^{T-1} e^{-i\omega jt} y(t, \mathbf{x}) \\ &= \textstyle\sum_{m=1}^{M} A_m(\omega_j) \exp[-i\boldsymbol{\kappa}_m(\omega_j)'\mathbf{x}] + u(\omega_j, \mathbf{x}), \\ &\qquad\qquad j = 1, \cdots, \tfrac{1}{2}(T-1),\end{aligned}$$

where for ω_j and $\mathbf{x}$, $u(\omega_j, \mathbf{x}) = u_1(\omega_j, \mathbf{x}) - iu_2(\omega_j, \mathbf{x})$ is a complex normal random variable given by (2.3), and the moments of the $u(\omega_j, \mathbf{x})$'s are given by (2.4). As before, since the $u(\omega_j, \mathbf{x})$'s are independent for different values of j, we fix j and write

$$(5.2)\qquad z(\mathbf{x}) = \textstyle\sum_{m=1}^{M} A_m e^{-i\boldsymbol{\kappa}_m'\mathbf{x}} + u(\mathbf{x}),$$

where $A_m = \frac{1}{2}(a_m + ib_m)$ and the variables $u(\mathbf{x}) = u_1(\mathbf{x}) - iu_2(\mathbf{x})$ are independent and normal and $u_1(\mathbf{x})$ and $u_2(\mathbf{x})$ are independent, each having mean 0 and variance $\frac{1}{2}\sigma^2/T$. Here we denote $\boldsymbol{\kappa}_m = (\kappa_{m1}, \cdots, \kappa_{mr})'$, $m = 1, \cdots, M$. The maximum likelihood estimators are those which minimize

$$\begin{aligned}(5.3)\qquad &\textstyle\sum_{x_1,\cdots,x_r=1}^{n} |z(\mathbf{x}) - \sum_{m=1}^{M} A_m e^{-i\boldsymbol{\kappa}_m'\mathbf{x}}|^2 \\ &= \textstyle\sum_{x_1,\cdots,x_r=1}^{n} |z(\mathbf{x})|^2 - 2\mathscr{R} \sum_{m=1}^{M} A_m \sum_{x_1,\cdots,x_r=1}^{n} \bar{z}(\mathbf{x}) e^{-i\boldsymbol{\kappa}_m'\mathbf{x}} \\ &\quad + n^r \textstyle\sum_{m=1}^{M} |A_m|^2 .\end{aligned}$$

We assume

$$(5.4)\qquad \min_{j=1,\cdots,r} \min_{1\le m\ne l\le M} |\kappa_{mj} - \kappa_{lj}| > 0, \quad \boldsymbol{\kappa}_m \in K^I, \qquad m = 1, \cdots, M,$$

where K^I is defined at the beginning of Section 3. We denote the set of $(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M)$ satisfying (5.4) by K^{IM}. We cannot minimize (5.3) in unrestricted fashion. Let R_n denote the set of $(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M)$ satisfying

$$(5.5)\qquad \lim_{n\to\infty} \min_{j=1,\cdots,r} \min_{1\le m\ne l\le M} n|\kappa_{mj} - \kappa_{lj}| = \infty, \quad \boldsymbol{\kappa}_m \in K, \qquad m = 1, \cdots, M.$$

We minimize (5.3) subject to (5.5), which is comparable to the condition of Walker [19] when $r = 1$. If the minimization is not so restricted, estimators of $\boldsymbol{\kappa}_m$ for more than one m will converge to the same value, when, in fact, $(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M) \in K^{IM}$, i.e., the $\boldsymbol{\kappa}_m$'s are separated.

Minimization of (5.3) with respect to a_m, b_m, $\boldsymbol{\kappa}_m$, $m = 1, \cdots, M$, then yields

$$(5.6) \qquad \hat{A}_m = \tfrac{1}{2}(\hat{a}_m + i\hat{b}_m) = \frac{1}{n^r} \sum_{x_1,\cdots,x_r=1}^{n} e^{i\hat{\boldsymbol{\kappa}}_m'\mathbf{x}} z(\mathbf{x}) , \qquad m = 1, \cdots, M ,$$

and $\hat{\boldsymbol{\kappa}}_m = (\hat{\kappa}_{m1}, \cdots, \hat{\kappa}_{mr})$, $m = 1, \cdots, M$, such that

$$(5.7) \qquad \sum_{m=1}^{M} I_n(\hat{\boldsymbol{\kappa}}_m) = \sup_{(\boldsymbol{\kappa}_1,\cdots,\boldsymbol{\kappa}_M)\in R_n} \sum_{m=1}^{M} I_n(\boldsymbol{\kappa}_m) .$$

That is, we find the M greatest maxima of $I_n(\boldsymbol{\kappa})$ subject to the separation condition (5.5). In the following the subscript 0 will denote the true value of the corresponding parameter. Thus we assume $(\boldsymbol{\kappa}_{10}, \cdots, \boldsymbol{\kappa}_{M0}) \in K^{IM}$.

From (2.10) and (5.2)

$$(5.8) \qquad (2\pi n)^r \sum_{m=1}^{M} I_n(\boldsymbol{\kappa}_m) = \sum_{m=1}^{M} \left| \sum_{x_1,\cdots,x_r=1}^{n} e^{i\boldsymbol{\kappa}_m'\mathbf{x}} u(\mathbf{x}) + \sum_{l=1}^{M} A_{l0} D_n(\boldsymbol{\kappa}_m - \boldsymbol{\kappa}_{l0}) \right|^2 .$$

When $(\boldsymbol{\kappa}_{10}, \cdots, \boldsymbol{\kappa}_{M0}) \in K^{IM}$ and $(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M) \in R_n$ only M of the M^2 differences $\boldsymbol{\kappa}_m - \boldsymbol{\kappa}_{l0}$ can have components which are $O(n^{-1})$. We label the $\boldsymbol{\kappa}_m$'s so that the differences $\boldsymbol{\kappa}_m - \boldsymbol{\kappa}_{m0}$ have components which can be $O(n^{-1})$. Then from (5.8) we obtain

$$\sup_{(\boldsymbol{\kappa}_1,\cdots,\boldsymbol{\kappa}_M)\in R_n} \left|(2\pi)^r \sum_{m=1}^{M} I_n(\boldsymbol{\kappa}_m) - n^{-r} \sum_{m=1}^{M} |A_{m0}|^2 |D_n(\boldsymbol{\kappa}_m - \boldsymbol{\kappa}_{m0})|^2\right| = o_p(n^r) ,$$

which is the analogue of (3.4). Denote $R_{n,\delta} = R_n \cap \{(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M) : |\kappa_{mj} - \kappa_{mj0}| \geqq n^{-1}\delta,\ j = 1, \cdots, r,\ m = 1, \cdots, M\}$. Then if δ is small and n is large it follows that

$$\sup_{(\boldsymbol{\kappa}_1,\cdots,\boldsymbol{\kappa}_M)\in R_{n,\delta}} \left(\frac{2\pi}{n}\right)^r \sum_{m=1}^{M} I_n(\boldsymbol{\kappa}_m) \leqq \sum_{m=1}^{M} |A_{m0}|^2 \left(\frac{\sin \frac{1}{2}\delta}{\frac{1}{2}\delta}\right)^{2r} + o_p(1) .$$

Since

$$(5.9) \qquad p \lim_{n\to\infty} \left(\frac{2\pi}{n}\right)^r \sum_{m=1}^{M} I_n(\boldsymbol{\kappa}_{m0}) = \sum_{m=1}^{M} |A_{m0}|^2 ,$$

we obtain the generalization of Theorem 3.1 for $M > 1$,

$$(5.10) \qquad \hat{\kappa}_{mj} - \kappa_{mj0} = o_p(n^{-1}) , \qquad j = 1, \cdots, r , \quad m = 1, \cdots, M ,$$

as $n \to \infty$. However, it is still necessary to align the $\hat{\boldsymbol{\kappa}}_m$'s and the $\boldsymbol{\kappa}_m$'s properly. By the law of the mean

$$I_n(\hat{\boldsymbol{\kappa}}_m) = I_n(\boldsymbol{\kappa}_{m0}) + \sum_{j=1}^{r} \left.\frac{\partial I_n(\boldsymbol{\kappa})}{\partial \kappa_j}\right|_{\boldsymbol{\kappa}_m^*} (\hat{\kappa}_{mj} - \kappa_{mj0}) , \qquad m = 1, \cdots, M ,$$

for an appropriate $\boldsymbol{\kappa}_m^*$. From (2.11), Lemma 2.1, Lemma 2.2, (5.9), and (5.10), we conclude

$$p \lim_{n\to\infty} (2\pi/n)^r I_n(\hat{\boldsymbol{\kappa}}_m) = |A_{m0}|^2 .$$

If the subscripts are arranged so that $|A_{10}|^2 \geqq \cdots \geqq |A_{M0}|^2$, then the probability tends to 1 as $n \to \infty$ that $I_n(\hat{\boldsymbol{\kappa}}_1) \geqq \cdots \geqq I_n(\hat{\boldsymbol{\kappa}}_M)$. That is, if $\hat{\boldsymbol{\kappa}}_m$ determines the

mth largest maximum of $I_n(\boldsymbol{\kappa})$, subject to (5.5), it corresponds to the mth largest harmonic component in the estimation procedure.

The remaining results in Section 3 follow in similar fashion, with appropriate generalization. The estimators (5.6) are consistent,

$$p\lim_{n\to\infty} \hat{A}_m = A_{m0}\,, \qquad m = 1, \cdots, M\,. \tag{5.11}$$

For each $m = 1, \cdots, M$ there is a set of equations of the form (3.6), with U_n defined by (5.3). These lead to the following generalization of Theorem 3.3.

THEOREM 5.1. *Let* $z(\mathbf{x}) = \sum_{m=1}^{M} A_m \exp(-i\boldsymbol{\kappa}_m{}'\mathbf{x}) + u(\mathbf{x})$, *where* $(\boldsymbol{\kappa}_1, \cdots, \boldsymbol{\kappa}_M) \in K^{IM}$, $A_m = \frac{1}{2}(a_m + ib_m)$, *and the variables* $u(\mathbf{x}) = u_1(\mathbf{x}) - iu_2(\mathbf{x})$ *are independently and normally distributed, with* $u_1(\mathbf{x})$ *and* $u_2(\mathbf{x})$ *being independent and each having mean* 0 *and variance* $\frac{1}{2}\sigma^2/T$. *If* $\hat{a}_m, \hat{b}_m, \hat{\boldsymbol{\kappa}}_m$ *are defined by* (5.6) *and* (5.7) *then as* $n \to \infty$ $(n^{\frac{1}{2}r}(\hat{a}_m - a_m), n^{\frac{1}{2}r}(\hat{b}_m - b_m), n^{\frac{1}{2}r+1}(\hat{\boldsymbol{\kappa}}_m - \boldsymbol{\kappa}_m)')'$ *have independent limiting normal distributions with mean vectors* $\mathbf{0}$ *and covariance matrices* $(2\sigma^2/T)\boldsymbol{\Sigma}_m^{-1}$, $m = 1, \cdots, M$, *where* $(2\sigma^2/T)\boldsymbol{\Sigma}_m^{-1}$ *is defined by* (3.19) *with* a, b, *and* A *replaced by* a_m, b_m, *and* A_m, *respectively.*

The independence for different values of m is a consequence of the separation conditions (5.4). Theorem 3.4 also generalizes in the same manner.

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