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A STOCHASTIC PARTICLE METHOD WITH RANDOM WEIGHTS FOR THE COMPUTATION OF STATISTICAL SOLUTIONS OF MCKEAN–VLASOV EQUATIONS

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We are interested in statistical solutions of McKean–Vlasov–Fokker–Planck equations. An example of motivation is the Navier–Stokes equation for the vorticity of a two-dimensional incompressible fluid flow. We propose an original and efficient numerical method to compute moments of such solutions. It is a stochastic particle method with random weights. These weights are defined through nonparametric estimators of a regression function and convey the uncertainty on the initial condition of the considered equation. We prove an existence and uniqueness result for a class of nonlinear stochastic differential equations (SDEs), and we study the relationship between these nonlinear SDEs and statistical solutions of the corresponding McKean–Vlasov equations. This result forms the foundation of our stochastic particle method where we estimate the convergence rate in terms of the numerical parameters: the number of simulated particles and the time discretization step.

1. Introduction. Partial differential equations (PDEs) with random initial conditions are possible models for some complex physical phenomena such as turbulence (see, e.g., Monin and Yaglom [21] and Vishik and Fursikov [28]). They also can express a lack of information on the initial state of a system, as in weather forecasting (see Chorin, Kast and Kupferman [6]) where data are collected from a finite and relatively small number of meteorological stations. Of course, there are many functions fitting such a finite set of values; therefore, sparse data often lead to statistical models.

In both cases, one can only simulate mean quantities over the set of initial conditions of the model. However, it is often difficult to estimate the accuracy of usual related numerical methods such as closure models (see, e.g., Fox [13], Mohammadi and Pironneau [20] and Vishik and Fursikov [28]). Following a quite different approach, we propose here an original stochastic particle method with random weights to compute

$$\langle M_1(t), f\rangle_{L^2(\mathbb{R})} := \mathbb{E}\int_{\mathbb{R}} p(t,x,\omega) f(x)\,dx, \tag{1}$$

where f is a given test function and $p(t,x,\omega)$ is the solution of a McKean–Vlasov

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equation with random initial condition. Our motivation comes from the fact that the viscous Burgers equation and the two-dimensional incompressible Navier–Stokes equation for the vorticity belong to the class of McKean–Vlasov equations and thus have a probabilistic interpretation in terms of stochastic particle systems (see, e.g., Sznitman [22]).

We now fix some notation and consider the McKean–Vlasov equation

$$
\begin{aligned}
\frac{\partial p}{\partial t}(t,x,\omega) &= -\frac{\partial}{\partial x}\big(u_b(t,x,\omega)p(t,x,\omega)\big) \\
&\qquad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\big(u_\sigma(t,x,\omega)\big)^2 p(t,x,\omega)\big), \\
p(0,x,\omega) &= p_0(x,\omega), \\
u_b(t,x,\omega) &:= \int_{\mathbb{R}} b(x,y)p(t,y,\omega)\,dy, \\
u_\sigma(t,x,\omega) &:= \int_{\mathbb{R}} \sigma(x,y)p(t,y,\omega)\,dy,
\end{aligned}
\tag{2}
$$

where b and σ are smooth and bounded functions from $\mathbb{R}^2$ to $\mathbb{R}$. For technical reasons we thereafter suppose that the possible initial conditions of (2) are parametrized by realizations $\theta(\omega)$ of a real-valued random variable θ with law ν concentrated on a closed interval of $\mathbb{R}$, say $[-1,1]$.

The paper is organized as follows:

First, we briefly outline the theory of *statistical solutions* of an evolution problem, especially the notion of moments of a statistical solution, and we present some known results on the two-dimensional incompressible Navier–Stokes equation. We then study statistical solutions of the model problem (2) and their moments. In particular, we show that the identity (1) defines the first moment of the statistical solution of (2).

Second, we prove an original probabilistic interpretation of the moments. To this end, we prove the following result, which is interesting in itself. Consider the nonlinear stochastic differential equation

$$
\begin{cases}
dX_t = \mathbb{E}\big[b(x,X_t)\mid\theta\big]\big|_{x=X_t}\,dt + \mathbb{E}\big[\sigma(x,X_t)\mid\theta\big]\big|_{x=X_t}\,dW_t, \qquad t\le T,\\
(X_0,\theta) \text{ with law } \big[\Phi(a)\big](x)\,dx\,\nu(da),\\
\theta \text{ random variable independent of } W.
\end{cases}
$$

Under appropriate hypotheses on the kernels b and σ, we show that this equation has a unique weak solution, and we describe the relationship between this solution and the moment defined in (1).

Third, we develop the following stochastic particle method:

$$
\begin{aligned}
\overline{X}^{i,N}_{(k+1)\Delta t} = \overline{X}^{i,N}_{k\Delta t} &+ \sum_{j=1}^{N} \alpha_{ij} b\Big(\overline{X}^{i,N}_{k\Delta t}, \overline{X}^{j,N}_{k\Delta t}\Big)\Delta t \\
&+ \sum_{j=1}^{N} \alpha_{ij} \sigma\Big(\overline{X}^{i,N}_{k\Delta t}, \overline{X}^{j,N}_{k\Delta t}\Big)\big(W^i_{(k+1)\Delta t} - W^i_{k\Delta t}\big), \\
\overline{X}^{i,N}_0 = X^i_0, &
\end{aligned}
$$

where the $W^i_{\cdot}$, $1 \le i \le N$ are independent real Brownian motions. The weights α_{ij} are defined from nonparametric estimators of the functions u_b and u_σ. We show that

$$
\langle M_1(t), f\rangle_{L^2(\mathbb{R})} \simeq \frac{1}{N}\sum_{i=1}^{N} f\Big(\overline{X}^{i,N}_t\Big).
$$

Finally, we prove estimates on the convergence rate of the above approximation in terms of the number N of simulated particles and the time discretization step Δt.

REMARK 1.1. Admittedly, our technical assumptions on the functions b and σ exclude the singular interaction kernels corresponding to the Burgers and Navier–Stokes equations: our results will hopefully be extended to these singular kernels in the future.

A summary of the results of this paper has appeared in [26].

NOTATION. For $k \in \mathbb{N}$, $C^k_{\mathrm{b}}(\mathbb{R}^n)$ is the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$ whose partial derivatives up to order k are continuous and uniformly bounded over $\mathbb{R}^n$.

$\mathbb{E}$ is the expectation operator under the law $\mathbb{P}_W$ of a real-valued Brownian motion W. and, for any probability measure ν on $[-1, 1]$, $\mathbb{E}^\nu$ is the expectation operator under the product measure $\mathbb{P}_W \otimes \nu$.

C, $C(T)$ are strictly positive real constants that can change from line to line.

2. Statistical solution of a Cauchy problem: application to the model equation (2). In this section, we define the notion of statistical solution and of moments of such a solution. We give assumptions under which a statistical solution and moments exist for (2).

The notion of statistical solution was first proposed by Hopf [17] to describe turbulence. This approach has then been studied by several authors, in particular, Foias [10], Foias and Temam [11, 12], and Vishik and Fursikov [28]. A somewhat different notion of statistical solution has been studied by Carraro and Duchon [4] (see also the references therein) for the inviscid Burgers equation.

Consider an evolution equation on a strip $[0, T] \times \mathbb{R}^n$, $n \leq 3$:

$$\begin{aligned} &\frac{du}{dt} + Au = 0, \\ &u|_{t=0} = u_0 \in L^2(\mathbb{R}^n), \end{aligned} \tag{3}$$

where u_0 is a random variable with law μ.

DEFINITION 2.1 (Vishik and Fursikov [28], page 87). The Cauchy problem (3) is said to have a "low Reynolds number" by analogy with fluid mechanics if, for each initial condition u_0 in the support of μ, (3) has a unique solution $Su_0 \in C([0, T], L^2(\mathbb{R}^n))$. In this case, the statistical solution of (3) with initial condition μ is the probability measure on $L^2(0, T; L^2(\mathbb{R}^n))$ defined by

$$m := \mu \circ S^{-1}.$$

REMARK 2.2. The notion of statistical solution may be defined in a more general setting (see [28], Definition 1.1, page 122). Indeed, the Cauchy problem (3) may have a statistical solution even if it does not have a low Reynolds number, but we do not consider this situation here.

Suppose now that (3) has a statistical solution m whose marginal, or *spatial statistical solution*, at time t satisfies

$$\text{for some integer } k \geq 1, \qquad \int_{L^2(\mathbb{R}^n)} \|u\|^k_{L^2(\mathbb{R}^n)} m_t(du) < +\infty. \tag{4}$$

In particular, if $k = 1$, we assume that the mean energy at time t is finite. For any $p \leq k$, consider the linear form on $L^2(p) := \otimes^p L^2(\mathbb{R}^n)$ defined by

$$F_p : \phi \in L^2(p) \mapsto \int_{L^2(\mathbb{R}^n)} \left\langle \bigotimes^p u, \phi \right\rangle_{L^2(p)} m_t(du).$$

By the Cauchy–Schwarz inequality and assumption (4), the application F_p is continuous. Hence, by the Riesz representation theorem, there exists a unique element $M_p(t) \in L^2(p)$ such that

$$\begin{aligned} &\forall p \leq k,\ \forall \phi \in L^2(p), \\ &\qquad \langle M_p(t), \phi \rangle_{L^2(p)} = \int_{L^2(\mathbb{R}^n)} \left\langle \bigotimes^p u, \phi \right\rangle_{L^2(p)} m_t(du). \end{aligned} \tag{5}$$

By definition, $M_p(t)$ is the pth *moment* of the measure m_t.

We now illustrate these notions in the particular case of the Navier–Stokes equation:

$$
\begin{aligned}
&\partial_t \omega(t,x,\omega_0) = -\nabla \cdot \big(\mathbf{u}(t,x,\omega_0)\omega(t,x,\omega_0)\big) + \frac{\sigma^2}{2}\Delta\omega(t,x,\omega_0),\\
&\mathbf{u} = \mathbf{K} * \omega \qquad \text{where } \mathbf{K}(\mathbf{y}) = \frac{1}{2\pi|\mathbf{y}|^2}(-y_2, y_1),\\
&\omega(0,x,\omega_0) = \omega_0(x).
\end{aligned}
\tag{6}
$$

If the law μ of the initial condition of (6) is concentrated on $L^1(\mathbb{R}^2) \cap L^\infty_{\mathrm{c}}(\mathbb{R}^2)$ [i.e., the subspace of $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ of functions with compact support], Constantin and Wu [9] have shown that (6) has a unique statistical solution with initial condition μ. Moreover, if μ is concentrated on a closed ball of $L^1(\mathbb{R}^2) \cap L^\infty_{\mathrm{c}}(\mathbb{R}^2)$, the spatial correlations of the velocity $\mathbf{u}$ are related to moments (see Vaillant [27]):

$$
\int \mathbf{u}_i(t,x,\omega_0)\mu(d\omega_0) = \langle M_1(t), \mathbf{K}_i(x-\cdot)\rangle_{L^2(\mathbb{R}^2)}, \qquad i = 1,2
$$

(mean velocity),

$$
\tfrac{1}{2}\int \mathbf{u}^2(t,x,\omega_0)\mu(d\omega_0) = \tfrac{1}{2}\langle M_2(t), \mathbf{K}(x-\cdot)\cdot\mathbf{K}(x-\cdot)\rangle_{L^2(\mathbb{R}^2)\otimes L^2(\mathbb{R}^2)}
$$

(mean kinetic energy).

This representation mainly explains our interest in simulating moments.

We now turn to the statistical study of the model equation (2). Remember that the uncertainty on initial conditions of (2) is supposed to come from a random parameter θ with law ν. Admittedly, this assumption is quite restrictive from a physical point of view. Nevertheless, it is very useful for numerical reasons: generally, the law of the random initial condition of (2) is a measure on an infinite-dimensional functional space; the parametrization allows us to reduce it to the law of a finite-dimensional random variable.

From now on, we denote by Φ the one-to-one application assigning to any parameter $a \in [-1,1]$ an initial condition $\Phi(a) := p_0(\cdot, a)$.

We now prove that the model problem (2) has a low Reynolds number in the sense of Definition 2.1. This is a straightforward consequence of the following proposition.

PROPOSITION 2.3. *Suppose that*

(H1) $\exists\, \varepsilon \in]0,1[$ *such that*

$$
\begin{aligned}
&b \in C^{2+\varepsilon}_{\mathrm{b}}(\mathbb{R}^2), \qquad \sigma \in C^{2+\varepsilon}_{\mathrm{b}}(\mathbb{R}^2) \quad \textit{and},\\
&\textit{for any } (x,y) \in \mathbb{R}^2, \qquad \sigma(x,y) \geq \sigma_* > 0,
\end{aligned}
\tag{7}
$$

where, for $k \in \mathbb{N}$, $C_{\mathrm{b}}^{k+\varepsilon}(\mathbb{R}^n)$ *denotes Hölder spaces of functions* (*see*, *e.g*., [18]). *In particular, there exists a strictly positive constant* L *such that*

$$\begin{aligned}\forall\,(x,y,z,u)\in\mathbb{R}^4,\\ |b(x,y)-b(z,u)|+|\sigma(x,y)-\sigma(z,u)|\le L(|x-z|+|y-u|).\end{aligned}\tag{8}$$

Suppose also that the function Φ *satisfies*:

(H2) $\Phi([-1\ ,\ 1])\subset C_{\mathrm{b}}^{2+\varepsilon}(\mathbb{R})\cap W^{2,1}(\mathbb{R})$ *and* $\Phi([-1\ ,\ 1])$ *is a set of probability density functions*;

(H3) Φ *is Lipschitz continuous for the norm in* $L^1(\mathbb{R})$;

(H4) Φ *is such that*

$$\sup_{a\in[-1,1]}\|p_0(\cdot,a)\|_{W^{2,1}(\mathbb{R})}<+\infty.$$

Then, for any $a\in[-1,1]$, *the equation*

$$\begin{aligned}\frac{\partial p(t,x,a)}{\partial t} &= -\frac{\partial}{\partial x}\big(u_b(t,x,a)p(t,x,a)\big)\\ &\qquad+\frac{1}{2}\frac{\partial^2}{\partial x^2}\big((u_\sigma(t,x,a))^2p(t,x,a)\big),\\ p(0,x,a) &= p_0(x,a),\\ u_b(t,x,a) &:= \int_{\mathbb{R}} b(x,y)p(t,y,a)\,dy,\\ u_\sigma(t,x,a) &:= \int_{\mathbb{R}} \sigma(x,y)p(t,y,a)\,dy,\end{aligned}\tag{9}$$

has a unique solution in the set of probability densities, *and this solution satisfies*

$$p(\cdot,\cdot,a)=:(S\circ\Phi)(a)\in C_{\mathrm{b}}^{1,2+\varepsilon}\big([0,T]\times\mathbb{R}\big)\cap C\big([0,T],L^2(\mathbb{R})\big),$$

where $C_{\mathrm{b}}^{1,2+\varepsilon}([0,T]\times\mathbb{R})$ *is the set of functions from* $[0,T]\times\mathbb{R}$ *to* $\mathbb{R}$ *of class* C^1 *in time and* $C_{\mathrm{b}}^{2+\varepsilon}$ *in space*.

PROOF. As functions b and σ are Lipschitz continuous and bounded, there is a unique solution $\mathbb{P}$ to the nonlinear martingale problem with coefficients $\int b(x,y)\mathbb{P}_t(dy)$ and $\int\sigma(x,y)\mathbb{P}_t(dy)$ and initial condition $p_0(x,a)\,dx$ (see Sznitman [22]). Moreover, any density $p(t,\cdot,a)$ solution to (9) is the density of $Y_t(a)$, where the process $(Y_t(a))_{t\in[0,T]}$ satisfies the following stochastic differential equation:

$$\begin{cases} dY_t(a)=\left(\displaystyle\int_{\mathbb{R}} b(x,y)p(t,y,a)\,dy\right)\Big|_{x=Y_t(a)}dt\\ \qquad\qquad+\left(\displaystyle\int_{\mathbb{R}}\sigma\big(x,y\big)p(t,y,a)\,dy\right)\Big|_{x=Y_t(a)}dW_t,\\ Y_0(a)\text{ with law }\big[\Phi(a)\big](x)\,dx.\end{cases}$$

Thus, $p(t,\cdot,a)$ is also the density of $X_t(a)$, where the process $(X_t(a))_{t\in[0,T]}$ satisfies the following nonlinear SDE:

$$\begin{cases} dX_t(a) = \mathbb{E}\big(b(x, X_t(a))\big)\big|_{x=X_t(a)}\,dt + \mathbb{E}\big(\sigma(x, X_t(a))\big)\big|_{x=X_t(a)}\,dW_t, \\ X_0(a) \text{ with law } \big[\Phi(a)\big](x)\,dx. \end{cases} \tag{10}$$

Then the uniqueness of $\mathbb{P}$ implies the uniqueness of the density solution of (9). Moreover, from (10), the functions u_b and u_σ satisfy

$$u_b(t,x,a) = \mathbb{E}b\big(x, X_t(a)\big), \qquad u_\sigma(t,x,a) = \mathbb{E}\sigma\big(x, X_t(a)\big). \tag{11}$$

Using equalities (11) and hypothesis (7), one easily checks that the functions u_b and u_σ are of class C_{b}^2 in x. They are also Hölder continuous of order $1/2$ in t. Indeed, for any $(s,t,x,a) \in [0,T]^2 \times \mathbb{R} \times [-1,1]$,

$$\begin{aligned} \mathbb{E}|X_t(a) - X_s(a)|^2 &\le 2\mathbb{E}\bigg\{\int_s^t \mathbb{E}b\big(x, X_\tau(a)\big)\,d\tau\bigg\}^2 \\ &\qquad + 2\mathbb{E}\bigg\{\int_s^t \mathbb{E}\sigma\big(x, X_\tau(a)\big)\,dW\tau\bigg\}^2, \\ &\le 2\|b\|^2_{L^\infty(\mathbb{R}^2)}|t-s|^2 + 2\|\sigma\|^2_{L^\infty(\mathbb{R}^2)}|t-s|. \end{aligned}$$

Hence, as b and σ are Lipschitz functions,

$$\begin{aligned} &|u_b(t,x,a) - u_b(s,x,a)| + |u_\sigma(t,x,a) - u_\sigma(s,x,a)| \\ &\quad = \big|\mathbb{E}b\big(x, X_t(a)\big) - \mathbb{E}b\big(x, X_s(a)\big)\big| \\ &\qquad + \big|\mathbb{E}\sigma\big(x, X_t(a)\big) - \mathbb{E}\sigma\big(x, X_s(a)\big)\big|, \\ &\quad \le L\mathbb{E}|X_t(a) - X_s(a)| \\ &\quad \le L\sqrt{2T\|b\|^2_{L^\infty(\mathbb{R}^2)} + 2\|\sigma\|^2_{L^\infty(\mathbb{R}^2)}}\sqrt{|t-s|}. \end{aligned}$$

Then, as $\Phi(a) \in C_{\mathrm{b}}^{2+\varepsilon}(\mathbb{R})$, Theorem 5.1.9 in Lunardi [18] shows that (9) has a unique solution in $C_{\mathrm{b}}^{1,2+\varepsilon}([0,T]\times\mathbb{R})$. We can then define the operator

$$\begin{aligned} S : \Phi\big([-1,1]\big) &\to C_{\mathrm{b}}^{1,2+\varepsilon}\big([0,T]\times\mathbb{R}\big), \\ p_0 = \Phi(a) &\mapsto Sp_0 := p(\cdot,\cdot,a). \end{aligned} \tag{12}$$

Moreover, observe that $\Phi([-1,1]) \in C_b(\mathbb{R}) \cap W^{2,1}(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Then, owing to assumption (7), the Fokker–Planck equation (9) has a unique solution in $C([0,T]; L^2(\mathbb{R}))$ (see, e.g., Cessenat et al. [5], page 89), and

$$S \circ \Phi\big([-1,1]\big) \subset C_{\mathrm{b}}^{1,2+\varepsilon}\big([0,T]\times\mathbb{R}\big) \cap C\big([0,T]; L^2(\mathbb{R})\big). \qquad \square$$

Before turning our attention to the question of moments of the statistical solution, we state and prove a technical regularity result which will be needed in the proof of Lemma A.1.

PROPOSITION 2.4. *Under the hypotheses of Proposition* 2.3, *the mapping*

$$a \in [-1, 1] \mapsto (S \circ \Phi)(a) \in L^\infty([0, T]; L^1(\mathbb{R}))$$

is Lipschitz continuous.

PROOF. We sketch the proof in Vaillant [27]. Choose $a^1, a^2 \in [-1, 1]$ and, for $i = 1$ or 2, set

$$\begin{aligned}
p_t^i &= p(T - t, \cdot, a^i),\\
u_b^i(t) &= \int_{\mathbb{R}} b(\cdot, y) p_t^i(y)\, dy, \qquad v_\sigma^i(t) = \tfrac{1}{2}\left(\int_{\mathbb{R}} \sigma(x, y) p_t^i(y)\, dy\right)^2,\\
L_t^i &= v_\sigma^i(t)\partial_{xx}^2 + \big(2\partial_x v_\sigma^i(t) - u_b^i(t)\big)\partial_x,\\
k_t^i &= \partial_x u_b^i(t) - \partial_{xx}^2 v_\sigma^i(t),\\
F_t &= \big(v_\sigma^1(t) - v_\sigma^2(t)\big)\partial_{xx}^2 p_t^1\\
&\quad + \big\{\big(u_b^2(t) - u_b^1(t)\big) + 2\big(\partial_x v_\sigma^1(t) - \partial_x v_\sigma^2(t)\big)\big\}\partial_x p_t^1\\
&\quad + \big\{\partial_x u_b^2 - \partial_x u_b^1 + \partial_{xx}^2 v_\sigma^1(t) - \partial_{xx}^2 v_\sigma^2(t)\big\} p_t^1.
\end{aligned} \tag{13}$$

Apply the Feynman–Kac formula to (9): for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$p_t^i(x) = \mathbb{E}\left\{p_0^i(X_T^{i,t,x}) \exp\left(-\int_t^T k_\theta^i(X_\theta^{i,t,x})\, d\theta\right)\right\}, \tag{14}$$

$$\begin{aligned}
q_t(x) &:= p_t^2(x) - p_t^1(x)\\
&= \mathbb{E}\left\{\big(p_0^2(X_T^{2,t,x}) - p_0^1(X_T^{2,t,x})\big) \exp\left(-\int_t^T k_\theta^2(X_\theta^{2,t,x})\, d\theta\right)\right\}\\
&\quad - \mathbb{E}\left\{\int_t^T F_s(X_s^{2,t,x}) \exp\left(-\int_t^s k_\theta^2(X_\theta^{2,t,x})\, d\theta\right) ds\right\},
\end{aligned} \tag{15}$$

where $X_\cdot^{i,t,x}$ is the Markov process whose infinitesimal generator is L_t^i and such that $X_t^{i,t,x} = x$ a.s. Moreover, as $b \in C_b^2(\mathbb{R}^2)$, for $\alpha \in \{0, 1, 2\}$ and $i = 1$ or 2,

$$\begin{aligned}
\|u_b^i(t)\|_{L^\infty(\mathbb{R})} &= \left\|\partial_x^\alpha \int_{\mathbb{R}} \sigma(\cdot, y) p_t^i(y)\, dy\right\|_{L^\infty(\mathbb{R})}\\
&\le \|\partial_x^\alpha \sigma\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\partial_x^\alpha v_t^i\|_{L^\infty(\mathbb{R})} < +\infty, \qquad \sup_{t\in[0,T]} \|\partial_x^\alpha u_t^i\|_{L^\infty(\mathbb{R})} < +\infty,\\
&\|\partial_x^\alpha v_\sigma^1(t) - \partial_x^\alpha v_\sigma^2(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^\alpha u_b^1(t) - \partial_x^\alpha u_b^2(t)\|_{L^\infty(\mathbb{R})}\\
&\qquad \le C\|q_t\|_{L^1(\mathbb{R})},\\
&|F_t(x)| \le C\|q_t\|_{L^1(\mathbb{R})}\big(|p_t^1(x)| + |\partial_x p_t^1(x)| + |\partial_{xx}^2 p_t^1(x)|\big).
\end{aligned} \tag{16}$$

In view of these bounds and of equality (15), we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}} |q_t(x)|\,dx \\
&\quad \le C(T)\int_{\mathbb{R}}\Big\{\mathbb{E}\big|p_0^2\big(X_T^{2,t,x}\big)-p_0^1\big(X_T^{2,t,x}\big)\big|+\int_t^T \mathbb{E}\big|F_s\big(X_s^{2,t,x}\big)\big|\,ds\Big\}\,dx \qquad (17)\\
&\quad =: C(T)\big(A(t)+B(t)\big).
\end{aligned}
$$

Moreover, owing to hypothesis (7), we have already observed that the functions u_b and u_σ are of class C_{b}^2 in x and Hölder continuous of order $1/2$ in t. Hence, the density $\Gamma(s,y,t,x)$ of the law of $X_s^{t,x}$ is exponentially bounded (see Friedman [14], page 139–150):

$$
\forall \overline{\sigma} > \|\sigma\|_{L^\infty(\mathbb{R}^2)},\ \exists C_0>0, \qquad \Gamma(s,y,t,x) \le \frac{C_0}{\sqrt{s-t}}\exp\Big(-\frac{(x-y)^2}{2\overline{\sigma}(s-t)}\Big). \qquad (18)
$$

In view of inequalities (16), the constant C_0 can be chosen uniform in $a\in[-1,1]$. Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}\big|p_0^2\big(X_T^{2,t,x}\big)-p_0^1\big(X_T^{2,t,x}\big)\big|\,dx \\
&\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} |q_T(y)|\Gamma(T,y,t,x)\,dy\,dx \\
&\qquad \le \int_{\mathbb{R}}\Big\{|q_T(y)|\int_{\mathbb{R}}\frac{C_0}{\sqrt{T-t}}\exp\Big(-\frac{(x-y)^2}{2\overline{\sigma}(T-t)}\Big)\,dx\Big\}\,dy \\
&\qquad \le C\int_{\mathbb{R}} |q_T(y)|\,dy.
\end{aligned}
$$

Thus

$$
A(t) \le C\|p_0^2-p_0^1\|_{L^1(\mathbb{R})}. \qquad (19)
$$

Before estimating $B(t)$, observe that the hypotheses (H1) and (H2) ensure that, for any $a\in[-1,1]$, the solution $p(\cdot,\cdot,a)$ to (9) belongs to $C([0,T],W^{2,1}(\mathbb{R}))$ (see Cannarsa and Vespri [3]). Moreover, from the Feynman–Kac formula (14) and inequality (18), one easily shows that

$$
\sup_{a\in[-1,1]}\Big(\sup_{t\in[0,T]}\|p(t,\cdot,a)\|_{L^1(\mathbb{R})}\Big) < +\infty.
$$

Then observe that $\partial_x p(t,\cdot,a)$ and $\partial_{xx}^2 p(t,\cdot,a)$ satisfy PDEs similar to (9) so that, repeating the same kind of arguments and using hypothesis (H4), one can show that

$$
\sup_{a\in[-1,1]}\Big(\sup_{t\in[0,T]}\|p(t,\cdot,a)\|_{W^{2,1}(\mathbb{R})}\Big) < +\infty. \qquad (20)
$$

We can now estimate $B(t)$ in the same way than $A(t)$. Indeed, using (18) and the third line of (16), one has

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\int_t^T \mathbb{E}\left|F_s(X_s^{t,x})\right| ds\right) dx \\
&\quad\le C\int_{\mathbb{R}}\int_t^T \|q_s\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}\sum_{i=0}^{2}|\partial_y^i p_t^1(y)|\frac{1}{\sqrt{s-t}}\exp\left(-\frac{(x-y)^2}{\overline{\sigma}(s-t)}\right) dx\,dy \\
&\quad= C\int_t^T \|q_s\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}\sum_{i=0}^{2}|\partial_y^i p_t^1(y)|\left\{\int_{\mathbb{R}}\frac{1}{\sqrt{s-t}}\exp\left(-\frac{(x-y)^2}{\overline{\sigma}(s-t)}\right) dx\right\} dy\,ds \\
&\quad\le C\int_t^T \|q_s\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}\sum_{i=0}^{2}|\partial_y^i p_t^1(y)|\,dy\,ds \\
&\quad\le C\sup_{t\in[0,T]}\|p_t^1\|_{W^{2,1}(\mathbb{R})}\int_t^T \|q_s\|_{L^1(\mathbb{R})}\,ds \\
&\quad\le C\sup_{a\in[-1,1]}\left(\sup_{t\in[0,T]}\|p(t,\cdot,a)\|_{W^{2,1}(\mathbb{R})}\right)\int_t^T \|q_s\|_{L^1(\mathbb{R})}\,ds.
\end{aligned}
$$

We now use property (20) and get that, for any $t\in[0,T]$,

$$
\begin{aligned}
&\|p(t,\cdot,a^1)-p(t,\cdot,a^2)\|_{L^1(\mathbb{R})} \\
&\qquad\le C\left(\|p_0^2-p_0^1\|_{L^1(\mathbb{R})}+\int_0^t \|p(s,\cdot,a^1)-p(s,\cdot,a^2)\|_{L^1(\mathbb{R})}\,ds\right).
\end{aligned}
$$

Finally, we deduce from the previous inequality that

$$
\begin{aligned}
&\|S\circ\Phi(a^1)-S\circ\Phi(a^2)\|_{L^\infty([0,T],L^1(\mathbb{R}))} \\
&\qquad=\sup_{t\in[0,T]}\|p(t,\cdot,a^1)-p(t,\cdot,a^2)\|_{L^1(\mathbb{R})} \\
&\qquad\le C\|\Phi(a^2)-\Phi(a^1)\|_{L^1(\mathbb{R})} \\
&\qquad\quad+C\int_0^T \|S\circ\Phi(a^1)-S\circ\Phi(a^2)\|_{L^\infty([0,t],L^1(\mathbb{R}))}\,dt.
\end{aligned}
$$

The Lipschitz continuity of $S\circ\Phi$ then follows from the Gronwall lemma and the Lipschitz continuity of the function Φ. $\square$

REMARK 2.5. In the preceding proof, the main step is the proof of inequality (20). This inequality is easy to get if the kernel σ is constant, as in the case of the Burgers or the vorticity equations. Indeed, in that case, we can write (9) in an integral form:

$$
p(t,x,a)=G_t*p_0(x,a)-\int_0^t G_{t-s}*\partial_x\big(u_b(s,x,a)p(s,x,a)\big)\,ds,
$$

where

$$G_t(x) = (2\pi\sigma^2 t)^{-1} \exp\left(-\frac{x^2}{2\sigma^2 t}\right)$$

is the heat kernel. By classical properties of G_t and inequalities (16), it is then easy to deduce that

$$\|p(t,\cdot,a^1) - p(t,\cdot,a^2)\|_{L^1(\mathbb{R}^2)} \le C(T)\|p_0(\cdot,a^1) - p_0(\cdot,a^2)\|_{L^1(\mathbb{R}^2)}.$$

In view of Definition 2.1 and Proposition 2.3, the unique statistical solution of (2) with initial condition $\mu = \nu \circ \Phi^{-1}$ is

$$m := \mu \circ S^{-1} = \nu \circ (S \circ \Phi)^{-1}. \tag{21}$$

The following proposition gives conditions under which the time marginals of the measure m has moments up to order k.

PROPOSITION 2.6. *Assume that the hypotheses of Proposition* 2.3 *hold. Assume also that the measure* ν *and the function* Φ *satisfy*

$$\exists k \in \mathbb{N}, \qquad \int_{-1}^{1} \|\Phi(a)\|^k_{L^2(\mathbb{R})} \nu(da) < +\infty. \tag{22}$$

Then, for any $t \in [0,T]$, *the measure* $m_t := \mu \circ S_t$ *has moments up to order* k.

PROOF. We need to prove that the measure m_t satisfies condition (4); that is,

$$\int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})} m_t(dp) < +\infty.$$

By the definition of m_t, one has

$$\begin{aligned}\int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})} m_t(dp) &= \int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})} \big(\nu \circ (S_t \circ \Phi)^{-1}\big)(dp) \\ &= \int_{-1}^{1} \|(S_t \circ \Phi)(a)\|^k_{L^2(\mathbb{R})} \nu(da).\end{aligned}$$

As in the proof of Lemma 2.4, the solution $(S_t \circ \Phi)(a)$ to (9) satisfies a Feynman–Kac formula, from which inequalities (16) and (18) easily lead to

$$\|(S_t \circ \Phi)(a)\|_{L^2(\mathbb{R})} \le C\|\Phi(a)\|_{L^2(\mathbb{R})}.$$

The result follows from assumption (22). □

In view of (5), the first moment of the measure m_t is then defined by

$$\begin{aligned}\forall f \in L^2(\mathbb{R}), \qquad \langle M_1(t), f\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \langle S_t p_0, f\rangle_{L^2(\mathbb{R})} \mu(dp_0) \\ &= \int_{-1}^{1} \langle S_t \circ \Phi(a), f\rangle_{L^2(\mathbb{R})} \nu(da).\end{aligned} \tag{23}$$

To develop an approximation method of the moments, we now prove a probabilistic representation of these moments.

3. A probabilistic representation of the moments. Our objective is now to give a probabilistic representation of the moments of the statistical solution of (2) in order to be able to approximate these moments by a stochastic particle method. For the sake of simplicity, we limit ourselves to considering the first moment. The extension to moments of higher order is straightforward (see Corollary 3.4).

Remember that, for any $a \in [-1, 1]$, the solution $(S_t \circ \Phi)(a)$ of (9) is the density of the random variable $X_t(a)$, where $X.(a)$ is the weak solution of the stochastic differential equation (10). Hence, for any bounded and measurable function f,

$$\langle (S_t \circ \Phi)(a), f \rangle_{L^2(\mathbb{R})} = \mathbb{E} f\big(X_t(a)\big),$$

and we have

$$\begin{aligned} \langle M_1(t), f \rangle_{L^2(\mathbb{R})} &= \int_{-1}^{1} \langle (S_t \circ \Phi)(a), f \rangle_{L^2(\mathbb{R})} \nu(da), \\ &= \int_{-1}^{1} \mathbb{E} f\big(X_t(a)\big) \nu(da). \end{aligned}$$

This representation suggests the following naive method of simulation:

1. Consider N_1 realizations a_l, $1 \le l \le N_1$, of a random variable θ with law ν.
2. For each initial parameter a_l, consider N_2 i.i.d. random variables $X_0^i(a_l)$, with common law $[\Phi(a_l)](x)\,dx$, and define the particle system

$$(24) \qquad \begin{aligned} dX_t^{i,N_2}(a_l) &= \frac{1}{N_2} \sum_{j=1}^{N_2} b\Big(X_t^{i,N_2}(a_l), X_t^{j,N_2}(a_l)\Big)\,dt \\ &\quad + \frac{1}{N_2} \sum_{j=1}^{N_2} \sigma\Big(X_t^{i,N_2}(a_l), X_t^{j,N_2}(a_l)\Big)\,dW_t^i, \\ X_0^{i,N_2}(a_l) &= X_0^i(a_l) \qquad \text{for all } 1 \le i \le N_2. \end{aligned}$$

   Then

$$(25) \qquad \mathbb{E} f\big(X_t(a_l)\big) \simeq \frac{1}{N_2} \sum_{i=1}^{N_2} f\Big(X_t^{i,N_2}(a_l)\Big).$$

   This latter approximation is understood in the following sense: the measure-valued process $1/N_2 \sum_{i=1}^{N_2} \delta_{X_\cdot^{i,N_2}}$ converges in law to $\delta_{\mathbb{P}_{X.(a_l)}}$ as N_2 tends to ∞. This type of convergence is known as the *propagation of chaos*. We refer to Méléard [19] or Sznitman [22] for further details.

We finally get

$$(26) \qquad \langle M_1(t), f \rangle_{L^2(\mathbb{R})} \simeq \frac{1}{N_1} \sum_{l=1}^{N_1} \frac{1}{N_2} \sum_{i=1}^{N_2} f\Big(X_t^{i,N_2}(a_l)\Big).$$

This approximation is numerically very expensive as it requires $N_1 \times N_2$ simulations of paths of stochastic processes. The method with a much lower cost which is developed in Section 4 relies on the construction of a stochastic process whose marginal laws are equal to $[(S_t \circ \Phi)(a)](x)\,dx \otimes \nu(da)$. In view of (10) and (11), a natural candidate is the weak solution (if it exists) of the following stochastic differential equation:

$$
\begin{cases} dX_t = u_b(t, X_t, \theta)\,dt + u_\sigma(t, X_t, \theta)\,dW_t, \\ \forall A \in \mathcal{B}(\mathbb{R} \times [-1,1]), \quad \mathbb{P}\big((X_0, \theta) \in A\big) = \int_A \big[\Phi(a)\big](x)\,dx\,\nu(da), \\ \qquad\qquad\qquad\qquad \theta \text{ random variable independent of } W, \end{cases} \tag{27}
$$

where the functions u_b and u_σ are the coefficients of (9). However, we cannot deduce a numerical method from (27). Indeed, we cannot simulate the solution of (27) since we do not know the coefficients $u_b(t,x,a)$ and $u_\sigma(t,x,a)$ without solving (9). To overcome this problem, we will observe that, for any function $g \in C_b(C([0,T],\mathbb{R}),\mathbb{R})$,

$$
\mathbb{E}^\nu g(X.) = \int_{-1}^{1} \mathbb{E} g\big(X.(a)\big)\nu(da), \tag{28}
$$

where $X.(a)$ is the solution of (10) (see Theorem 3.1). From this property, we will prove that $X.$ is the unique solution of the nonlinear SDE

$$
\begin{cases} dX_t = \mathbb{E}\big[b(x, X_t) \mid \theta\big]\big|_{x=X_t}\,dt + \mathbb{E}\big[\sigma(x, X_t) \mid \theta\big]\big|_{x=X_t}\,dW_t, \qquad t \le T, \\ (X_0, \theta) \text{ with law } \big[\Phi(a)\big](x)\,dx\,\nu(da), \\ \theta \text{ random variable independent of } W \end{cases} \tag{29}
$$

(see Theorem 3.2). We now prove our conjectures.

THEOREM 3.1. *Suppose that the hypotheses of Proposition* 2.3 *hold. Then*:

(i) *the stochastic differential equation* (27) *has a unique weak solution*;
(ii) *for any function* $g \in C_b(C([0,T],\mathbb{R}),\mathbb{R})$,

$$
\mathbb{E}^\nu g(X.) = \int_{-1}^{1} \mathbb{E} g\big(X.(a)\big)\nu(da), \tag{30}
$$

where $X.(a)$ *is the solution of* (10) *and, if* $\mathbb{P}_W$ *stands for the law of the Brownian motion* $W.$, $\mathbb{E}^\nu$ *is the expectation operator under the product measure* $\mathbb{P}_W \otimes \nu$.

PROOF. We first need to check that the mapping $a \in [-1,1] \mapsto \mathbb{E}g(X.(a))$ is Lebesgue measurable. It is actually continuous, as proven in Lemma A.1 whose proof is given in the Appendix. $\square$

PROOF OF (i). Observe that (27) can be rewritten as

$$\begin{cases} dX_t = u_b(t, X_t, \theta_t)\,dt + u_\sigma(t, X_t, \theta_t)\,dW_t, \\ d\theta_t = 0, \\ \forall A \in \mathcal{B}\big(\mathbb{R} \times [-1, 1]\big), \qquad \mathbb{P}\big((X_0, \theta_0) \in A\big) = \int_A \big[\Phi(a)\big](x)\,dx\,\nu(da). \end{cases} \tag{31}$$

In the proof of Proposition 2.3, we have shown that hypothesis (7) ensures that the functions u_b and u_σ are Hölder continuous in t. Similarly, one easily checks that hypothesis (8) implies boundedness and Lipschitz continuity in x of $u_b(t, x, a)$ and $u_\sigma(t, x, a)$. Finally, Lemma 2.4 ensures that u_b and u_σ also are Lipschitz continuous in a. Thus, (31) and, of course, (27), has a unique solution in law. □

PROOF OF (ii). We first fix some notation.

- $E = C([0, T], \mathbb{R} \times [-1, 1])$ and (y_t) is the canonical process on E.
- For each $a \in [-1, 1]$, $c.(a)$ denotes the constant application $t \in [0, T] \mapsto a$.
- For each $a \in [-1, 1]$ and any function $\phi \in C^2(\mathbb{R})$, we set

$$L_t^a \phi(x) = u_b(t, x, a)\phi'(x) + \tfrac{1}{2} u_\sigma^2(t, x, a)\phi''(x), \tag{32}$$

and, for each function $\psi \in C^2(\mathbb{R} \times [-1, 1])$,

$$A_t \psi(x, a) = \big(L_t^a \psi(\cdot, a)\big)(x). \tag{33}$$

Observe that A_t is the infinitesimal generator of the Markov process $(X., \theta.)$, unique solution in law of (31). In other words, the law of $(X., \theta.)$ is the unique solution of the martingale problem associated with operator A_t, which is the only probability measure $\mathbb{P}^\nu$ on E satisfying the following properties:

(a) $\mathbb{P}^\nu \circ y(0)^{-1} = [\Phi(a)](x)\,dx \otimes \nu(da)$;
(b) for any function $\psi \in C_b^2(\mathbb{R} \times [-1, 1])$, the process $M.(\psi, A)$, defined by

$$M_t(\psi, A) = \psi\big(y(t)\big) - \psi\big(y(0)\big) - \int_0^t A_s \psi\big(y(s)\big)\,ds \tag{34}$$

is a $\mathbb{P}^\nu$-martingale.

By uniqueness, equality (30) will thus be proved if we show that the probability measure $\mathcal{Q}$, defined by

$$\begin{gathered} \forall g \in C_b\big(C\big([0, T], \mathbb{R} \times [-1, 1]\big), \mathbb{R}\big), \\ \langle \mathcal{Q}, g \rangle := \int_{-1}^1 \mathbb{E} g\big(X.(a), c.(a)\big)\nu(da), \end{gathered} \tag{35}$$

also satisfies properties (a) and (b).

By definition of the process $X_\cdot(a)$, the solution of (10), $\mathcal{Q}$ obviously satisfies property (a). Now let $p \in \mathbb{N}$, $h \in C_{\mathrm{b}}((\mathbb{R} \times [-1,1])^p)$, $\psi \in C^2(\mathbb{R} \times [-1,1])$ and $(t_1, \dots, t_p, s, t) \in [0,T]^p$, such that

$$0 \le t_1 \le \cdots \le t_p \le s \le t.$$

To show that $\mathcal{Q}$ satisfies property (b), it is sufficient to check that

$$\mathbb{E}^{\mathcal{Q}}\big[h\big(y(t_1), \dots, y(t_p)\big)\big(M_t(\psi, A) - M_s(\psi, A)\big)\big] = 0.$$

Observe that $L^a_\cdot$ is the infinitesimal generator of the process $X_\cdot(a)$. Thus, applying Itô's formula to the function ψ and using definitions (33) and (35), we get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{Q}}\big[h\big(y(t_1), \dots, y(t_p)\big)\big(M_t(\psi, A) - M_s(\psi, A)\big)\big]\\
&\quad = \int_{-1}^{1} \mathbb{E}\bigg[h\big((X_{t_1}(a), a), \dots, (X_{t_p}(a), a)\big)\\
&\qquad\qquad \times \int_s^t u_\sigma\big(\tau, X_\tau(a), a\big)\partial_x \psi\big(\tau, X_\tau(a), a\big)\, dW_\tau\bigg]\nu(da)\\
&\quad = 0
\end{aligned}$$

since u_σ is a bounded function. $\square$

We now identify the solution of (27) as the unique solution of a nonlinear SDE, which will allow us to develop our stochastic particle method. This is the main result of this section.

THEOREM 3.2. *Suppose that the hypotheses of Proposition* 2.3 *hold. Then there exists a unique weak solution to the SDE* (29), *where* $(t,x) \mapsto \mathbb{E}[b(x, X_t) \mid \theta]$ *and* $(t,x) \mapsto \mathbb{E}[\sigma(x, X_t) \mid \theta]$ *stand for continuous modifications of the conditional expectation processes. Moreover, the law of the solution is* $\mathbb{P}_{X_\cdot(a)} \otimes \nu(da)$, *where* $X_\cdot(a)$ *is the unique weak solution of* (10).

PROOF. The existence of a solution results from Theorem 3.1, since equalities (11) and (28) imply that

$$u_b(t,x,a) = \mathbb{E}\big[b(x, X_t) \mid \theta = a\big], \qquad u_\sigma(t,x,a) = \mathbb{E}\big[\sigma(x, X_t) \mid \theta = a\big], \tag{36}$$

and that continuous modifications of $(\mathbb{E}[b(x, X_t) \mid \theta])$ and $(\mathbb{E}[\sigma(x, X_t) \mid \theta])$ exist in view of the Kolmogorov–Centsov criterion.

The uniqueness of a solution is an easy consequence of Theorem 3.1. $\square$

REMARK 3.3. Equation (29) reduces to the classical nonlinear stochastic differential equation (10) when ν is a Dirac measure.

Finally, in view of definition (5), a straightforward consequence of Theorems 3.1 and 3.2 is that moments of a statistical solution of the McKean–Vlasov equation (2) have the following probabilistic representation:

COROLLARY 3.4. *Suppose that the hypotheses of Proposition* 2.6 *hold. Let* $(X^i_\cdot,\ 1 \le i \le k)$ *be independent copies of the processes* $X_\cdot$ *solution of* (29). *Then, for any continuous and bounded fonction* $f \in L^2(k) = \bigotimes^k L^2(\mathbb{R})$,

$$\langle M_k(t), f\rangle_{L^2(k)} = \mathbb{E}^\nu f(X^1_t, \dots, X^k_t). \tag{37}$$

4. Approximation of the moments: the stochastic particle method. We now use the nonlinear SDE (29) to construct a particle system $(X^{i,N}_\cdot,\ 1 \le i \le N)$, which coincides with the system (24) when ν is a Dirac measure and such that

$$\mathbb{E}^\nu \left| \langle M_1(t), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f(X^{i,N}_t) \right| \underset{N\to+\infty}{\longrightarrow} 0. \tag{38}$$

First, consider the case where ν is a Dirac measure in a point $a \in [-1, 1]$. The SDE (29) reduces to (10), which is the limit equation for the particle system (24). Formally, this system can be constructed from the SDE (10) as follows: consider N independent copies $(X^i_\cdot(a),\ 1 \le i \le N)$ of the solution of (10) and replace the coefficients of this equation by estimates gotten from a Monte Carlo method:

$$\mathbb{E}\big[b\big(x, X_t(a)\big)\big] \simeq \frac{1}{N}\sum_{i=1}^N b\big(x, X^i_t(a)\big),$$

$$\mathbb{E}\big[\sigma\big(x, X_t(a)\big)\big] \simeq \frac{1}{N}\sum_{i=1}^N \sigma\big(x, X^i_t(a)\big).$$

To simulate the SDE (29), we generalize this method: we replace coefficients $\mathbb{E}[b(x, X_t) \mid \theta = a]$ and $\mathbb{E}[\sigma(x, X_t) \mid \theta = a]$ by estimates constructed with N independent copies $(X^i_t, \theta^i),\ 1 \le i \le N$, of the pair (X_t, θ). There is an extensive literature about such estimators of regression functions: we refer, for example, to Chu and Marron [7] Hardle [15], Hardle, Kerkyacharian, Picard and Tsybakov [16], or Benveniste et al. [1]. We consider in this article the following *kernel* estimators:

- If the measure ν is discrete, that is, $\nu = \sum_{\ell=1}^M p_\ell \delta_{a_\ell}$, we consider the regressogram estimator (see, e.g., Bouleau and Lépingle [2]):

$$\mathbb{E}\big[b(x, X_t) \mid \theta = a_\ell\big] \simeq \sum_{i=1}^N \frac{\mathbb{I}(\theta^i = a_\ell)}{\sum_{k=1}^N \mathbb{I}(\theta^k = a_\ell)} b(x, X^i_t). \tag{39}$$

- If the measure ν has a density, we consider the two following estimators: The Nadaraya–Watson estimator (see, e.g., Hardle [15]):

$$\mathbb{E}\big[b(x, X_t) \mid \theta = a\big] \simeq \sum_{i=1}^{N} \frac{G((\theta^i - a)/h_N)}{\sum_{k=1}^{N} G\big((\theta^k - a)/h_N\big)} b(x, X_t^i), \tag{40}$$

where $h_N > 0$ and G is, for example, a Gaussian density.
The approximate regressogram estimator:

$$\mathbb{E}\big[b(x, X_t) \mid \theta = a\big] \simeq \sum_{i=1}^{N} \frac{\mathbb{I}(\tilde{\theta}^i = a)}{\sum_{k=1}^{N} \mathbb{I}(\tilde{\theta}^k = a)} b(x, \tilde{X}_t^i), \tag{41}$$

where $\tilde{\nu}$ is a discrete probability measure approximating ν and the $(\tilde{X}^i, \tilde{\theta}^i)$, $1 \le i \le N$, are independent copies of the process $(\tilde{X}, \tilde{\theta})$, weak solution of (29) with initial law $[\Phi(a)](x)\,dx\,\tilde{\nu}(da)$.

REMARK 4.1. The above choices of the estimators are mainly motivated by the simplicity of formulas (39) and (40), which allows us to estimate the convergence rate of the particle method that we define below. However, we think that our results should easily be extended to more accurate estimators, for example, those developed in the references mentioned above or wavelet estimators.

Replacing, in the SDE (29), the exact coefficients by one of the formulas (39) or (40), we get the particle system:

$$\begin{aligned} dX_t^{i,N} &= \sum_{j=1}^{N} \alpha_{ij} b\big(X_t^{i,N}, X_t^{j,N}\big)\,dt + \sum_{j=1}^{N} \alpha_{ij} \sigma\big(X_t^{i,N}, X_t^{j,N}\big)\,dW_t^i, \\ X_t^{i,N}|_{t=0} &= X_0^i \qquad \text{for all } 1 \le i \le N, \end{aligned} \tag{42}$$

where the (X_0^i, θ^i) are independent copies with common law $[\Phi(a)](x)\,dx\,\nu(da)$ and

$$\alpha_{ij} = \frac{\mathbb{I}(\theta^i = \theta^j)}{\sum_{k=1}^{N} \mathbb{I}(\theta^i = \theta^k)} \qquad \text{(regressogram estimator for a discrete } \nu), \tag{43}$$

$$\alpha_{ij} = \frac{G((\theta^i - \theta^j)/h_N)}{\sum_{k=1}^{N} G((\theta^i - \theta^k)/h_N)} \tag{44}$$

(Nadaraya–Watson estimator for an absolutely continuous ν),

or

$$\alpha_{ij} = \frac{\mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^j)}{\sum_{k=1}^{N} \mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^k)} \tag{45}$$

(approximate regressogram estimator for an absolutely continuous ν).

Observe that the system (42) generalizes the system (24). Indeed, if ν is a Dirac measure, the weights α_{ij} reduce to the usual value $1/N$.

We deduce an approximation formula for the first moment from Corollary 3.4:

$$\langle M_1(T), f\rangle_{L^2(\mathbb{R})} = \mathbb{E}^\nu f(X_T) \simeq \frac{1}{N}\sum_{i=1}^N f\Big(\overline{X}_T^{i,N}\Big), \tag{46}$$

where $\overline{X}_\cdot^{i,N}$, $1 \le i \le N$, is defined by discretizing the SDE (42) by the Euler scheme with constant step $\Delta t = T/K$ $(t_k = k\Delta t, 0 \le k \le K)$:

$$\begin{aligned} \overline{X}_{t_{k+1}}^{i,N} &= \overline{X}_{t_k}^{i,N} + \sum_{j=1}^N \alpha_{ij} b\Big(\overline{X}_{t_k}^{i,N}, \overline{X}_{t_k}^{j,N}\Big)\Delta t \\ &\qquad + \sum_{j=1}^N \alpha_{ij}\sigma\Big(\overline{X}_{t_k}^{i,N}, \overline{X}_{t_k}^{j,N}\Big)(W_{t_{k+1}}^i - W_{t_k}^i), \\ \overline{X}_0^{i,N} &= X_0^i. \end{aligned} \tag{47}$$

We aim to estimate the accuracy of this particle method. To this end, we introduce the Euler scheme for the SDE (29)

$$\begin{aligned} \overline{X}_{t_{k+1}} &= \overline{X}_{t_k} + u_b(t_k, \overline{X}_{t_k}, \theta)\Delta t + u_\sigma(t_k, \overline{X}_{t_k}, \theta)(W_{t_{k+1}} - W_{t_k}), \\ \overline{X}_0 &= X_0. \end{aligned} \tag{48}$$

Considering N independent copies $\overline{X}_\cdot^i$, $1 \le i \le N$, of the process $\overline{X}_\cdot$, we split the convergence error of the particle method into three parts:

$$\begin{aligned} &\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f\Big(\overline{X}_T^{i,N}\Big) \\ &\qquad = \langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \mathbb{E}^\nu\big[f(\overline{X}_T)\big] + \mathbb{E}^\nu\big[f(\overline{X}_T)\big] \\ &\qquad\quad - \frac{1}{N}\sum_{i=1}^N f\Big(\overline{X}_T^i\Big) + \frac{1}{N}\sum_{i=1}^N \Big(f(\overline{X}_T^i) - f(\overline{X}_T^{i,N})\Big). \end{aligned} \tag{49}$$

In view of (1), the first term on the right-hand side of (49) is a time discretization error. We estimate it in Section 5, owing to the results of Talay [23]. The second one is a statistical error. Indeed, in view of the strong law of large numbers,

$$\mathbb{E}^\nu\left|\mathbb{E}^\nu\big[f(\overline{X}_T)\big] - \frac{1}{N}\sum_{i=1}^N f(\overline{X}_T^i)\right| \le \frac{\|f\|_{L^\infty(\mathbb{R})}}{\sqrt{N}}. \tag{50}$$

The last term is an error related to the propagation of chaos of the particle system (42). In Sections 6 and 7, we estimate it successively for the three families of weights (43), (45) and (44). For a review of error estimates on stochastic particle methods for McKean–Vlasov nonlinear PDEs with deterministic initial conditions, see Talay [24].

5. Discretization of the limit process $X_\cdot$.

PROPOSITION 5.1. *Suppose that the hypotheses of Theorem* 3.1 *hold. In addition, suppose that the functions b and σ are in $C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R}^2)$. Then, for any test function $f \in C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})$,*

$$|\mathbb{E}^{\nu}[f(X_T)] - \mathbb{E}^{\nu}[f(\overline{X}_T)]| \le C\,\Delta t.$$

The proof of Proposition 5.1 relies on the following technical lemma:

LEMMA 5.2. *For any $a \in [-1, 1]$, let $\mathcal{L}_t^a$ be the operator defined by*

$$\forall \phi \in C_{\mathrm{b}}^2(\mathbb{R}), \qquad \mathcal{L}_t^a \phi(x) = \tfrac{1}{2} u_\sigma^2(t, x, a)\phi''(x) + u_b(t, x, a)\phi'(x).$$

Then, for any function $f \in C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})$, the PDE

$$\begin{aligned} \partial_t v(t, x, a) + \mathcal{L}_t^a v(t, x, a) &= 0, \\ v(T, x, a) &= f(x), \end{aligned} \tag{51}$$

has a unique solution in the space $C_{\mathrm{b}}^{2,4+\varepsilon}([0, T] \times \mathbb{R})$. Moreover, the mapping

$$\begin{aligned} a \in [-1, 1] \mapsto \sup_{t\in[0,T]} \sum_{i=1}^{4} \bigl(&\|\partial_x^{(i)} u_b(t, \cdot, a)\|_{L^\infty(\mathbb{R})} + \|\partial_x^{(i)} u_\sigma(t, \cdot, a)\|_{L^\infty(\mathbb{R})} \\ &+ \|\partial_x^{(i)} v(t, \cdot, a)\|_{L^\infty(\mathbb{R})}\bigr) \end{aligned}$$

is bounded.

PROOF. We only sketch the proof detailed in Vaillant [27]. As the functions b and σ are in $C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R}^2)$, one easily checks that, for any differentiation of order $q \le 4$,

$$\begin{aligned} &\sup_{a\in[-1,1]} \sup_{t\in[0,T]} \bigl(\|\partial_x^{(q)} u_b(t, \cdot, a)\|_{L^\infty(\mathbb{R})} + \|\partial_x^{(q)} u_\sigma(t, \cdot, a)\|_{L^\infty(\mathbb{R})}\bigr) < +\infty, \\ &\sup_{a\in[-1,1]} \bigl(\|u_b(t, \cdot, a) - u_b(s, \cdot, a)\|_{L^\infty(\mathbb{R})} + \|u_\sigma(t, \cdot, a) - u_\sigma(s, \cdot, a)\|_{L^\infty(\mathbb{R})}\bigr) \\ &\qquad \le C\sqrt{t - s}. \end{aligned} \tag{52}$$

Lemma 5.2 is then a consequence of Theorem 5.1.9 in Lunardi [18]: (51) has a unique solution $v(t, x, a)$ in $C_{\mathrm{b}}^{2,4+\varepsilon}([0, T] \times \mathbb{R})$ and

$$\|v(\cdot, \cdot, a)\|_{C_{\mathrm{b}}^{2,4+\varepsilon}([0,T]\times\mathbb{R})} \le C\|f\|_{C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})}.$$

The constant C is uniform in a owing to inequalities (52). □

PROOF OF PROPOSITION 5.1. For any $a \in [-1, 1]$, let $(\overline{X}_{t_k}(a))_{k \le K}$ be the process defined by the Euler scheme, with constant discretization step $\Delta t = T/K$, applied to the SDE (10). The law of the process $\overline{X}.$ satisfies

$$\mathbb{P}_{\overline{X}.} = \mathbb{P}_{\overline{X}_{\cdot(a)}} \otimes \nu(da).$$

Consequently,

$$\mathbb{E}^{\nu}[f(X_T)] - \mathbb{E}^{\nu}[f(\overline{X}_T)] = \int_{-1}^{1} \{\mathbb{E}[f(X_T(a))] - \mathbb{E}[f(\overline{X}_T(a))]\} \nu(da). \tag{53}$$

Thus, we have to prove that, for any $a \in [-1, 1]$,

$$|\mathbb{E}[f(X_T(a))] - \mathbb{E}[f(\overline{X}_T(a))]| \le C \, \Delta t,$$

with a constant C independent of a.

By the Feynman–Kac formula, the solution of (51) defined by Lemma 5.2 is given by

$$v(t, x, a) = \mathbb{E}[f(X_T^{t,x}(a))],$$

where $X_{\cdot}^{t,x}(a)$ is the Markov process whose generator is $\mathcal{L}_t^a$ and such that $X_t^{t,x}(a) = x$ a.s. Hence,

$$\begin{aligned}
\mathbb{E} f(X_T(a)) - \mathbb{E} f(\overline{X}_T(a)) &= \mathbb{E} v(0, X_0(a), a) - \mathbb{E} v(T, \overline{X}_T(a), a) \\
&= \sum_{k=1}^{K} [\mathbb{E} v(t_{k-1}, \overline{X}_{t_{k-1}}(a), a) - \mathbb{E} v(t_k, \overline{X}_{t_k}(a), a)].
\end{aligned}$$

From this representation of the discretization error, Talay [23] showed that

$$\mathbb{E} f(X_T(a)) - \mathbb{E} f(\overline{X}_T(a)) \le \mathcal{C}(T, a) \, \Delta t,$$

where $a \mapsto \mathcal{C}(T, a)$ is a sum of terms of the type $\partial_x^{(i)} u_b \, \partial_x^{(j)} u_\sigma \, \partial_x^{(k)} v$, $i, j, k \le 4$. We conclude by using our Lemma 5.2. Estimate (53) is thus proved. □

We now give estimates for the third term of (49), namely,

$$\frac{1}{N} \sum_{i=1}^{N} \Big(f(\overline{X}_T^i) - f(\overline{X}_T^{i,N}) \Big), \tag{54}$$

depending on the choice of the random weights (43), (44) or (45).

6. Global error estimates for the stochastic particle system with the regressogram estimator. We distinguish two cases: the probability measure ν is discrete, or it is absolutely continuous. First, suppose that the measure ν is discrete:

$$\nu = \sum_{\ell=1}^{M} p_\ell \delta_{a_\ell}, \qquad a_\ell \in [-1, 1] \ \forall \, \ell \le M, \text{ and } a_\ell \ne a_m \text{ if } \ell \ne m, \tag{55}$$

and define the random weights α_{ij} by (43).

PROPOSITION 6.1. *Suppose that the hypotheses of Theorem* 3.1 *hold. Moreover, suppose that the functions b and σ are in $C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R}^2)$. Then*

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}^{\nu}\left|\overline{X}_T^{i}-\overline{X}_T^{i,N}\right|^2\le C\left(\frac{M}{N}+M(\Delta t)^2\right). \tag{56}$$

PROOF. It is convenient to rewrite the left-hand side of (56) after having gathered particles having the same initial law, that is, to split the particle system (42) into M independent subsystems. To this end, set

$$\mathcal{C}(M,N):=\{(c(1),\dots,c(N))\in\mathbb{N}^N\mid\forall\, 1\le i\le N,\ c(i)\le M\}.$$

We identify an element of this set and a random choice according to the law ν. For any $c=(c(1),\dots,c(N))\in\mathcal{C}(M,N)$, define the processes $\overline{X}^{i}(c)$ and $\overline{X}^{i,N}(c)$ by

$$\begin{cases}\overline{X}^{i}_{t_{k+1}}(c)=\overline{X}^{i}_{t_k}(c)+u_b\big(t_k,\overline{X}^{i}_{t_k}(c),a_{c(i)}\big)\,\Delta t\\ \qquad\qquad +u_\sigma\big(t_k,\overline{X}^{i}_{t_k}(c),a_{c(i)}\big)\big(W^i_{t_{k+1}}-W^i_{t_k}\big),\\ \overline{X}^{i}_0(c)\text{ with law }\big[\Phi(a_{c(i)})\big](x)\,dx,\\ \alpha_{ij}(c)=\dfrac{\mathbb{I}(c(i)=c(j))}{\sum_{k=1}^N\mathbb{I}(c(k)=c(i))},\\ \overline{X}^{i,N}_{t_{k+1}}(c)=\overline{X}^{i,N}_{t_k}(c)+\Delta t\displaystyle\sum_{j=1}^N\alpha_{ij}(c)b\big(\overline{X}^{i,N}_{t_k}(c),\overline{X}^{j,N}_{t_k}(c)\big)\\ \qquad\qquad +\displaystyle\sum_{j=1}^N\alpha_{ij}(c)\sigma\big(\overline{X}^{i,N}_{t_k}(c),\overline{X}^{j,N}_{t_k}(c)\big)\big(W^i_{t_{k+1}}-W^i_{t_k}\big),\\ \overline{X}^{i,N}_0(c)=\overline{X}^{i}_0(c).\end{cases} \tag{57}$$

Setting $E_\ell(c)=\{j\in\mathbb{N},\ 1\le j\le N\mid c(j)=c(\ell)\}$, $\underline{\theta}=(\theta^1,\dots,\theta^N)$ and $\underline{a}_c=\big(a_{c(1)},\dots,a_{c(N)}\big)$, we observe that

$$\begin{aligned}&\mathbb{E}^{\nu}\left\{\frac{1}{N}\sum_{i=1}^{N}\left|\overline{X}^{i}_{t_k}-\overline{X}^{i,N}_{t_k}\right|^2\right\}\\ &\quad=\sum_{c\in\mathcal{C}(M,N)}\mathbb{P}(\underline{\theta}=\underline{a}_c)\left\{\frac{1}{N}\sum_{\ell=1}^{M}\sum_{j\in E_\ell(c)}\mathbb{E}\left|\overline{X}^{j}_{t_k}(c)-\overline{X}^{j,N}_{t_k}(c)\right|^2\right\}.\end{aligned} \tag{58}$$

Suppose that

$$\forall \ell \le M,\ \forall j \in E_\ell(c), \qquad \mathbb{E}\Big[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\Big]^2 \le C\bigg(\frac{1}{\sharp E_\ell(c)} + (\Delta t)^2\bigg). \tag{59}$$

Then

$$\mathbb{E}^\nu\bigg\{\frac{1}{N}\sum_{i=1}^N\Big|\overline{X}^i_{t_k} - \overline{X}^{i,N}_{t_k}\Big|^2\bigg\} \le \sum_{c\in\mathcal{C}(M,N)} \mathbb{P}(\underline{\theta} = \underline{a}_c)\bigg\{\frac{C}{N}\sum_{\ell=1}^M \big(1 + \sharp E_\ell(c)\ (\Delta t)^2\big)\bigg\}$$
$$\le C\bigg\{\frac{M}{N} + M(\Delta t)^2\bigg\}.$$

PROOF OF ESTIMATE (59). For any $\ell \le M$ and $j \in E_\ell(c)$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\overline{X}^j_{t_{k+1}}(c) - \overline{X}^{j,N}_{t_{k+1}}(c)\Big]^2 \\
&\quad = \mathbb{E}\Big[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\Big]^2 \\
&\qquad + (\Delta t)^2\mathbb{E}\bigg[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} b\Big(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\Big) - u_b\Big(t_k, \overline{X}^j_{t_k}(c), a_{c(\ell)}\Big)\bigg]^2 \\
&\qquad + \Delta t\mathbb{E}\bigg[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} \sigma\Big(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\Big) - u_\sigma\Big(t_k, \overline{X}^j_{t_k}(c), a_{c(\ell)}\Big)\bigg]^2 \\
&\qquad + 2\Delta t\mathbb{E}\bigg[\Big(\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\Big)\bigg(\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} b\Big(\overline{X}^{j,N}_{k\Delta t}(c), \overline{X}^{i,N}_{t_k}(c)\Big) \\
&\qquad\qquad - u_b\Big(t_k, \overline{X}^j_{k\Delta t}(c), a_{c(\ell)}\Big)\bigg)\bigg].
\end{aligned} \tag{60}$$

Observe that, owing to Theorem 3.1 and equalities (11),

$$u_b(t, x, a_{c(j)}) = \mathbb{E}\big[b\big(x, X_t(a_{c(j)})\big)\big] \quad \text{and} \quad u_\sigma(t, x, a_{c(j)}) = \mathbb{E}\big[\sigma\big(x, X_t(a_{c(j)})\big)\big].$$

Then, inserting $\mathbb{E}[b(x, \overline{X}^j_{t_k}(c))]\,|_{x=\overline{X}^j_{t_k}(c)}$ and $\mathbb{E}[\sigma(x, \overline{X}^j_{t_k}(c))]\,|_{x=\overline{X}^j_{t_k}(c)}$ in equality (60), we have

$$\begin{aligned}
\mathbb{E}\Big[\overline{X}^j_{t_{k+1}}(c) - \overline{X}^{j,N}_{t_{k+1}}(c)\Big]^2 &\le (1 + C\Delta t)\mathbb{E}\Big[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\Big]^2 \\
&\quad + C\,\Delta t A_1(j, l, c, t_k) + C\,\Delta t A_2(j, l, c, t_k),
\end{aligned} \tag{61}$$

with

$$
\begin{aligned}
&A_1(j,l,c,t_k)\\
&\quad = \mathbb{E}\Bigg[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} b\Big(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\Big) - \mathbb{E}\Big[b\Big(x, \overline{X}^{j}_{t_k}(c)\Big)\Big]\Big|_{x=\overline{X}^{j}_{t_k}(c)}\Bigg]^2\\
&\qquad + \mathbb{E}\Bigg[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} \sigma\Big(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\Big) - \mathbb{E}\Big[\sigma\Big(x, \overline{X}^{j}_{t_k}(c)\Big)\Big]\Big|_{x=\overline{X}^{j}_{t_k}(c)}\Bigg]^2,\\
&A_2(j,l,c,t_k)\\
&\quad = \Big|\mathbb{E}\big[b\big(x, X_{t_k}(a_{c(j)})\big)\big]\big|_{x=\overline{X}^{j}_{t_k}(c)} - \mathbb{E}\Big[b\Big(x, \overline{X}^{j}_{t_k}(c)\Big)\Big]\Big|_{x=\overline{X}^{j}_{t_k}(c)}\Big|^2\\
&\qquad + \Big|\mathbb{E}\big[\sigma\big(x, X_{t_k}(a_{c(j)})\big)\big]\big|_{x=\overline{X}^{j}_{t_k}(c)} - \mathbb{E}\Big[\sigma\Big(x, \overline{X}^{j}_{t_k}(c)\Big)\Big]\Big|_{x=\overline{X}^{j}_{t_k}(c)}\Big|^2.
\end{aligned}
$$

To estimate $A_1(j,l,c,t_k)$, observe that the particles $(\overline{X}^{i,N}_{\boldsymbol{\cdot}})_{i\in E_\ell(c)}$ have the same weight $1/(\sharp E_\ell(c))$ and the same initial law $p_0(x, a_{c(\ell)})\,dx$. Thus, using the symmetry of the particle system $(\overline{X}^{i,N}_{\boldsymbol{\cdot}})_{i\in E_\ell(c)}$ and the Lipschitz property of functions b and σ, one shows that (see, e.g., Sznitman [22])

$$
A_1(j,l,c,t_k) \le C\bigg(\mathbb{E}\Big[\overline{X}^{j}_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\Big]^2 + \frac{1}{\sharp E_\ell(c)}\bigg). \tag{62}
$$

Furthermore, in view of Proposition 5.1,

$$
A_2(j,l,c,t_k) \le C(\Delta t)^2. \tag{63}
$$

Owing to inequalities (61), (62) and (63), we deduce estimate (59) by induction. □

We are now in a position to estimate the accuracy of the particle method. This is a straightforward consequence of (49), (50) and Propositions 5.1 and 6.1.

THEOREM 6.2 (Discrete case). *Suppose that the probability measure ν is of the form* (55) *and that the hypotheses of Theorem* 3.1 *hold. In addition, suppose that the functions b and σ are in $C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R}^2)$. Consider the particle system* (47) *with weights* (43). *Then, for any function $f \in C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})$,*

$$
\mathbb{E}^{\nu}\bigg|\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^{N} f\big(\overline{X}^{i,N}_T\big)\bigg|^2 \le C\bigg(\frac{M}{N} + M(\Delta t)^2\bigg).
$$

We now study the convergence rate of the particle method with weights (45) when the measure ν has a density w.r.t. Lebesgue's measure.

THEOREM 6.3. *Let ν be a probability measure on $[-1,1]$ with density q and distribution function V. We define the weights of the particle method by*

$$\alpha_{ij} = \frac{\mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^j)}{\sum_{k=1}^N \mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^k)},$$

where the independent random variables $\tilde{\theta}^i$, $1 \le i \le N$, have the common law

$$\nu_M = \frac{1}{M}\sum_{\ell=1}^M \delta_{V^{-1}(\ell/M)}, \qquad M \in \mathbb{N}.$$

Suppose that:

(i) *the hypotheses of Theorem* 6.2 *hold*;
(ii) *there exists a strictly positive constant q_* such that*

$$\forall a \in [-1,1], \qquad q(a) \ge q_* > 0.$$

Then, for any test function $f \in C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})$,

$$\mathbb{E}^{\nu_M}\left|\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}_T^{i,N}\right)\right|^2 \le C\left(\frac{M}{N} + M(\Delta t)^2 + \frac{1}{M^2}\right).$$

PROOF. Owing to Theorem 6.2, we already know that

$$\mathbb{E}^{\nu_M}\left|\mathbb{E}^{\nu_M} f(X_T) - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}_T^{i,N}\right)\right|^2 \le C\left(\frac{M}{N} + M(\Delta t)^2\right).$$

We thus have to prove that

$$\left|\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \mathbb{E}^{\nu_M} f(X_T)\right|^2 \le \frac{C}{M^2},$$

that is,

$$|\mathbb{E}^{\nu} f(X_T) - \mathbb{E}^{\nu_M} f(X_T)|^2 \le \frac{C}{M^2}. \tag{64}$$

Let $X_\cdot(V^{-1}(y))$ denote the solution of the SDE (10) with initial law $[\Phi(V^{-1}(y))](x)\,dx$. Set

$$F_T(f,\cdot)y \in [0,1] \mapsto \mathbb{E}\left[f\left(X_T\left(V^{-1}(y)\right)\right)\right].$$

As V is the distribution function of the measure ν, one has

$$\mathbb{E}^{\nu} f(X_T) - \mathbb{E}^{\nu_M} f(X_T) = \int_0^1 F_Y(f,y)\,dy - \frac{1}{M}\sum_{\ell=1}^M F_T\left(f, \frac{\ell}{M}\right).$$

Observe that, for any $(y_1, y_2) \in [0, 1]^2$,

$$F_T(f, y_1) - F_T(f, y_2) = \int_{\mathbb{R}} f(x)\big(p\big(t, x, V^{-1}(y_1)\big) - p\big(t, x, V^{-1}(y_2)\big)\big)\, dx,$$

where $p(t, x, V^{-1}(y_i))$ is the solution of the PDE (9) with initial condition $\Phi(V^{-1}(y_i))$.

In view of Proposition 2.2 of [25], we know that the mapping

$$a \in [-1, 1] \mapsto p(t, \cdot, a) \in L^1(\mathbb{R})$$

is Lipschitz continuous. Thus, we deduce that

$$|F_T(f, y_1) - F_T(f, y_2)| \le C|V^{-1}(y_1) - V^{-1}(y_2)|.$$

In addition, the mapping V^{-1} is Lipschitz continuous since

$$\sup_{y\in[0,1]}\left|\frac{d}{dy}V^{-1}(y)\right| = \sup_{y\in[0,1]}\left|\frac{1}{q(V^{-1}(y))}\right| \le \frac{1}{q_*}.$$

Consequently,

$$|F_T(f, y_1) - F_T(f, y_2)| \le C|y_1 - y_2|. \tag{65}$$

Thus, estimate (64) readily follows from (65). □

REMARK 6.4. We can deduce from Theorem 6.3 that, in this case, the particle method converges if

$$\lim_{N\to+\infty} \frac{M}{N} = 0 \quad \text{and} \quad \lim_{\Delta t\to 0} M(\Delta t)^2 = 0.$$

The first condition is natural: the measure ν_M, concentrated in M points, can be well approximated by the empirical measure of $(\theta^1, \dots, \theta^N)$ only if $N \gg M$. The second condition is a relationship between the time and space discretization steps, Δt and $1/M$. It is implied by $M\Delta t =$ constant, which is a c.f.l. condition, implying the stability of the numerical scheme.

7. Global error estimates for the particle system with the Nadaraya–Watson estimator. In this section, we suppose that the probability measure ν is supported in $[-1, 1]$ and has a strictly positive Lipschitz continuous density q. The weights α_{ij} of the particle method are defined by (44):

$$\alpha_{ij} = \frac{G_N(\theta^i - \theta^j)}{\sum_{k=1}^N G_N(\theta^i - \theta^k)},$$

where $G_N(\cdot) := (1/h_N)G(\cdot/h_N)$, $h_N > 0$ and G is a Gaussian density on $\mathbb{R}$.

We gather estimates for G_N is the following lemma.

LEMMA 7.1. *One has*

$$\forall N \in \mathbb{N},\ \forall x \in \mathbb{R}, \qquad \mathbb{E}^{\nu} G_N(\theta - x) \leq \|q\|_{L^\infty(\mathbb{R})} \tag{66}$$

and

$$\forall x \in \mathbb{R}, \qquad \mathbb{E}^{\nu} G_N(x - \theta) \geq q_* \int_0^1 G(z)\,dz. \tag{67}$$

Moreover, for any Lipschitz-continuous bounded real function ϕ on $[-1, 1]$ with Lipschitz constant L_ϕ, one has

$$\int_{-1}^{1} \left| \int_{\mathbb{R}} G_N(z - x)\phi(z)\,dz - \phi(x) \right|^2 dx \leq C\big(L_\phi^2 h_N^2 + \|\phi\|_\infty^2 T_G(N)\big), \tag{68}$$

where

$$\begin{aligned} T_G(N) := \int_{-1}^{1} \bigg[\int_{-\infty}^{(x-1)/h_N} &\big(G(z) + G(z)^2\big)\,dz \\ &+ \int_{(x+1)/h_N}^{+\infty} \big(G(z) + G(z)^2\big)\,dz \bigg]^2 dx. \end{aligned} \tag{69}$$

In addition,

$$T_G(N) \leq C h_N \quad \text{and} \quad \int_{-1}^{1} \left| \int_{\mathbb{R}} G_N(x - z)\phi(z)\,dz - \phi(x) \right|^2 dx \leq C h_N. \tag{70}$$

PROOF. Inequality (66) results from

$$\begin{aligned} \mathbb{E}^{\nu} G_N(\theta - x) &= \int_{-1}^{1} \frac{1}{h_N} G\Big(\frac{x - z}{h_N}\Big)\phi(z)\,dz \\ &= \int_{(x-1)/h_N}^{(x+1)/h_N} G(z)\phi(x - z h_N)\,dz \\ &\leq \int_{\mathbb{R}} G(z)\phi(x - z h_N)\,dz. \end{aligned}$$

Inequality (67) results from $h_N \leq 1$.

Now observe

$$\begin{aligned} &\int_{\mathbb{R}} G_N(x - z)\phi(z)\,dz - \phi(x) \\ &\quad = \int_{(x-1)/h_N}^{(x+1)/h_N} G\Big(\frac{z}{h_N}\Big)\phi(x - z)\,dz - \int_{\mathbb{R}} G(z)\phi(x)\,dz \\ &\quad = \int_{(x-1)/h_N}^{(x+1)/h_N} G(z)\big(\phi(x - z h_N) - \phi(x)\big)\,dz \\ &\qquad - \phi(x)\bigg[\int_{-\infty}^{(x-1)/h_N} G(z)\,dz + \int_{(x+1)/h_N}^{+\infty} G(z)\,dz\bigg]. \end{aligned}$$

Inequality (68) readily follows.

Finally, inequality (70) follows from

$$\forall z > 0, \qquad \int_z^{+\infty} \exp(-x^2)\,dx \le C\exp(-z^2). \qquad \square$$

Our next statement provides a propagation of a chaos-type estimate.

PROPOSITION 7.2. *Suppose that*:

(i) *the hypotheses of Theorem* 3.1 *hold*;
(ii) *in addition, the interaction kernels b and σ are in $C_b^{4+\varepsilon}(\mathbb{R}^2)$, $0 < \varepsilon < 1$*;
(iii) *the sequence (h_N) tends to* 0 *and* $\lim_{N\to+\infty} \log(N)/(Nh_N^2) = 0$;
(iv) $\sup_{a\in[-1,1]} \int_{\mathbb{R}} x^4 p_0(x,a)\,dx < +\infty$;
(v) *the probability measure ν has a strictly positive Lipschitz-continuous density q on* $[-1,1]$.

Then there exist a strictly positive constant C, independent of N and Δt, and an integer N_0 such that, for any $N \ge N_0$,

$$\frac{1}{N}\sum_{i=1}^N \left(\mathbb{E}^\nu|\overline{X}_T^i - \overline{X}_T^{i,N}|^2\right) \le C\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N} + (\Delta t)^2\right).$$

PROOF. Similarly to what we got in the proof of Proposition 6.1, for all indices $i \le N$ and $k \le K$, we have

$$\begin{aligned}\mathbb{E}^\nu\left|\overline{X}_{t_{k+1}}^i - \overline{X}_{t_{k+1}}^{i,N}\right|^2 &\le (1 + C\Delta t)\mathbb{E}^\nu\left|\overline{X}_{t_k}^i - \overline{X}_{t_k}^{i,N}\right|^2 \\ &\quad + C\,\Delta t\big(A_1(i,t_k) + A_2(i,t_k)\big),\end{aligned} \tag{71}$$

where

$$\begin{aligned}A_1(i,t_k) &= \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij} b\left(\overline{X}_{t_k}^{i,N}, \overline{X}_{t_k}^{j,N}\right) - \mathbb{E}^\nu\left[b\left(x, \overline{X}_{t_k}^i\right) \mid \theta^i\right]\Big|_{x=\overline{X}_{t_k}^i}\right]^2 \\ &\quad + \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij}\sigma\left(\overline{X}_{t_k}^{i,N}, \overline{X}_{t_k}^{j,N}\right) - \mathbb{E}^\nu\left[\sigma\left(x, \overline{X}_{t_k}^i\right) \mid \theta^i\right]\Big|_{x=\overline{X}_{t_k}^i}\right]^2,\end{aligned}$$

and

$$\begin{aligned}A_2(i,t_k) &= \left|\mathbb{E}^\nu[b(x, X_{t_k}^i) \mid \theta^i]\big|_{x=\overline{X}_{t_k}^i} - \mathbb{E}^\nu\left[b\left(x, \overline{X}_{t_k}^i\right) \mid \theta^i\right]\Big|_{x=\overline{X}_{t_k}^i}\right|^2 \\ &\quad + \left|\mathbb{E}^\nu[\sigma(x, X_{t_k}^i) \mid \theta^i]\big|_{x=\overline{X}_{t_k}^i} - \mathbb{E}^\nu\left[\sigma\left(x, \overline{X}_{t_k}^i\right) \mid \theta^i\right]\Big|_{x=\overline{X}_{t_k}^i}\right|^2.\end{aligned}$$

In view of Proposition 5.1,

$$A_2(i,t_k) \le C(\Delta t)^2. \tag{72}$$

We first consider $A_1(i,t_k)$. We insert $b(\overline{X}^i_{t_k}, \overline{X}^j_{t_k})$ and use the Lipschitz property of b. As

$$\forall\, 1 \le i, j \le N, \qquad \alpha_{ij} > 0 \quad \text{and} \quad \sum_{j=1}^{N} \alpha_{ij} = 1, \tag{73}$$

Jensen's inequality and easy computations lead to

$$\begin{aligned}
A_1(i,t_k) \le{}& C\left(\mathbb{E}^\nu |\overline{X}^{i,N}_{t_k} - \overline{X}^i_{t_k}|^2 + \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij} |\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}|^2\right]\right) \\
&+ C\Big(\mathbb{E}^\nu[\alpha_{ii} b(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2 + \mathbb{E}^\nu[\alpha_{ii}\sigma(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2\Big) \\
&+ C\mathbb{E}^\nu\left[\sum_{j=1, j\ne i}^N \alpha_{ij} b(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}) - \mathbb{E}^\nu[b(x, \overline{X}^i_{t_k}) \mid \theta^i]\big|_{x=\overline{X}^i_{t_k}}\right]^2 \\
&+ C\mathbb{E}^\nu\left[\sum_{j=1, j\ne i}^N \alpha_{ij} \sigma(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}) - \mathbb{E}^\nu[\sigma(x, \overline{X}^i_{t_k}) \mid \theta^i]\big|_{x=\overline{X}^i_{t_k}}\right]^2.
\end{aligned} \tag{74}$$

Set

$$\begin{aligned}
S_N(t_k) &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}^\nu |\overline{X}^{i,N}_{t_k} - \overline{X}^i_{t_k}|^2, \\
\delta_0(t_k) &= \frac{1}{N}\sum_{i=1}^N \Big(\mathbb{E}^\nu[\alpha_{ii} b(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2 + \mathbb{E}^\nu[\alpha_{ii}\sigma(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2\Big), \\
\Delta_b(t_k) &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}^\nu\left[\sum_{j=1, j\ne i}^N \alpha_{ij} b(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}) - \mathbb{E}^\nu[b(x, \overline{X}^i_{t_k}) \mid \theta^i]\big|_{x=\overline{X}^i_{t_k}}\right]^2, \\
\Delta_\sigma(t_k) &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}^\nu\left[\sum_{j=1, j\ne i}^N \alpha_{ij} \sigma(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}) - \mathbb{E}^\nu[\sigma(x, \overline{X}^i_{t_k}) \mid \theta^i]\big|_{x=\overline{X}^i_{t_k}}\right]^2.
\end{aligned}$$

In view of (71), (72) and (74), we finally get

$$\begin{aligned}
S_N(t_{k+1}) \le{}& (1 + C\Delta t) S_N(t_k) \\
&+ C\,\Delta t\big(\Delta t + \delta_0(t_k) + \Delta_b(t_k) + \Delta_\sigma(t_k)\big) \\
&+ \frac{C\Delta t}{N}\sum_{j=1}^N \mathbb{E}^\nu\left(|\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}|^2 \sum_{i=1}^N \alpha_{ij}\right).
\end{aligned} \tag{75}$$

The terms $\Delta_b(t_k)$ and $\Delta_\sigma(t_k)$ characterize the accuracy of the approximation of the regression functions $\mathbb{E}^\nu[b(x, \overline{X}^i_{t_k}) \mid \theta^i = a]$ and $\mathbb{E}^\nu[\sigma(x, \overline{X}^i_{t_k}) \mid \theta^i = a]$ by the Nadaraya–Watson estimator (40). Indeed, using results of Collomb [8] (we refer to Vaillant [27], Section 4.3.3 for the easy and lengthy modification of Collomb's calculation—here the assumption that q is Lipschitz is used in force) and (68) one can check that

$$\Delta_b(t_k) + \Delta_\sigma(t_k) \leq C\left(\frac{1}{Nh_N} + h_N^2 + T_G(N)\right). \tag{76}$$

In view of (70), we deduce that

$$\Delta_b(t_k) + \Delta_\sigma(t_k) \leq C\left(\frac{1}{Nh_N} + h_N\right). \tag{77}$$

The term

$$\frac{1}{N}\sum_{j=1}^N \mathbb{E}^\nu\left(|\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}|^2 \sum_{i=1}^N \alpha_{ij}\right)$$

characterizes the uncertainty on the initial condition of the PDE (9). Indeed, if the measure ν were a Dirac mass, all the weights α_{ij} would be equal to $1/N$ and

$$\frac{1}{N}\sum_{j=1}^N \mathbb{E}^\nu\left(|\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}|^2 \sum_{i=1}^N \alpha_{ij}\right) = S_N(t_k).$$

Suppose that we have shown Lemma 7.3. Owing to estimates (77) and (78), Proposition 7.2 then follows from the induction (75). □

LEMMA 7.3. *Suppose that the hypotheses of Proposition* 7.2 *hold. Then there exist a strictly positive constant C, independent of N and Δt, and an integer N_0 such that, for any $N \geq N_0$,*

$$\delta_0(t_k) + \frac{1}{N}\sum_{j=1}^N \mathbb{E}^\nu\left(\sum_{i=1}^N \alpha_{ij} |\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}|^2\right) \leq C\left(S_N(t_k) + \frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N}\right). \tag{78}$$

PROOF. As noticed before, if the measure ν were a Dirac mass, all the weights α_{ij} would be equal to $1/N$. We thus naturally found it useful to rewrite $\sum_{i=1}^N \alpha_{ij}$ in order to separately estimate the different sources of fluctuation around the value 1:

$$\begin{aligned}\sum_{i=1}^N \alpha_{ij} &= \sum_{i=1}^N \frac{G_N(\theta^i - \theta^j)}{G(0)/h_N + \sum_{k=1,k\neq i}^N G_N(\theta^i - \theta^k)} \\ &=: 1 + A_1(j) + A_2(j) + A_3(j) + A_4(j) + A_5(j) + A_6(j),\end{aligned} \tag{79}$$

with

$$A_1(j) = \mathbb{E}^\nu\Big(\frac{G_N(\theta^i - x)}{q(\theta^i)}\Big)\Big|_{x=\theta^j} - 1,$$

$$A_2(j) = \mathbb{E}^\nu\Big(\frac{G_N(\theta^i - x)}{G(0)/((N-1)h_N) + q(\theta^i)}\Big)\Big|_{x=\theta^j} - \mathbb{E}^\nu\Big(\frac{G_N(\theta^i - x)}{q(\theta^i)}\Big)\Big|_{x=\theta^j},$$

$$A_3(j) = \frac{1}{N-1}\sum_{i=1,\ i\neq j}^{N} \frac{G_N(\theta^i - \theta^j)}{G(0)/((N-1)h_N) + q(\theta^i)} - \mathbb{E}^\nu\Big(\frac{G_N(\theta^i - x)}{G(0)/((N-1)h_N) + q(\theta^i)}\Big)\Big|_{x=\theta^j},$$

$$A_4(j) = \frac{1}{N-1}\sum_{i=1,\ i\neq j}^{N}\Big(\frac{G_N(\theta^i - \theta^j)}{G(0)/((N-1)h_N) + \mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^i}} - \frac{G_N(\theta^i - \theta^j)}{G(0)/((N-1)h_N) + q(\theta^i)}\Big),$$

$$A_5(j) = \frac{1}{N-1}\sum_{i=1,\, i\neq j}^{N}\Big(\frac{G_N(\theta^i - \theta^j)}{G(0)/((N-1)h_N) + 1/(N-1)\sum_{k=1,k\neq i}^N G_N(\theta^i - \theta^k)} - \frac{G_N(\theta^i - \theta^j)}{G(0)/((N-1)h_N) + \mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^i}}\Big),$$

$$\begin{aligned} A_6(j) = {} & \frac{1}{N-1}\Big(\frac{G_N(0)}{G(0)/((N-1)h_N) + 1/(N-1)\sum_{k=1,k\neq j}^N G_N(\theta^j - \theta^k)} - \frac{G_N(0)}{G(0)/((N-1)h_N) + \mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^j}}\Big) \\ & + \frac{1}{N-1}\Big(\frac{G_N(0)}{G(0)/((N-1)h_N) + \mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^j}} - \frac{G_N(0)}{G(0)/((N-1)h_N) + q(\theta^j)}\Big) \\ & + \frac{1}{N-1}\frac{G_N(0)}{G(0)/((N-1)h_N) + q(\theta^j)}. \end{aligned}$$

Under hypothesis (iv) of Proposition 7.2, the random variables X_0^i, $1 \le i \le N$, have moments up to order 4. Hence, since the functions b and σ are bounded and the weights α_{ij} satisfy (73),

$$\sup_{t_k\in[0,T]}\sup_{1\le i\le N}\big(\mathbb{E}^\nu[\overline{X}^i_{t_k}]^4+\mathbb{E}^\nu[\overline{X}^{i,N}_{t_k}]^4\big)<+\infty.$$

Hence, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^N\mathbb{E}^\nu\left\{\left(\sum_{i=1}^N\alpha_{ij}\right)|\overline{X}^{j,N}_{t_k}-\overline{X}^j_{t_k}|^2\right\}\\
&\quad=S_N(t_k)+\frac{1}{N}\sum_{j=1}^N\mathbb{E}^\nu\left(\sum_{k=1}^6 A_k(j)|\overline{X}^{j,N}_{t_k}-\overline{X}^j_{t_k}|^2\right)\\
&\quad\le S_N(t_k)+C\frac{1}{N}\sum_{j=1}^N\Bigg(\mathbb{E}^\nu|A_4(j)|\\
&\qquad\qquad+\sqrt{\mathbb{E}^\nu|\overline{X}^{j,N}_{t_k}-\overline{X}^j_{t_k}|^4}\left(\sum_{k=1,\ k\neq4}^6\sqrt{\mathbb{E}^\nu A_k(j)^2}\right)\Bigg).
\end{aligned}$$

Estimate (78) then results from estimates (80)–(82), (84), (89) and (91) below. □

Estimate for the second moment of $A_1(j)$. We have

$$\begin{aligned}
\mathbb{E}^\nu|A_1(j)|^2&=\int_{-1}^1\left|\mathbb{E}^\nu\left(\frac{G_N(\theta-x)}{q(\theta)}\right)-1\right|^2q(x)\,dx\\
&=\int_{-1}^1\left|\int_{\mathbb{R}}G_N(x-z)\mathbb{I}_{[-1,1]}(z)\,dz-\mathbb{I}_{[-1,1]}(x)\right|^2q(x)\,dx\\
&\le\|q\|_{L^\infty(\mathbb{R})}\int_{-1}^1\left|\int_{\mathbb{R}}G_N(x-z)\mathbb{I}_{[-1,1]}(z)\,dz-\mathbb{I}_{[-1,1]}(x)\right|^2dx.
\end{aligned}$$

In view of Lemma 7.1, we conclude

$$\mathbb{E}^\nu|A_1(j)|^2\le Ch_N. \tag{80}$$

Estimate for the second moment of $A_2(j)$. As q is a strictly positive continuous function on the compact set $[-1,1]$, there exists a strictly positive constant q_* such that, for any $y\in[-1,1]$, $q(y)\ge q_*>0$. Fix $x\in[-1,1]$. We have

$$\begin{aligned}
&\mathbb{E}^\nu\left(\frac{G_N(\theta-x)}{q(\theta)}\right)-\mathbb{E}^\nu\left(\frac{G_N(\theta-x)}{G(0)/((N-1)h_N)+q(\theta)}\right)\\
&\quad=\mathbb{E}^\nu\left(\frac{G_N(\theta-x)}{q(\theta)(G(0)/((N-1)h_N)+q(\theta))}\right)\frac{G(0)}{(N-1)h_N}\\
&\quad\le\frac{\mathbb{E}^\nu G_N(\theta-x)}{q_*^2}\frac{G(0)}{(N-1)h_N}.
\end{aligned}$$

In view of Lemma 7.1, we finally get

$$
\begin{aligned}
\mathbb{E}^\nu|A_2(j)|^2 = \int_{-1}^{1}\bigg|\mathbb{E}^\nu\Big(\frac{G_N(\theta-x)}{q(\theta)}\Big) \\
- \mathbb{E}^\nu\Big(\frac{G_N(\theta-x)}{G(0)/((N-1)h_N)+q(\theta)}\Big)\bigg|^2 q(x)\,dx \\
\le \frac{C}{[(N-1)h_N]^2}.
\end{aligned}
\tag{81}
$$

Estimate for the second moment of $A_3(j)$. As the random variables θ^i, $1 \le i \le N$, are independent and identically distributed, we have

$$
\begin{aligned}
\mathbb{E}^\nu|A_3(j)|^2 &= \int_{-1}^{1}\mathbb{E}^\nu\bigg|\frac{1}{N-1}\sum_{i=1,i\neq j}^{N}\frac{G_N(x-\theta^i)}{q(\theta^i)+G(0)/((N-1)h_N)} \\
&\qquad - \mathbb{E}^\nu\Big(\frac{G_N(x-\theta^i)}{q(\theta^i)+G(0)/((N-1)h_N)}\Big)\bigg|^2 q(x)\,dx \\
&\le \int_{-1}^{1}\frac{1}{N-1}\mathbb{E}^\nu\Big(\frac{G_N(x-\theta)}{q(\theta)+G(0)/((N-1)h_N)}\Big)^2 q(x)\,dx.
\end{aligned}
$$

For any $x\in[-1,1]$, we have

$$
\begin{aligned}
\mathbb{E}^\nu\Big(\frac{G_N(x-\theta)}{q(\theta)+G(0)/((N-1)h_N)}\Big)^2 &\le \mathbb{E}^\nu\Big(\frac{G_N(x-\theta)}{q(\theta)}\Big)^2 \\
&= \int_{-1}^{1}\frac{1}{h_N^2}\frac{G^2((x-y)/h_N)}{q^2(y)}q(y)\,dy \\
&= \frac{1}{h_N}\int_{(x-1)/h_N}^{(x+1)/h_N}\frac{G^2(z)}{q(x-zh_N)}\,dz \\
&\le \frac{\|G\|^2_{L^2(\mathbb{R})}}{q_*}\,\frac{1}{h_N}.
\end{aligned}
$$

Consequently,

$$
\mathbb{E}^\nu|A_3(j)|^2 \le \frac{C}{(N-1)h_N}. \tag{82}
$$

Contribution of $A_4(j)$ *to the convergence error.* Setting $\underline{\theta} = (\theta^1,\dots,\theta^N)$, we have

$$
\mathbb{E}^\nu\big(|A_4(j)|\,|\overline{X}^{j,N}_{t_k} - \overline{X}^{j}_{t_k}|^2\big) = \mathbb{E}^\nu\big(|A_4(j)|\mathbb{E}^\nu[|\overline{X}^{j,N}_{t_k} - \overline{X}^{j}_{t_k}|^2|\underline{\theta}]\big),
$$

since $A_4(j)$ is $\sigma(\underline{\theta})$-measurable. Moreover, owing to the boundedness of functions b and σ, hypothesis (iv) of Proposition 7.2 and (73), there exists a strictly positive constant C such that

$$\forall N \geq 1,\ \forall j = 1, \dots, N,\ \forall k \leq K, \qquad \mathbb{E}\Big[\big|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\big|^2 \underline{\theta}\Big] \leq C \text{a.s.}$$

Hence, we have

$$\mathbb{E}^{\nu}\Big(|A_4(j)|\,\big|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\big|^2\Big) \leq C\mathbb{E}^{\nu}|A_4(j)|. \tag{83}$$

Then, as the random variables θ^i, $1 \leq i \leq N$, are i.i.d.,

$$\begin{aligned}
\mathbb{E}^{\nu}|A_4(j)| \leq C\mathbb{E}^{\nu}\Bigg[&\frac{1}{N-1}\sum_{i=1, i\neq j}\frac{G_N(\theta^i - \theta^j)}{q(\theta^i)[\mathbb{E}^{\nu}G_N(x-\theta^j)]\,|_{x=\theta^i}} \\
&\qquad \times \big|[\mathbb{E}^{\nu}G_N(x-\theta^j)]\,\big|_{x=\theta^i} - q(\theta^i)\big|\Bigg] \\
\leq C\mathbb{E}^{\nu}&\Bigg[\frac{G_N(\theta^1-\theta^2)}{q(\theta^1)[\mathbb{E}^{\nu}G_N(x-\theta^2)]\,|_{x=\theta^1}}\big|[\mathbb{E}^{\nu}G_N(x-\theta^2)]\,\big|_{x=\theta^1} - q(\theta^1)\big|\Bigg].
\end{aligned}$$

In view of (67), it becomes

$$\begin{aligned}
\mathbb{E}^{\nu}|A_4(j)| &\leq \left((q_*)^2\int_0^1 G(z)\,dz\right)^{-1}\mathbb{E}^{\nu}\big[G_N(\theta^1-\theta^2) \\
&\qquad\qquad \times \big|\,[\mathbb{E}^{\nu}G_N(x-\theta^2)]\,\big|_{x=\theta^1} - q(\theta^1)\big|\big] \\
&= \left((q_*)^2\int_0^1 G(z)\,dz\right)^{-1}\mathbb{E}^{\nu}\Big\{\mathbb{E}^{\nu}[G_N(\theta^1-\theta^2)\mid\theta^1] \\
&\qquad\qquad \times \big|\,[\mathbb{E}^{\nu}G_N(x-\theta^2)]\,\big|_{x=\theta^1} - q(\theta^1)\big|\Big\} \\
&\leq C\mathbb{E}^{\nu}\big|[\mathbb{E}^{\nu}G_N(x-\theta^2)]\,\big|_{x=\theta^1} - q(\theta^1)\big| \qquad \text{[owing to inequality (66)]} \\
&\leq C\sqrt{h_N} \qquad \text{(owing to Lemma 7.1).}
\end{aligned}$$

Therefore, in view of inequality (83), it holds that

$$\mathbb{E}\big|A_4(j)\big(X_t^{j,N} - X_t^{j}\big)^2\big| \leq C\sqrt{h_N}. \tag{84}$$

Estimate for the second moment of $A_5(j)$. The term $A_5(j)$ measures the convergence rate of the denominator of α_{ij} toward its mean value:

$$\begin{aligned}
&\frac{G_N(\theta^i-\theta^j)}{G(0)/((N-1)h_N)+1/((N-1)h_N)\sum_{k=1,k\neq i}^N G_N(\theta^i-\theta^k)}\\
&\quad-\frac{G_N(\theta^i-\theta^j)}{G(0)/((N-1)h_N)+\mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^i}}\\
&\quad=\frac{G_N(\theta^i-\theta^j)}{D_1(N)D_2(N)}\\
&\qquad\times\left(\mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^i}-\frac{1}{(N-1)h_N}\sum_{k=1,k\neq i}^N G_N(\theta^i-\theta^k)\right),
\end{aligned}$$

where

$$D_1(N)=\frac{G(0)}{(N-1)h_N}+\frac{1}{(N-1)h_N}\sum_{k=1,k\neq i}^N G_N(\theta^i-\theta^k),$$

$$D_2(N)=\frac{G(0)}{(N-1)h_N}+\mathbb{E}^\nu G_N(x-\theta)\,|_{x=\theta^i}.$$

Owing to the lower bound (67), we see that $D_2(N)$ is bounded from below by a strictly positive constant independent of N. This property does not hold for $D_1(N)$. We thus use a localization argument by introducing the event

$$\left[\left|\frac{1}{N-1}\sum_{k=1,k\neq i}^N G_N(x-\theta^k)-\mathbb{E}^\nu G_N(x-\theta)\right|\geq\eta(N,a)\right].$$

We start by showing that there exists $\eta(N,a)>0$ such that

$$\mathbb{P}^\nu\left(\left|\frac{1}{N-1}\sum_{k=1,k\neq i}^N G_N(x-\theta^k)-\mathbb{E}^\nu G_N(x-\theta)\right|\geq\eta(N,a)\right)\leq\frac{1}{N^a}\tag{85}$$

for all $(N,a)\in\mathbb{N}\times\mathbb{R}_*^+$, and that

$$\lim_{N\to+\infty}\eta(N,a)=0\qquad\text{if }\lim_{N\to+\infty}\frac{\log(N)}{Nh_N^2}=0.\tag{86}$$

Indeed, the random variables

$$Y_k(N,x):=h_NG_N(x-\theta^k)-\mathbb{E}^\nu G_n(x-\theta^k),\qquad 1\leq k\leq N,$$

are i.i.d. and bounded by $2\|G\|_{L^\infty(\mathbb{R})}$. Then Hoeffding's inequality implies

$$\mathbb{P}^\nu\left(\left|\frac{1}{N-1}\sum_{k=1,\,k\neq i}^{N} G_N(x-\theta^k)-\mathbb{E}^\nu G_N(x-\theta)\right|\geq \eta(N,a)\right)$$

$$=\mathbb{P}^\nu\left(\left|\sum_{k=1}^{N-1} Y_k(N,x)\right|\geq (N-1)\eta(N,a)h_N\right)$$

$$\leq \exp\left(-\frac{(N-1)(\eta(N,a)h_N)^2}{2\|G\|^2_{L^\infty(\mathbb{R})}}\right).$$

Hence, inequality (85) and the limits in (86) hold for

$$\eta(N,a)=\sqrt{2\|G\|^2_{L^\infty(\mathbb{R})}a\frac{\log N}{(N-1)h_N^2}}.$$

We are now in a position to estimate the second moment of $A_5(j)$. For $x\in[-1,1]$ and $a>0$, set

$$E(x,N,a)=\left\{\omega\Big/\left|\frac{1}{N-1}\sum_{k=1,k\neq i}^{N} G_N\big(x-\theta^k(\omega)\big)-\mathbb{E}^\nu G_N(x-\theta)\right|\leq \eta(N,a)\right\},$$

$$A_5(j,x)=\frac{G_N(x-\theta^j)}{G(0)/((N-1)h_N)+1/(N-1)\sum_{k=1,k\neq i}^{N} G_N(x-\theta^k)}$$

$$-\frac{G_N(x-\theta^j)}{G(0)/((N-1)h_N)+\mathbb{E}^\nu G_N(x-\theta)}.$$

As $\eta(N,a)$ tends to 0 when N tends to ∞, we have, for N large enough,

$$\eta(N,a)\leq \tfrac{1}{2}q_*\int_0^1 G(z)\,dz.$$

Thus, in view of (67), we have

$$\frac{1}{N-1}\sum_{k=1,k\neq i}^{N} G_N(x-\theta^k)\geq \frac{1}{2}q_*\int_0^1 G(z)\,dz$$

on the event $E(x,N,a)$. Therefore,

$$\mathbb{E}^\nu\{|A_5(j,x)|^2\mathbb{I}\big(E(x,N,a)\big)\}$$

(87)

$$\leq C\mathbb{E}^\nu\left\{G_N(x-\theta^j)\left|\frac{1}{N-1}\sum_{k=1,k\neq i}^{N} G_N(x-\theta^k)-\mathbb{E}^\nu G_N(x-\theta^i)\right|\right\}^2.$$

We then distinguish two cases:

- $j = i$. As the random variables θ^k are independent and the sum only concerns subscripts different from i, the random variables $G_N(x-\theta^i)$ and $1/(N-1) \times \sum_{k=1,k\neq i}^N G_N(x-\theta^k) - \mathbb{E}^\nu[G_N(x-\theta)]$ are independent. Thus,

$$\begin{aligned}
&\mathbb{E}^\nu\{\mid A_5(i,x)\mid^2 \mathbb{I}(E(x,\eta,N))\}\\
&\quad \le \mathbb{E}^\nu[G_N^2(x-\theta^i)]\mathbb{E}^\nu\left|\frac{1}{N-1}\sum_{k=1,k\neq i}^N G_N(x-\theta^k) - \mathbb{E}^\nu[G_N(x-\theta)]\right|^2\\
&\quad \le C\mathbb{E}^\nu[G_N^2(x-\theta^i)]\frac{1}{N-1}\mathbb{E}^\nu[G_N^2(x-\theta)]\\
&\quad \le \frac{C}{(N-1)h_N^2}.
\end{aligned}$$

- $j \neq i$. We isolate the term $G_N(x-\theta^i)$ from the rest of the sum. A computation similar to the case $j = i$ leads to

$$\mathbb{E}^\nu\{|A_5(j,x)|^2 \mathbb{I}(E(x,\eta,N))\} \le C\left(\frac{1}{Nh_N^2} + \frac{1}{(N-1)^2 h_N^2}\right).$$

Finally, it becomes

$$\mathbb{E}^\nu\{|A_5(j,x)|^2 \mathbb{I}(E(x,N,a))\} \le \frac{C}{Nh_N^2}. \tag{88}$$

On the other hand, roughly bounding G_N by $1/h_N$ and then $A_5(j,x)$ by CN and using (85), we have

$$\mathbb{E}^\nu|A_5(j,x)|^2 \le \frac{C}{Nh_N^2}$$

for a and $\eta(N,a)$ suitably chosen and any $j \le N$.

Observing that the constant C does not depend on $x \in [-1,1]$, we conclude

$$\mathbb{E}^\nu|A_5(j)|^2 = \frac{1}{N-1}\sum_{i=1,i\neq j}^N \int_{-1}^1 \mathbb{E}^\nu|A_5(j,x)|^2 q(x)\,dx \le \frac{C}{Nh_N^2}. \tag{89}$$

Estimate for $\mathbb{E}^\nu[A_6(j)]^2$ *and* $\delta_0(t_k)$. These two terms concern the interaction of a particle with itself. For the first term of $A_6(j)$, we observe that it looks like $A_5(j)$, except that the numerator is constant and of order $1/h_N$; for the two last terms, we use Lemma 7.1 and get

$$\mathbb{E}^\nu|A_6(j)|^2 \le C\left(\frac{1}{N^2h_N^3} + \frac{1}{N^2h_N^2}\right) \le 2C\left(\frac{1}{N^2h_N^3}\right). \tag{90}$$

Moreover, as the random variables $(\overline{X}^i_{t_k}, \theta^i)$, $1 \le i \le N$, are i.i.d.,

$$\begin{aligned}
\delta_0(t_k) &= \frac{1}{N}\sum_{i=1}^{N}\big(\mathbb{E}^\nu[\alpha_{ii} b(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2 + \mathbb{E}^\nu[\alpha_{ii}\sigma(\overline{X}^i_{t_k}, \overline{X}^i_{t_k})]^2\big)\\
&= \mathbb{E}^\nu[\alpha_{11} b(\overline{X}^1_{t_k}, \overline{X}^1_{t_k})]^2 + \mathbb{E}^\nu[\alpha_{11}\sigma(\overline{X}^1_{t_k}, \overline{X}^1_{t_k})]^2\\
&\le \big(\|b\|^2_{L^\infty(\mathbb{R}^2)} + \|\sigma\|^2_{L^\infty(\mathbb{R}^2)}\big)\mathbb{E}^\nu[\alpha_{11}],\\
&\le C\mathbb{E}^\nu\left|\alpha_{11} - \frac{1}{N-1}\right| + \frac{1}{N-1}.
\end{aligned}$$

Proceeding as in the preceding steps, we get

$$\delta_0(t_k) \le C\left(\frac{1}{N}\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N}\right) + \frac{1}{N-1}\right). \tag{91}$$

We can finally estimate the accuracy of the particle method. This is a straightforward consequence of (49), (50), (77), Proposition 5.1 and Lemma 7.3.

THEOREM 7.4. *Suppose that the hypotheses of Proposition* 7.2 *hold. Then there exists an integer N_0 such that, for any $N \ge N_0$ and any test function* $f \in C_{\mathrm{b}}^{4+\varepsilon}(\mathbb{R})$, $0 < \varepsilon < 1$,

$$\mathbb{E}^\nu\left|\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^{N} f(\overline{X}^{i,N}_T)\right|^2 \le C\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N} + (\Delta t)^2\right).$$

8. Conclusion. We have constructed an original and efficient stochastic method to compute moments of statistical solutions of McKean–Vlasov equations, and we have analyzed the convergence rate of the method. Several extensions should be studied in the future, for example: first, the cases of nonsmooth interaction kernels and, in particular, the cases of the Burgers and Navier–Stokes equations; second, the use of random weights other than ours, for example, weights resulting from conditional expectation estimators using wavelets.

In [25] we present numerical results describing the satisfying performances of our particle method for the computation of statistical solutions of Burger's and the Navier–Stokes equations.

APPENDIX

LEMMA A.1. *Suppose that the hypotheses of Proposition* 2.3 *hold. Then, for any function* $g \in C_{\mathrm{b}}(C([0,T],\mathbb{R}),\mathbb{R})$, *the mapping*

$$a \in [-1,1] \mapsto \mathbb{E}g\big(X_\cdot(a)\big)$$

is continuous.

PROOF. Let $g \in C_{\mathrm{b}}(C([0,T],\mathbb{R}))$. For any $a \in [-1,1]$, let $\mathbb{P}_{X.(a)}$ denote the law of the process $X.(a)$. Let a sequence $(a_n) \subset [-1,1]$ converge to a. We have to verify that the sequence $(\mathbb{P}_{X.(a_n)})$ weakly converges to $\mathbb{P}_{X.(a)}$. Owing to the boundedness of the functions b and σ, it is clear that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left\{\sup_{t\in[0,T]} |X_t(a_n) - X_s(a_n)|^4\right\} \le C(T)(t-s)^2.$$

The sequence $(\mathbb{P}_{X.(a_n)})$ is thus tight, so there exists a subsequence of $(\mathbb{P}_{X.(a_n)})$, which we abusively denote by $(\mathbb{P}_{X.(a_n)})$, that weakly converges to a probability measure $\mathbb{P}^\infty$ on $C([0,T],\mathbb{R})$. It remains to prove that $\mathbb{P}^\infty$ is equal to $\mathbb{P}_{X.(a)}$. As (10) has a unique solution in law, this is equivalent to showing that $\mathbb{P}^\infty$ is the unique solution of the martingale problem associated with the operator L_t^a defined in (32).

So let $\psi \in C_{\mathrm{b}}^2(\mathbb{R})$, $p \in \mathbb{N}$, $h \in C_{\mathrm{b}}(\mathbb{R}^p)$ and $(t_1,\dots,t_p,s,t) \in [0,T]^p$, such that

$$0 \le t_1 \le \cdots \le t_p \le s \le t.$$

For any $\alpha \in [-1,1]$, we set

$$M_t(\psi, L^\alpha) = \psi\bigl(x(t)\bigr) - \psi\bigl(x(0)\bigr) - \int_0^t L_\tau^\alpha \psi\bigl(x(\tau)\bigr)\,d\tau, \tag{92}$$

where $x(\cdot)$ is the canonical process on $C([0,T],\mathbb{R})$. For any probability measure m on $C([0,T],\mathbb{R})$, we set

$$\Delta(m,\alpha) := \mathbb{E}^m\bigl[h\bigl(x(t_1),\dots,x(t_p)\bigr)\bigl(M_t(\psi,L^\alpha) - M_s(\psi,L^\alpha)\bigr)\bigr]. \tag{93}$$

We have to prove:

(a) $\forall\, \tilde\psi \in C_b(\mathbb{R})$, $\mathbb{E}^{\mathbb{P}^\infty}[\tilde\psi(x(0))] - \int_{\mathbb{R}} \tilde\psi(x)[\Phi(a)](x)\,dx = 0$;
(b) $\Delta(\mathbb{P}^\infty, a) = 0$.

As $(\mathbb{P}_{X.(a_n)})$ weakly converges to $\mathbb{P}^\infty$, for any function $\tilde\psi \in C_b(\mathbb{R})$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\infty}[\tilde\psi(x_0)] - \int_{\mathbb{R}} \tilde\psi(x)[\Phi(a)](x)\,dx \\
&\quad = \lim_{n\to+\infty}\left\{\mathbb{E}^{\mathbb{P}_{X.(a_n)}}[\tilde\psi(x_0)] - \int_{\mathbb{R}} \tilde\psi(x)[\Phi(a)](x)\,dx\right\} \\
&\quad = \lim_{n\to+\infty}\int_{\mathbb{R}} \tilde\psi(x)\bigl\{[\Phi(a_n)](x) - [\Phi(a)](x)\bigr\}\,dx \\
&\quad \le \|\tilde\psi\|_{L^\infty(\mathbb{R})} \lim_{n\to+\infty}\bigl\|[\Phi(a_n)] - [\Phi(a)]\bigr\|_{L^1(\mathbb{R})} \\
&\quad = 0.
\end{aligned}$$

It now remains to prove property (b). One has

$$\begin{aligned}
\Delta(\mathbb{P}^\infty, a) &= \Delta(\mathbb{P}^\infty, a) - \Delta(\mathbb{P}_{X.(a_n)}, a) + \Delta(\mathbb{P}_{X.(a_n)}, a) \\
&\quad - \Delta(\mathbb{P}_{X.(a_n)}, a_n) + \Delta(\mathbb{P}_{X.(a_n)}, a_n).
\end{aligned}$$

As $(\mathbb{P}_{X.(a_n)})$ weakly converges to $\mathbb{P}^\infty$,

$$\lim_{n\to+\infty}\{\Delta(\mathbb{P}^\infty, a) - \Delta(\mathbb{P}_{X.(a_n)}, a)\} = 0. \tag{94}$$

Moreover, the measure $\mathbb{P}_{X.(a_n)}$ is a solution of the martingale problem associated with the operator $L_t^{a_n}$, so

$$\Delta(\mathbb{P}_{X.(a_n)}, a_n) = 0 \qquad \forall\, n \in \mathbb{N}. \tag{95}$$

Finally, one has

$$\begin{aligned}
&\Delta(\mathbb{P}_{X.(a_n)}, a) - \Delta(\mathbb{P}_{X.(a_n)}, a_n)\\
&\quad = \mathbb{E}\biggl[h\bigl(X_{t_1}(a_n), \dots, X_{t_p}(a_n)\bigr)\int_s^t (L_\tau^a - L_\tau^{a_n})\psi\bigl(X_\tau(a_n)\bigr)\,d\tau\biggr]\\
&\quad = \mathbb{E}\biggl[h\bigl(X_{t_1}(a_n), \dots, X_{t_p}(a_n)\bigr)\\
&\qquad\quad \times \int_s^t \bigl(u_b\bigl(\tau, X_\tau(a_n), a\bigr) - u_b\bigl(\tau, X_\tau(a_n), a_n\bigr)\bigr)\psi'\bigl(X_\tau(a_n)\bigr)\,d\tau\biggr]\\
&\qquad + \mathbb{E}\biggl[h\bigl(X_{t_1}(a_n), \dots, X_{t_p}(a_n)\bigr)\\
&\qquad\qquad \times \int_s^t \tfrac{1}{2}\bigl(u_\sigma^2\bigl(\tau, X_\tau(a_n), a\bigr) - u_\sigma^2\bigl(\tau, X_\tau(a_n), a_n\bigr)\bigr)\psi''\bigl(X_\tau(a_n)\bigr)\,d\tau\biggr].
\end{aligned}$$

By boundedness of the functions ψ, ψ', ψ'' and h and properties (7) and (8) of b and σ, it is easy to check that

$$\begin{aligned}
&\Delta(\mathbb{P}_{X.(a_n)}, a) - \Delta(\mathbb{P}_{X.(a_n)}, a_n)\\
&\qquad \le C \sup_{\tau\in[0,T]} \|(S_\tau \circ \Phi)(a_n) - (S_\tau \circ \Phi)(a)\|_{L^1(\mathbb{R})},
\end{aligned} \tag{96}$$

(7), where the operator $S \circ \Phi$ has been defined in Proposition 2.3. Then, owing to Proposition 2.4,

$$\lim_{n\to+\infty}\{\Delta(\mathbb{P}_{X.(a_n)}, a) - \Delta(\mathbb{P}_{X.(a_n)}, a_n)\} = 0. \tag{97}$$

Property (b) results from (94), (95) and (97). $\square$

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