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ON THE ASYMPTOTIC PATTERNS OF SUPERCRITICAL BRANCHING PROCESSES IN VARYING ENVIRONMENTS

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Let $\{Z_n\}$ be a branching process whose offspring distributions vary with n. It is shown that the sequence $\{\max_{i>0} P(Z_n = i)\}$ has a limit. Denote this limit by M. It turns out that M is positive only if the offspring variables rapidly approach constants. Let $\{c_n\}$ be a sequence of constants and $W_n = Z_n/c_n$. It will be proven that $M = 0$ is necessary and sufficient for the limit distribution functions of all convergent $\{W_n\}$ to be continuous on $(0, \infty)$. If $M > 0$ there is, up to an equivalence, only one sequence $\{c_n\}$ such that $\{W_n\}$ has a limit distribution with jump points in $(0, \infty)$. Necessary and sufficient conditions for continuity of limit distributions are derived in terms of the offspring distributions of $\{Z_n\}$.

1. Introduction and results. A branching process in varying environments $\{Z_n\}$ is a sequence of nonnegative integer-valued random variables $\{Z_n\}$ defined inductively by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \sum_{k=1}^{Z_n} X_{n,k}, & \text{if } Z_n \geq 1, \\ 0, & \text{if } Z_n = 0, \end{cases} \tag{1}$$

where $\{X_{n,k};\ k = 1, 2, \dots\}$, the offspring variables of the nth generation, are for each n independent and identically distributed given Z_n. The term *varying environments* refers to the fact that, unlike the classical Galton–Watson process, the probability distributions of $\{X_{n,k}\}$ are allowed to vary with n. Let X_n be a random variable distributed like $X_{n,1}$. Write $M_n = \max_{i>0} P(Z_n = i)$ and 1_A for the indicator function of the set A. We say that the sequences $\{a_n\}$ and $\{b_n\}$ are equivalent and write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = \gamma$ for $\gamma \in (0, \infty)$. In what follows convergence to a variable W includes the case when the limit is defective. That is, $P(W = \infty) > 0$ is allowed.

The limit behavior of the branching process in varying environments in the case $P(\lim_{n\to\infty} Z_n > 0) > 0$ was studied under two (not mutually incompatible) conditions: (i) $P(0 < \lim_{n\to\infty} Z_n < \infty) > 0$ and (ii) $P(\lim_{n\to\infty} Z_n = \infty) > 0$. In the first case Church [3] proved that $\sum_{n=1}^{\infty}(1 - P(X_n = 1)) < \infty$ is necessary and sufficient for $\{Z_n\}$ to converge in distribution to a nondegenerate limit. Lindvall [12] strengthened this result to a.s. convergence. There are

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a number of results in case (ii) centering on some norming constants $\{c_n\}$ tending to ∞ such that $\{W_n\}$, with $W_n = Z_n/c_n$, converges to a nondegenerate limit. Write F for the limit distribution of $\{W_n\}$. A number of papers have dealt with the asymptotic behavior of $\{W_n\}$. We mention the basic paper by Goettge [8] and more recent papers of Biggins and D'Souza [2], D'Souza and Biggins [7] and D'Souza [6]. For a survey of earlier literature, see [1] and [10].

In this paper, the aspect of the limit behavior of $\{Z_n\}$ which concerns us is the continuity or presence of jump points in $(0, \infty)$ in the limit distribution of $\{W_n\}$. In sharp contrast to the case of sums of independent random variables (take, e.g., the law of large numbers), for branching processes in varying environments $\lim_{n\to\infty}[\max_{i>0} P(Z_n = i)] = 0$ turns out to be necessary and sufficient for the limit distribution functions of all convergent $\{W_n\}$ to be continuous on $(0, \infty)$. Sufficient conditions for the limit of $\{W_n\}$ to be continuous outside 0 were given by Cohn and Schuh [5] and Cohn [4] in the one-type case and by Jones [11] in the multitype setting. Hattori, Hattori and Watanabe [9] studied the support of the limit distribution of the multitype process.

Define k_n by $P(X_n = k_n) = \max_{i>0} P(X_n = i)$ and $i_n = \prod_{j=0}^{n-1} k_j$.

THEOREM 1. *The following statements are equivalent*:

(i) $\limsup_{n\to\infty} M_n > 0$;
(ii) *the sequence* $\{M_n\}$ *converges to a positive limit*;
(iii) $\sum_{n=1}^{\infty} i_n(1 - P(X_n = k_n)) < \infty$;
(iv) *there exist a sequence of positive integers* $\{m_n\}$ *and an event of positive probability*, Λ, *such that* $\lim_{n\to\infty} 1_{\{Z_n = m_n\}} = 1_\Lambda$ *a.s.* (*Here* Λ *may be chosen to have probability* $\lim_{n\to\infty} M_n$.)

COROLLARY 2. *The sequence* $\{M_n\}$ *converges.*

THEOREM 3. *Suppose that* $\sum_{n=1}^{\infty} i_n(1 - P(X_n = k_n)) < \infty$. *Then the following statements hold*:

(i) $\{Z_n/i_n\}$ *converges a.s. to a limit* W *with* $\max_{x>0} P(W = x) > 0$;
(ii) *if* $\{Z_n/c_n\}$ *converges a.s. to a limit* W' *with* $\max_{x>0} P(W' = x) > 0$, *for some constants* $\{c_n\}$, *then* $c_n \sim i_n$.

COROLLARY 4. *The following conditions are equivalent*:

(i) $\sum_{n=1}^{\infty} i_n(1 - P(X_n = k_n)) = \infty$;
(ii) *any a.s. convergent* $\{Z_n/c_n\}$ *has a continuous limit distribution function in* $(0, \infty)$.

It seems rather surprising that continuity may fail only if the offspring variables approach constants very rapidly. In the case of the classical Galton–Watson process, it is sufficient to assume that the offspring distribution is not concentrated in one point, that is, to exclude the deterministic case, when of course F is not continuous. Another consequence of Theorem 3 in the case when there are two or more nonequivalent rates of convergence for $\{Z_n\}$ (see [13] and [6]) is that there is essentially only one convergent $\{W_n\}$ with limit distribution admitting jump points in $(0, \infty)$. Take the branching process considered by MacPhee and Schuh [13] with offspring generating functions

$$f_n(s) = (1 - 4^{-(n+1)})s^2 + 4^{-(n+1)}s\big[(m-2)4^{n+1} + 2\big], \qquad s \in [0,1],\ n = 0, 1, \dots . \tag{2}$$

By Theorem 2 of [13], if $m > 4$, there are two rates of growth: $\{2^n\}$ and $\{m^n\}$. In the first case we get $k_n = 2$ and $i_n = 2^n$ with

$$\sum_{n=1}^{\infty} i_n(1 - P(X_n = k_n)) = \sum_{n=1}^{\infty} 2^{-n} < \infty$$

and Theorem 3(i) implies that the limit distribution of $\{Z_n/2^n\}$ must have jump points in $(0, \infty)$. By Theorem 3(ii) the limit corresponding to $\{Z_n/m^n\}$ is continuous outside 0.

2. Proofs. We shall need a number of lemmas.

LEMMA 5. *Suppose that $\{Z_n\}$ is a branching process in varying environments and $\{c_{n_k}\}$ is a sequence of constants such that $\{Z_{n_k}/c_{n_k}\}$ converges weakly as $k \to \infty$ to a nondegenerate limit W. Then there exist some random variables $\{W_i^{(n)}\}$ such that*

$$W = \sum_{i=1}^{Z_n} W_i^{(n)} \quad a.s., \tag{3}$$

where $W_i^{(n)}$, $i = 1, 2, \dots,$ are independent and identically distributed given Z_n.

PROOF. Notice that for $n < n_k$,

$$Z_{n_k} = \begin{cases} \sum_{i=1}^{Z_n} Z^{(i)}_{n_k - n, n}, & \text{if } Z_n \geq 1, \\ 0, & \text{if } Z_n = 0, \end{cases} \tag{4}$$

where $Z^{(i)}_{m,n}$ is the number of the mth generation offspring of the ith individual of the nth generation. The random variables $\{Z^{(i)}_{n_k - n, n};\ i = 1, \dots, Z_n\}$ are independent and identically distributed given Z_n. Since $\{Z_{n_k}/c_{n_k}\}$ converges weakly as $k \to \infty$, so does $\{Z_{n_k - n}/c_{n_k}\}$ for $i = 1, 2, \dots$. This may be shown by using Laplace transforms. Indeed, write $V_k = Z_{n_k}/c_{n_k}$, $\hat{V}_k = Z_{n_k - n, n}/c_{n_k}$, $\phi_k(t) = E[\exp(-tV_k)]$, $\hat{\phi}_k(t) = E[\exp(-t\hat{V}_k)]$ and $f_n(t) = \sum_{i=0}^{\infty} t^i P(Z_n = i)$. Then (4) yields

$$\phi_k(t) = f_n\big(\hat{\phi}_k(t)\big). \tag{5}$$

Using in (5) that $\lim_{k \to \infty} \phi_k(t)$ exists for all t and that $f_n(t)$ is continuous and strictly increasing in t implies that $\lim_{k \to \infty} \hat{\phi}_k(t)$ must also exist for all t and therefore $\{\hat{V}_k\}$ converges weakly as $k \to \infty$. Notice now that a subsequence of a branching process in varying environments $\{Z_{n_k}\}$ is also a branching process in varying environments. Thus Theorem 29 of [8] applies to yield that $\{\hat{V}_k\}$ converges a.s. as $k \to \infty$. Now dividing (4) by c_{n_k} and letting $k \to \infty$ completes the proof. □

LEMMA 6. *If $\{c_{n_k}\}$ is a sequence of constants such that $\{Z_{n_k}/c_{n_k}\}$ converges in distribution as $k \to \infty$ to a nondegenerate limit, then there exists a whole sequence $\{c_n\}$ such that $\{Z_n/c_n\}$ converges a.s. as $n \to \infty$.*

PROOF. As was noticed in the course of the proof of Lemma 6, $\{Z_{n_k}\}$ is also a branching process in varying environments. Thus Theorem 16 of [8] applies and yields $c_{n_k} \sim c/h_{n_k}(s_0)$ for some s_0, where $h_n(s) = -\log f_n^{-1}(s)$, f_n being the generating function of Z_n. However, according to Theorem 17 of [8], $\{h_n(s_0)Z_n\}$ converges in distribution as $n \to \infty$ and Theorem 29 of [8] completes the proof. □

LEMMA 7. *If $\{Y_i^{(n)}, i = 1, 2, \dots\}$ are, for each n, nonnegative, independent and identically distributed random variables such that $\lim_{n\to\infty} P(\sum_{i=1}^{m_n} Y_i^{(n)} = c_n) = 1$, for some constants $\{c_n\}$ and $\{m_n\}$, then $\lim_{n\to\infty} P(Y_i^{(n)} = c_n/m_n) = 1$.*

PROOF. Notice first that the result is elementary in the case when $\{m_n\}$ are bounded. Let $\xi_1, \dots, \xi_n$ be some independent and identically distributed random variables, $S_n = \xi_1 + \cdots + \xi_n$ and $p = \sup_x P(\xi_1 = x)$. Define the concentration function of the random variable X by $Q(X; \lambda) = \sup_x P(x \le X \le x + \lambda)$. Then by a result on concentration functions of sums of independent random variables [see, e.g., [14], page 68, equation (2.58)], for any $\lambda > 0$,

$$Q(S_n; \lambda) \le An^{-1/2}\big(1 - Q(\xi_1; \lambda)\big)^{-1/2},$$

where A is an absolute constant. Notice that letting λ tend to 0 yields

$$\sup_x P(S_n = x) \le A\big(n(1-p)\big)^{-1/2}. \tag{6}$$

Let x_n be such that $P(Y_i^{(n)} = x_n) = \sup_x P(Y_i^{(n)} = x)$. By (6) we get

$$\limsup_{n\to\infty} m_n\big(1 - P\big(Y_i^{(n)} = x_n\big)\big) \le A^2. \tag{7}$$

On the other hand,

$$P\left(\sum_{i=1}^{m_n} Y_i^{(n)} = m_n x_n\right) \ge \big(P\big(Y_1^{(n)} = x_n\big)\big)^{m_n}. \tag{8}$$

By (7) the right-hand side of (8) is bounded away from 0. Letting now $n \to \infty$ in (8) yields $x_n = c_n/m_n$ for n large enough and completes the proof. □

LEMMA 8. *Suppose that $\{c_k\}$ is a sequence of constants such that $\{Z_{n_k}/c_k\}$ converges a.s. as $k \to \infty$ to a limit W with $P(W = c) > 0$ and $c > 0$. Then there exist some positive integers $\{m_n\}$ such that*:

(i) $\lim_{n\to\infty} 1_{\{Z_n = m_n\}} = 1_{\{W=c\}}$ *a.s.*;
(ii) $m_{n+1}/m_n = k_n$ *for n large enough.*

PROOF. By Lemma 5 we get that

$$P(W = c \mid Z_n) = P\left(\sum_{i=1}^{Z_n} W_i^{(n)} = c \,\middle|\, Z_n \right), \tag{9}$$

and by the martingale convergence theorem there must exist some $\{m_n\}$ such that

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{m_n} W_i^{(n)} = c \right) = 1. \tag{10}$$

By Lemma 7 this can only happen when $\lim_{n\to\infty} P(W_i^{(n)} = c/(m_n)) = 1$. Thus, in this case,

$$\begin{aligned} &\lim_{n\to\infty} P\left(\sum_{i=1}^{m_n+1} W_m^{(n)} = c\left(1 + m_n^{(-1)}\right) \right) = 1, \\ &\lim_{n\to\infty} P\left(\sum_{i=1}^{m_n-1} W_i^{(n)} = c\left(1 - m_n^{(-1)}\right) \right) = 1. \end{aligned} \tag{11}$$

From (11) it follows that for $\delta \in (0, 0.5)$, n large enough, $l > m_n$ and $j < m_n$,

$$P\left(\sum_{i=1}^{l} W_i^{(n)} > c \right) > 1 - \delta, \qquad P\left(\sum_{i=1}^{j} W_i^{(n)} < c \right) > 1 - \delta. \tag{12}$$

The martingale convergence theorem applied to $\{P(W = c \mid Z_n)\}$ yields $\lim_{n\to\infty} P(W = c \mid Z_n) = 1_{\{W=c\}}$ a.s., which is equivalent to

$$\lim_{n\to\infty} 1_{\{Z_n \in A_n\}} = 1_{\{W=c\}} \quad \text{a.s.} \tag{13}$$

for $A_n = \{j\colon\ P(W = c \mid Z_n = j) > 1 - \delta\}$. However, $P(W = c \mid Z_n = j) = P(\sum_{i=1}^{j} W_i^{(n)} = c)$ follows from (9), and using (12) yields $A_n = \{m_n\}$ for n large enough. Finally, (13) completes the proof of (i). Notice now that by (i) $\lim_{n\to\infty} P(Z_{n+1} = m_{n+1} \mid Z_n = m_n) = 1$, which implies $\lim_{n\to\infty} P(X_{n,1} + \cdots + X_{m_n} = m_{n+1}) = 1$. By Lemma 6, this entails $P(X_n = m_{n+1}/m_n) = 1$, which can happen only if $m_{n+1}/m_n = k_n$ for n large, and the proof is complete. □

LEMMA 9. *The following conditions are equivalent*:

(i) $\sum_{n=1}^{\infty}(1 - P(X_{n,1} + \cdots + X_{n,m_n} = m_n k_n)) < \infty$;
(ii) $\sum_{n=1}^{\infty} m_n(1 - P(X_n = k_n)) < \infty$.

PROOF. If (i) holds, then

$$\begin{aligned} &\left(1 - P\left(X_{n,1} + \cdots + X_{n,m_n} = m_n k_n\right)\right) \\ &\quad = P\left(X_{n,1} + \cdots + X_{n,m_n} \neq m_n k_n\right) \\ &\quad \geq m_n\left(1 - P(X_n = k_n)\right)\left(P(X_n = k_n)\right)^{m_n - 1}. \end{aligned}$$

Since under (i), $\lim_{n\to\infty} P(X_{n,1} + \cdots + X_{n,m_n} = m_n k_n) = 1$, we invoke the argument used in (7) to deduce that $(P(X_n = k_n))^{m_n - 1}$ is bounded away from 0 as $n \to \infty$, and (ii) follows.

If (ii) holds, then

$$\begin{aligned}\left(1 - P\left(X_{n,1} + \cdots + X_{n,m_n} = m_n k_n\right)\right) &\le \left(1 - \left(P\left(X_{n,1} = k_n\right)\right)^{m_n}\right)\\ &\le m_n\left(1 - P\left(X_n = k_n\right)\right)\end{aligned}$$

and (i) follows. □

Proof of Theorem 1. Assume (i) and write $\alpha = \limsup_{n\to\infty} M_n > 0$. Choose a sequence $\{n_k\}$ with $\lim_{k\to\infty} M_{n_k} = \alpha$ and define $\{c_k\}$ such that $M_{n_k} = P(Z_{n_k} = c_k)$. Two cases are possible: (a) when $\{c_k\}$ is bounded and (b) when $\limsup_{k\to\infty} c_{n_k} = \infty$. The proof that we give here holds in general. However, we wish to mention that in case (a), (ii) and (iv) follow immediately from a result of Lindvall [12] asserting that $\{Z_n\}$ converges a.s. Indeed, this implies that there must exist some i^* such that $\alpha = \lim_{n\to\infty} P(Z_n = i^*)$, proving (ii). It is easy to see that (iv) follows now from a.s. convergence. Let us assume the general case. Then $\{Z_{n_k}/c_{n_k}\}$ or a subsequence thereof converges weakly to limit distribution F which is nondegenerate since $F(1) \ge F(0) + \alpha$. By Theorem 29 of [8] weak convergence implies a.s. convergence; denote the almost sure limit by W. According to Lemma 8(i) there exists a sequence $\{m_n\}$ with $1_{\{W=1\}} = \lim_{n\to\infty} 1_{\{Z_n = m_n\}}$ a.s. This proves (iv).

Dominated convergence and the definition of α and M_n give

$$\lim_{n\to\infty} P(Z_n = m_n) = P(W = 1) \ge \alpha = \limsup_{n\to\infty} \max_{i>0}\{P(Z_n = i)\},$$

which proves (ii).

Notice now that (iv) implies

$$\lim_{n\to\infty} P(Z_n = m_n, Z_{n+1} = m_{n+1}, \dots) = \lim_{n\to\infty} P(Z_n = m_n) > 0, \tag{14}$$

which leads to

$$\lim_{n\to\infty} P(Z_{n+1} = m_{n+1} | Z_n = m_n) P(Z_{n+2} = m_{n+2} | Z_{n+1} = m_{n+1}) \cdots = 1. \tag{15}$$

This in turn entails

$$\sum_{n=1}^{\infty} \left(1 - P\left(X_{n,1} + \cdots + X_{n,m_n} = m_{n+1}\right)\right) < \infty. \tag{16}$$

However, by Lemma 8(ii), $m_{n+1}/m_n = k_n$ for n large using this in (16) together with Lemma 9 yields $\sum_{n=1}^{\infty} m_n(1 - P(X_n = k_n)) < \infty$. Since $m_n \sim i_n$, we get $\sum_{n=1}^{\infty} i_n(1 - P(X_n = k_n)) < \infty$ and (iii) is proved. Assume now that (iii) holds. By Lemma 9,

$$\sum_{n=1}^{\infty} \left(1 - P\left(X_{n,1} + \cdots + X_{n,i_n} = k_n i_n\right)\right) < \infty. \tag{17}$$

Further, it is easy to see that $P(Z_n = i_n) > 0$ for all n which makes (17) imply (14) with m_n replaced by i_n. Since (14) implies (i), the proof is complete. □

PROOF OF THEOREM 3. Theorem 1(iv) together with Theorem 29 of Goettge [8] imply that $\{Z_n/m_n\}$ converges a.s. to a limit W'' with $P(W'' = 1) > 0$. Indeed, any convergent subsequence of $\{Z_n/m_n\}$ has a nondegenerate limit. Furthermore by Lemma 6 there are some constants $\{c_n\}$ such that $\{Z_n/c_n\}$ converges a.s. to a nondegenerate limit, where a subsequence of $\{c_n\}$ is equivalent to a subsequence of $\{m_n\}$. It is now easy to see, from Theorem 1(iv), that $\{m_n\}$ are necessarily, up to an equivalence, the norming constants making $\{Z_n/m_n\}$ a.s. convergent. Using now Lemma 8(ii) yields $m_n \sim i_n$, and (i) follows. However, the same argument based on Lemma 8 yields $c_n \sim i_n$, which implies (ii). This completes the proof. □

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