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OPTIMAL INSURANCE DEMAND UNDER MARKED POINT PROCESSES SHOCKS

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We study the stochastic control problem of maximizing expected utility from terminal wealth, when the wealth process is subject to shocks produced by a general marked point process; the problem of the agent is to derive the optimal allocation of his wealth between investments in a nonrisky asset and in a (costly) insurance strategy which allows "lowering" the level of the shocks. The agent's optimization problem is related to a suitable dual stochastic control problem in which the constraint on the insurance strategy disappears. We establish a general existence result for the dual problem as well as the duality between both problems. We conclude by some applications in the context of power (and logarithmic) utility functions and linear insurance premium which show, in particular, the existence of two critical values for the insurance premium: below the lower critical value, agents prefer to be completely insured, whereas above the upper critical value they take no insurance.

1. Introduction. We study the optimal insurance demand problem of an agent whose wealth is subject to shocks produced by some marked point process. Such a problem is formulated in continuous-time with Poisson shocks in Bryis (1986). Gollier (1994) studies a similar problem where shocks are not proportional to wealth. An explicit solution to the problem is provided by Briys by writing formally the Hamilton–Jacobi–Bellman equation.

An important feature of Briys' (1986) and Gollier's (1994) analysis is that no constraint on the insurance strategy is imposed, which is not realistic. In real cases, the insurance strategy is restricted to the interval $[0, 1]$. Also, in both papers, the insurance premium is assumed to be an affine function of the insurance strategy.

In this paper, we account explicitly for the constraint on the insurance strategy and we consider a more general convex insurance premium. We provide a dual formulation of the optimal insurance demand problem inspired from the usual technique in portfolio optimization theory; see Karatzas, Lehoczky and Shreve (1987), Karatzas (1989), Cox and Huang (1989) and Cvitanič and Karatzas (1992). In the case of Poisson shocks, unconstrained insurance strategy and affine insurance premium, the dual opitimization problem is degenerate and provides directly an explicit solution of the problem.

In the general case, the dual optimization problem cannot be solved explicitly but does not present any constraint on the controls. The proofs of the dual formulation of the problem are inspired from Cvitanič and Karatzas (1992)

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with some simplified arguments; see, for example, the proof of Lemma 6.3. Theorem 5.1 appeals to the optional decomposition Theorem of Föllmer and Kramkov (1997), which was first established by El Kaoui and Quenez (1995) and Cvitanič and Karatzas (1992) in a Brownian filtration framework. We also provide an existence theorem for the dual optimization problem which does not satisfy the usual conditions for application of general existence theorems as in Ekeland and Temam (1976).

Finally, we observe that the optimal insurance demand problem is closely related to the portfolio optimization problem in a large investor setting as in Cuocco and Cvitanič (1998) who studied the Brownian filtration case.

In this paper, we only discuss the problem of maximizing expected utility from terminal wealth. As in Karatzas, Lehoczky, Shreve and Xue (1991) and Cvitanič and Karatzas (1992), the model can be easily extended to allow for intertemporal consumption.

The paper is organized as follows. Section 2 provides the (explicit) solution of the optimal unconstrained insurance demand problem with Poisson shocks. Section 3 describes the general (constrained) insurance demand problem under marked point processes shocks. In Section 4, we introduce an appropriate set of auxiliary local martingales which is the basic tool for the dual formulation of the problem. In Section 5, we characterize the set of attainable terminal wealth processes which is now well understood to be closely related to the problem of maximizing expected utility from terminal wealth. Section 6 contains the main result of the paper which provides a dual formulation of the problem in terms of the auxiliary local martingales introduced in Section 4 under an existence assumption on the dual problem. In Section 7, we provide sufficient conditions which ensure the existence of a solution to the dual problem. We conclude by some examples in Section 8 concerning logarithmic and power utility functions, constant coefficients of the model and Poisson shocks.

2. Poisson shocks, unconstrained insurance strategy and linear insurance premium. In this section, we present a slight generalization of Briys (1986) which motivates the passage to the dual formulation of the problem. The reader interested in the general model of this paper can skip this section. We fix throughout this section a complete probability space $(\Omega, \mathscr{F}, P)$, a finite time horizon T, and a Poisson process $\{v(t),\ 0 \le t \le T\}$ with predictable intensity process $\{m(t),\ 0 \le t \le T\}$ with $m(t) \ge \eta,\ 0 \le t \le T$ P-a.s. for some constant $\eta > 0$. The process $\tilde{v}(t) = v(t) - \int_0^t m(u)\,du$ denotes the compensated Poisson process. Let $\mathbb{F} = \{\mathscr{F}(t),\ 0 \le t \le T\}$ the (P-completion) of the filtration generated by $\{v(t),\ 0 \le t \le T\}$ and assume $\mathscr{F}(T) = \mathscr{F}$.

Let $\beta = \{\beta(t), 0 \le t \le T\}$ be some $\mathbb{F}$-predictable process, and consider the associated wealth process $X^{x,\beta}$ defined by

$$X^{x,\beta}(0) = x \quad \text{and} \quad dX^{x,\beta}(t) = X^{x,\beta}(t-)\big[(r(t)-b\beta(t))\,dt-(1-\beta(t))\gamma(t)\,dv(t)\big],$$

where $\{r(t),\ 0 \le t \le T\}$ is an $\mathbb{F}$-predictable process satisfying $\int_0^T r(u)\,du < \infty$, b is some positive constant and $\gamma = \{\gamma(t),\ 0 \le t \le T\}$ is an $\mathbb{F}$-predictable positive process with $\gamma(t) \ge \eta,\ 0 \le t \le T$, and $\int_0^T \gamma(t)m(t)\,dt < \infty$ P-a.s.

Here $r(t)$ is the instantaneous interest rate process at time t; the process $\beta = \{\beta(t),\ 0 \le t \le T\}$ is called an unconstrained insurance strategy; that is $\beta(t)$ is the level of insurance demanded by the agent at time t. The function $\pi(x) = bx$ is the insurance premium, to be paid by the agent, for a level of insurance x, see Section 3 for a deeper description of the model. Notice that, as in Bryis (1986), we do not impose that the level of insurance must be in the interval $[0, 1]$; we shall take this constraint into account in the subsequent sections of the paper.

An unconstrained insurance strategy β is said to be admissible if the associated wealth process $X^{x,\beta}$ is nonnegative. We shall denote by $\mathscr{B}_0$ the set of all such admissible unconstrained insurance strategies.

The preferences of the agent are described by a utility function $U: (0,\infty) \to \mathbb{R}$ assumed to be increasing, strictly concave, of class C^1, and satisfies $U'(0+) = +\infty$ and $U'(+\infty) = 0$.

The (unconstrained) optimal insurance demand problem of the agent is

$$\sup_{\beta \in \mathscr{B}_0} E\big[U\big(X^{x,\beta}(T)\big)\big].$$

In this section, we solve the above optimization problems by a method similar to that introduced by Karatzas (1989) in the theory of continuous trading with complete market.

We shall denote I the (continuous strictly increasing) inverse of U' and we introduce the Legendre–Fenchel transform of $-U(-x)$ defined by

$$\tilde{U}(y) := \sup_{x>0}\{U(x) - xy\} = U(I(y)) - yI(y); \qquad y > 0.$$

Denote by q the process defined by

$$q(t) = \frac{b}{\gamma(t)m(t)}, \qquad 0 \le t \le T,$$

and consider the Doléans-Dade exponential local martingale $\hat{Z}$,

$$\hat{Z}(t) = \mathscr{E}\Big(\int_0^t (q(u) - 1)\, d\tilde{v}(u)\Big)$$

From our assumptions on the coefficients, $\hat{Z}$ is a P-martingale. Then we can define the probability measure $\hat{P} \sim P$ by $\hat{P}(A) = E\big[\hat{Z}(T)1_A\big]$ for all $A \in \mathscr{F}$. By the Girsanov theorem for Poisson processes, the predictable intensity of $\{v(t),\ 0 \le t \le T\}$ under the probability measure $\hat{P}$ is given by $\int_0^t m(u)q(u)\,du$. Denoting by $\hat{v}$ the $\hat{P}$-compensated Poisson process, we see that the wealth process associated to some admissible insurance strategy β satisfies

$$\begin{aligned} d\Big(X^{x,\beta}(t)\exp\big(-\int_0^t r(u)\,du\big)\Big) \\ = \Big(X^{x,\beta}(t-)\exp\big(-\int_0^t r(u)\,du\big)\Big)(1 - \beta(t))\gamma(t)\,d\hat{v}(t). \end{aligned}$$

Then $X^{x,\beta}\exp\big(-\int_0^{\bullet} r(u)\,du\big)$ is a $\hat{P}$-supermartingale as a nonnegative $\hat{P}$-local martingale. Now, by definition of $\tilde{U}$, we have

$$\tilde{U}\big(y\hat{Z}(T)\big) \geq U\big(X^{x,\beta}(T)\big) - y\hat{Z}(T)X^{x,\beta}(T), \quad P\text{-a.s.} \tag{2.1}$$

for all $x > 0$ and $\beta \in \mathscr{B}_0$. Furthermore, from the $\hat{P}$-supermartingale property of $X^{x,\beta}$, we have

$$E\big[\hat{Z}(T)X^{x,\beta}(T)\big] \leq x. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$E\big[\tilde{U}(y\hat{Z}(T))\big] \geq \sup_{\beta\in\mathscr{B}_0} E\big[U\big(X^{x,\beta}(T)\big)\big] - xy. \tag{2.3}$$

Now, observe that, in order to have equality in (2.3), it suffices to have equality in (2.1) and (2.2) for some $\hat{y} > 0$ and some $\hat{\beta} \in \mathscr{B}_0$. By definition of $\tilde{U}$, we have equality in (2.1) if and only if

$$I\big(y\hat{Z}(T)\big) = X^{x,\hat{\beta}}(T), \quad P\text{-a.s.} \tag{2.4}$$

By the local martingale representation theorem for Poisson processes [see Brémaud (1981), Theorem 9, page 64], it is possible to find such a process $\hat{\beta}$. Next, in order to have equality in (2.2), we need to define $\hat{y}$ by

$$E\big[\hat{Z}(T)I\big(\hat{y}\hat{Z}(T)\big)\big] = x. \tag{2.5}$$

By Fatou's lemma, we see that the left-hand side of (2.5) tends to $+\infty$ as $y \searrow 0$. Moreover, assuming that $E\big[\hat{Z}(T)I\big(y_0\hat{Z}(T)\big)\big] < \infty$ for some $y_0 > 0$, it follows from the decrease of $I(\cdot)$ that $\hat{Z}(T)I\big(y\hat{Z}(T)\big) \leq \hat{Z}(T)I\big(y_0\hat{Z}(T)\big)$ for $y \geq y_0$, and therefore, we see that the left-hand side of (2.5) tends to zero as $y \nearrow \infty$ by the dominated convergence theorem. Therefore by the strict decrease of I, (2.5) defines a unique $\hat{y} > 0$.

We have then proved that

$$E\big[\tilde{U}(\hat{y}\hat{Z}(T))\big] = \sup_{\beta\in\mathscr{B}_0} E\big[U\big(X^{x,\beta}(T)\big)\big] - xy$$

and the optimal insurance strategy is characterized by (2.4). Hence the dual formulation of the unconstrained optimal insurance problem solves the problem explicitly.

3. The general model. In this section, we present the model and the stochastic control problem considered throughout the rest of the paper. Our purpose is to obtain a dual formulation of the optimal insurance demand problem as in the previous section.

3.1. *The general framework.* Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and T a finite time horizon. We consider an integer valued random measure $v(dt, dz)$, defined on $(\Omega, \mathscr{F}, P)$, associated to the marked point process $(N, \{Y(n),\ n \in \mathbb{N}\})$; that is, $\{N(t)\ t \geq 0\}$ is a counting process corresponding to the random time points $\{T_n,\ n \in \mathbb{N}\}$, and $\{Y(n),\ n \in \mathbb{N}\}$ is a sequence of random variables with values in the mark space D, a Borel subset of $\mathbb{R}_+$.

As usual, v and $(N, \{Y(n),\ n \in \mathbb{N}\})$ are identified by the formula

$$v([0, t] \times B) = \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_B(Y_n) \quad \text{for all } t \in [0, T] \text{ and } B \in \mathscr{D},$$

where $\mathscr{D}$ is the Borel σ-field on D. We denote by $\mathbb{F} = \{\mathscr{F}(t),\ 0 \leq t \leq T\}$ the P-completed filtration generated by the random measure $v(dt, dz)$. We assume that $\mathscr{F}_0$ is trivial and $\mathscr{F}_T = \mathscr{F}$.

The random measure $v(dt, dz)$ is assumed to have a predictable intensity kernel $m_t(dz)$ with $\int_D m_t(dz) < \infty$ which means that there is a finite number of jumps during any finite time interval. By definition of the intensity kernel $m_t(dz)$, the compensated jump process

$$\tilde{v}(dt, dz) = v(dt, dz) - m_t(dz)\, dt$$

is such that $\{\tilde{v}([0, t] \times B),\ 0 \leq t \leq T\}$ is a $(P, \mathbb{F})$ martingale for all $t \in [0, T]$ and $B \in \mathscr{D}$. We shall assume throughout the paper that the following nondegeneracy condition holds:

$$\int_D m_t(dz) \geq \eta \quad \text{for all } t \in [0, T) \quad P\text{-a.s.}$$

for some $\eta > 0$ and that $E[\int_0^T \int_D m_t(dz)]$.

3.2. *The wealth process.* In this paper, we consider the problem of optimal insurance demand of an agent whose wealth process is subject to (negative jumps) produced by the random measure $v(dt, dz)$. We first describe the agent wealth process in the absence of insurance. Let $\{r(t), 0 \leq t \leq T\}$ be an $\mathbb{F}$-predictable nonnegative bounded process. Here, $r(t)$ is the instantaneous interest rate at time t. The wealth process evolves according to

$$X^{x,0}(0) = x,$$

$$dX^{x,0}(t) = X^{x,0}(t-)\Big[r(t)dt - \int_D \gamma_t(z) v(dt, dz)\Big],$$

where $\{\gamma_t(z)\}$ is a predictable D-marked process satisfying

$$\eta \leq \gamma_t(z) < 1 \quad \text{for all } (t, z) \in [0, T) \times D, \quad P\text{-a.s.}$$

for some $\eta > 0$. The condition $\gamma_t(z) < 1$ guarantees positivity of the wealth process $X^{x,0}$ for any initial data $x > 0$.

Now, suppose that the agent has the possibility of "lowering" the shocks on his wealth by buying an insurance.

An *insurance strategy* is an $\mathbb{F}$-predictable process $\{\beta(t),\ 0 \leq t \leq T\}$ valued in $[0, 1]$. At each time t, $\beta(t)$ is the rate of insurance decided by the agent; that is, if the agent is subject to some accident which costs an amount of money C, then the insurance company reimburses the amount $\beta(t)C$ so that the shock is reduced from C to $(1-\beta(t))C$. We shall denote by $\mathscr{B}$ the set of all insurance strategies.

We denote by π the insurance premium rate per unit of insured capital. We assume that

$$\pi\colon [0,1] \to \mathbb{R}_+ \text{ is strictly increasing, convex and } \pi(1) < \infty.$$

Given an insurance strategy β, the wealth process evolves according to

$$X^{x,\beta}(0) = x,$$
$$dX^{x,\beta}(t) = X^{x,\beta}(t-)\Big\{[r(t) - \pi(\beta(t))]\,dt - [1-\beta(t)]\int_D \gamma_t(z)v(dt,dz)\Big\}.$$

We shall denote by $\tilde{\pi}$ the Legendre–Fenchel transform of the function $\pi(x+1)$ (after extending the convex function π to the whole real line by assigning the value $+\infty$ outside the interval $[0,1]$) defined by

$$\begin{aligned}\tilde{\pi}(y) &= \sup_{x\in\mathbb{R}}\{xy - \pi(x+1)\} = \sup_{-1\le x\le 0}\{xy - \pi(x+1)\}\\ &= \sup_{0\le x\le 1}\{-(1-x)y - \pi(x)\}; \qquad y \in \mathbb{R}_+.\end{aligned}$$

Then it is easily checked that

$$\tilde{\pi} \text{ is convex, nonincreasing on } \mathbb{R}_+ \text{ and } \tilde{\pi}(\cdot) \ge -\pi(1). \tag{3.1}$$

3.3. *Utility functions.* A function $U\colon (0,\infty) \to \mathbb{R}$ will be called a utility function if it is increasing, strictly concave, of class C^1, and satisfies

$$U(\infty) := \lim_{x\nearrow\infty} U(x) = +\infty, \tag{3.2}$$

$$U'(0+) := \lim_{x\searrow 0} U'(x) = \infty \quad\text{and}\quad U'(\infty) := \lim_{x\nearrow\infty} U'(x) = 0. \tag{3.3}$$

We shall denote by I the (continuous strictly decreasing) inverse of the function U'. It is easily checked that I maps $(0,\infty)$ onto itself and satisfies $I(0+) = \infty$ and $I(\infty) = 0$. We also introduce the Legendre–Fenchel transform of $-U(-x)$ defined by

$$\tilde{U}(y) := \sup_{x>0}\{U(x) - xy\} = U(I(y)) - yI(y); \qquad 0 < y < \infty.$$

Then $\tilde{U}$ is strictly convex, decreasing with

$$\tilde{U}'(y) = -I(y); \qquad 0 < y < \infty$$

and satisfies

$$\tilde{U}(0+) = U(\infty) = +\infty \quad\text{and}\quad \tilde{U}(\infty) = U(0+). \tag{3.4}$$

We shall also assume the condition

$$y \mapsto \tilde{U}(e^y) \text{ is convex on } \mathbb{R}. \tag{3.5}$$

REMARK 3.1. As observed in Karatzas, Lehoczky, Shreve and Xu (1991), conditions (3.2) and (3.5) are implied by the condition $x \mapsto U'(x)$ is nondecreasing on $(0, \infty)$, which is equivalent to $y \mapsto yI(y)$ is nonincreasing on $(0, \infty)$. The latter (stronger) condition is satisfied by power utility functions as well as logarithmic ones.

Finally, we need the condition

$$\exists\, \alpha \in (0,1) \text{ and } \gamma > 1 \text{ such that } U'(\gamma x) \le \alpha U'(x) \text{ for all } x > 0, \tag{3.6}$$

which is equivalent to

$$\exists\, \alpha \in (0,1) \text{ and } \gamma > 1 \text{ such that } I(\alpha y) \le \gamma I(y) \text{ for all } y > 0. \tag{3.7}$$

By iterating (3.7), we obtain the apparently stronger statement

$$\forall\, \alpha \in (0,1),\ \exists\, \gamma \in (1,\infty) \text{ such that } I(\alpha y) \le \gamma I(y) \text{ for all } y > 0. \tag{3.8}$$

Condition (3.6) will be used in order to connect the solution of the dual problem of Section 6 to some attainable terminal wealth; see Lemma 6.3.

3.4. *The insurance demand problem.* In this paper, we consider the problem of optimal insurance demand of the agent faced to shocks on his wealth produced by the random measure $v(dt, dz)$. For all $\beta \in \mathscr{B}$, we introduce

$$J(x, \beta) := E\Big[U\big(X^{x,\beta}(T)\big)\Big].$$

REMARK 3.2. Function $J(x, \beta)$ is well defined for all $x > 0$ and $\beta \in \mathscr{B}$ and takes values in $\mathbb{R} \cup \{-\infty\}$. Indeed, we clearly have $0 \le X^{x,\beta}(T) \le x \exp\big(\int_0^T r(u)\, du\big)$; then since U is increasing, we have $E\big[U\big(X^{x,\beta}(T)\big)^+\big] < \infty$.

The agent optimal insurance demand problem is then to maximize the expected terminal wealth utility over all admissible insurance strategies, that is,

$$V(x) := \sup_{\beta \in \mathscr{B}} J(x, \beta).$$

From Remark 3.2, we see that $V(x) < \infty$.

4. Auxiliary local martingales. In order to solve the optimization problem $V(x)$ introduced in the previous section, we introduce a set of exponential local martingales which preserve a supermartingale property for the wealth process. We shall denote by $\mathscr{H}$ the set of all predictable D-marked processes θ satisfying

$$\int_0^T \int_D \big(|\theta_t(Z)| + \exp(\theta_t(z))\big) m_t(dz)\, dt < \infty, \quad P\text{-a.s.}$$

For each $\theta \in \mathscr{H}$, we define the Doléans–Dade exponential

$$
\begin{aligned}
Z^{\theta}(t) &:= \mathscr{E}\Big(\int_0^t \int_D \big(\exp(\theta_s(z)) - 1\big)\tilde{v}(ds, dz)\Big) \\
&= \exp\Big(\int_0^t \int_D \theta_s v(ds, dz) - \int_0^t \int_D \big(\exp(\theta_s(z)) - 1\big) m_s(dz)\, ds\Big),
\end{aligned}
$$

where the last equality follows from the exponential formula; see, for example, Brémaud (1981). In the following, we shall use the notations

$$
\begin{aligned}
\nu^{\theta}(t) &:= \int_D \gamma_t(z) \exp(\theta_t(z)) m_t(dz), \\
H^{\theta}(t) &:= Z^{\theta}(t) \exp - \int_0^t \Big[r(s) + \tilde{\pi}\big(\nu^{\theta}(s)\big)\Big]\, ds.
\end{aligned}
$$

We then have the following result.

PROPOSITION 4.1. *Let $x > 0$ and $\beta \in \mathscr{B}$ be some initial wealth and insurance strategy. Then for any $\theta \in \mathscr{H}$, the process $\{H^{\theta}(t) X^{x,\beta}(t),\ 0 \le t \le T\}$ is a P-supermartingale.*

PROOF. By Itô's lemma [see, e.g., Jacod and Shiryaev (1987), page 57], it follows that

$$
\begin{aligned}
d\big[H^{\theta}(t) X^{x,\beta}(t)\big] &= H^{\theta}(t-)\, dX^{x,\beta}(t) + X^{x,\beta}(t-)\, dH^{\theta}(t) + \Delta H^{\theta}(t) \Delta X^{x,\beta}(t) \\
&= H^{\theta}(t-)\, dX^{x,\beta}(t) + X^{x,\beta}(t-)\, dH^{\theta}(t) \\
&\quad + (1 - \beta(t)) \int_D \gamma_t(z) \big(\exp(\theta_t(z)) - 1\big) v(dt, dz) \\
&= H^{\theta}(t-) X^{x,\beta}(t-) \Big\{ -[\tilde{\pi}\big(\nu^{\theta}(t)\big) + \pi(\beta(t)) + (1 - \beta(t)) \nu^{\theta}(t)]\, dt \\
&\quad + \int_D \big(\exp(\theta_t(z)) - 1 + [1 - \beta(t)] \gamma_t(z) \exp(\theta_t(z))\big) \tilde{v}(dt, dz) \Big\}.
\end{aligned}
$$

Notice that, by definition of $\tilde{\pi}$, we have $\tilde{\pi}(y) + \pi(x) + (1 - x)y \ge 0$ for all $x \in [0, 1]$. Then process $H^{\theta} X^{x,\beta}$ is a local supermartingale. Since $H^{\theta} X^{x,\beta}$ is nonnegative, the required result follows from Fatou's lemma. $\square$

5. Attainable terminal wealth. In this section, we provide a characterization of the set of attainable terminal wealth processes which is related to the problem of maximizing expected utility from terminal wealth; see Karatzas, Lehoczky, Shreve and Xu (1991).

Let B be any $\mathscr{F}$-measurable nonnegative random variable with

$$
0 < \sup_{\theta \in \mathscr{H}} E\big[H^{\theta}(T) B\big] < \infty.
$$

Our interest is in the problem of minimal initial wealth in order to achieve a terminal wealth which dominates the random variable B, that is,

$$Q(0) := \inf\{x > 0\colon \exists\ \beta \in \mathscr{B},\ X^{x,\beta}(T) \geq B \text{ a.s.}\}.$$

The random variable B is said to be attainable if there exists some initial wealth $x > 0$ and some admissible insurance strategy β such that $X^{x,\beta}(T) = B$ a.s. The basic result of this paragraph for the rest of the paper consists in a dual characterization of the set of all such attainable random variables. Similar characterizations have been obtained in the theory of portfolio optimization by El Karoui and Quenez (1995), in the context of incomplete markets, and Cvitanič and Karatzas (1992) in the context of portfolio constraints.

Let us introduce the subset of $\mathscr{H}$,

$$\mathscr{H}_0 = \{\theta \in \mathscr{H}\colon E[Z^\theta(T)] = 1\}.$$

For all $\theta \in \mathscr{H}_0$, we can define a probability measure P^θ equivalent to P by

$$P^\theta(A) = E[Z^\theta(T)1_A]; \qquad A \in \mathscr{F}.$$

LEMMA 5.1. *Let Y be a nonnegative process. Suppose that the process*

$$\left\{Y(t)\exp\left(-\int_0^t \tilde{\pi}(\nu^\theta(u))\,du\right),\ 0 \leq t \leq T\right\}$$

is a P^θ-supermartingale for all $\theta \in \mathscr{H}_0$. Then there exists an insurance strategy $\beta \in \mathscr{B}$ and an optional nondecreasing process C, with $C(0) = 0$, such that

$$Y(t) \leq X^{Y(0),\beta}(t)\exp\left(-\int_0^t r(u)\,du\right), \qquad 0 \leq t \leq T,\ P\text{-a.s.}$$

PROOF. The proof is a consequence of Theorem 3.1 (and its corollary) in Föllmer and Kramkov (1997). We first introduce some notations. We denote by $\mathscr{C}$ the set of all nondecreasing cad-lag predictable processes with $C(0) = 0$. Given $\beta \in \mathscr{B}$ and $C \in \mathscr{C}$, we introduce the process $S^{\beta,C}$ defined by

$$S^{\beta,C}(t) = -\int_0^t (\pi(\beta(u)) - \pi(1))\,du - C(t) - \int_0^t (1 - \beta(u)) \int_D \gamma_u(z) v(du, dz).$$

We denote $\mathscr{S} := \{S^{\beta,C}\colon (\beta, C) \in \mathscr{B} \times \mathscr{C}\}$. Observe that $S^{0,0} = 0 \in \mathscr{S}$ and that, by convexity of $\pi(\cdot)$, $\mathscr{S}$ is predictably convex, that is, for $S_1, S_2 \in \mathscr{S}$ and for any predictable process h with $0 \leq h \leq 1$, the process $\int_0^\cdot h(u-)\,dS_1(u) + \int_0^\cdot (1 - h(u-))\,dS_2(u)$ is in $\mathscr{S}$. Next, we introduce the set $\mathscr{P}(\mathscr{S})$ defined by

$$\mathscr{P}(\mathscr{S}) = \{P' \sim P\colon \exists\ A \in \mathscr{C}, \forall S \in \mathscr{S}, S - A \text{ is a } P'\text{-local supermartingale}\}$$

and we denote by $A^{\mathscr{S}}_{P'}$ the upper variation process of $\mathscr{S}$ under P' as defined in Föllmer and Kramkov (1997). Clearly, we have $\mathscr{P}(\mathscr{S}) = \{P^\theta\colon \theta \in \mathscr{H}_0\}$ and

$$A^{\mathscr{S}}_{P^\theta}(t) = \int_0^t [\pi(1) + \tilde{\pi}(\nu^\theta(u))]\,du, \qquad 0 \leq t \leq T$$

for all $\theta \in \mathscr{H}_0$; recall that $\tilde{\pi}(y) \geq \pi(1)$ for all $y > 0$. We now claim that

$$\text{If } (S_n)_n \subset \mathscr{S}, \text{ uniformly bounded from below and } S_n \to S \text{ in the semimartingale topology, then we have } S \in \mathscr{S}; \tag{5.1}$$

this is Assumption 3.1 in Föllmer and Kramkov (1997). Before proving this, let us show how to complete the proof. Let $\tilde{Y}(t) := Y(t)\exp(\pi(1)t)$; $0 \leq t \leq T$. Then, from the condition of the lemma on the process Y, we deduce that the process $\{\tilde{Y}(t)\exp(-A^{\mathscr{S}}_{P^\theta}(t)), 0 \leq t \leq T\}$ is a nonnegative P^θ-supermartingale for all $\theta \in \mathscr{H}_0$. Then, from Corollary 3.1 in Föllmer and Kramkov (1997), we have $\tilde{Y} = Y(0)\mathscr{E}(S-C)$ for some optional nondecreasing process C and $S \in \mathscr{S}$, which provides the required result by stochastic composition.

It remains to prove (5.1). Let (β_n, C_n) be corresponding to S_n, that is, $S_n = S^{\beta_n, C_n}$. From Theorem II.3 in Mémin (1980), there is a subsequence [also denoted $(S_n)_n$] and a probability measure $Q \sim P$ with bounded density dQ/dP, such that $(S_n)_n$ is a Cauchy sequence in $\mathscr{M}^2(Q) \oplus \mathscr{A}(Q)$ where $\mathscr{M}^2(Q)$ is the Banach space of Q-square integrable martingales and $\mathscr{A}(Q)$ is the Banach space of predictable processes with finite Q-integrable variation. Then $S_n \to S = M + A$ in $\mathscr{M}^2(Q) \oplus \mathscr{A}(Q)$. Using Corollary III.4 in Mémin (1980), this proves that the limit process S is a semimartingale in the form

$$S(t) = -A(t) - \int_0^t (1-\beta(u)) \int_D \gamma_t(z) v(dt, dz), \qquad 0 \leq t \leq T,$$

where $\beta \in \mathscr{B}$ and A is a predictable process with finite variation. From the convergence of $(S_n)_n$ to S in $\mathscr{M}^2(Q) \oplus A(Q)$, it also follows that the local martingale part of (S_n) converges to the local martingale part of (S) in $\mathscr{M}^2(Q)$, and therefore

$$E^Q\left(\int_0^T |\beta_n(t) - \beta(t)| \int_D \gamma_t(z) m_t(dz)\, dt\right) \to 0.$$

By the nondegeneracy assumption on $\gamma_t(z)$ and $m_t(dz)$, it follows that $\beta_n \to \beta\, l \otimes P$-a.s. possibly after passing to a subsequence, where l is the Lebesgue measure on $[0, T]$. Since $0 \leq \pi(\beta_n(t)) \leq \pi(1)$, we can conclude from the dominated convergence theorem that

$$E^Q\left|\int_0^T \left[\pi(\beta_n(t)) - \pi(\beta(t))\right] dt\right| \to 0.$$

This implies that

$$C_n(t) \to C(t) := A(t) - \int_0^t \left[\pi(\beta(u)) - \pi(1)\right] du, \quad P\text{-a.s.}$$

along some subsequence. Clearly, the predictable process C inherits the increase of the processes C_n and therefore

$$S(t) = -\int_0^t \left[\pi(\beta(u)) - \pi(1)\right] du - C(t) - \int_0^t (1-\beta(u)) \int_D \gamma_t(z) v(dt, dz),$$

with $(\beta, C) \in \mathscr{B} \times \mathscr{C}$, proving that the limit $S \in \mathscr{S}$. $\square$

Next, for all stopping time τ valued in $[0, T]$, we define

$$\tilde{Q}(\tau) := \operatorname*{ess\,sup}_{\theta \in \mathscr{H}} \tilde{I}(\tau, \theta) \quad \text{with } \tilde{I}(\tau, \theta) := E\left[\frac{H^\theta(T)}{H^\theta(\tau)} B \,\middle|\, \mathscr{F}(\tau) \right].$$

In order to connect the "dual" problem $\tilde{Q}$ to Q, we need to establish a dynamic programming result. This is obtained by the general method developed by Neveu (1975) in discrete time and adapted to continuous time by El Karoui and Quenez (1995) in the context of portfolio optimization in incomplete markets.

LEMMA 5.2. *For all stopping time τ valued in $[0, T]$, and for all $\theta^1, \theta^2 \in \mathscr{H}$, there exists $\hat{\theta} \in \mathscr{H}$ such that $\tilde{I}(\tau, \hat{\theta}) = \max\{\tilde{I}(\tau, \theta^1), \tilde{I}(\tau, \theta^2)\}$.*

PROOF. Take two arbitrary elements θ and θ' in $\mathscr{H}$ and define

$$A := \{\omega \in \Omega: \tilde{I}(\tau, \theta) \geq \tilde{I}(\tau, \theta')\};$$

observe that A is $\mathscr{F}(\tau)$ measurable. Next define $\hat{\theta} := \theta 1_A + \theta' 1_{A^c}$. Then,

$$\begin{aligned}
\tilde{I}(\tau, \hat{\theta}) &= E\left[\frac{H^{\hat{\theta}}(T)}{H^{\hat{\theta}}(\tau)} B 1_A \,\middle|\, \mathscr{F}(\tau) \right] + E\left[\frac{H^{\hat{\theta}}(T)}{H^{\hat{\theta}}(\tau)} B 1_{A^c} \,\middle|\, \mathscr{F}(\tau) \right] \\
&= E\left[\frac{H^\theta(T)}{H^\theta(\tau)} B \,\middle|\, \mathscr{F}(\tau) \right] 1_A + E\left[\frac{H^{\theta'}(T)}{H^{\theta'}(\tau)} B \,\middle|\, \mathscr{F}(\tau) \right] 1_{A^c} \\
&= \max\{\tilde{I}(\tau, \theta), \tilde{I}(\tau, \theta')\}.
\end{aligned}$$

□

LEMMA 5.3 (Dynamic programming). *Let τ and ζ be two $[0, T]$-valued stopping times with $\zeta \geq \tau$ a.s. Then, we have*

$$\tilde{Q}(\tau) = \operatorname*{ess\,sup}_{\theta \in \mathscr{H}} E\left[\frac{H^\theta(\zeta)}{H^\theta(\tau)} \tilde{Q}(\zeta) \,\middle|\, \mathscr{F}(\tau) \right].$$

PROOF. By simple conditioning we have

$$\tilde{Q}(\tau) = \operatorname*{ess\,sup}_{\theta \in \mathscr{H}} E\left[\frac{H^\theta(\zeta)}{H^\theta(\tau)} \tilde{I}(\zeta, \theta) \,\middle|\, \mathscr{F}(\tau) \right].$$

Notice that $\tilde{I}_\theta(\zeta)$ depends on θ only through its realizations in the stochastic interval $[\zeta, T]$, and the first term inside the expectation (on the right-hand side of the last equation) depends on θ only through its realizations in the stochastic interval $[\tau, \zeta]$. Then we have

$$\tilde{Q}(\tau) \leq \operatorname*{ess\,sup}_{\theta \in \mathscr{H}} E\left[\frac{H^\theta(\zeta)}{H^\theta(\tau)} \tilde{Q}(\zeta) \,\middle|\, \mathscr{F}(\tau) \right].$$

We now prove the opposite inequality. Let θ be an arbitrary element in $\mathscr{H}$ and denote by $\mathscr{H}_\theta(\tau, \zeta) := \{\mu \in \mathscr{H}: \mu = \theta \text{ on } [\tau, \zeta]\}$. From Lemma 5.2, the family $\{\tilde{I}(\zeta, \theta), \theta \in \mathscr{H}\}$ is directed upwards. Then there exists a sequence $(\theta^n)_{n\geq 0} \subset \mathscr{H}$ such that

$$\tilde{Q}(\zeta) = \lim_{n\to\infty} \uparrow \tilde{I}(\zeta, \theta^n) \quad \text{a.s.}$$

See Neveu [(1975), page 121]. Moreover, since $\tilde{I}(\zeta, \theta^n)$ depends on θ^n only through its realizations on the stochastic interval $[\zeta, T]$, we may chose $(\theta^n)_{n\geq 0} \subset \mathscr{H}_\theta(\tau, \zeta)$. Now, for all $n \geq 0$, we have

$$\tilde{Q}(\tau) \geq E\left[\frac{H^{\theta^n}(\zeta)}{H^{\theta^n}(\tau)}\tilde{I}(\zeta, \theta^n)\middle|\mathscr{F}(\tau)\right] = E\left[\frac{H^{\theta}(\zeta)}{H^{\theta}(\tau)}\tilde{I}(\zeta, \theta^n)\middle|\mathscr{F}(\tau)\right],$$

and therefore

$$\tilde{Q}(\tau) \geq \lim_{n\to\infty} \uparrow E\left[\frac{H^{\theta}(\zeta)}{H^{\theta}(\tau)}\tilde{I}(\zeta, \theta^n)\middle|\mathscr{F}(\tau)\right] = E\left[\frac{H^{\theta}(\zeta)}{H^{\theta}(\tau)}\tilde{Q}(\zeta)\middle|\mathscr{F}(\tau)\right]$$

by monotone convergence. □

We now can state the main result of this section.

THEOREM 5.1. *We have*

$$Q(0) = \tilde{Q}(0) = \sup_{\theta\in\mathscr{H}} E\big[H^\theta(T)B\big].$$

Moreover, for all $\theta \in \mathscr{H}$, the following statements are equivalent:

(i) *θ achieves the supremum in $\tilde{Q}(0) = \sup_{\theta\in\mathscr{H}} E[H^\theta(T)B]$.*

(ii) *B is attainable (by some $\beta \in \mathscr{B}$) and the corresponding process $\{H^\theta(t)X^{Q(0),\beta}(t), 0 \leq t \leq T\}$ is a P-martingale.*

PROOF. *Step* 1. We first prove that $Q(0) \geq \tilde{Q}(0)$. Let $x > 0$ be an arbitrary initial wealth such that there exists an insurance strategy $\beta \in \mathscr{B}$ satisfying $X^{x,\beta}(T) \geq B$ a.s. Then, for all $\theta \in \mathscr{H}$, we have

$$H^\theta(T)X^{x,\beta}(T) \geq H^\theta(T)B \quad \text{a.s.}$$

Taking expectation on both sides of the last inequality and using Proposition 4.1 provides

$$x \geq E\big[H^\theta(T)B\big],$$

which provides the required inequality.

Step 2. We now prove the opposite inequality $Q(0) \leq \tilde{Q}(0)$. From the dynamic programming equation of Lemma 5.3, it follows that the process

$$\left\{Z^\theta(t)\tilde{Q}(t)\exp\left(-\int_0^t (r(u) + \tilde{\pi}(\nu^\theta(u)))\,du\right), 0 \leq t \leq T\right\}$$

is a P-supermartingale for each $\theta \in \mathscr{H}$. Moreover, as in El Karoui and Quenez (1995) and Cvitanič and Karatzas (1992), it is possible to show that there exists a right continuous with left limits supermartingale [still denoted $\tilde{Q}(t)$] which coincides a.s. with $\tilde{Q}$. It then follows that, for all $\theta \in \mathscr{H}_0$, the process

$$\left\{\tilde{Q}(t)\exp\left(-\int_0^t (r(u)+\tilde{\pi}(\nu^\theta(u)))\,du\right), 0 \le t \le T\right\}$$

is a P^θ-supermartingale for all $\theta \in \mathscr{H}_0$. By Lemma 5.1, we deduce that there exists an insurance strategy $\beta \in \mathscr{B}$, such that

$$\tilde{Q}(t) \le X^{\tilde{Q}(0),\beta}(t), \qquad 0 \le t \le T, \ P\text{-a.s.} \tag{5.2}$$

Since $\tilde{Q}(T) = B$, it follows from the definition of $Q(0)$ that $\tilde{Q}(0) \ge Q(0)$.

Step 3. It remains to prove the equivalence between statements (i) and (ii) of the theorem. (ii) $\Rightarrow$ (i) is trivial. Now, suppose that there exists some $\hat{\theta} \in \mathscr{H}$ such that

$$\tilde{Q}(0) = E\big[H^{\hat{\theta}}(T)B\big].$$

Then since the process $H^{\hat{\theta}}\tilde{Q}$ is a P-supermartingale,

$$E\big[H^{\hat{\theta}}(t)\tilde{Q}(t)\big] \ge E\big[E\big(H^{\hat{\theta}}(T)\tilde{Q}(T)|\mathscr{F}(t)\big)\big] = E\big[H^{\hat{\theta}}(T)B\big] = \tilde{Q}(0)$$

and

$$E\big[H^{\hat{\theta}}(t)\tilde{Q}(t)\big] \le H^{\hat{\theta}}(0)\tilde{Q}(0) = \tilde{Q}(0).$$

Hence, $H^{\hat{\theta}}\tilde{Q}$ is a P-supermartingale with constant expectation and therefore a P-martingale. It follows that the increasing process C appearing in the proof of Lemma 5.1 must be zero. This proves that $X^{Q(0),\beta}(T) = B$ P-a.s. $\square$

6. Dual optimization problem. We now introduce the following optimization problem:

$$\tilde{V}(y) = \inf_{\theta\in\mathscr{H}} \tilde{J}(y,\theta),$$
$$\tilde{J}(y,\theta) = E\big[\tilde{U}(yH^\theta(T))\big]$$

and

$$H^\theta(T) = Z^\theta(T)\exp\left(-\int_0^T (r(u)+\tilde{\pi}(\nu^\theta(u)))\,du\right).$$

The following result shows (in particular) that function $\tilde{J}$ is well defined and takes values in the extended real line $\mathbb{R} \cup \{+\infty\}$.

LEMMA 6.1. *For all $y > 0$, the family $\{\tilde{U}(yH^\theta(T))^-, \ \theta \in \mathscr{H}\}$ is uniformly integrable.*

PROOF. If $U(0+) \ge 0$ then the result is trivial. Then we concentrate on the case $U(0+) < 0$. The following argument is reported from Kramkov and

Schachermayer (1997). Let g: $\big(-\tilde{U}(0), -\tilde{U}(\infty)\big) \to (0, \infty)$ denote the inverse of $-\tilde{U}$. Notice that $0 \in \big(-\tilde{U}(0), -\tilde{U}(\infty)\big) = \big(-U(\infty), -U(0+)\big)$; see (3.4). The function g is strictly increasing and

$$\begin{aligned} E\big[g\big(\tilde{U}(yH^\theta(T))^-\big)\big] &\le E\big[g\big(-\tilde{U}(yH^\theta(T))\big)\big] + g(0) \\ &\le Cy + g(0), \end{aligned}$$

where we used the fact that $E[Z^\theta(T)] \le 1$ and C is a lower bound of $\exp\big(-\int_0^T (r(u) + \tilde{\pi}(\nu^\theta(u)))\big)$; recall that the process r is bounded and $\tilde{\pi}(\cdot) \ge -\pi(1)$. Now, from l'Hôpital's rule, we have

$$\lim_{x \to -\tilde{U}(\infty)} \frac{g(x)}{x} = \lim_{y \to \infty} \frac{y}{-\tilde{U}(y)} = \lim_{y \to \infty} \frac{1}{I(y)} = +\infty;$$

the required result then follows from the de la Valée–Poussin theorem; see Dellacherie and Meyer (1975). □

In order to relate the optimization problems V and $\tilde{V}$, we need the following condition.

ASSUMPTION 6.1. There exists some $\hat{y} > 0$ such that $\tilde{V}(\hat{y}) < \infty$.

REMARK 6.1. Assumption 6.1 is equivalent to $\tilde{V}(y) < \infty$ for all $y > 0$. Indeed, for $y \ge \hat{y}$, this follows from the decrease of $\tilde{U}$. Next consider some $\alpha \in (0, 1)$. As observed in Kramkov and Schachermayer (1997), condition (3.6) implies that there exists some $y_0 > 0$ and some $C < \infty$ such that $\tilde{U}(\alpha z) < C\tilde{U}(z)$ for all $z < y_0$. Then, it follows from the decrease of $\tilde{U}$ that, for all $\theta \in \mathscr{H}$,

$$\tilde{U}(\alpha \hat{y} H^\theta(T)) \le C\tilde{U}(\hat{y}H^\theta(T))1_{\{\hat{y}H^\theta(T) \le y_0\}} + \big|\tilde{U}(\alpha y_0)\big|$$

and therefore $\tilde{V}(\alpha\hat{y}) < \infty$ whenever $\tilde{V}(\hat{y}) < \infty$.

LEMMA 6.2. *Function $\tilde{V}$ is convex and satisfies*

$$\tilde{V}(y) \ge \sup_{\xi > 0}\{V(\xi) - \xi y\} \quad \textit{for all } y > 0.$$

PROOF. (i) We first prove the convexity of $\tilde{V}$. Fix $\lambda \in (0, 1)$ and $y, y' > 0$. Let θ and θ' be two arbitrary elements in $\mathscr{H}$ and define

$$\begin{aligned} G(t) &:= \lambda y H^\theta(t) + (1-\lambda) y' H^{\theta'}(t), \\ \exp(\hat{\theta}_t(z)) &:= \lambda y \exp(\theta_t(z)) \frac{H^\theta(t-)}{G(t-)} + (1-\lambda) y' \exp(\theta'_t(z)) \frac{H^{\theta'}(t-)}{G(t-)}, \\ \mu(t) &:= \lambda y \tilde{\pi}(\nu^\theta(t)) \frac{H^\theta(t-)}{G(t-)} + (1-\lambda) y' \tilde{\pi}(\nu^{\theta'}(t)) \frac{H^{\theta'}(t-)}{G^\varepsilon(t-)}. \end{aligned}$$

By Itô's lemma, we get

$$dG(t) = G(t-)\left[\int_D (\exp(\hat{\theta}_t(z)) - 1)\tilde{v}(dt, dz) - \mu(t)dt\right].$$

Notice that, from the convexity of $\tilde{\pi}$, we have

$$\mu(t) \geq \tilde{\pi}\left(\lambda y \nu^{\theta}(t)\frac{H^{\theta}(t-)}{G(t-)} + (1-\lambda)y'\nu^{\theta'}(t)\frac{H^{\theta'}(t-)}{G(t-)}\right) = \tilde{\pi}\big(\nu^{\hat{\theta}}(t)\big).$$

Since $dH^{\hat{\theta}}(t) = H^{\hat{\theta}}(t-)\big[\int_D(\exp(\hat{\theta}_t(z)) - 1)\,\tilde{v}(dt, dz) - \tilde{\pi}(\nu^{\theta'}(t))\,dt\big]$, it follows from the comparison theorem [see, e.g., Protter (1990), Theorem 54, page 268] that

$$G(t) \leq (\lambda y + (1-\lambda)y')H^{\hat{\theta}}(t), \qquad 0 \leq t \leq T \text{ a.s.}$$

Therefore, by convexity and decrease of $\tilde{U}$, we see that

$$\tilde{U}\Big((\lambda y + (1-\lambda)y')H^{\hat{\theta}}(T)\Big) \leq \lambda\tilde{U}\big(yH^{\theta}(T)\big) + (1-\lambda)\tilde{U}\big(y'H^{\theta'}(T)\big);$$

then the required result follows from the arbitrariness of θ and θ' in $\mathscr{H}$.

(ii) We now prove the last claim of the lemma. Consider some ξ, $y > 0$, $\beta \in \mathscr{B}$ and $\theta \in \mathscr{H}$. Then, by the definition of $\tilde{U}$, we have

$$\tilde{U}(yH^{\theta}(T)) \geq U(X^{\xi,\beta}(T)) - yH^{\theta}(T)X^{\xi,\beta}(T).$$

From the supermartingale property established in Proposition 4.1, we have $E[H^{\theta}(T)X^{\xi,\beta}(T)] \leq \xi$ and therefore

$$E\big[\tilde{U}(yH^{\theta}(T))\big] \geq E\big[U(X^{\xi,\beta}(T))\big] - y\xi;$$

the required result is obtained by taking supremum over $\beta \in \mathscr{B}$ and $\xi > 0$ on the right-hand side and infinimum over $\theta \in \mathscr{H}$ on the left-hand side of the last inequality. □

We leave the discussion of the existence problem in $\tilde{V}(y)$ for the next section and we simply assume it in the present section.

ASSUMPTION 6.2. For all $y > 0$, there exists $\theta(y) \in \mathscr{H}$ such that

$$\tilde{V}(y) = \tilde{J}(y, \theta(y)).$$

In the next section, we shall provide conditions under which existence in $\tilde{V}(y)$ indeed holds.

REMARK 6.2. Under Assumption 6.1 and 6.2, we have that

$$E\big[H^{\theta(y)}(T)I(yH^{\theta(y)}(T))\big] < \infty \quad \text{for all } y > 0.$$

To see this, we use the fact that condition (3.6) implies that there exists some $y_0 > 0$ and some constant $C < \infty$ such that $zI(z) < C\tilde{U}(z)$ for all $z \in (0, y_0)$;

see Kramkov and Schachermayer (1997). Then, by the decrease of I, it is easily checked that

$$H^{\theta(y)}(T)I\big(yH^{\theta(y)}(T)\big) \le H^{\theta(y)}(T)I(y_0) + \frac{C}{y}\tilde{U}\big(yH^{\theta(y)}(T)\big)1_{\{yH^{\theta(y)}(T)<y_0\}},$$

which proves the announced claim.

LEMMA 6.3. *Let Assumptions* 6.1 *and* 6.2 *hold. Then,*

$$E\big[H^{\theta(y)}(T)I(yH^{\theta(y)}(T))\big] = \sup_{\theta\in\mathscr{H}} E\big[H^{\theta}(T)I(yH^{\theta(y)}(T))\big].$$

PROOF. In order to establish the above result, we use a variations calculus argument to obtain a characterization of the optimality of $\theta(y)$ for the dual problem $\tilde{V}(y)$. For ease of notation, we set $\hat{\theta} = \theta(y)$. Fix some $\varepsilon > 0$ and $\theta \in \mathscr{H}$ and define

$$\begin{aligned} G^{\varepsilon}(t) &:= (1-\varepsilon)H^{\hat{\theta}}(t) + \varepsilon H^{\theta}(t), \\ \exp(\theta_t^{\varepsilon}(z)) &:= (1-\varepsilon)\exp(\hat{\theta}_t(z))\frac{H^{\hat{\theta}}(t-)}{G^{\varepsilon}(t-)} + \varepsilon\exp(\theta_t(z))\frac{H^{\theta}(t-)}{G^{\varepsilon}(t-)}, \\ \mu^{\varepsilon}(t) &:= (1-\varepsilon)\tilde{\pi}(\nu^{\hat{\theta}}(t))\frac{H^{\hat{\theta}}(t-)}{G^{\varepsilon}(t-)} + \varepsilon\tilde{\pi}(\nu^{\theta}(t))\frac{H^{\theta}(t-)}{G^{\varepsilon}(t-)}. \end{aligned}$$

By the same argument as in the proof of Lemma 6.2, we see that

$$G^{\varepsilon}(t) \le H^{\theta^{\varepsilon}}(t), \qquad 0 \le t \le T \text{ a.s.}$$

Then, from the optimality of $\hat{\theta}$ and the decrease of $\tilde{U}$, this provides

$$E\big[\tilde{U}(yH^{\hat{\theta}}(T)) - \tilde{U}(yG^{\varepsilon}(T))\big] \le E\big[\tilde{U}(yH^{\hat{\theta}}(T)) - \tilde{U}(yH^{\theta^{\varepsilon}}(T))\big] \le 0;$$

hence

$$E\left[\frac{1}{\varepsilon}\big(\tilde{U}\big(yH^{\hat{\theta}}(T)\big) - \tilde{U}(yG^{\varepsilon}(T))\big)\right] \le 0. \tag{6.1}$$

Since $\tilde{U}$ is C^1 there exists a random variable F^{ε} between $yG^{\varepsilon}(T)$ and $yH^{\hat{\theta}}(T)$ such that $\tilde{U}\big(yH^{\hat{\theta}}(T)\big) - \tilde{U}(yG^{\varepsilon}(T)) = \tilde{U}'(F^{\varepsilon})y\big(H^{\hat{\theta}}(t) - G^{\varepsilon}(T)\big)$ and therefore

$$E\big[I(F^{\varepsilon})\big(H^{\theta}(T) - H^{\hat{\theta}}(T)\big)\big] \le 0 \quad \text{for all } \varepsilon > 0.$$

Notice that $F^{\varepsilon} \to yH^{\hat{\theta}}(T)$ P-a.s. Therefore, in order to obtain the required result it suffices to prove that

$$\begin{aligned} &\liminf_{\varepsilon\to 0} E\big[I(F^{\varepsilon})\big(H^{\theta}(T) - H^{\hat{\theta}}(T)\big)\big] \\ &\qquad \ge E\Big[\liminf_{\varepsilon\to 0} I(F^{\varepsilon})\big(H^{\theta}(T) - H^{\hat{\theta}}(T)\big)\Big]. \end{aligned} \tag{6.2}$$

To see this, write

$$\begin{aligned}E\big[I(F^\varepsilon)\big(H^\theta(T)-H^{\hat\theta}(T)\big)\big] &= E\big[I(F^\varepsilon)\big(H^\theta(T)-H^{\hat\theta}(T)\big)^+\big]\\ &\quad+E\big[I(F^\varepsilon)H^\theta(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}}\big]\\ &\quad-E\big[I(F^\varepsilon)H^{\hat\theta}(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}}\big].\end{aligned}$$

The first two terms on the right-hand side are handled by Fatou's lemma. As for the last term, observe that $F^\varepsilon \ge yG^\varepsilon(T) \ge y(1-\varepsilon)H^{\hat\theta}(T)$ on $\{H^{\hat\theta}(T)\ge H^\theta(T)\}$; by the decrease of I it then follows that

$$\begin{aligned}I(F^\varepsilon)H^{\hat\theta}(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}} &\le I\big(y(1-\varepsilon)H^{\hat\theta}(T)\big)H^{\hat\theta}(T)\\ &\le I\big(\alpha yH^{\hat\theta}(T)\big)H^{\hat\theta}(T)\end{aligned}$$

for all $\varepsilon\le 1-\alpha$, where α is an arbitrary value in (0, 1). Now, from (3.7), this provides

$$I(F^\varepsilon)H^{\hat\theta}(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}} \le \gamma I\big(yH^{\hat\theta}(T)\big)H^{\hat\theta}(T),$$

which is an integrable random variable; see Remark 6.2. Then by the dominated convergence theorem, we see that

$$\lim_{\varepsilon\to 0}E\big[I(F^\varepsilon)H^{\hat\theta}(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}}\big] = E\big[I\big(yH^{\hat\theta}(T)\big)H^{\hat\theta}(T)1_{\{H^{\hat\theta}(T)\ge H^\theta(T)\}}\big]$$

and (6.2) follows. □

COROLLARY 6.1. *Let Assumptions* 6.1 *and* 6.2 *hold*. *Then*, *for all* $y>0$, *the random variable* $I(yH^{\theta(y)}(T))$ *is attainable starting from the initial wealth*

$$x(y) := E\big[H^{\theta(y)}(T)I(yH^{\theta(y)}(T))\big],$$

that is, *there exists an insurance strategy* $\beta(y)\in\mathscr{B}$ *such that*

$$X^{x(y),\,\beta(y)}(T) = I\big(yH^{\theta(y)}(T)\big)\quad a.s.$$

Furthermore, $E[U(X^{x(y),\,\beta(y)}(T))] > -\infty$.

PROOF. The first part of the claim is a direct consequence of Lemma 6.3 and Theorem 5.1(ii). The second part follows from the fact that

$$U\big(X^{x(y),\beta(y)}(T)\big)=U\big(I\big(yH^{\theta(y)}(T)\big)\big)=\tilde U\big(yH^{\theta(y)}(T)\big)+yH^{\theta(y)}(T)I\big(yH^{\theta(y)}(T)\big);$$

both terms on the right-hand side are integrable by Assumptions 6.1, 6.2 and Remark 6.2. □

LEMMA 6.4. *Let Assumptions* 6.1 *and* 6.2 *hold*. *Then for all* $x>0$ *there exists some* $y(x)>0$ *such that*

$$\tilde V(y(x)) + xy(x) = \inf_{y>0}\tilde V(y)+xy. \tag{6.3}$$

Furthermore, $y(x)$ *satisfies*

$$x = E\Big[H^{\theta(y(x))}(T)I\Big(y(x)H^{\theta(y(x))}(T)\Big)\Big]. \tag{6.4}$$

PROOF. (i) The proof of (6.3) is the same as in Cvitanič and Karatzas (1992) and is reported here only for completeness. From Lemma 6.2, we have $\tilde{V}(y)+xy \geq V(x/2)+(x/2)y$ for all $x>0$, and therefore $\lim_{y\to\infty} \tilde{V}(y)+xy=+\infty$.

Moreover, since $\tilde{\pi}(\cdot) \geq -\pi(1)$ and the process $\{r(t),\ 0\leq t\leq T\}$ is bounded, we get with some positive constant C,

$$E\big[\tilde{U}\big(yH^{\theta}(T)\big)\big] \geq E\big[\tilde{U}\big(yCZ^{\theta}(T)\big)\big] \geq \tilde{U}\big(yCE[Z^{\theta}(T)]\big)$$

by the convexity of $\tilde{U}$. Now recall that Z^{θ} is a supermartingale (as a nonnegative local martingale) and therefore $E[Z^{\theta}(T)] \leq 1$. Then by the decrease of $\tilde{U}$, we get

$$\tilde{V}(y) \geq \tilde{U}(yC),$$

which proves that $\tilde{V}(0+)=+\infty$. Hence the function $f_x\colon y \mapsto \tilde{V}(y)+xy$ is convex (see Lemma 6.2) and tends to infinity as $y\searrow 0$ or $y\nearrow\infty$. This proves the existence of $y(x)>0$ achieving the minimum of f_x over $(0,\infty)$.

(ii) It remains to prove the last part of the lemma. Notice that

$$\begin{aligned}\inf_{\zeta>0}\big\{\zeta y(x)x+\tilde{J}(\zeta y(x),\theta(y(x)))\big\} &= \inf_{\zeta>0}\big\{\zeta x+\tilde{J}(\zeta,\theta(y(x)))\big\}\\ &\geq \inf_{\zeta>0}\big\{\zeta x+\tilde{V}(\zeta)\big\}\\ &= f_x(y(x)) = xy(x)+\tilde{V}(y(x)).\end{aligned}$$

Hence $\zeta=1$ attains infinimum of $G(\zeta) := \zeta y(x)x+\tilde{J}(\zeta y(x),\theta(y(x)))$. In the case $U(0+)>-\infty$, it is proved in Karatzas, Lehoczky, Shreve and Xu (1991) that the function $G(\zeta)$ is well defined and differentiable at $\zeta=1$; writing that $G'(1)=0$ then provides (6.4). We then concentrate on the case $U(0+)<0$ [which includes $U(0+)=-\infty$] and we prove that the last statement still holds; that is,

$$\begin{gathered}G(\zeta) \text{ is well defined for } \zeta>0, \text{ differentiable at } \zeta=1\\ \text{and } G'(1)=y(x)x-y(x)E\big[H^{\theta(y(x))}(T)I\big(y(x)H^{\theta(y(x))}(T)\big)\big].\end{gathered}$$

Denote by y^0 the positive real parameter defined by $\tilde{U}(y^0)=0$. From Lemma 6.1 we have $E\big[\tilde{U}\big(\zeta y(x)H^{\theta(y(x))}(T)\big)^{-}\big]<\infty$. As for the positive part, we consider separately the cases $\zeta\geq 1$ and $\zeta<1$. In the first case, it follows from the decrease of $\tilde{U}$ that $\tilde{U}(\zeta y(x)H^{\theta(y(x))}(T))^{+} \leq \tilde{U}(y(x)H^{\theta(y(x))}(T))^{+}$ which is integrable by Assumption 6.2. Next, for the case $\zeta<1$, we adapt the argument of Karatzas, Lehoczky, Shreve and Xu (1991),

$$\tilde{U}(\zeta y) = \int_{\zeta y}^{y^0} I(u)\,du = \frac{1}{\zeta}\int_{y}^{y^0/\zeta} I(\zeta u)\,du \leq \frac{\gamma}{\zeta}\Big[\tilde{U}(y)-\tilde{U}(y^0/\zeta)\Big],$$

where we used (3.8). Then, for all $\zeta \in (0, 1)$, we have

$$\tilde{U}\big(\zeta y(x) H^{\theta(y(x))}(T)\big)^+ \leq \zeta\gamma\big|\tilde{U}\big(y(x)H^{\theta(y(x))}(T)\big) - \tilde{U}(\zeta y^0)\big|$$

and therefore $E\big[\tilde{U}\big(\zeta y(x) H^{\theta(y(x))}(T)\big)^+\big] < \infty$ by Assumption 6.2.

The proof of differentiability of G at $\zeta = 1$ and the expression of the derivative is obtained as in Karatzas, Lehoczky, Shreve and Xu (1991) by a dominated convergence argument using (3.7). □

We now can state the basic result of this section relating problems $V(x)$ and $\tilde{V}(y)$.

THEOREM 6.1. (i) *Let Assumptions* 6.1 *and* 6.2 *hold. Then the optimization problem $V(x)$ has a solution $\hat{\beta} \in \mathscr{B}$ and*

$$V(x) = \tilde{V}(y(x)) + xy(x),$$

where $y(x)$ is defined in Lemma 6.4.

(ii) *The optimal insurance strategy $\hat{\beta}$ is characterized by*

$$X^{x,\hat{\beta}}(T) = I\big(y(x)H^{\theta(y(x))}(T)\big).$$

PROOF. The existence of $\hat{\beta}$ in (ii) is ensured by Corollary 6.1. By the definition of $\tilde{U}$, we have

$$\tilde{U}\big(y(x)H^{\theta}(T)\big) \geq U\big(X^{x,\beta}(T)\big) - y(x)H^{\theta}(T)X^{x,\beta}(T) \tag{6.5}$$

for all $\theta \in \mathscr{H}$ and $\beta \in \mathscr{B}$. Moreover, from the supermartingale property established in Proposition 4.1, we have

$$E\big[H^{\theta}(T)X^{x,\beta}(T)\big] \leq x. \tag{6.6}$$

By taking expectation in (6.5) and plugging (6.6), we get

$$\tilde{J}(y(x), \theta) \geq J(x, \beta) - xy(x) \quad \text{for all } \theta \in \mathscr{H} \text{ and } \beta \in \mathscr{B}.$$

Now, with $\theta = \theta(y(x))$ and $\beta = \hat{\beta}$, we have equality in (6.5) by definition of $\tilde{U}$ and $\hat{\beta}$ via Corollary 6.1. By Lemma 6.4, equality also holds in (6.6). Therefore

$$\tilde{J}(y(x), \theta(y(x))) = J(x, \hat{\beta}) - xy(x).$$ □

7. Existence in the dual optimization problem. In this paragraph, we provide sufficient conditions under which Assumption 6.2 holds.

From the convexity of the function $x \mapsto |x| + e^x$, it follows that the set $\mathscr{H}$ is convex. In order to ensure the convexity of $\tilde{J}(y, \cdot)$, we need the following assumption.

ASSUMPTION 7.1. For all $t \in [0, T]$, the function

$$\theta \mapsto \int_D e^{\theta(z)} m_t(dz) + \tilde{\pi}\left(\int_D \gamma_t(z) e^{\theta(z)} m_t(dz)\right)$$

is convex.

REMARK 7.1. Suppose that function π is an affine function defined by $\pi(x) = a + bx$, $x \in [0, 1]$ [this framework encompasses the case considered in Briys (1986)]. Then Assumption 7.1 holds. Indeed, it is easily checked that

$$\tilde{\pi}(y) = -y \wedge b - a \quad \text{for all } y > 0,$$

and the function which appears in Assumption 7.1 is given by

$$g(\theta) = \begin{cases} -a + \int_D (1 - \gamma_t(z)) e^{\theta(z)} m_t(dz), & \text{if } \int_D e^{\theta(z)} \gamma_t(z) m_t(dz) \le b, \\ -b - a + \int_D e^{\theta(z)} m_t(dz), & \text{if } \int_D e^{\theta(z)} \gamma_t(z) m_t(dz) \ge b, \end{cases}$$

which is clearly a convex function; recall that $\gamma_t(z) < 1$ for all $(t, z) \in [0, T) \times D$.

REMARK 7.2. Suppose that function π is C^1, strictly convex and satisfies $\pi'(0) = 0$ and $\pi'(1) = +\infty$. Then Assumption 7.1 holds. Indeed, by direct computation, we get

$$\tilde{\pi}(y) = y[(\pi')^{-1}(y) - 1] - \pi\big((\pi')^{-1}(y)\big); \qquad y > 0.$$

Denoting by g the function appearing in Assumption 7.1 and by ∇g its Gâteaux derivative, it is easily checked that

$$\langle \nabla g(\theta) - \nabla g(\mu),\ \theta - \mu \rangle := \int_D (\theta(z) - \mu(z))[h(t, z, \theta) - h(t, z, \mu)] m_t(dz),$$

where $h(t, z, \zeta) = e^{\zeta(z)}[1 - \gamma_t(z) + \gamma_t(z)(\pi')^{-1}(\nu^{\zeta}(t))]$. Since $(\pi')^{-1}$ is increasing (as inverse of an increasing function), each term inside the integral is nonnegative [recall that $0 < \gamma_t(z) < 1$], we have $\langle \nabla g(\theta) - \nabla g(\mu), \theta - \mu \rangle \ge 0$ and therefore g is convex.

LEMMA 7.1. *Under Assumption* 7.1, $\tilde{J}(y, \cdot)$ *is convex for all* $y > 0$.

PROOF. Fix some $\lambda \in [0, 1]$ and consider two elements θ_1 and θ_2 in $\mathscr{H}$. By Assumption 7.1, we get

$$H^{\lambda\theta_1 + (1-\lambda)\theta_2}(T) \ge \exp\big[\lambda \ln H^{\theta_1}(T) + (1 - \lambda) \ln H^{\theta_2}(T)\big].$$

The result then follows from the fact that $\tilde{U}$ is nonincreasing and $\tilde{U} \circ e^{\cdot}$. □

ASSUMPTION 7.2. For all $t \in [0, T)$, the range of $\gamma_t(\cdot)$ is finite.

LEMMA 7.2. *Suppose that Assumption* 7.2 *holds. Let* $(\theta^n)_{n \ge 0}$ *be a sequence in* $\mathscr{H}$ *which converges a.s. to some* $\theta \in \mathscr{H}$ *and such that*

$$E \int_0^T \int_D \big|\theta_t^n(z)^+ - \theta_t(z)^+\big| m_t(dz)\, dt \to 0 \quad \text{as } n \to \infty.$$

Then, for all $y > 0$, *we have*

$$\liminf_{n \to \infty} \tilde{J}(y, \theta^n) \ge \tilde{J}(y, \theta).$$

PROOF. *Step* 1. We first prove that $\limsup_{n\to\infty} H^{\theta^n}(T) \le H^\theta(T)$, or equivalently,

$$\limsup_{n\to\infty} \ln Z^{\theta^n}(T) - \int_0^T \tilde{\pi}(\nu^{\theta^n}(u))\,du \le \ln Z^\theta(T) - \int_0^T \tilde{\pi}(\nu^\theta(u))\,du. \tag{7.1}$$

(i) Using Fatou's lemma and the fact that $\tilde{\pi}$ is nonincreasing, we see that

$$\begin{aligned}\liminf_{n\to\infty} \int_0^T \tilde{\pi}(\nu^{\theta^n}(u))\,du &\ge \int_0^T \liminf_{n\to\infty} \tilde{\pi}(\nu^{\theta^n}(u))\,du \\ &= \int_0^T \tilde{\pi}\Big(\limsup_{n\to\infty} \nu^{\theta^n}(u)\Big)\,du.\end{aligned}$$

Now recall that $\nu^{\theta^n}(t) = \int_D \gamma_t(z)\exp(\theta_t^n(z))m_t(dz)$ and therefore, by Assumption 7.2, we get

$$\liminf_{n\to\infty} \int_0^T \tilde{\pi}(\nu^{\theta^n}(u))\,du \ge \int_0^T \tilde{\pi}(\nu^\theta(u))\,du. \tag{7.2}$$

(ii) Since $e^x - 1 \ge -1$, we get by Fatou's lemma,

$$\begin{aligned}&\liminf_{n\to\infty} \int_0^T \int_D (\exp(\theta_t^n(z)) - 1)m_t(dz)\,ds \\ &\quad\ge \int_0^T \int_D (\exp(\theta_t(z)) - 1)m_t(dz)\,ds.\end{aligned} \tag{7.3}$$

(iii) By direct computation, we see that

$$\begin{aligned}&E\left|\int_0^T \int_D \theta_t^n(z)^+ v(dt,dz) - \int_0^T \int_D \theta_t(z)^+ v(dt,dz)\right| \\ &\quad\le E\int_0^T \int_D \left|\theta_t^n(z)^+ - \theta_t(z)^+\right| v(dt,dz) \\ &\quad= E\int_0^T \int_D \left|\theta_t^n(z)^+ - \theta_t(z)^+\right| m_t(dz)\,dt,\end{aligned}$$

which proves that $\int_0^T \int_D \theta_t^n(z)^+ v(dt,dz)$ converges to $\int_0^T \int_D \theta_t(z)^+ v(dt,dz)$ a.s. possibly along some subsequence. Now,

$$\begin{aligned}&\limsup_{n\to\infty} \int_0^T \int_D \theta_t^n(z) v(dt,dz) \\ &\quad= \int_0^T \int_D \theta_t(z)^+ v(dt,dz) - \liminf_{n\to\infty} \int_0^T \int_D \theta_t^n(z)^- v(dt,dz) \\ &\quad\le \int_0^T \int_D \theta_t(z) v(dt,dz)\end{aligned} \tag{7.4}$$

by Fatou's lemma.

(iv) The result announced in (7.1) follows from (7.2), (7.3), (7.4) and the definition of $Z^\theta(T)$.

Step 2. We shall consider two cases.

(i) Assume that $U(0+) \geq 0$; then $\tilde{U}$ is nonnegative [see (3.4)] and nonincreasing. It follows from Fatou's lemma that

$$\liminf_{n\to\infty} \tilde{J}(y, \theta^n) \geq E\big[\tilde{U}\big(y \limsup_{n\to\infty} H^{\theta^n}(T)\big)\big],$$

and the required result follows from the first step of this proof and the decrease of $\tilde{U}$.

(ii) Assume that $U(0+) < 0$. Then from Lemma 6.1, the sequence $(\tilde{U}(yH^{\theta^n}(T))^-)_{n\geq 0}$ is uniformly integrable. Then, by Fatou's lemma,

$$\begin{aligned}\liminf_{n\to\infty} \tilde{J}(y, \theta^n) &= \liminf_{n\to\infty} E\big[\tilde{U}\big(yH^{\theta^n}(T)\big)^+\big] + \liminf_{n\to\infty} -E\big[\tilde{U}\big(yH^{\theta^n}(T)\big)^-\big]\\ &\geq E\big[\tilde{U}\big(yH^{\theta}(T)\big)\big]\end{aligned}$$

by the first step of this proof. □

We now can state the existence theorem for the dual problem.

THEOREM 7.1. *Suppose that Assumptions* 7.1 *and* 7.2 *hold. Then, for all* $y > 0$, *there exists a solution* $\theta(y)$ *to the dual problem* $\tilde{V}(y)$ [*i.e.*, $\theta(y) \in \mathscr{H}$ *and* $\tilde{V}(y) = \tilde{J}(y, \theta(y))$] *satisfying*

$$E\left[\int_0^T \int_D (\exp(\theta(y)_t(z)) + |\theta(y)_t(z)|) m_t(dz)\, dt\right] < \infty. \tag{7.5}$$

PROOF. First notice that $\tilde{\pi}(y) \geq -\pi(1) > -\infty$ and the process $\{r(t), 0 \leq t \leq T\}$ is nonnegative. Let $(\theta^n)_{n\geq 0}$ be a minimizing sequence of $\tilde{V}(y)$ with $\tilde{J}(y, \theta^n) \leq \tilde{V}(0) + 1$. Then,

$$\begin{aligned}\tilde{V}(0) + 1 &\geq J(y, \theta^n)\\ &\geq E\Big[\tilde{U}\Big(y \exp\big(T\pi(1) + \ln Z^{\theta^n(T)}\big)\Big)\Big]\\ &\geq \tilde{U}\Big(ye^{T\pi(1)} \exp\big(E\big[\ln Z^{\theta^n(T)}\big]\big)\Big)\end{aligned}$$

since $\tilde{U} \circ e^{\cdot}$ is convex. Now, since $\tilde{U}$ is continuous and strictly decreasing, it is invertible with strictly decreasing inverse. This provides

$$E\left[\int_0^T \int_D \exp(\theta_t^n(z)) m_t(dz)\, dt - \int_0^T \int_D \theta_t^n(z) v(dt, dz)\right] \leq \text{Const.} \tag{7.6}$$

and then

$$\begin{aligned}&E\left[\int_0^T \int_D \exp(\theta_t^n(z)^+ \wedge k) m_t(dz)\, dt - \int_0^T \int_D \theta_t^n(z)^+ v(dt, dz)\right]\\ &\quad = E\left[\int_0^T \int_D \exp(\theta_t^n(z)^+ \wedge k) m_t(dz)\, dt\right]\\ &\qquad - E\left[\int_0^T \int_D \theta_t^n(z)^+ v(dt, dz)\right] \leq \text{Const.}\end{aligned}$$

for all integer k. Now using Theorem T3, page 235 of Brémaud (1981) and sending k to infinity, it follows from Fatou's lemma that

$$E\left[\int_0^T \int_D \left(\exp(\theta_t^n(z)^+) - \theta_t^n(z)^+\right) m_t(dz)\, dt\right] \le \text{Const.} \tag{7.7}$$

Therefore, it follows from (7.6) that

$$\begin{aligned} E\left[\int_0^T \int_D \left(\exp(\theta_t^n(z)^+) - \theta_t^n(z)^+\right) m_t(dz)\, dt\right] & \\ + E\left[\int_0^T \int_D \theta_t^n(z)^- m_t(dz)\, dt\right] &\le \text{Const.} \end{aligned}$$

and then

$$E\left[\int_0^T \int_D \theta_t^n(z)^- m_t(dz)\, dt\right] \le \text{Const.} \tag{7.8}$$

Since the function $x \mapsto e^x - x$ defined on $\mathbb{R}_+$ satisfies $(e^x - x)/x \to \infty$ as $x \to \infty$, inequality (7.7) proves that the sequence $(\theta^{n+})_{n\ge 0}$ is uniformly integrable by the La Vallée–Poussin theorem; then by the Dunford–Petis compacity criterion, we see that by possibly passing to a subsequence, there exists a convex combination $\hat\theta_t^n(z)^+ := \sum_{k\ge n} \lambda_k^n \theta_t^n(z)^+$ such that

$$E\int_0^T \int_D \left|\hat\theta_t^n(z)^+ - \hat\theta_t^+(z)\right| m_t(dz)\, dt \to 0$$

for some $\hat\theta^+$; see Dellacherie and Meyer [(1975), Theorem 25, page 43]. Moreover, by possibly passing to a subsequence, the last convergence result holds in the a.s. sense.

Next, inequality (7.8) says that the sequence $(\theta^{n-})_{n\ge 0}$ is bounded in L^1. From the Kómlos theorem [see Hall and Heyde (1980), Theorem 7.3, page 205], by possibly passing to a subsequence, there exists a convex combination $\hat\theta_t^n(z)^- := \sum_{k\ge n} \lambda_k^n \theta_t^n(z)^-$ such that $\hat\theta_t^n(z)^- \to \hat\theta_t^-(z)$ a.s. for some $\hat\theta^-$ (Notice that we can take the same convex combination as before by possibly composing both convex combinations). Defining $\hat\theta := \hat\theta^+ - \hat\theta^-$, we conclude that

$$\hat\theta_t^n(z) \to \hat\theta_t(z) \quad \text{a.s.}$$

Moreover, by Fatou's lemma and the convexity of $e^{\cdot}$, we see that

$$\begin{aligned} \text{Const.} &\ge \liminf_{n\to\infty} \int_0^T \int_D \left(\exp(\hat\theta_t^n(z)) - \hat\theta_t^n(z)\right) m_t(dz)\, dt \\ &\ge \int_0^T \int_D \left(\exp(\hat\theta_t(z)) - \hat\theta_t(z)\right) m_t(dz)\, dt, \end{aligned}$$

which implies (7.5) and therefore $\hat\theta \in \mathscr{H}$. Hence the sequence $(\hat\theta^n)_{n\ge 0}$ meets the conditions of Lemma 7.2 which proves that

$$\liminf_{n\to\infty} \tilde J(y, \hat\theta^n) \ge \tilde J(y, \hat\theta).$$

In order to conclude the proof, it remains to show that the sequence $(\hat{\theta}^n)_n$ is a minimizing sequence. But this is a direct consequence of Lemma 7.1. □

8. Examples

8.1. *Logarithmic utility and Poisson shocks.* In this paragraph, we solve explicitly the optimal insurance demand problem in the case $U(x) = \ln x$, the random measure $v(dt, dy)$ is a Poisson process, denoted dv_t, with constant intensity m and the jump size is a constant $\gamma \in (0, 1)$. The insurance premium function is given by $\pi(x) = bx$.

Direct computation shows that $\tilde{U}(y) = -1 - \ln y$ and $\tilde{\pi}(y) = -b \wedge y$. The dual optimization problem in this case is

$$\tilde{V}(y) = -1 - \ln y - E\left[\int_0^T r(t)\,dt\right] + \inf_{\theta \in \mathscr{H}} E\left[\int_0^T \left(e^{\theta_t} - 1 - \theta_t - (\gamma e^{\theta_t}) \wedge \frac{b}{m}\right) m\,dt\right];$$

recall (7.5). It follows that the solution to the dual problem does not depend on y and, since $y(x)$ achieves the minimum of $\tilde{V}(y) + xy$ over all $y > 0$ (see Lemma 6.4), we have

$$y(x) = \frac{1}{x} \quad \text{for all } x > 0.$$

Next, we turn to the solution of the dual problem.

Case 1. $b \le \gamma m$. (since γm is the expected relative jump of the wealth process, the insurance premium is said to be *fair* in the case $b = \gamma m$). Then it is easily checked that the constant process $\hat{\theta} := 0$ solves the dual problem. From Theorem 6.1, the wealth process associated to the initial capital x and the optimal insurance strategy $\hat{\beta}$ is related to $\hat{\theta}$ by

$$X^{x,\hat{\beta}}(T) = \frac{x}{H^{\hat{\theta}}(T)} = \frac{x}{H^0(T)} = x \exp\left(-bT + \int_0^T r(t)\,dt\right);$$

then the optimal insurance strategy is given by the constant process $\hat{\beta} = 1$.

Case 2. $\gamma m \le b \le \gamma m/(1-\gamma)$. Then it is easily checked that the constant process $\hat{\theta} = \ln(b/\gamma m)$ solves the dual problem. The wealth process associated to the initial capital x and the optimal insurance strategy $\hat{\beta}$ is related to $\hat{\theta}$ by

$$X^{x,\hat{\beta}}(T) = \frac{x}{H^{\hat{\theta}}(T)} = x \exp\left(-bT + \int_0^T r(t)\,dt\right)\left[\mathscr{E}\left(\int_0^T (e^{\hat{\theta}_t} - 1)\,d\tilde{v}_t\right)\right]^{-1}.$$

Applying Itô's lemma allows us to identify the optimal insurance strategy

$$\hat{\beta}(t) = 1 - \frac{1}{\gamma} + \frac{m}{b}, \qquad 0 \le t \le T.$$

Hence, the insurance strategy is a decreasing function of b and tends to 0 as b approaches the value $\gamma m/(1-\gamma)$.

We now clarify the above identification. Let Y be the process defined by $Y(t) = [Z^{\theta}(t)]^{-1-\delta}$; $\delta \ge 0$ (the case $\delta = 0$ is needed here, whereas the case $\delta > 0$ will be used in the subsequent paragraph). Then, it follows from Itô's lemma that

$$dY(t) = (1+\delta)Y(t)(e^{\theta_t} - 1)m\,dt + Y(t) - Y(t-).$$

Next observe that $Y(t) - Y(t-) = Y(t-)\big[(1 + (e^{\theta_t} - 1)\,dv_t)^{-1-\delta} - 1\big] = Y(t-)$ $(\exp(-(1+\delta)\theta_t) - 1)\,dv_t$ and therefore

$$dY(t) = (1+\delta)Y(t)(e^{\theta_t} - 1)m\,dt + Y(t-)\big(\exp(-(1+\delta)\theta_t) - 1\big)\,dv_t.$$

Case 3. $b \ge \gamma m/(1-\gamma)$. Then it is easily checked that the constant process $\hat{\theta} = -\ln(1-\gamma)$ solves the dual problem. By writing the wealth process, associated to the initial capital x and the optimal insurance strategy $\hat{\beta}$, in terms of $\hat{\theta}$, and applying Itô's lemma, we see that the optimal insurance strategy is the constant process $\hat{\beta} = 0$; that is, the agent does not demand any insurance.

8.2. *Power utility and Poisson shocks.* In this paragraph, we consider the same framework as in the previous application except that the utility function is given by

$$U(x) = \frac{x^p}{p}; \qquad x > 0$$

for some $p \in (0,1)$. We also take the interest rate to be constant for simplicity. Then it is easily checked that function $\tilde{U}$ is given by

$$\tilde{U}(y) = \frac{y^{-q}}{q}; \qquad y > 0 \text{ where } q = \frac{p}{1-p}.$$

Define

$$\tilde{V}(t,y) = \inf_{\theta \in \mathscr{H}} E\left[\tilde{U}\left(y\frac{H^{\theta}(T)}{H^{\theta}(t)}\right)\right].$$

Of course $\tilde{V}(y) = \tilde{V}(0,y)$. In order to solve the above stochastic control problem, we write formally the associated Hamilton–Jacobi–Bellman equation and derive a smooth solution to it; we then conclude that the (classical) solution of the HJB equation indeed solves the control problem $\tilde{V}$ by a verification theorem argument.

The HJB equation together with the terminal condition associated to our control problem is

$$\inf_{\theta\in\mathbb{R}} \mathscr{L}^{\theta} v(t,y) = 0; \qquad (t,y) \in [0,T) \times (0,\infty), \tag{8.9}$$

$$v(T,y) = \tilde{U}(y); \qquad y \in (0,\infty), \tag{8.10}$$

where

$$\mathscr{L}^{\theta} v(t,y) = \frac{\partial v}{\partial t}(t,y) + \left[-r - \tilde{\pi}(e^{\theta} m\gamma) - m(e^{\theta}-1)\right] y \frac{\partial v}{\partial y}(t,y)$$
$$+ m\left[v(t, ye^{\theta}) - v(t,y)\right].$$

As in the previous application, we have $\tilde{\pi}(y) = -b \wedge y$; $y > 0$. Clearly, given the form of $\tilde{U}$, the value function of the dual problem is of the form $\tilde{V}(t,y) = f(t)y^{-q}$ for all $(t,y) \in [0,T] \times (0,\infty)$. Function $f(t)$ is determined by plugging $\tilde{V}(t,y)$ in the HJB equation; then we get

$$f'(t) + \left[rq - m + \inf_{\theta\in\mathbb{R}} h(\theta)\right] f(t) = 0,$$
$$f(T) = \frac{1}{q},$$

where

$$h(\theta) = me^{-q\theta} + mq(e^{\theta}-1) - qb \wedge (e^{\theta} m\gamma).$$

By direct computation, we see that the value of θ which attains the minimum of $h(\theta)$ is given by

$$\hat{\theta} = \begin{cases} 0, & \text{if } b \le m\gamma, \\ \ln\left(\dfrac{b}{m\gamma}\right), & \text{if } m\gamma \le b \le m\gamma\left(\dfrac{1}{1-\gamma}\right)^{1+q}, \\ \dfrac{-\ln(1-\gamma)}{1+q}, & \text{if } b \ge m\gamma\left(\dfrac{1}{1-\gamma}\right)^{1+q}. \end{cases}$$

Hence the function

$$W(t,y) = \frac{y^{-q}}{q} \exp((rq - m + h(\hat{\theta}))(T-t)); \qquad (t,y) \in [0,T] \times (0,\infty)$$

is a classical solution to the HJB equation (8.9)–(8.10).

PROPOSITION 8.1. *The value function of the dual optimization problem is given by*

$$\tilde{V}(y) = W(0,y) = \frac{y^{-q}}{q} \exp((rq - m + h(\hat{\theta}))T) \quad \textit{for all } y > 0.$$

PROOF. Denote by $\mathscr{H}_b$ the subset of $\mathscr{H}$ consisting of all bounded elements, and define $\tilde{V}_b(y) = \inf_{\theta\in\mathscr{H}_b} E[\tilde{U}(yH^{\theta}(T))]$. Clearly, we have $\tilde{V}_b(y) \ge \tilde{V}(y)$.

(i) We first prove that $\tilde{V}_b = W$ and therefore $\tilde{V} \geq W$. Let $(\theta^n)_{n\geq 0}$ be a minimizing sequence of $\tilde{V}_b(y)$ in $\mathscr{H}_b$. By Itô's lemma and the terminal condition satisfied by W, we have

$$\begin{aligned} E\left[\tilde{U}\left(y\frac{H^{\theta^n}(T)}{H^{\theta^n}(t)}\right)\right] - W(t,y) &= E\left[W\left(T, y\frac{H^{\theta^n}(T)}{H^{\theta^n}(t)}\right) - W(t,y)\right] \\ &= E\left[\int_t^T \mathscr{L}^{\theta^n(u)} W\left(u, y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)}\right) du\right] \\ &\quad + E\left[\int_t^T W_y\left(u, y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)}\right) y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)} \right. \\ &\qquad \left. \times \left(\exp(\theta^n(u-)) - 1\right) d\tilde{v}_u\right], \end{aligned}$$

where $d\tilde{v}_t = dv_t - m\,dt$ is the compensated Poisson process. Since $\mathscr{L}^\theta W(t,z) \geq 0$ for all $\theta \in \mathbb{R}$, by definition of W, this provides

$$\begin{aligned} &E\left[\tilde{U}\left(y\frac{H^{\theta^n}(T)}{H^{\theta^n}(t)}\right)\right] - W(t,y) \\ &\quad \geq E\left[\int_t^T W_y\left(u, y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)}\right) y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)} \left(\exp(\theta^n(u)) - 1\right) d\tilde{v}_u\right] \end{aligned}$$

In order to prove that $W(t,y) = \tilde{V}_b(t,y)$, we have to show that

$$E\left[\int_t^T W_y\left(u, y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)}\right) y\frac{H^{\theta^n}(u-)}{H^{\theta^n}(t)} \left(\exp(\theta^n(u)) - 1\right) d\tilde{v}_u\right] = 0. \tag{8.11}$$

Indeed, the last claim implies that

$$W(t,y) \leq E\left[\tilde{U}\left(y\frac{H^{\theta^n}(T)}{H^{\theta^n}(t)}\right)\right]$$

and with $\theta = \hat{\theta}$ the above inequality is in fact an equality. To prove (8.11), we show that the process appearing inside the expectation is a martingale. To see this, observe that from the bound on θ^n and the form of W it follows that

$$\begin{aligned} &E\left[\int_t^T \left| W_y\left(u, y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)}\right) y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)} \left(\exp(\theta^n(u)) - 1\right)\right| m\,du\right] \\ &\qquad \leq \text{Const. } E\left[\int_t^T \tilde{U}\left(y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)}\right) du\right]. \end{aligned}$$

Moreover, from the fact that $\tilde{\pi}(\cdot) \geq -\pi(1)$, $E\left[Z^{\theta^n}(T)/Z^{\theta^n}(t)\right] \leq 1$ and the decrease of $\tilde{U}$, we see that $E\left[\tilde{U}(yH^{\theta^n}(u)/H^{\theta^n}(t))\right] \leq$ Const. $E\left[\tilde{U}(yH^{\theta^n}(T)/\right.$

$H^{\theta^n}(t))]$. We then get

$$E\left[\int_t^T \left| W_y\left(u, y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)}\right) y\frac{H^{\theta^n}(u)}{H^{\theta^n}(t)}(\exp(\theta^n(u)) - 1)\right| m\, du\right]$$
$$\leq \text{Const. } E\left[\tilde{U}\left(y\frac{H^{\theta^n}(T)}{H^{\theta^n}(t)}\right)\right]$$
$$< \infty,$$

since $(\theta^n)_n$ is a minimizing sequence; this provides (8.11).

(ii) We now prove that $\tilde{V} \geq W$. Denote by $\tilde{V}_*$ the lower semicontinuous (l.s.c.) envelope of $\tilde{V}$, that is, the largest l.s.c. envelope dominated by $\tilde{V}$. Notice that $\tilde{V}(t, y)$ is continuous in y and therefore the l.s.c. envelope concerns only the t variable. We only highlight the main steps of the argument. The value function $\tilde{V}$ satisfies the dynamic programming equation

$$\tilde{V}(t, y) = E\left[\tilde{V}\left(t+\tau, y\frac{H^{\hat{\theta}}(t+\tau)}{H^{\hat{\theta}}(t)}\right)\right]$$

for all nonnegative stopping time $\tau \leq T - t$, where $\hat{\theta}$ is the solution of $\tilde{V}(t, y)$; actually we only need that the left-hand side term be larger than the right-hand side one in the dynamic programming equation. By Fatou's lemma and the definition of $\tilde{V}_*$, this provides

$$\begin{aligned}
\tilde{V}_*(t, y) &= \liminf_{t'\to t} \tilde{V}_*(t', y) = \liminf_{t'\to t} E\left[\tilde{V}\left(t'+\tau, y\frac{H^{\hat{\theta}}(t'+\tau)}{H^{\hat{\theta}}(t')}\right)\right] \\
&\geq E\left[\liminf_{t'\to t} \tilde{V}\left(t'+\tau, y\frac{H^{\hat{\theta}}(t'+\tau)}{H^{\hat{\theta}}(t')}\right)\right] \qquad (8.12) \\
&= E\left[\tilde{V}_*\left(t+\tau, y\frac{H^{\hat{\theta}}(t+\tau)}{H^{\hat{\theta}}(t)}\right)\right].
\end{aligned}$$

Next let φ be an arbitrary $C^1([0, T]\times\mathbb{R})$ function such that $0 = (\tilde{V}_* - \varphi)(t, y) = \min(\tilde{V}_* - \varphi)$. Then it follows from (8.12) that

$$\varphi(t, y) \geq E\left[\varphi\left(t+\tau, y\frac{H^{\hat{\theta}}(t+\tau)}{H^{\hat{\theta}}(t)}\right)\right].$$

Applying Itô's lemma and choosing the stopping time appropriately (in order for the local martingale term to be a martingale), we see that for all $h > 0$,

$$\frac{1}{h}E\left[-\int_t^{t+\tau\wedge h} \mathscr{L}^{\hat{\theta}(u)}\varphi\left(u, y\frac{H^{\hat{\theta}}(u)}{H^{\hat{\theta}}(t)}\right) du\right] \geq 0,$$

which provides by passing to the limit as $h \searrow 0$,

$$\sup_{\theta\in\mathbb{R}} -\mathscr{L}^{\theta}\varphi(t, y) \geq 0.$$

Hence $\tilde{V}_*$ is a viscosity supersolution of the HJB equation (8.9)–(8.10), and therefore $\tilde{V} \geq \tilde{V}_* \geq W$ by the comparison theorem for viscosity solutions. □

The solution of the dual problem is given by $\hat{\theta}$ and does not depend on y. Since $y(x)$ achieves the minimum of $\tilde{V}(y)+xy$ over all $y > 0$ (see Lemma 6.4), we have

$$y(x)^{1/(p-1)} = x\exp(-(rq - m + h(\hat{\theta}))T); \qquad x > 0.$$

Now we can deduce an explicit solution to the optimal insurance demand problem.

Case 1. $b \leq \gamma m$. Then the solution of the dual problem is $\hat{\theta} = 0$ and the optimal insurance strategy $\hat{\beta}$ is related to $\hat{\theta}$ by

$$\begin{aligned} X^{x,\hat{\beta}}(T) &= I\big(y(x)H^{\hat{\theta}}(T)\big) = \big(y(x)H^0(T)\big)^{1/(p-1)} \\ &= x\exp((r-b)T). \end{aligned}$$

Hence, the optimal insurance strategy is given by the constant process

$$\hat{\beta} = 1.$$

Case 2. $m\gamma \leq b \leq m\gamma(1/(1-\gamma))^{1+q}$. Then the solution of the dual problem is $\hat{\theta} = \ln(b/m\gamma)$ and the optimal insurance strategy is characterized by

$$\begin{aligned} X^{x,\hat{\beta}}(T) &= I\big(y(x)H^{\hat{\theta}}(T)\big) \\ &= (y(x))^{1/(p-1)}(\exp(-r+b)T)^{1/(p-1)}\bigg[\mathscr{E}\bigg(\int_0^T (e^{\hat{\theta}} - 1)\,d\tilde{v}\bigg)\bigg]^{1/(p-1)}. \end{aligned}$$

Applying Itô's lemma, we see that

$$\begin{aligned} &\bigg[\mathscr{E}\bigg(\int_0^T (e^{\hat{\theta}} - 1)\,d\tilde{v}\bigg)\bigg]^{1/(p-1)} \\ &\quad = \exp\bigg(\frac{m}{1-p}(e^{\hat{\theta}} - 1)T\bigg)\mathscr{E}\bigg(\int_0^T (\exp(\hat{\theta}/(p-1)) - 1)\,dv\bigg). \end{aligned}$$

Then, direct identification shows that the constant insurance strategy

$$\hat{\beta} = 1 - \frac{1}{\gamma} + \frac{1}{\gamma}\bigg(\frac{b}{m\gamma}\bigg)^{-q-1}$$

is the optimal insurance strategy for the the problem $V(x)$.

Case 3. $b \geq m\gamma(1/(1-\gamma))^{1+q}$. Then the solution of the dual problem is $\hat{\theta} = [\ln(1-\gamma)]/(1+q)$. Proceeding as in the second case, we see that the optimal insurance strategy is characterized by

$$\begin{aligned} X^{x,\hat{\beta}}(T) &= I\big(y(x)H^{\hat{\theta}}(T)\big) \\ &= xe^{-rT}\mathscr{E}\bigg(-\int_0^T \gamma\,dv_t\bigg) \end{aligned}$$

and therefore the optimal insurance strategy is given by the constant process

$$\hat{\beta} = 0.$$

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