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DUAL FORMULATION OF THE UTILITY MAXIMIZATION PROBLEM UNDER TRANSACTION COSTS

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In the context of a general multivariate financial market with transaction costs, we consider the problem of maximizing expected utility from terminal wealth. In contrast with the existing literature, where only the liquidation value of the terminal portfolio is relevant, we consider general utility functions which are only required to be consistent with the structure of the transaction costs. An important feature of our analysis is that the utility function is not required to be C^1. Such nonsmoothness is suggested by major natural examples. Our main result is an extension of the well-known dual formulation of the utility maximization problem to this context.

1. Introduction. We consider a general multivariate financial market with transaction costs as in Kabanov (1999), and we analyze the stochastic control problem of maximizing expected utility from terminal wealth.

The existing literature in this framework only considers an utility function defined on the liquidation value of the terminal portfolio; see for example, Davis, Panas and Zariphopoulou (1993), Cvitanić and Karatzas (1996), Kabanov (1999), Cvitanić and Wang (1999). This is of course not consistent with economic intuition which suggests that agents prefer holding the portfolio to its liquidation value. Indeed, once the portfolio is liquidated, its liquidation value does not allow financing it because of the presence of transaction costs.

Instead, we introduce an utility function U defined on $\mathbb{R}^{d+1}$, where $d+1$ is the number of tradable assets in the financial market. For the sake of consistency with the structure of transaction costs, the function U is required to be increasing in the sense of the partial ordering induced by the transaction costs. This natural economic condition turns out to be crucial. Also by examining some natural examples of such utility functions, it turns out that the usual smoothness condition fails to hold.

The main result of this paper is to obtain a dual formulation of the utility maximization problem as it was established in the frictionless markets literature by Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and the recent paper by Kramkov and Schachermayer (1999). In particular, we require a natural extension, to our multivariate framework, of the important condition on the asymptotic elasticity introduced by Kramkov and Schachermayer.

In the presence of transaction costs, such a dual formulation has been derived by Cvitanić and Karatzas (1996) and Kabanov (1999) under the assum-

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ption of existence for the dual problem. Recently, Cvitanić and Wang (1999) proved the dual formulation, without appealing to such existence assumption. This was achieved by suitably enlarging the set of controls of the dual problem, as in Kramkov and Schachermayer (1999). However, as mentioned above, Cvitanić and Wang only considered the one-dimensional ($d = 1$) problem of maximizing expected utility of the liquidation value of the terminal wealth, with smooth utility function defined on $\mathbb{R}_+$.

An important feature of our analysis is that neither the utility function U, nor the Legendre–Fenchel transform $\widetilde{U}$ of $-U(-\cdot)$ are required to be smooth. We then use different arguments from those of Kramkov and Schachermayer (1999). In particular, we introduce an approximation of function $\widetilde{U}$ by quadratic inf-convolution, and then pass to the limit.

Let us mention that Cvitanić (1998) dealt with a nonsmooth utility maximization problem of the form $\inf_{x\in C} F(x)$ for some convex subset C of a Banach space, and lower semicontinuous convex function F. In this case, it was possible to apply directly the classical Kuhn–Tucker conditions in Banach spaces established in the context of nonsmooth convex problems [see, e.g., Aubin and Ekeland (1984)]. Our dual optimization problem is naturally set in the Banach space L^1. However, the classical result of this theory requires that 0 lie in the interior of the set $\mathrm{dom}(F) - C$, which fails to hold for our dual optimization problem.

The paper is organized as follows. Section 2 contains the exact formulation of the utility maximization problem. Section 3 introduces the main polar transformations of the variables and functions involved in the problem. It also contains some preliminary results on these transformations. The main duality result together with the precise assumptions are stated in Section 4. Section 5 contains three natural examples of utility functions consistent with the structure of transaction costs, which are naturally nonsmooth. The proof of the main theorem is reported in Section 9 after some preparation in Sections 6, 7 and 8. Finally, we report some useful results concerning the notion of asymptotic elasticity in the Appendix.

2. The utility maximization problem. In this section, we formulate the utility maximization problem under proportional transaction costs. In contrast with the usual literature in this area [see, e.g., Cvitanić and Karatzas (1996), Kabanov (1999)], the utility function will be defined on the vector terminal wealth and not on the liquidation value of the terminal wealth.

2.1. *The financial market.* Let T be a finite time horizon and let $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\le T}, P)$ be a stochastic basis with the trivial σ-algebra $\mathscr{F}_0$. Let $S := (S^0, \dots, S^d)$ be a continuous semimartingale with strictly positive components; the first component is assumed to be constant over time $S^0(\cdot) = 1$. With the interpretation of S as a price process, this means that the first security ("cash") is taken as the numéraire.

A *trading strategy* is an adapted, right-continuous, (componentwise) nondecreasing process L taking values in $\mathbb{M}_+^{d+1}$, the set of $(d+1)\times(d+1)$-matrices with nonnegative entries; L_t^{ij} is the cumulative net amount of funds transferred from the asset i to the asset j up to the date t; this process may have a jump at the origin $\Delta L_0^{ij} = L_0^{ij}$ corresponding to the initial transfer. Constant proportional transaction costs are described by a matrix $(\lambda^{ij}) \in \mathbb{M}_+^{d+1}$ with zero diagonal and which satisfy

$$(1 + \lambda^{ij}) \leq (1 + \lambda^{ik})(1 + \lambda^{kj}) \quad \forall\, i, j, k \in \{0, \dots, d\}. \tag{2.1}$$

The above condition says that no transaction costs can be saved by any artificial transit through another account, and is only needed in order to obtain the interesting characterization of Remark 4.3 below.

Given an initial holdings vector $x \in \mathbb{R}^d$ and a strategy L, the portfolio holdings $X = X^{x,L}$ are defined by the dynamics,

$$X_t^i = x + \widehat{X}_-^i \cdot S_t^i + \sum_{j=0}^{d} \big(L_t^{ji} - (1 + \lambda^{ij}) L_t^{ij}\big), \tag{2.2}$$

where $\widehat{X}^i := X^i/S^i$ (i.e., $\widehat{X}$ is the process X divided by the process S componentwise), and $X_-^i \cdot S_t^i$ is the stochastic integral of X_-^i with respect to S^i.

Alternatively, the wealth process X can be defined by

$$\widehat{X}_t^i = \hat{x} + \sum_{j=0}^{d} \int_0^t \big(dL_s^{ji} - (1 + \lambda^{ij}) dL_s^{ij}\big)/S_s^i.$$

This equivalent definition shows that the stochastic integration with respect of S is not essential for the definition of the wealth process. In particular, the condition that S is a semimartingale is not needed. However, we shall need to apply a duality result due to Kabanov and Last (1998), which requires S to be a continuous semimartingale. This is the only reason why our price process is assumed to be a continuous semimartingale.

2.2. *Admissible strategies.* Following Kabanov (1999), we define the *solvency region*,

$$K := \left\{ x \in \mathbb{R}^{d+1} \colon \exists\, a \in \mathbb{M}_+^{d+1},\ x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij}) a^{ij}) \geq 0;\ i = 0, \dots, d \right\}.$$

The elements of K can be interpreted as the vectors of portfolio holdings such that the no-bankruptcy condition is satisfied: the liquidation value of the portfolio holdings x, through some convenient transfers, is nonnegative. In particular, K contains the positive orthant $\mathbb{R}_+^{d+1} = [0, \infty)^{d+1}$.

Clearly, the set K is a closed convex cone containing the origin. We can then define the partial ordering $\succeq$ induced by K:

$$x_1 \succeq x_2 \quad \text{if and only if } x_1 - x_2 \in K.$$

Let $\kappa \geq 0$ be some given constant. A trading strategy L is said to be *κ-admissible* for the initial holdings $x \in K$ if the no-bankruptcy condition

$$X^{x,L}(\cdot) \succeq -\kappa S(\cdot) \tag{2.3}$$

holds. We shall denote by $\mathscr{A}_\kappa(x)$ the set of all κ-admissible trading strategies for the initial holdings $x \in K$, and we introduce the set

$$\underline{\mathscr{X}}(x) := \Big\{X \in L^0(\mathbb{R}^{d+1}, \mathscr{F}_T)\colon X = X_T^{x,L} \text{ for some } L \in \bigcup_{\kappa \geq 0} \mathscr{A}_\kappa(x)\Big\}.$$

2.3. *The problem formulation.* Throughout this paper, we consider a *utility function* U mapping $\mathbb{R}^{d+1}$ into $\mathbb{R}$ with effective domain $\mathrm{dom}(U) \subset K$, and satisfying the conditions

$$\begin{gathered} U(0) = 0, \quad U \text{ is concave on } K, \\ U(x_1) \geq U(x_2) \quad \text{for all } x_1 \succeq x_2 \succeq 0. \end{gathered} \tag{2.4}$$

The third condition says that the agent preferences are monotonic in the sense of the partial ordering $\succeq$. The second condition is the concavity of the preferences of the agent. As will be clear from the definition of the utility maximization problem, the first condition can be relaxed by only requiring $U(0) > -\infty$. The case $U(0) = -\infty$ was solved by Kramkov and Schachermayer (1999) in the one-dimensional frictionless framework. We leave this problem for future research in order to simplify the (already complex) framework of this paper.

Notice that the utility function is neither required to be differentiable, nor strictly concave and strictly increasing.

Our interest is on the stochastic control problem,

$$V(x) := \sup_{X \in \underline{\mathscr{X}}(x)} EU(X)$$

of maximizing expected utility from terminal wealth. Since $\mathrm{dom}(U) \subset K$, the above maximization can be restricted to the $\succeq$-nonnegative elements of $\underline{\mathscr{X}}(x)$,

$$V(x) := \sup_{X \in \mathscr{X}(x)} EU(X) \quad \text{with } \mathscr{X}(x) := \{X \in \underline{\mathscr{X}}(x)\colon X \succeq 0,\ P\text{-a.s.}\}.$$

The chief goal of this paper is to derive a dual formulation of this problem in the spirit of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and the recent paper of Kramkov and Schachermayer (1999), KS99 hereafter.

REMARK 2.1. In the frictionless case, the above problem can be reduced to the framework of a classical utility function defined on the positive real line. Indeed, if $\lambda = 0$, the solvency region $K = \{x \in \mathbb{R}^{d+1}\colon \bar{x} := \sum_{i=0}^d x^i \geq 0\}$. Clearly, $x \succeq (\bar{x}, 0, \dots, 0)$ and $(\bar{x}, 0, \dots, 0) \succeq x$. From the increase of U in the sense of the partial ordering $\succeq$ in Condition (2.4), this proves that $U(x) = u(\bar{x}) := U(\bar{x}, 0, \dots, 0)$.

3. Preliminaries: polar transformations.

3.1. *Solvency region.* We shall frequently make use of the positive polar cone associated to K defined as usual by $K^* = \{y \in \mathbb{R}^{d+1}\colon xy \geq 0$, for all $x \in K\}$; here xy is the canonical scalar product of $\mathbb{R}^{d+1}$. It is easily checked that K^* is the polyhedral cone defined by

$$K^* = \{y \in \mathbb{R}_+^{d+1}\colon y^j - (1+\lambda^{ij})y^i \leq 0 \text{ for all } 0 \leq i, j \leq d\} \tag{3.1}$$

[see Kabanov (1999)]. In particular, this shows that

$$K^* \backslash \{0\} \subset (0,\infty)^{d+1} \subset K.$$

An alternative characterization of K relies on the function

$$l(x) := \inf_{y \in K_0^*} xy \quad \text{where } K_0^* := \{y \in K^*\colon y^0 = 1\}.$$

Then, we have clearly,

$$x \succeq 0 \quad \text{if and only if } l(x) \geq 0.$$

REMARK 3.1. It follows from the definition of K_0^* and (3.1) that, for all $y \in K_0^*$, we have

$$\underline{\lambda} := \min_{0\leq i\leq d}(1+\lambda^{i0})^{-1} \leq (1+\lambda^{j0})^{-1} \leq y^j \leq (1+\lambda^{0j}) \leq \max_{0\leq i\leq d}(1+\lambda^{0i}) =: \bar{\lambda}.$$

Let $\mathbf{1}_0$ be the vector of $\mathbb{R}^{d+1}$ with components $\mathbf{1}_0^i = 0$ for all $i = 1,\ldots,d$ and $\mathbf{1}_0^0 = 1$. It is proved in Bouchard (1999) that

$$l(x) = \sup\{w \in \mathbb{R}\colon x \succeq w\mathbf{1}_0\};$$

that is, $l(x)$ is the liquidation value (on the bank account) of the portfolio x. We shall refer to l as the *liquidation function*.

REMARK 3.2. Existence holds for the last formulation of the liquidation function $l(x)$; that is, $x \succeq l(x)\mathbf{1}_0$ for all $x \in \mathbb{R}^{d+1}$. This follows from the fact that the set $\{w \in \mathbb{R}\colon x \succeq w\mathbf{1}_0\} = \{w \in \mathbb{R}\colon (x - w\mathbf{1}_0)y \geq 0 \text{ for all } y \in K^*\}$ is closed.

Another interesting property of the liquidation function is the following characterization of the boundary ∂K of K.

LEMMA 3.1. $\partial K = \{x \in K\colon l(x) = 0\}$.

PROOF. Let x be in $\operatorname{int}(K)$. From Remark 3.1, there exists some positive scalar $\varepsilon > 0$ such that $x - \varepsilon y \in K$ for all $y \in K_0^*$. Then, $(x - \varepsilon y)y \geq 0$. Using again Remark 3.1, we see that $xy \geq \varepsilon|y|^2 \geq \varepsilon(d+1)\underline{\lambda}^2$, and therefore $l(x) > 0$.

Conversely assume that $l(x) > 0$ and set $r := l(x)/[(d+1)\bar{\lambda}^2]^{1/2}$. By definition of the liquidation function, it follows from the Cauchy–Schwarz inequality that, for all $z \in B(x, r)$,

$$zy = xy + (z - x)y \geq l(x) - |z - x| \cdot |y| \geq 0 \quad \text{for all } y \in K_0^*.$$

This proves that $l(z) \geq 0$. Then $B(x, r) \subset K$ and $x \in \text{int}(K)$. □

We shall also make use of the partial ordering $\succeq_*$ induced by K^* defined by

$$y_1 \succeq_* y_2 \quad \text{if and only if } y_1 - y_2 \in K^*.$$

Then, by introducing the function

$$l^*(y) := \inf_{x \in K,\, |x|=1} xy,$$

we obtain an alternative characterization of the partial ordering $\succeq_*$ (or equivalently, of the polar cone K^*),

$$y \succeq_* 0 \quad \text{if and only if } l^*(y) \geq 0.$$

By similar arguments as in the proof of Lemma 3.1, we prove the following characterization of the boundary ∂K^* of K^*.

LEMMA 3.2. $\partial K^* = \{y \in K^*: l^*(y) = 0\}$.

We shall need the following easy result on the function l^*.

LEMMA 3.3. *Let $b > 0$. Then, there exists $y(b) \in \text{int}(K^*)$ such that*

$$\textit{for all } y \in K^*, \quad l^*(y) \geq b \quad \Longrightarrow \quad y \succeq_* y(b).$$

PROOF. Suppose the contrary. This means that for all $z \in \text{int}(K^*)$, there exists $y(z) \in K^*$, with $l^*(y(z)) \geq b$, such that $y(z) - z \notin K^*$; that is, $l^*(y(z) - z) < 0$. Now by definition of function l^*, we easily see that $l^*(y(z)) \leq l^*(y(z) - z) + |z|$. We obtain therefore $b < |z|$ for all $z \in \text{int}(K^*)$. Sending z to 0 leads to a contradiction. □

3.2. *Utility function.* Define the Legendre–Fenchel transform

$$\widetilde{U}(y) := \sup_{x \in K}(U(x) - xy) \quad \text{for all } y \in \mathbb{R}^{d+1}.$$

Then $\widetilde{U}$ is a convex function from $\mathbb{R}^{d+1}$ into the extended real line $\mathbb{R} \cup \{+\infty\}$. We shall denote by $\partial\widetilde{U}$ the subgradient of $\widetilde{U}$.

From the definition of K^*, for all $y \in \mathbb{R}^{d+1} \backslash K^*$, there exists some $x_0 \in K$ such that $x_0 y < 0$. Then, for all integer n, we have $\widetilde{U}(y) \geq -n x_0 y$ and therefore,

$$\text{dom}(\widetilde{U}) \subset K^*. \tag{3.2}$$

Moreover, whenever U is unbounded, we clearly have $\widetilde{U}(0) = +\infty$. More information on the domain of $\widetilde{U}$ will be obtained later on (see Lemma 4.2).

We now state an important property of function $\widetilde{U}$ which follows immediately from its definition as the Legendre–Fenchel transform of the $\succeq$-increasing function U.

LEMMA 3.4. *Function $\widetilde{U}$ is decreasing in the sense of the partial ordering $\succeq_*$, that is,*

$$\textit{for all } y_1 \succeq_* y_2 \succeq_* 0 \quad \textit{we have } \widetilde{U}(y_2) \geq \widetilde{U}(y_1).$$

PROOF. Let $y_1 \succeq_* y_2 \succeq_* 0$. Then $y_1 - y_2 \in K^*$ and $U(x) - xy_1 \leq U(x) - xy_2$ for all $x \in K$. The required result follows by taking supremum over $x \in K$ in the last inequality. □

4. The main result.

4.1. *Assumptions.* For ease of exposition, we collect and comment the assumptions of the main result of the paper in this subsection. Recall that conditions (2.4) are assumed to hold throughout the paper. We first start by the following technical condition which is needed for the proof of Lemma 8.3.

ASSUMPTION 4.1. *For all convex subset C of K, the set $\partial U(C)$ is convex.*

Notice that Assumption 4.1 is always true for convex functions defined on the real line. Example 5.3 provides an interesting utility function which does not satisfy the last assumption. Unfortunately, we are not able to prove whether this assumption is necessary for the main theorem of this paper to hold.

We shall also appeal to the following stringent condition.

ASSUMPTION 4.2. $\sup_{x \in K} U(x) = +\infty$.

Under this assumption, $\widetilde{U}(0) = +\infty$, and the solution of the dual problem $W(x)$ defined in (4.2) is guaranteed to be strictly positive P-a.s. We shall see that, whenever Assumption 4.2 does not hold, our main duality result remains valid provided that function $\widetilde{U}$ satisfies the Inada condition.

ASSUMPTION 4.3. $\sup_{x \in K} U(x) < \infty$ *and* $\liminf_{|y| \to 0} \inf_{q \in -\partial \widetilde{U}(y)} l(q) = +\infty$.

REMARK 4.1. In the one-dimensional smooth case with strictly concave utility function U, the second requirement of Assumption 4.3 is equivalent to the condition $U'(\infty) = 0$ (assumed in KS99), and holds whenever U is bounded. When U is not strictly concave, this is no longer true, as one can check easily in the example $U(x) = x \wedge a + \chi_{[0,\infty)}$ for some $a > 0$, $\widetilde{U}(y) = a(1-y)^+ + \chi_{[0,\infty)}$, where χ is the indicator function in the sense of convex analysis.

Another technical condition needed for the proof of our main result (precisely in Lemma 8.3) is the following.

ASSUMPTION 4.4. *Function $\widetilde{U}$ satisfies one of the following conditions*:

(A1) $\widetilde{U}(y) = \infty$ *for all* $y \in \partial K^*$. *In this case, set* $H := K^*$.

(A2) $\widetilde{U}$ *can be extended to an open convex cone* H *of* $\mathbb{R}^{d+1}$, *with* $K^*\backslash\{0\} \subset H \subset K$, *in such a way that the extended* $\widetilde{U}$ *on* H *is convex*, *bounded from below by* 0 *and decreasing in the sense of the partial ordering* $\succeq_*$.

Observe that the above condition (A2) is trivially satisfied in the one-dimensional case $d+1=1$. Indeed, in this case $K = K^* = \mathbb{R}_+$, and the only possible choice for H is $(0,\infty) = \operatorname{int}(K)$.

Unfortunately, we have not been able to remove this technical condition in the general multidimensional case, and we leave this issue as another challenging open problem. In Section 5, we shall see that Examples 5.2 and 5.3 satisfy (A1), while Example 5.1 satisfies (A2).

Our last assumption is a natural extension to the multidimensional framework of the asymptotic elasticity condition introduced by KS99. Consider the function

$$\delta_{-\partial\widetilde{U}}(y) := \sup_{q\in-\partial\widetilde{U}(y)} (qy)$$

and define the asymptotic elasticity of the convex function $\widetilde{U}$ by

$$AE(\widetilde{U}) = \limsup_{l^*(y)\to 0} \frac{\delta_{-\partial\widetilde{U}}(y)}{\widetilde{U}(y)}.$$

ASSUMPTION 4.5. $AE(\widetilde{U}) < \infty$.

We postpone the discussion of this assumption after the proof of Lemma 4.2 below, and we start by providing its relevant implications for the subsequent analysis of the paper.

LEMMA 4.1. $AE(\widetilde{U}) < \infty$ *if and only if there exist two parameters* $b, \beta > 0$ *such that*

(4.1) $\widetilde{U}(\mu y) < \mu^{-\beta}\widetilde{U}(y)$ *for all* $\mu \in (0,1]$ *and* $y \in K^*$ *with* $l^*(y) \le b$.

For the proof, see the Appendix.

Combining Lemmas 3.3 and 4.1, we obtain the following easy consequence.

COROLLARY 4.1. *Let condition* $AE(\widetilde{U}) < \infty$ *hold. Then*, *there exist constants* $C \ge 0$ *and* $\beta > 0$ *such that*, *for all* $\mu \in (0,1]$,

$$\widetilde{U}(\mu y) \le \mu^{-\beta}[C + \widetilde{U}(y)] \quad \textit{for all } y \in K^*.$$

Characterization (4.1) of Assumption 4.5 provides more specific information about the domain of $\widetilde{U}$.

LEMMA 4.2. *Let Assumption* 4.5 *hold*. *Then*:

(i) $\mathrm{int}(K^*) \subset \mathrm{dom}(\widetilde{U})$ *and therefore* $\mathrm{int}[\mathrm{dom}(\widetilde{U})] = \mathrm{int}(K^*)$,
(ii) *For all* $y \in \mathrm{int}(K^*)$, *we have* $\partial\widetilde{U}(y) \subset -K$.

PROOF. (i) Since U is a proper convex function, so is $\widetilde{U}$. Let $y_0 \in K^*\backslash\{0\}$ be such that $\widetilde{U}(y_0) < \infty$. Consider an arbitrary $y \in \mathrm{int}(K^*)$. For all $\varepsilon > 0$, observe that $l^*(y - \varepsilon y_0) \geq l^*(y) + \varepsilon l^*(-y_0)$ so that $\liminf_{\varepsilon \searrow 0} l^*(y - \varepsilon y_0) \geq l^*(y) > 0$ by Lemma 3.2. This proves that $y \succeq_* \varepsilon y_0$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 3.4, we see that $\widetilde{U}(y) \leq \widetilde{U}(\varepsilon y_0)$. Using Corollary 4.1, this proves that $\widetilde{U}(y) \leq \mu^{-\beta}[C + \widetilde{U}(y_0)] < \infty$. Hence $\mathrm{int}(K^*) \subset \mathrm{dom}(\widetilde{U})$. In view of (3.2), this proves that $\mathrm{int}[\mathrm{dom}(\widetilde{U})] = \mathrm{int}(K^*)$.

(ii) Let p be any element in $\partial\widetilde{U}(y)$ for some $y \in \mathrm{int}[\mathrm{dom}(\widetilde{U})]$. By definition, this means that $\widetilde{U}(z) \geq \widetilde{U}(y) + p(z - y)$ for all $z \in \mathbb{R}^{d+1}$. Set $z := y + h$ for some $h \succeq_* 0$. Then, it follows from (i) that

$$0 \geq \widetilde{U}(y + h) - \widetilde{U}(y) \geq ph \quad \text{for all } h \in K^*,$$

which ends the proof. □

We now turn to the discussion of Assumption 4.5. By analogy to $\widetilde{U}$, we define the asymptotic elasticity of the concave function U by

$$AE(U) := \limsup_{l(x)\to\infty} \frac{\delta_{\partial U}(x)}{U(x)} \quad \text{where } \delta_{\partial U}(x) := \sup_{p\in\partial U(x)} (px).$$

REMARK 4.2. From Remark 2.1, it is clear that above notion of asymptotic elasticity coincides with that of KS99 in the smooth case.

As in KS99, the following result states the equivalence between the conditions $AE(\widetilde{U}) < \infty$ and $AE(U) < 1$, under *Inada-type* conditions on U and $\widetilde{U}$.

PROPOSITION 4.1. (i) *Suppose that* $\limsup_{l(x)\to\infty} \sup_{p\in\partial U(x)} |p| = 0$. *Then*

$$AE(\widetilde{U}) < \infty \quad \Longrightarrow \quad AE(U) < 1.$$

(ii) *Suppose that* $\liminf_{|y|\to 0} \inf_{q\in -\partial\widetilde{U}(y)} l(q) = \infty$. *Then*

$$AE(U) < 1 \quad \Longrightarrow \quad AE(\widetilde{U}) < \infty.$$

For the proof, see the Appendix.

In the smooth one-dimensional framework, we have

$$\limsup_{l(x)\to\infty} \sup_{p\in\partial U(x)} |p| = U'(\infty) \quad \text{and} \quad \liminf_{|y|\to 0} \inf_{q\in -\partial \widetilde{U}(y)} l(q) = -\widetilde{U}'(0).$$

If in addition U is strictly concave, we have $\widetilde{U}' = -(U')^{-1}$, and the conditions $U'(\infty) = 0$ and $\widetilde{U}'(0) = -\infty$ are equivalent. Hence, Proposition 4.2 provides the equivalence between $AE(U) < 1$ and $AE(\widetilde{U}) < \infty$ under the Inada condition $U'(\infty) = 0$.

4.2. *Dual formulation.* We first recall an important result on the problem of super-replication. Denoting by $\mathscr{M}(P)$ the set of all P-martingales, we introduce the set

$$\mathscr{D} := \{Z \in \mathscr{M}(P)\colon \widehat{Z}_t \in K^*, 0 \le t \le T \ P\text{-a.s.}\},$$

which plays the same role as the set of equivalent martingale measures in frictionless financial markets. For some (positive) contingent claim $C \in L^0(K, \mathscr{F}_T)$, let

$$\Gamma(C) := \{x \in \mathbb{R}^{d+1}\colon X \succeq C \text{ for some } X \in \underline{\mathscr{X}}(x)\}.$$

THEOREM 4.1 [Kabanov and Last (1998)]. *Let S be a continuous process in $\mathscr{M}(Q)$ for some $Q \sim P$. Suppose further that* $\operatorname{int}(K^*) \neq \emptyset$, *Then*

$$\Gamma(C) = D(C) := \{x \in \mathbb{R}^{d+1}\colon E\widehat{Z}_T C - \widehat{Z}_0 x \le 0 \textit{ for all } Z \in \mathscr{D}\}.$$

REMARK 4.3. It is an easy exercise to check that, under condition (2.1), $\operatorname{int}(K^*) \neq \emptyset$ if an only if $\lambda^{ij} + \lambda^{ji} > 0$ for all $i, j = 0, \dots, d$.

For the purpose of this paper, we need to define a suitable extension of the set $\mathscr{D}$ and given some $y \in K^*$, we define the set

$$\mathscr{Y}(y) := \{Y \in L^0(K^*, \mathscr{F}_T)\colon EXY \le xy \text{ for all } x \in K \text{ and } X \in \mathscr{X}(x)\}.$$

REMARK 4.4. From the no-bankruptcy condition (2.3), it is easily checked that $\{\widehat{Z}_T\colon Z \in \mathscr{D} \text{ and } \widehat{Z}_0 = y\} \subset \mathscr{Y}(y)$.

We can now define the candidate dual problem,

$$W(x) := \inf_{y\in K^*,\, Y\in\mathscr{Y}(y)} (E\widetilde{U}(Y) + xy). \tag{4.2}$$

Since

$$\widetilde{U}(Y) \ge U(X) - XY \quad \text{for all } X \in \mathscr{X}(x),\ y \in K^* \text{ and } Y \in \mathscr{Y}(y),$$

it follows from the definition of the dual control set $\mathscr{Y}(y)$ that

$$V(x) \le W(x). \tag{4.3}$$

This proves in particular that the condition $W(x) < \infty$ guarantees that $V(x) < \infty$. The following is the main result of this paper.

THEOREM 4.2. *Let U be a utility function satisfying* (2.4) *together with Assumptions* 4.1, 4.2, 4.4 *and* 4.5. *Suppose further that the conditions of Theorem* 4.1 *hold*.

Let x be any initial wealth in $\operatorname{int}(K)$ *with* $W(x) < \infty$. *Then*:

(i) *Existence holds for the optimization problem* (4.2), *that is*,

$$W(x) = E\widetilde{U}(Y_*) + xy_* \quad \text{for some } y_* \in K^* \text{ and } Y_* \in \mathscr{Y}(y_*),$$

moreover, $P[Y_* = 0] = 0$.

(ii) *There exists some X_* valued in $-\partial\widetilde{U}(Y_*)$ such that*

$$X_* \in \mathscr{X}(x) \quad \text{and} \quad V(x) = EU(X_*),$$

(iii) $V(x) = W(x)$.

(iv) *Suppose that*

$$\mathscr{Y}(y_+) \cap L^0(K^*\backslash\{0\}, \mathscr{F}_T) \neq \varnothing \quad \text{for some } y_+ \in K^*. \tag{4.4}$$

Then the above claims (i)–(iii) *are still valid if Assumption* 4.3 *is substituted for Assumption* 4.2.

REMARK 4.5. The conditions of Theorem 4.1 are needed in Theorem 4.2 only in order to apply Theorem 4.1 directly. It is still a challenging open problem to derive Theorem 4.1 under weaker assumptions.

REMARK 4.6. Consider the following stronger version of (ii):

(ii′) For all random variable X_* valued in $-\partial\widetilde{U}(Y_*)$,

$$X_* \in \mathscr{X}(x) \quad \text{and} \quad V(x) = EU(X_*).$$

It is again a challenging open problem to prove that (ii′) holds. We thank D. Ocone for this interesting comment.

REMARK 4.7. In the frictionless case, $\lambda = 0$, (4.4) is implied by the existence of an equivalent local martingale measure for the price process S, that is,

$$S \in \mathscr{M}_{\text{loc}}(Q) \quad \text{for some } Q \sim P. \tag{4.5}$$

This condition is also sufficient in order for the result $\Gamma(C) = D(C)$ of Theorem 4.1 to hold; see Delbaen and Schachermayer (1998). Therefore, under (4.5), Theorem 4.2 is valid without the conditions of Theorem 4.1. Finally, recall that the utility function can be reduced to a function defined on the positive real line (see Remark 2.1), and therefore:

Assumptions 4.1 and 4.4 are trivially satisfied,
In the case of a strictly concave utility function, either Assumption 4.2 or Assumption 4.3 is trivially satisfied.

In summary, when $\lambda = 0, U$ is a strictly concave function satisfying (2.4), and S satisfies (4.5), statements (i)–(iii) of Theorem 4.2 are valid under Assumption 4.5 on the asymptotic elasticity of $\widetilde{U}$.

The details of the proof will be reported in the following sections. For the convenience of the reader, we present here its main steps. The main difficulty arises from the nonsmoothness of the utility function and its Legendre–Fenchel transform. We then start in Section 6 by introducing a suitable approximation $\widetilde{U}^n$ of $\widetilde{U}$. By substituting $\widetilde{U}^n$ with $\widetilde{U}$, we define a sequence of approximate dual problems W^n. Let $\mathscr{S}(x)$ [resp. $\mathscr{S}^n(x)$] denote the set of all possible solutions of the optimization problem $W(x)$ [resp. $W^n(x)$]. We proceed as follows.

(i) For each n, we prove in Section 7 that $\mathscr{S}^n(x) \neq \emptyset$; that is, $W^n(x) = E\widetilde{U}^n(Y^n) + xy^n$ for some $y^n \in K^*$ and $Y^n \in \mathscr{Y}(y^n)$.

(ii) By means of a calculus of variations technique, we find in Section 8 that the optimality of (y^n, Y^n) leads to the existence of a sequence $(Z^n)_n$, and the r.v. $X^n = -D\widetilde{U}^n(Y^n) \in (\partial\widetilde{U} + N_{\overline{H}})(Z^n)$ such that X^n is "approximately" in $\mathscr{X}(x)$. After passing to appropriate convex combinations, we prove that the sequence $(Z^n)_n$ converges to some $Y_* \in \mathscr{S}(x)$, and $X^n \to X_* \in -\partial\widetilde{U}(Y_*)$ P-a.s. We then show that X_* lies in $\mathscr{X}(x)$ by using Theorem 4.1.

(iii) Now, the proof of Theorem 4.2 is easily completed in the last section. Indeed, optimality of X_* for the initial optimization problem $V(x)$ is now a direct consequence of the Kuhn–Tucker system. Thus equality between $V(x)$ and $W(x)$ follows and duality holds.

5. Main examples. We now provide three natural examples of utility functions consistent with the condition of $\succeq$-increase. The first example is the usual utility of the liquidation value of the terminal wealth process, in which U is not smooth. The second one shows that the presence of constraints in the definition of $\widetilde{U}$ produces a lack of regularity even in the case where U is smooth. In the third example, both U and $\widetilde{U}$ are smooth. The first two examples will be shown to satisfy all the conditions of Theorem 4.2, while the last example does not satisfy Assumption 4.1.

We shall use the characterization of function $\widetilde{U}$ by means of Lagrange multipliers. Denoting by $-\partial U$ the subgradient of the convex function $-U$, it follows from the classical Kuhn–Tucker theory that, for all $y \in \mathrm{dom}(\widetilde{U})$, the supremum in the definition of $\widetilde{U}(y)$ is attained at by some $x_y^* \in K$ characterized by the following system:

$$y - \mu^* \in \partial U(x_y^*) \quad \text{for some } \mu^* \in K^* \text{ with } \mu^* x_y^* = 0. \tag{5.1}$$

Conversely, if $x_y^* \in K$ satisfies (5.1), then it is a point of maximum in the definition of $\widetilde{U}(y)$, and

$$\widetilde{U}(y) = U(x_y^*) - yx_y^*.$$

For ease of exposition, we only work out these examples for the one-dimensional case $d = 1$. Then, it is easily checked that the solvency region is the closed convex cone generated by the $\mathbb{R}^2$ vectors

$$v_1 := \alpha_1\big(1, -(1+\lambda^{10})^{-1}\big) \quad \text{and} \quad v_2 := \alpha_2(-1, 1+\lambda^{01}),$$

where $\alpha_1 := [1-(1+\lambda^{10})^{-1}(1+\lambda^{01})^{-1}]^{-1}$ and $\alpha_2 := [-1+(1+\lambda^{10})(1+\lambda^{01})]^{-1}$. We denote by (v_1^*, v_2^*) the dual basis of (v_1, v_2) in $\mathbb{R}^2$, that is, $v_i^* v_j = \delta_{ij}$. Direct computation provides

$$v_1^* = \big(1, (1+\lambda^{01})^{-1}\big) \quad \text{and} \quad v_2^* = (1, 1+\lambda^{10}).$$

Clearly, the positive polar cone K^* is generated by (v_1^*, v_2^*). We shall assume that K^* has nonempty interior or, equivalently, $\lambda^{10} + \lambda^{01} > 0$.

EXAMPLE 5.1. Let $u\colon \mathbb{R}_+ \to \mathbb{R}$ be a C^1 increasing and strictly concave function with $u(0) = 0, u(+\infty) = +\infty, u'(0) = +\infty$ and $u'(+\infty) = 0$. Following Cvitanić and Karatzas (1996), Kabanov (1999) and Cvitanić and Wang (1999), we consider the utility function,

$$U(x) := u(l(x)) = u(\min(xv_1^*, xv_2^*)) = u\big(xv_1^*\mathbf{1}_{\{x^1\ge 0\}} + xv_2^*\mathbf{1}_{\{x^1<0\}}\big) \quad \text{for all } x \in K.$$

Observe that U is not differentiable along the half line $\{x \in K\colon x^1 = 0\} = \{(x^0, 0)\colon x^0 \ge 0\}$. In order to compute explicitly the Legendre–Fenchel transform $\widetilde{U}$, we solve the Kuhn–Tucker system (5.1), that is, find $(x, \mu_1, \mu_2) \in K \times \mathbb{R}_+^2$ such that

$$y - \mu_1 v_1^* - \mu_2 v_2^* \in \partial U(x) \quad \text{and} \quad \mu_1 x v_1^* + \mu_2 x v_2^* = 0.$$

(i) Suppose that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Then, $xv_1^* = xv_2^* = 0$ and then $x = 0$, which leads to a contradiction since $l(0) = 0$ and $u'(0) = +\infty$.

(ii) Suppose that $\mu_1 = 0$ and $\mu_2 \neq 0$. Then $xv_2^* = 0$ and therefore $x \in \operatorname{cone}(v_1) \subset \partial K$. It follows that $l(x) = 0$ and the Kuhn–Tucker system cannot be satisfied because of the condition $u'(0) = +\infty$.

(iii) The case $\mu_2 = 0$ and $\mu_1 \neq 0$ is similar to the previous one and leads to the same conclusion.

(iv) From the previous cases, we see that we must have $\mu_1 = \mu_2 = 0$ in order for the pair (x, μ) to solve the Kuhn–Tucker system. We now consider three cases depending on the sign of x^1.

Suppose that $x^1 > 0$. Then U is differentiable at the point x and the Kuhn–Tucker system reduces to $y = u'(l(x))v_1^*$. Then, direct calculation shows that

$$y = y^0 v_1^* \quad \text{and} \quad \widetilde{U}(y) = \tilde{u}(y^0) \quad \text{for all } y^0 > 0,$$

where $\tilde{u}$ is the one-dimensional Legendre–Fenchel transform as in the previous example.

The case $x^1 < 0$ is treated by analogy with the previous one and provides

$$y = y^0 v_2^* \quad \text{and} \quad \widetilde{U}(y) = \tilde{u}(y^0) \quad \text{for all } y^0 > 0,$$

where $\tilde{u}$ is the one-dimensional Fenchel–Legendre transform as in the previous example.

Finally suppose that $x^1 = 0$. Then $\partial l(x) = \{(1, \rho): (1+\lambda^{10})^{-1} \le \rho \le 1+\lambda^{01}\}$. By direct calculation, we see that

$$y = y^0(1, \rho) \quad \text{and} \quad \widetilde{U}(y) = \tilde{u}(y^0) \quad \text{for all } y^0 > 0.$$

In conclusion, the function $\widetilde{U}$ is finite on $K^*\setminus\{0\}$, and

$$\widetilde{U}(y) = \tilde{u}(y^0) \quad \text{for all } y \in K^*\setminus\{0\}.$$

Clearly, Assumptions 4.1, 4.2 and 4.4(A2) are satisfied. To see that Assumption 4.5 holds, we compute that $\widetilde{U}$ has a singular gradient given by

$$D\widetilde{U}(y) = \tilde{u}'(y^0)\mathbf{1}_0.$$

This shows that $AE(\widetilde{U})$ is finite since $AE(\tilde{u})$ is finite or equivalently $AE(u)$ is strictly smaller than 1.

Let us conclude the discussion of this example by comparing our main Theorem 4.2 to Theorem 2.1 in Cvitanić and Wang (1999), CW hereafter. CW derived the dual formulation of the utility maximization problem under the condition $(*)$ $wu'(w) \le a + (1-b)u(w)$ for all $w > 0$, for some $a > 0$ and $0 < b \le 1$. From Lemmas 6.2 and 6.3 in KS99, observe that condition $(*)$ implies that $AE(u) = 1 - b < 1$. Hence Assumption 4.5 is weaker than condition $(*)$ in the one-dimensional case $(d = 1)$ studied by CW.

EXAMPLE 5.2. Let r be an arbitrary element of $\operatorname{int}(K^*)$ and let

$$\rho_i := (rv_i)^{-1}; \quad i = 1, 2 \text{ so that } r = \rho_1^{-1}v_1^* + \rho_2^{-1}v_2^*.$$

Consider the utility function

$$U(x) = u(rx) \quad \text{for all } x \in K,$$

where $u: \mathbb{R}_+ \to \mathbb{R}$ is a C^1 increasing, strictly concave function satisfying $u'(0+) = +\infty$ and $u'(+\infty) = 0$. Clearly, U is strictly concave and increasing in the sense of the partial ordering $\succeq$, and Assumption 4.1 holds. We further impose the conditions $u(0) = 0$ and $u(\infty) = \infty$ in order to satisfy the requirement of (2.4) and Assumption 4.2.

It remains to check that Assumptions 4.4 and 4.5 hold. In order to compute explicitly the Legendre–Fenchel transform $\widetilde{U}$, we solve the Kuhn–Tucker system (5.1). Denote by $\tilde{u}$ the one-dimensional Legendre–Fenchel transform $\tilde{u}(\zeta) = \sup_{\xi \ge 0}(u(\xi) - \xi\zeta)$.

(i) If μ_1 and μ_2 are both nonzero, then $x_y^* v_1^* = x_y^* v_2^* = 0$, which cannot happen unless $x_y^* = 0$, but this does not solve the first-order condition.

(ii) If $\mu_1 = \mu_2 = 0$, then $y = \lambda r$ for some $\lambda > 0$ and $\widetilde{U}(y) = \tilde{u}(\lambda) = \tilde{u}(|r|^{-2} yr)$.

(iii) If $\mu_i = 0$ and $\mu_{i-1} > 0$ for $i = 1, 2$, then $x_y^* = \xi v_i$ for some $\xi > 0$, and $y = \mu_{i-1} v_{i-1}^* + u'(r x_y^*) r$. This proves that $y \in \text{cone}(r, v_{i-1}^*)$, and provides $\xi = \rho_i (u')^{-1}(\rho_i y v_i)$, by taking scalar product with v_i.

Hence,

$$\widetilde{U}(y) = \tilde{u}(\rho_i y v_i) \quad \text{for all } y \in K^* \backslash \text{cone}(r, v_i^*).$$

By continuity, this clearly defines function $\widetilde{U}$ for all $y \in K^* \backslash \{0\}$. In particular, $\widetilde{U}(\lambda r) = \tilde{u}(|r|^{-2} yr)$ for all $\lambda > 0$. Observe that:

$\widetilde{U}(y) = +\infty$ for all $y \in \partial K^*$ so that condition (A1) of Assumption 4.4 holds.

$\widetilde{U}$ is not differentiable at any element of cone(r), and

$$\partial \widetilde{U}(y) = \begin{cases} \tilde{u}'(\rho_i y v_i) \rho_i v_i, & \text{for } y \in \text{int}(K^* \backslash \text{cone}(r, v_i^*)), \\ \tilde{u}'(\lambda)[\rho_1 v_1, \rho_2 v_2], & \text{for } y = \lambda r; \lambda > 0, \end{cases}$$

where $[\rho_1 v_1, \rho_2 v_2] = \{\mu \rho_1 v_1 + (1-\mu)\rho_2 v_2 \colon 0 \le \mu \le 1\}$. Since

$$\sup_{q \in -\partial \widetilde{U}(\lambda r)} q \lambda r = \sup_{0 \le \mu \le 1} -\tilde{u}'(\lambda)(\mu \rho_1 v_1 + (1-\mu)\rho_2 v_2)\lambda r = -\tilde{u}'(\lambda)\lambda \quad \text{for all } \lambda > 0,$$

it follows that

$$AE(\widetilde{U}) = AE(\tilde{u}) = \limsup_{\zeta \to 0} \frac{-\zeta \tilde{u}'(\zeta)}{\tilde{u}(\zeta)}.$$

Hence, from Lemma 6.3 in KS99, Assumption 4.5 is satisfied in this example whenever $AE(u) < 1$.

EXAMPLE 5.3. Consider the utility function

$$U(x) = u_1(x v_1^*) + u_2(x v_2^*) \quad \text{for all } x \in K,$$

where for $j = 1, 2$, $u_j \colon \mathbb{R}_+ \to \mathbb{R}$ is a C^1 increasing, strictly concave function satisfying $u_j'(0+) = +\infty$, $u_j'(+\infty) = 0$, $u_j(0) = 0$, and $u_j(\infty) = \infty$. Clearly, U is strictly concave and increasing in the sense of the partial ordering $\succeq$, and conditions (2.4) together with Assumption 4.2 are satisfied.

We compute explicitly the Legendre–Fenchel transform $\widetilde{U}$ by solving the Kuhn–Tucker system (5.1). It turns out that the Lagrange multiplier is zero so that the Kuhn–Tucker system reduces to

$$y = \sum_{j=1,2} u_j'(x v_j^*) v_j^*.$$

Since $(v_1^{**}, v_2^{**}) = (v_1, v_2)$, it follows from uniqueness of the representation of y in the basis (v_1^*, v_2^*) of $\mathbb{R}^2$ that $u'_j(xv_j^*) = yv_j$, and therefore

$$\widetilde{U}(y) = \tilde{u}_1(yv_1) + \tilde{u}_2(yv_2),$$

where $\tilde{u}_j$ is the one-dimensional Legendre–Fenchel transform of $-u_j(-\cdot)$.

Clearly, condition (A1) of Assumption 4.4 is satisfied. Moreover, $\widetilde{U}$ is differentiable and

$$\widetilde{U}'(y) = \sum_{j=1,2} \tilde{u}'_j(yv_j)v_j$$

so that Assumption 4.5 is satisfied whenever $AE(u_j) < 1$ for $j = 1, 2$. However, Assumption 4.1 is not satisfied. Indeed, take two arbitrary vectors x_1 and x_2 in $\operatorname{int}(K)$, and compute for $\lambda \in (0, 1)$,

$$\lambda U'(x_1) + (1-\lambda)U'(x_2) = \sum_{j=1,2} \big[\lambda u'_j(x_1 v_j^*) + (1-\lambda)u'_j(x_2 v_j^*)\big]v_j^*.$$

Suppose to the contrary that Assumption 4.1 holds. Then

$$\begin{aligned}\sum_{j=1,2} \big[\lambda u'_j(x_1 v_j^*) + (1-\lambda)u'_j(x_2 v_j^*)\big]v_j^* &= U'(\mu x_1 + (1-\mu)x_2)\\ &= \sum_{j=1,2} u'_j\big(\mu x_1 v_j^* + (1-\mu)x_2 v_j^*\big)v_j^*.\end{aligned}$$

Setting $\xi_{ij} := v_j^* x_i$, and recalling that $x_i = \xi_{i1}v_1 + \xi_{i2}v_2$, this provides

$$\lambda u'_j(\xi_{1j}) + (1-\lambda)u'_j(\xi_{2j}) = u'_j(\mu\xi_{1j} + (1-\mu)\xi_{2j}) \quad \text{for } j = 1, 2.$$

Since μ does not depend on j, it is easy to build examples of functions u_j so that these equalities cannot hold simultaneously.

6. Approximation by quadratic inf-convolution. Let H be the open convex cone introduced in Assumption 4.4; that is, $H = \operatorname{int}(K^*)$ under (A1) and $K^* \subset H$ under (A2).

Let $n \geq 1$ be an arbitrary integer. Following Aubin (1984) or Clarke, Ledyaev, Stern and Wolenski (1998), we define the quadratic inf-convolution approximation of $\widetilde{U}$ by

$$\widetilde{U}^n(y) := \inf_{z \in \overline{H}} \left(\widetilde{U}(z) + \frac{n}{2}|z-y|^2 \right) \quad \text{for all } y \in \mathbb{R}^{d+1},$$

where $\overline{H}$ is the closure of H in $\mathbb{R}^{d+1}$. For each $n \geq 1$, $\widetilde{U}^n$ is finite on $\mathbb{R}^{d+1}$, and strictly convex in there. Since $\widetilde{U}$ is nonnegative, we have

$$0 \leq \widetilde{U}^n(y) \leq \widetilde{U}(y) \quad \text{for all } y \in \mathbb{R}^{d+1}. \tag{6.1}$$

In order to handle the nonsmoothness of the utility function U, we define the approximate dual problems,

$$W^n(x) := \inf_{y \in K^*,\, Y \in \mathscr{Y}(y)} \big(E\widetilde{U}^n(Y) + xy\big).$$

From (6.1), we have

$$W^n(x) \le W(x) \quad \text{for all } x \in K.$$

In the remaining part of this section, we state several properties of $\widetilde{U}^n$ which are extremely important for the subsequent analysis.

PROPERTY 1. *For all $y \in \mathbb{R}^{d+1}$, there exists a unique $z^n(y) \in \overline{H}$ such that*

$$\widetilde{U}^n(y) = \widetilde{U}(z^n(y)) + \frac{n}{2}|z^n(y) - y|^2.$$

PROOF. This follows by direct application of Theorem 2.2, page 21, in Aubin (1984) to the function $F(z) = \widetilde{U}(z) + \chi_{\overline{H}}(z)$ where $\chi_{\overline{H}}(z) = 0$ on $\overline{H}$ and $+\infty$ otherwise, is the characteristic function of $\overline{H}$ in the sense of convex analysis. □

PROPERTY 2. (i) *For all $x \in K$ and $y \in \mathrm{dom}(\widetilde{U}^n)$, we have $|z^n(y) - y|^2 \le \frac{4}{n}[\widetilde{U}^n(y) + xy + C]$, for some constant C.*

(ii) *Let $(y^n)_n$ be a sequence converging to $y \in \mathrm{dom}(\widetilde{U})$. Then*

$$z^n(y^n) \to y.$$

(iii) *Let $(y^n)_n$ be a sequence converging to y. Suppose further that $z^n(y^n) \to y$. Then*

$$\widetilde{U}^n(y^n) \to \widetilde{U}(y).$$

For the proof, see the Appendix.

PROPERTY 3. *Function $\widetilde{U}^n$ is continuously differentiable on $\mathbb{R}^{d+1}$ and*

$$D\widetilde{U}^n(y) = n(y - z^n(y)) \in (\partial\widetilde{U} + N_{\overline{H}})(z^n(y)),$$

where $N_{\overline{H}}(z) := \{\xi \in \mathbb{R}^{d+1}\colon \xi z \ge \xi y \text{ for all } y \in \overline{H}\}$ is the normal cone to $\overline{H}$ at point z.

PROOF. Applying Theorem 5.2, page 66, of Aubin (1984) to the function $f(y) = \widetilde{U}(y) + \chi_{\overline{H}}(y)$, it follows that

$$D\widetilde{U}^n(y) = n(y - z^n(y)) \in \partial(\widetilde{U} + \chi_{\overline{H}})(z^n(y)).$$

The required result follows from Theorem 4.4, page 52 in Aubin (1984) and the definition of normal cones. □

PROPERTY 4. *Suppose that $AE(\widetilde{U}) < \infty$. Then, there exist positive constants $C \ge 0$ and $\beta > 0$ such that, for all $n \ge 1$,*

$$\widetilde{U}^n(\mu y) \le \mu^{-\beta}\big(C + \widetilde{U}^n(y)\big) \quad \text{for all } \mu \in (0,1] \text{ and } y \in \mathbb{R}^{d+1}.$$

PROOF. By a trivial change of variable, it follows from the cone property of H that:

$$\widetilde{U}^n(\mu y) = \mu^2 \inf_{z\in\overline{H}}\left(\mu^{-2}\widetilde{U}(\mu z) + \frac{n}{2}|z-y|^2\right).$$

Using Corollary 4.1, this provides

$$\widetilde{U}^n(\mu y) \le \mu^{-\beta}C + \mu^{-\beta}\inf_{z\in\overline{H}}\left(\widetilde{U}(z) + \mu^{\beta+2}\frac{n}{2}|z-y|^2\right)$$

and the required result from the fact that $\mu^{\beta+2} \le 1$. □

7. Existence for the dual problems. We recall the notation $\mathscr{S}^n(x)$ and $\mathscr{S}(x)$ for the set of all possible solutions of the optimization problems $W^n(x)$ and $W(x)$. We first show in Lemma 7.1 that for all $n \ge 0$, there exists a solution to problem $W^n(x)$. We then show in Lemma 7.2 the existence for the dual problem $W(x)$. In Corollary 7.2, we establish the convergence of the value functions $W^n(x)$ toward $W(x)$. We conclude this section by stating a stronger technical convergence result that will be needed in the following section.

LEMMA 7.1. *Consider some initial wealth x in* $\mathrm{int}(K)$ *satisfying* $W(x) < \infty$. *Then* $\mathscr{S}^n(x) \neq \varnothing$ *for all* $n \ge 1$.

PROOF. Let $n \ge 1$ be a fixed integer. Let $(y^k, Y^k)_k$ be a minimizing sequence of $W^n(x)$. If the set $\{k \ge 0\colon y^k = 0\}$ is infinite, then $(y^k, Y^k) \to (\tilde{y}, \widetilde{Y}) = 0$ along a subsequence, and the result of the lemma is trivial. We then specialize the discussion to the nontrivial case where $\{k \ge 0\colon y^k = 0\}$ is finite. By passing to a subsequence, we can assume this set to be empty.

Since $\widetilde{U}^n \ge 0$, it follows from (6.1) that $\infty > W(x) \ge W^n(x) \ge xy^k - 1 \ge w^k l(x) - 1$, where $w^k := (y^k)^0$ is the first component of the $\mathbb{R}^{d+1}$ vector y^k. Recall that $x \in \mathrm{int}(K)$. Then it follows from Lemma 3.1 that $l(x) > 0$ and therefore the sequence $(w^k)_k$ is bounded. Now observe that $\{y \in K^*\colon y^0 = 1\}$ is a compact subset of $\mathbb{R}^{d+1}$, which proves that the sequence $(y^k/w^k)_k$ is bounded, and therefore the sequence $(y^k)_k$ is bounded. By possibly passing to a subsequence, this implies the existence of $\tilde{y} \in K^*$ such that

$$y^k \to \tilde{y} \quad \text{as } k \to \infty.$$

Next, since $S_T = X_T^{S_0, 0} \in \mathscr{X}(S_0)$, it follows from the definition of the set $\mathscr{Y}(y^k)$ that $E|Y^k S_T| = EY^k S_T \le S_0 y^k$. Then, the sequence $(Y^k S_T)_k$ is bounded in L^1 norm. By Komlòs theorem [see, e.g., Hall and Heyde 1980], we deduce the existence of a sequence $\widetilde{Y}^k \in \mathrm{conv}(Y^j, j \ge k)$ such that

$$\widetilde{Y}^k \to \widetilde{Y} \quad P\text{-a.s.};$$

recall that $S_T^i > 0$ P-a.s. for all $i = 1, \dots, d$. Clearly, $\widetilde{Y}$ is valued in K^* and $\widetilde{Y}^k \in \mathscr{Y}(\tilde{y}^k)$, where $\tilde{y}^k$ is the corresponding convex combination of $(y^j, j \ge 0)$. By Fatou's lemma, we also have $EX\widetilde{Y} \le x\tilde{y}$ for all $X \in \mathscr{X}(x)$; recall that $X \in K$ and $\widetilde{Y}^k \in K^*$. Hence $\widetilde{Y} \in \mathscr{Y}(\tilde{y})$. Now, from the convexity of $(y, Y) \mapsto$

$\widetilde{U}^n(Y) + xy$, it follows that $(\tilde{y}^k, \widetilde{Y}^k)_k$ is also a minimizing sequence of W^n. Since $\widetilde{U} \geq 0$, we get by Fatou's lemma,

$$W^n(x) \leq E\widetilde{U}^n(\widetilde{Y}) + x\tilde{y} \leq \liminf_{k\to\infty} E\widetilde{U}^n(\widetilde{Y}^k) + x\tilde{y}^k = W^n(x).$$

This proves that $(\tilde{y}, \widetilde{Y}) \in \mathscr{S}^n(x)$. □

LEMMA 7.2. *Consider some initial wealth x in* $\operatorname{int}(K)$ *satisfying* $W(x) < \infty$. *For each $n \geq 1$, let (y^n, Y^n) be an arbitrary element of $\mathscr{S}^n(x)$. Then, there exists a sequence* $(\bar{y}^n, \overline{Y}^n) \in \operatorname{conv}((y^k, Y^k), k \geq n)$ *such that*

$$(\bar{y}^n, \overline{Y}^n) \to (y_*, Y_*) \in \mathscr{S}(x), \quad P\text{-a.s. and } E\widetilde{U}^n(\overline{Y}^n) \to E\widetilde{U}(Y_*).$$

PROOF. Since $\widetilde{U}^n \geq 0$, it follows from (6.1) that $\infty > W(x) \geq W^n(x) \geq xy^n \geq w^n l(x)$, where $w^n := (y^n)^0$ is the first component of the $\mathbb{R}^{d+1}$ vector y^n. By the same argument as in the previous proof, $y^n \to y_* \in K^*$ along a subsequence, and there exists a sequence $\overline{Y}^n \in \operatorname{conv}(Y^j, j \geq n)$ such that $\overline{Y}^n \to Y_*$ P-a.s. and $Y_* \in \mathscr{Y}(y_*)$.

Let $(\lambda^{n,j})_{j\geq n}$ be the coefficients of the above convex combination. From the convexity of $\widetilde{U}^n$ and the increase of $\widetilde{U}^n$ in n, we see that

$$\widetilde{U}^n(\overline{Y}^n) \leq \sum_{j\geq n} \lambda^{n,j} \widetilde{U}^n(Y^j) \leq \sum_{j\geq n} \lambda^{n,j} \widetilde{U}^j(Y^j).$$

Taking expectations, and using Property 1 of the quadratic inf-convolution approximation, as well as (6.1), we see that for $\overline{Y}^n$ and the corresponding convex combination $\bar{y}^n$ of $(y^j; j \geq n)$,

$$\begin{aligned} E\widetilde{U}\big(z^n(\overline{Y}^n)\big) + x\bar{y}^n &= E\widetilde{U}^n(\overline{Y}^n) - \frac{n}{2}|z^n(\overline{Y}^n) - \overline{Y}^n|^2 + x\bar{y}^n \\ &\leq E\widetilde{U}^n(\overline{Y}^n) + x\bar{y}^n \\ &\leq \sum_{j\geq n} \lambda^{n,j}\big[E\widetilde{U}^j(Y^j) + xy^j\big] \\ &= \sum_{j\geq n} \lambda^{n,j} W^j(x) \leq W(x). \end{aligned} \tag{7.1}$$

Using Property 2(i) of the inf-convolution approximation, we see that

$$E|z^n(\overline{Y}^n) - \overline{Y}^n|^2 \leq \frac{4}{n}[C + W(x)]$$

for some constant C. Therefore, $z^n(\overline{Y}^n) - \overline{Y}^n \to 0$ in L^2 norm. Since $\overline{Y}^n \to Y_*$ P-a.s. this proves that $z^n(\overline{Y}^n) \to Y_*$ P-a.s. along some subsequence. We now take limits in (7.1). In view of Property 2(iii), it follows from Fatou's lemma that $E\widetilde{U}(Y_*) + xy_* \leq W(x)$. Since $y_* \in K^*$ and $Y_* \in \mathscr{Y}(y_*)$, this proves that $(y_*, Y_*) \in \mathscr{S}(x)$. The previous inequalities also provide the convergence of $E\widetilde{U}^n(\overline{Y}^n)$ towards $E\widetilde{U}(Y_*)$. □

COROLLARY 7.1. *Let x in* $\operatorname{int}(K)$ *be such that* $W(x) < \infty$. *Then, the sequence $W^n(x)$ converges towards $W(x)$.*

PROOF. Observe that the sequence $(W^n(x))_n$ is increasing. Since $W^n(x) \le W(x)$ by (6.1), we have $W^n(x) \to W^\infty(x)$ for some $W^\infty(x) \le W(x)$. We now use the same argument as in the previous proof to get

$$E\widetilde{U}^n(\overline{Y}^n) + x\bar{y}^n \le \sum_{k\ge n} \lambda^{n,k} W^k(x) \le W(x).$$

Taking limits, it follows from the previous lemma that $W(x) \le W^\infty(x) \le W(x)$. Then $W^\infty(x) = W(x)$. □

COROLLARY 7.2. *Consider some initial wealth x in* int(K) *satisfying $W(x) < \infty$. For each n, let (y^n, Y^n) be an arbitrary element in $\mathscr{S}^n(x)$, and let $(y_*, Y_*) \in \mathscr{S}(x)$ be the limit defined in Lemma 7.2. Set $J^n := \widetilde{U}^n(Y^n)$.*

Then there exists a sequence $(y_*^n, Y_*^n, J_*^n) \in$ conv$((y^k, Y^k, J^k), k \ge n)$ *such that*

$$(y_*^n, Y_*^n) \to (y_*, Y_*),\ P\text{-a.s.} \quad \text{and} \quad J_*^n \to \widetilde{U}(Y_*) \text{ in } L^1(P).$$

PROOF. From Lemma 7.2, there exists a sequence $(\bar{y}^n, \overline{Y}^n) \in$ conv$((y^k, Y^k), k \ge n)$ which converges P-a.s. to $(y_*, Y_*) \in \mathscr{S}(x)$. Denote by $(\lambda^{n,k}, k \ge n)$ the coefficients defining the convex combination, and set $\overline{J}^n := \sum_{k\ge n} \lambda^{n,k} J^k$.

First, observe that $E\overline{J}^n + x\bar{y}^n = \sum_{k\ge n} \lambda^{n,k} W^k(x) \to W(x)$ by Corollary 7.1, and then $E\overline{J}^n \to E\widetilde{U}(Y_*)$. Since $\overline{J}^n \ge 0$ for all n, this proves that the sequence $(\overline{J}^n)_n$ is bounded in $L^1(P)$. From Komlòs theorem, we can then deduce the existence of a sequence $J_*^n \in$ conv$(\overline{J}^k, k \ge n) =$ conv$(J^k, k \ge n)$ and an integrable r.v. J_*, such that

$$J_*^n \to J_*, \quad P\text{-a.s. and } EJ_*^n \to E\widetilde{U}(Y_*),$$

where we used again Corollary 7.1. We shall denote by $(\lambda_*^{n,k}, k \ge n)$ the coefficients defining this new convex combination. Set $(y_*^n, Y_*^n) := \sum_{k\ge n} \lambda_*^{n,k}(y^k, Y^k)$. Since $(y_*^n, Y_*^n) \in$ conv$((\bar{y}^k, \overline{Y}^k), k \ge n)$, we have

$$(y_*^n, Y_*^n) \to (y_*, Y_*), \quad P\text{-a.s.}$$

Next, it follows from the increase of $\widetilde{U}^n$ in n, as well as the convexity of $\widetilde{U}^n$ that

$$J_*^n = \sum_{k\ge n} \lambda_*^{n,k} \widetilde{U}^k(Y^k) \ge \sum_{k\ge n} \lambda_*^{n,k} \widetilde{U}^n(Y^k) \ge \widetilde{U}^n(Y_*^n).$$

Using Property 2 of the quadratic inf-convolution (as in the end of the proof of Lemma 7.2), this proves that $J_* \ge \widetilde{U}(Y_*)$ P-a.s. On the other hand, it follows from Fatou's lemma that $E\widetilde{U}(Y_*) = \lim_n EJ_*^n \ge EJ_*$. This proves that $J_* = \widetilde{U}(Y_*)$ P-a.s.

We have then established that $J_*^n \to \widetilde{U}(Y_*)$ P-a.s. and $EJ_*^n \to E\widetilde{U}(Y_*)$. Since $J_*^n \ge 0$ P-a.s., this proves that $J_*^n \to \widetilde{U}(Y_*)$ in $L^1(P)$; see, for example, Shiryaev (1995).

8. Attainability. We first start by characterizing the optimality of $(y^n, Y^n) \in \mathscr{S}^n(x)$ by the classical technique of calculus of variation.

LEMMA 8.1. *Let Assumption* 4.5 *hold, and consider some initial wealth* $x \in \operatorname{int}(K)$ *satisfying* $W(x) < \infty$. *For each* n, *let* (y^n, Y^n) *be an arbitrary element of* $\mathscr{S}^n(x)$. *Set* $X^n := -D\widetilde{U}^n(Y^n) = n(z^n(Y^n) - Y^n)$; *see Property* 3. *Then*,

$$EX^n(Y - Y^n) \le x(y - y^n) \quad \textit{for all } y \in K^* \textit{ and } Y \in \mathscr{Y}(y).$$

PROOF. Let $y \in K^*$ and $Y \in \mathscr{Y}(y)$ be fixed. Set

$$(\zeta_\varepsilon^n, \xi_\varepsilon^n) := (1-\varepsilon)(y^n, Y^n) + \varepsilon(y, Y), \qquad Z_\varepsilon^n := z^n(\xi_\varepsilon^n)$$

and

$$X_\varepsilon^n := -D\widetilde{U}^n(\xi_\varepsilon^n) = n(Z_\varepsilon^n - \xi_\varepsilon^n).$$

Clearly, as $\varepsilon \searrow 0$, $\xi_\varepsilon^n \to Y^n$, $Z_\varepsilon^n \to Z^n := z^n(Y^n)$ and $X_\varepsilon^n \to X^n$ P-a.s.

By the optimality of (y^n, Y^n) for the problem $W^n(x)$ and the convexity of $\widetilde{U}^n$, we have

$$0 \ge E\big[\widetilde{U}^n(Y^n) - \widetilde{U}^n(\xi_\varepsilon^n)\big] + x(y^n - \zeta_\varepsilon^n) \ge -EX_\varepsilon^n(Y^n - \xi_\varepsilon^n) + x(y^n - \zeta_\varepsilon^n).$$

Dividing by ε, this provides

$$EX_\varepsilon^n(Y - Y^n) - x(y - y^n) \le 0.$$

In order to prove the required result, it remains to check that

$$\liminf_{\varepsilon \searrow 0} EX_\varepsilon^n(Y - Y^n) \ge EX^n(Y - Y^n).$$

To prove this, we intended to show that the sequence $(X_\varepsilon^n(Y - Y^n))_\varepsilon$ is bounded from below by some integrable random variable independent of ε, which allows applying Fatou's lemma.

Let $\alpha > 0$ be a given parameter. By convexity of $\widetilde{U}^n$, we see that

$$\widetilde{U}^n\big((1-\varepsilon-\alpha)Y^n\big) \ge \widetilde{U}^n\big(\xi_\varepsilon^n + \alpha(Y - Y^n)\big) - (\varepsilon+\alpha)YD\widetilde{U}^n\big(\xi_\varepsilon^n + \alpha(Y - Y^n)\big).$$

From Property 3 of the quadratic inf-convolution,

$$D\widetilde{U}^n\big(\xi_\varepsilon^n + \alpha(Y - Y^n)\big) \in \big(\partial\widetilde{U} + N_{\overline{H}}\big)\big(z^n(\xi_\varepsilon^n + \alpha(Y - Y^n))\big) \subset -K,$$

since $\widetilde{U}$ is decreasing in the sense of $\succeq_*$ on H (see Lemma 3.4 and Assumption 4.4) and by the definition of H. Then $YD\widetilde{U}^n(\xi_\varepsilon^n + \alpha(Y - Y^n)) \le 0$. Using again the convexity of $\widetilde{U}^n$, we get

$$\begin{aligned}\widetilde{U}^n\big((1-\varepsilon-\alpha)Y^n\big) &\ge \widetilde{U}^n\big(\xi_\varepsilon^n + \alpha(Y - Y^n)\big)\\ &\ge \widetilde{U}^n(\xi_\varepsilon^n) + \alpha D\widetilde{U}^n(\xi_\varepsilon^n)(Y - Y^n) \ge -\alpha X_\varepsilon^n(Y - Y^n),\end{aligned}$$

where we used the non-negativity of $\widetilde{U}^n$. Now, let $4\alpha \le 1$ and $\varepsilon \le 1-2\alpha$. Then, from Property 4, which is inherited from Assumption 4.5, this provides

$$\begin{aligned} X_\varepsilon^n(Y - Y^n) &\ge \frac{-1}{\alpha}\widetilde{U}^n\big((1-\varepsilon-\alpha)Y^n\big) \ge \frac{-(1-\varepsilon-\alpha)^{-\beta}}{\alpha}[C + \widetilde{U}^n(Y^n)] \\ &\ge -\alpha^{-\beta-1}[C + \widetilde{U}^n(Y^n)]. \end{aligned} \tag{8.1}$$

Now, observe that $E\widetilde{U}^n(Y^n) + xy^n = W^n(x) \to W(x)$, so that $\widetilde{U}^n(Y^n)$ is integrable for large n, and the proof is complete. □

The following result is an easy consequence of Komlòs theorem. We report it for completeness.

LEMMA 8.2. *Let $(\phi^n)_n$ be a sequence of r.v. in $L^0(\mathbb{R}^p, \mathscr{F})$. Suppose that*

$$\sup_n |\phi^n| < \infty, \quad P\text{-a.s.}$$

Then there exists a r.v. $\phi \in L^0(\mathbb{R}^p, \mathscr{F})$ such that, after possibly passing to a subsequence,

$$\frac{1}{n}\sum_{j=1}^n \phi^j \to \phi, \quad P\text{-a.s.}$$

PROOF. Set $\varphi := \sup_n |\phi^n|$ and define the probability measure P' by the density $dP'/dP = e^{-\varphi}/Ee^{-\varphi}$. Then, $P' \sim P$, and the sequence $(\phi^n)_n$ is bounded in $L^1(P')$. The required result follows from Komlòs theorem. □

LEMMA 8.3. *Let Assumptions* 4.1, 4.2, 4.4 *and* 4.5 *hold, and consider some $x \in \operatorname{int}(K)$ with $W(x) < \infty$.*

Let $(X^n)_n$ be the sequence introduced in Lemma 8.1, *and (y_*, Y_*) be the solution in $\mathscr{S}(x)$ introduced in Lemma* 7.2. *Then $P[Y_* = 0] = 0$, and there exist a sequence $X_*^n \in \operatorname{conv}(X^j, j \ge n)$ and X_* such that*

$$X_* \in -\partial\widetilde{U}(Y_*) \quad \textit{and} \quad X_*^n \to X_*,\ P\text{-a.s.}$$

Moreover, under condition (4.4), *the above statement still holds if Assumption* 4.3 *is substituted for Assumption* 4.2.

PROOF. (i) We first prove the required result when condition (A1) of Assumption 4.4 is satisfied. We shall use the notation of Lemma 8.1. Define the sequence $Z_*^n = \sum_{k\ge n} \lambda^{n,k} Z^k$, where $(\lambda^{n,k}, k \ge n)_n$ are the coefficients of the convex combination relating $(Y_*^n)_n$ to $(Y^n)_n$, and observe that

$E\widetilde{U}^n(Y^n) = E\widetilde{U}(Z^n) + \frac{n}{2}|Z^n - Y^n|^2 \to E\widetilde{U}(Y_*)$, so that $Z^n - Y^n \to 0$ P-a.s. after possibly passing to a subsequence. Then $Z_*^n = Y_*^n + \sum_{k\geq n} \lambda^{n,k}(Z^k - Y^k) \to Y_*$ P-a.s. Since $W(x) = E\widetilde{U}(Y_*) + xy_*$ is finite, it follows from condition (A1) that $Y_* \in \mathrm{int}(K^*)P$-a.s and the sequence $(Z_*^n(\omega))_n$ is valued in a compact subset $J(\omega)$ of $\mathrm{int}(K^*)$ for a.e. $\omega \in \Omega$. In particular, we have $N_H(Z_*^n) = \{0\}$ for large n.

By definition, $-X^n \in \partial\widetilde{U}(Z^n)$ P-a.s., or equivalently, $Z^n \in \partial U(X^n)$ P-a.s. From Assumption 4.1, there exists $\overline{X}^n = \sum_{k\geq n} \mu^{n,k} X^k \in \mathrm{conv}(X^k, k \geq n)$ such that $-\overline{X}^n \in \partial\widetilde{U}(Z_*^n)$. Since the sequence $(Z_*^n(\omega))_n$ is valued in a compact subset of $\mathrm{int}(K^*)$, it follows from the convexity of $\widetilde{U}$ that the sequence $\overline{X}^n \in -\partial\widetilde{U}(Z_*^n)$ is bounded P-a.s. We now use Lemma 8.2 to find a sequence $\overline{X}_*^n \in \mathrm{conv}(\overline{X}^k, k \geq n)$ which converges P-a.s. to some random variable X_*.

It remains to prove that $-X_* \in \partial\widetilde{U}(Y_*)$. Since $\overline{X}^n \in -\partial\widetilde{U}(Z_*^n)$, the definition of the subgradient provides

$$\widetilde{U}(z) \geq \widetilde{U}(Z_*^n) + \overline{X}^n(Z_*^n - z) \quad \text{for all } z \in K^*.$$

Let $(\lambda^{n,j})_{j\geq n}$ be the coefficients of the convex combination defining $(\overline{X}_*^n)$ from $(\overline{X}^n)$, and set $\overline{Z}_*^n := \sum_{j\geq n} \lambda^{n,j} Z_*^j$. By convexity of $\widetilde{U}$, the previous inequality implies that

$$\begin{aligned}\widetilde{U}(z) &\geq \widetilde{U}(\overline{Z}_*^n) + \sum_{j\geq n} \lambda^{n,j}\overline{X}^j(Z_*^j - z)\\ &= \widetilde{U}(\overline{Z}_*^n) + \overline{X}_*^n(\overline{Z}_*^n - z) + \sum_{j\geq n} \lambda^{n,j}\overline{X}^j(Z_*^j - \overline{Z}_*^n).\end{aligned}$$

Now, recall that $Z_*^n \to Y_*$ P-a.s. Then, $Z_*^j - \overline{Z}_*^n \to 0$ P-a.s. Since the sequence $(\overline{X}^n)$ is P-a.s. bounded, it follows that $\overline{X}^j(Z_*^j - \overline{Z}_*^n) \to 0$ P-a.s. and the same result prevails for the convex combination. Hence, by taking limits in the last inequality, we get

$$\widetilde{U}(z) \geq \widetilde{U}(Y_*) + X_*(Y_* - z) \quad \text{for all } z \in K^*,$$

proving that $-X_* \in \partial\widetilde{U}(Y_*)$.

(ii) Now suppose that condition (A2) of Assumption 4.4 is satisfied. As in part (i) of this proof, $Z_*^n \to Y_*$ P-a.s. We first prove that

$$P[Y_* = 0] = 0. \tag{8.2}$$

Consider first the case where Assumption 4.2 is satisfied, that is, $\sup_{x\in K} U(x) = +\infty$. Then, $\widetilde{U}(0) = +\infty$, and we obtain immediately (8.2) from the fact that $W(x) < \infty$. Next, suppose that condition (4.4) holds, and Assumption 4.3 is satisfied instead of Assumption 4.2. Let Y_+ be an element in $\mathscr{Y}(y+) \cap L^0(K^*\backslash\{0\}, \mathscr{F}_T)$, and observe that $Y_+^0 > 0$ P-a.s. Define the event set $A := \{Y_* = 0\}$. From Assumption 4.3, the sequence $(l(X^n))_n$ converges P-a.s. to $+\infty$ on A, since by definition $X^n := -D\widetilde{U}^n(Y^n) \in (\partial\widetilde{U} + N_{\overline{H}})(z^n(Y^n))$. But, from the first order condition of Lemma 8.1 together with the definition of the liquidation function l, we have

$$x(y_+ - y^n) \le EX^n(Y_+ - Y^n) \ge E\big[Y^0_+ l(X^n) - X^n Y^n\big].$$

Furthermore, since $AE(\widetilde{U}^n) < \infty$ by Assumption 4.5, and $\widetilde{U}$ is bounded (as a consequence of the boundedness of U), we see that $\sup_n EX^n Y^n < \infty$. Therefore, whenever $P[A] > 0$, the right-hand side of the last inequality explodes to $+\infty$, whereas the left-hand side remains bounded. This is the required contradiction, and the proof of (8.2) is complete.

Then, for n sufficiently large, Z^n_* is valued in the open domain H, and therefore $N_{\overline{H}}(Z^n_*) = \{0\}$. We then proceed as above to obtain the existence of a sequence $\overline{X}^n_* \in \mathrm{conv}(\overline{X}^k, k \ge n) = \mathrm{conv}(X^k; k \ge n)$ such that $\overline{X}^n_* \to X_* P$-a.s.

We now prove that $-X_* \in \partial\widetilde{U}(Y_*)$. Let us be more specific, and call $\overline{U}$ the extension of $\widetilde{U}$ to the open convex domain H. By the same argument as in (i), we see that $-X_* \in \partial\overline{U}(Y_*)$. By definition, $\widetilde{U} = \overline{U} + \chi_{K^*}$, where $\chi_{K^*} = 0$ on K^* and $+\infty$ otherwise. Then, $\partial\widetilde{U} = \partial\overline{U} + N_{K^*}$, and $\partial\overline{U}(Y_*) \subset \partial\widetilde{U}(Y_*)$. □

PROPOSITION 8.1. *Let Assumptions* 4.1, 4.2, 4.4 *and* 4.5 *hold, and consider some* $x \in \mathrm{int}(K)$ *with* $W(x) < \infty$. *Let* (y_*, Y_*) *be the solution of* $W(x)$ *introduced in Lemma* 7.2. *Then* $P[Y_* = 0] = 0$ (*Lemma* 8.3), *and there exists a r.v.* X_* *valued in* $-\partial\widetilde{U}(Y_*)$ *such that*

$$\text{(8.3)} \qquad EX_*(Y - Y_*) + x(y_* - y) \le 0 \quad \text{for all } y \in K^* \text{ and } Y \in \mathscr{Y}(y).$$

Moreover, *under condition* (4.4), *the above statement still holds if Assumption* 4.3 *is substituted for Assumption* 4.2.

PROOF. Let $(y^n, Y^n) \in \mathscr{S}^n(x)$, $X^n := -D\widetilde{U}^n(Y^n)$, $J^n := \widetilde{U}^n(Y^n)$, and $Z^n := z^n(Y^n)$. Let $(y^n_*, Y^n_*, X^n_*, J^n_*, Z^n_*) \in \mathrm{conv}((y^k, Y^k, X^k, J^k, Z^k), k \ge n)$ be as in Lemmas 7.2 and 8.3 and Corollary 7.2: $(y^n_*, Y^n_*, X^n_*) \to (y_*, Y_*, X_*)$ P-a.s. and $J^n_* \to \widetilde{U}(Y_*)$ in $L^1(P)$. We shall denote by $(\lambda^{n,k}, k \ge n)_n$ the coefficients of the last convex combination. From Lemma 8.1, we have

$$\text{(8.4)} \qquad \liminf_{n\to\infty} E \sum_{k\ge n} \lambda^{n,k} X^k(Y - Y^k) \le x(y - y_*).$$

By the same argument as in the proof of Lemma 8.1, we get the lower bound (8.1),

$$\text{(8.5)} \qquad \sum_{k\ge n} \lambda^{n,k} X^k(Y - Y^k) \ge \mathrm{Const}[1 + J^n_*].$$

The sequence $(J^n_*)_n$ is uniformly integrable as it converges in the $L^1(P)$ norm. Then we can apply Fatou's lemma in (8.4) and we get

$$\text{(8.6)} \qquad E \liminf_{n\to\infty} \sum_{k\ge n} \lambda^{n,k} X^k(Y - Y^k) \le x(y - y_*).$$

Now observe that $\sum_{k\geq n}\lambda^{n,k}X^kZ^k - X_*^nZ_*^n = \sum_{k\geq n}\lambda^{n,k}X^k(Z^k - Z_*^n) \leq 0$ since $X^k \in -\partial\widetilde{U}(Z^k)$ and $\widetilde{U}$ is convex. Then, inequality (8.6) provides

$$
\begin{aligned}
x(y-y_*) &\geq E\liminf_{n\to\infty}\left[X_*^n(Y-Z_*^n) + \sum_{k\geq n}\lambda^{n,k}X^k(Z^k-Y^k)\right] \\
&= E\left[X_*(Y-Y_*) + \liminf_{n\to\infty}\sum_{k\geq n}\lambda^{n,k}X^k(Z^k-Y^k)\right].
\end{aligned} \tag{8.7}
$$

Notice that $E\widetilde{U}^n(Y^n) = E\widetilde{U}(Z^n) + \frac{n}{2}|Z^n - Y^n|^2 \to E\widetilde{U}(Y_*)$. Then, $E|Z^n - Y^n|^2 \to 0$, and therefore $Z^n - Y^n \to 0$ P-a.s. after possibly passing to a subsequence. Since

$$
\left|\sum_{k\geq n}\lambda^{n,k}X^k(Z^k-Y^k)\right| \leq \sum_{k\geq n}\lambda^{n,k}|X^k|\sup_{k\geq n}|Z^k-Y^k| = |X_*^n|\sup_{k\geq n}|Z^k-Y^k|,
$$

this implies that $\sum_{k\geq n}\lambda^{n,k}X^k(Z^k-Y^k) \to 0$ P-a.s. Reporting this in (8.7) provides the result announced in the statement of the proposition. □

We now use Theorem 4.1 in order to derive a characterization of attainable contingent claims.

LEMMA 8.4. *Let the conditions of Theorem* 4.1 *hold. Let* $C \in L^0(K, \mathscr{F}_T)$ *and* $x \in K$ *be such that*

$$
\sup_{y\in K^*}\sup_{Y\in\mathscr{Y}(y)}(ECY - xy) = ECY_\circ - xy_\circ = 0
$$

for some $y_\circ \in K^*\backslash\{0\}$ *and* $Y_\circ \in \mathscr{Y}(y_\circ)$ *with* $P[Y_\circ = 0] = 0$. *Then* $C \in \mathscr{X}(x)$; *that is, the contingent claim* C *is attainable from the initial wealth* x.

PROOF. From Remark 4.4, we have $EC\widehat{Z}_T - x\widehat{Z}_0 \leq 0$ for all $Z \in \mathscr{D}$. This proves that $x \in D(C) = \Gamma(C)$ by Theorem 4.1. Hence, $X \succeq C$ (i.e., $X - C \in K$) P-a.s. for some $X = X_T^{x,L} \in \mathscr{X}(x)$. Since $Y_\circ \in K^*$ P-a.s., it follows from the definition of $\mathscr{Y}(y_\circ)$ and the condition of the lemma that

$$
0 \leq E(X-C)Y_\circ = EXY_\circ - xy_\circ \leq 0.
$$

This proves that $(X-C)Y_\circ = 0$ P-a.s. and therefore $X - C \in \partial K$ P-a.s. by the fact that $Y_\circ \neq 0$ P-a.s. Finally, from Lemma 3.1, we have $l(X-C) = 0$, and by Remark 3.2, there exists some random transfer matrix $a \in L^0(\mathbb{M}_+^{d+1}, \mathscr{F}_T)$ such that

$$
C^i = X^i + \sum_{j=0}^{d}\left[a^{ji} - (1+\lambda^{ij})a^{ij}\right] \quad \text{for all } i = 0, \dots, d.
$$

Now set $\widetilde{L} = L + a\mathbf{1}_{\{T\}}$. Clearly, $\widetilde{L} \in \mathscr{A}(x)$ and $C = X_T^{x,\widetilde{L}} \in \mathscr{X}(x)$. □

COROLLARY 8.1. *Let the conditions of Proposition* 8.1 *and Theorem* 4.1 *hold. Let* (y_*, Y_*) *be the solution of* $W(x)$ *introduced in Lemma* 7.2. *Then* $P[Y_* = 0] = 0$, *and there exists a r.v.* X_* *valued in* $-\partial\widetilde{U}(Y_*)$ *such that*

$$X_* \in \mathscr{X}(x) \quad \text{and} \quad EX_*Y_* = xy_*.$$

PROOF. By Proposition 8.1, $P[Y_* = 0] = 0$ and X_* is valued in $-\partial\widetilde{U}(Y_*)$. Then, X_* takes values in K P-a.s. by Lemma 4.2(ii). We now apply inequality (8.3) of Proposition 8.1 for $y = 2y_*$ and $Y = 2Y_*$ (resp. $y = y_*/2$ and $Y = Y_*/2$). This provides immediately $EX_*Y_* = xy_*$. Then, applying again inequality (8.3) provides

$$EX_*Y - xy_* \leq 0 = EX_*Y_* - xy_* \quad \text{for all } Y \in \mathscr{Y}(y_*).$$

Since $X_* \in L^0(K, \mathscr{F}_T)$, we are in the context Lemma 8.4, and the proof is complete. □

9. Proof of Theorem 4.2. Part (i) of the theorem is proved in Lemma 7.2. Let X_* be the contingent claim introduced in Corollary 8.1. We intend to prove the optimality of X_* for problem $V(x)$. Since X_* is valued in $-\partial\widetilde{U}(Y_*)$, it follows from the definition of the subgradient of the convex function $\widetilde{U}$ that

$$\widetilde{U}(Y_*) + X_*Y_* \leq \widetilde{U}(y) + X_*y \quad \text{for all } y \in K^*.$$

Then, from the duality relation between U and $\widetilde{U}$ [see, e.g., Rockafellar (1970)],

$$U(x) = \inf_{y \in K^*} \big(\widetilde{U}(y) + xy\big),$$

we deduce that

$$\widetilde{U}(Y_*) + X_*Y_* \leq U(X_*).$$

We now take expectations, and use Corollary 8.1 to get

$$W(x) = E\widetilde{U}(Y_*) + xy_* = E[\widetilde{U}(Y_*) + X_*Y_*] \leq EU(X_*) \leq V(x). \tag{9.1}$$

In view of (4.3), this provides

$$W(x) = V(x) = EU(X_*),$$

as announced in parts (ii), (iii) and (iv) of the theorem.

APPENDIX

A.1. Proof of Proposition 4.1. (i) Assume that $AE(\widetilde{U}) < \infty$ and

$$\limsup_{l(x)\to\infty} \sup_{p\in\partial U(x)} |p| = 0, \tag{A.1}$$

and let us prove that $AE(U) < 1$.

Let β be an arbitrary positive constant. Since $AE(\widetilde{U}) < \infty$, we have, for some $b > 0$,

$$qy - \beta\widetilde{U}(y) < 0 \text{ for all } q \in -\partial\widetilde{U}(y) \text{ and } y \in K^* \text{ with } l^*(y) \le b. \tag{A.2}$$

From the positive homogeneity of l^*, there exists some $y_0 \in \operatorname{int}(K^*)$ satisfying $l^*(y_0) = b$.

We now observe that there exists a constant $c > 0$ such that

$$\text{for all } x \succeq c\mathbf{1}_0 \text{ and } p \in \partial U(x), \text{ we have } y_0 \succeq_* p.$$

Indeed, if such a positive constant does not exist, then

$$\text{for all } n, \text{ there exist } x_n \succeq n\mathbf{1}_0 \text{ and } p_n \in \partial U(x_n) \text{ such that } y_0 - p_n \notin K^*.$$

Since $y_0 \in \operatorname{int}(K^*)$, this leads to a contradiction with (A.1).

Now, take $x \succeq c\mathbf{1}_0$, that is, $l(x) \ge c$. Let p be an arbitrary element in $\partial U(x)$. By the definition of $\widetilde{U}$, we have $x \in \partial\widetilde{U}(p)$ and

$$U(x) = \inf_{y\in\mathbb{R}^{d+1}} \big(\widetilde{U}(y) + xy\big) = \widetilde{U}(p) + xp. \tag{A.3}$$

Then, applying (A.2) with $y = p$ and $q = x$, we see that $\widetilde{U}(p) > xp/\beta$. Plugging the last inequality in (A.3), we get

$$U(x) > (1 + \beta^{-1})xp \quad \text{for all } x \in K \text{ with } l(x) \ge c.$$

The required result follows from the arbitrariness of p in $\partial U(x)$.

(ii) The second part can be proved similarly. Assume that $AE(U) < 1$ and

$$\liminf_{|y|\to 0} \inf_{q\in-\partial\widetilde{U}(y)} l(q) = \infty, \tag{A.4}$$

and let us prove that $AE(\widetilde{U}) < \infty$.

Let β be an arbitrary positive constant. Since $AE(U) < 1$, we have, for some $c > 0$,

$$\left(1 + \frac{1}{\beta}\right)px < U(x) \quad \text{for all } p \in \partial U(x) \text{ and } x \in K \text{ with } l(x) \ge c. \tag{A.5}$$

Set $x_0 := c\mathbf{1}_0$ so that $l(x_0) = c$ and $x_0 \in \operatorname{int}(K)$, and observe that there exists a constant $b > 0$ such that

$$\text{for all } y \preceq_* b\mathbf{1}_0 \text{ and } q \in -\partial\widetilde{U}(y), \qquad q \succeq x_0.$$

Indeed, if such a positive constant does not exist, then

$$\text{for all } n, \text{ there exist } y_n \preceq_* \frac{1}{n}\mathbf{1}_0 \text{ and } q_n \in -\partial\widetilde{U}(y_n) \text{ such that } q_n - x_0 \notin K.$$

By definition of the liquidation function l, this means that $l(q_n - x_0) = l(q_n) - c < 0$. Since $y_n \to 0$, this leads to a contradiction with (A.4).

Now, take $y \preceq_* b\mathbf{1}_0$, that is, $l^*(y) \leq b$. Let q be an arbitrary element in $-\partial\widetilde{U}(y)$. By the definition of $\widetilde{U}$, we have $y \in \partial U(q)$ and

$$\widetilde{U}(y) = U(q) - qy. \tag{A.6}$$

Then, applying (A.5) with $x = q$ and $p = y$, we see that $U(q) > (1 + \frac{1}{\beta})qy$. Plugging the last inequality in (A.6), we get

$$\widetilde{U}(y) > qy/\beta \quad \text{for all } y \in K^* \text{ with } l^*(y) \leq b \text{ and all } q \in -\partial\widetilde{U}(y).$$

The required result follows from the arbitrariness of q in $-\partial\widetilde{U}(y)$.

A.2. Proof of Lemma 4.1. (i) We first prove the necessary condition. The condition $AE(\widetilde{U}) < \infty$ means that there exist $b, \beta > 0$ such that

$$py - \beta\widetilde{U}(y) < 0 \quad \text{for all } y \in B \text{ and } p \in -\partial\widetilde{U}(y), \tag{A.7}$$

where $B = \{y \in K^*: l^*(y) \leq b\}$. Now fix some $y \in B$, and observe that $\mu y \in B$ for all $\mu \in (0, 1]$. Let F be the convex function defined on $(0, 1]$ by $F(\mu) := \widetilde{U}(\mu y)$. Then it follows from (A.7) that

$$-\mu q - \beta F(\mu) < 0 \quad \text{for all } \mu \in (0, 1] \text{ and } q \in \partial F(\mu). \tag{A.8}$$

Set $G(\mu) := \mu^{-\beta}\widetilde{U}(y)$. In order to complete the proof, we have to check that

$$(F - G)(\mu) \leq 0 \quad \text{for all } \mu \in (0, 1]. \tag{A.9}$$

Clearly, function G satisfies the first-order differential equation,

$$-\mu G'(\mu) - \beta G(\mu) = 0 \quad \text{for all } \mu \in (0, 1]. \tag{A.10}$$

Since $F(1) = G(1)$, it follows from (A.8) and (A.10) that $q > G'(1)$ for all $q \in \partial F(1)$. Then by closedness of the subgradient of the convex function F [see Clarke, Ledyaev, Stern and Wolenski (1998)], there exists a small parameter $\varepsilon > 0$ such that

$$q > G'(1) \quad \text{for all } q \in \bigcup_{1-\varepsilon \leq \mu \leq 1} \partial F(\mu).$$

Now, by convexity of F, we see that for all $\mu \in [1 - \varepsilon, 1)$ and $q \in \partial F(\mu)$,

$$F(\mu) \leq F(1) - q(1 - \mu) = G(1) - q(1 - \mu) < G(1) - G'(1)(1 - \mu) \leq G(\mu),$$

where the last inequality follows from the convexity of G. Hence

$$F < G \quad \text{on } [1 - \varepsilon, 1). \tag{A.11}$$

Next, set $\mu_0 := \sup\{\mu \in (0,1)\colon (F-G)(\mu) = 0\}$ with the usual convention $\sup\emptyset = -\infty$. In view of (A.11) and the continuity of F and G, the statement (A.9) is equivalent to $\mu_0 \le 0$. We then argue by contradiction, and assume that $\mu_0 \in (0,1)$. By definition of μ_0 and (A.11), we have $(F-G)(\mu_0) = 0$ and $F - G < 0$ on $(\mu_0, 1)$. This implies that $\partial(F-G)(\mu_0) \subset \mathbb{R}_-$ and therefore

$$q_0 \le G'(\mu_0) \quad \text{for all } q_0 \in \partial F(\mu_0).$$

On the other hand, turning back to (A.8) and (A.10) for $\mu = \mu_0$, we see that $q_0 > G'(\mu_0)$ which is the required contradiction.

(ii) We now prove sufficiency. Fix some $y \in K^*$ such that $l^*(y) \le b$, and set $F(\mu) := \widetilde{U}(\mu y)$, $G(\mu) := \mu^{-\beta}\widetilde{U}(y)$. Let q be an arbitrary element in $\partial F(1)$. Since F is convex, it follows from the definition of the subgradient and the fact that $F(1) = G(1)$ that

$$\varepsilon q \ge F(1) - F(1-\varepsilon) > G(1) - G(1-\varepsilon) \quad \text{for all } \varepsilon \in (0,1). \tag{A.12}$$

Dividing by ε and sending ε to zero provides $G'(1) \le q$ for all $q \in \partial F(1)$. This can be written equivalently in terms of $\widetilde{U}$ as

$$-\beta\widetilde{U}(y) \le -py \qquad \forall\, p \in -\partial\widetilde{U}(y),$$

which ends the proof. □

A.3. Proof of Property 2. This is an easy adaptation from Aubin (1984). By definition of $\widetilde{U}^n$ and $\widetilde{U}$, it follows that

$$\begin{aligned}
\widetilde{U}^n(y) &= \widetilde{U}(z^n(y)) + \frac{n}{2}|z^n(y) - y|^2 \\
&\ge U(x) - xy - x(z^n(y) - y) + \frac{n}{2}|z^n(y) - y|^2 \quad \text{for all } x \in K \\
&\ge U(x) - xy - \frac{|x|^2}{n} + \frac{n}{4}|z^n(y) - y|^2,
\end{aligned}$$

where we used the trivial inequality $ab \le n^{-1}|a|^2 + 4^{-1}n|b|^2$. Collecting terms and recalling that U is nonnegative, this provides

$$|z^n(y) - y|^2 \le \frac{4}{n}\left[\widetilde{U}^n(y) + xy + \frac{|x|^2}{n}\right].$$

This proves (i). The same inequality together with the observation that $\widetilde{U}^n \le \widetilde{U}$ provide (ii) by continuity of $\widetilde{U}$ on its domain.

It remains to prove (iii). To see this, observe that

$$\widetilde{U}(z^n(y^n)) = \widetilde{U}^n(y^n) - \frac{n}{2}|z^n(y^n) - y^n|^2 \le \widetilde{U}^n(y^n),$$

and therefore

$$\widetilde{U}(y) \le \liminf_{n\to\infty} \widetilde{U}^n(y^n).$$

On the other hand, since $\widetilde{U}^n \le \widetilde{U}$,

$$\limsup_{n\to\infty} \widetilde{U}^n(y^n) \le \lim_{n\to\infty} \widetilde{U}(y^n) = \widetilde{U}(y)$$

by continuity of $\widetilde{U}$. □

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