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Positivity functions for curves on algebraic varieties

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This is the second part of our work on Zariski decomposition structures, where we compare two different volume type functions for curve classes. The first function is the polar transform of the volume for divisor classes. The second function captures the asymptotic geometry of curves analogously to the volume function for divisors. We prove that the two functions coincide, generalizing Zariski's classical result for surfaces to all varieties. Our result confirms the log concavity conjecture of the first named author for weighted mobility of curve classes in an unexpected way, via Legendre–Fenchel type transforms. During the course of the proof, we obtain a refined structure theorem for the movable cone of curves.

1. Introduction

Let X be a smooth complex projective variety of dimension n. The Riemann–Roch problem asks whether one can determine the dimension of the space of sections of a holomorphic line bundle L on X. An important subtlety of this problem is that the answer is not determined by purely topological data — line bundles which share the same Chern class need not have isomorphic spaces of sections. In general, the problem only has a satisfactory answer for sufficiently ample line bundles, which exhibit a close relationship between geometry, cohomology, and intersection theory.

Over the past forty years, mathematicians have realized that one obtains a much richer theory by studying the asymptotic behavior of the space of sections of mL as m increases. Indeed, by working asymptotically, we can recover for general effective line bundles some of the same interplay between sheaf cohomology and intersection theory which undergirds the theory of ample line bundles. This point

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of view leads to many important positivity invariants for line bundles and linear systems. Perhaps the most important asymptotic invariant of a line bundle L is its volume,[1] defined as

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

When X is a surface, the volume of L can be calculated using intersection theory. The key construction is the Zariski decomposition [1962], which splits L into a "positive" part and a "rigid" part. In higher dimensions as well, there is a close relationship between the asymptotic geometry of divisors and intersection-theoretic positivity via volume-type functions.

Recently there has been interest in extending the theory of positivity to subvarieties of arbitrary codimension (see e.g., [Debarre et al. 2011; Lehmann 2016; Fulger and Lehmann 2017a; 2017b]). By analogy, one would like to study the asymptotic geometry of cycles and its relationship with numerical measures of positivity. In this paper we develop such a theory for curves: we show that the asymptotic enumerative geometry of curve classes is controlled by intersection-theoretic invariants.

Our comparison relies upon several natural volume-type functions for curve classes. The first function involves the numerical positivity of a curve class.

Definition 1.1 [Xiao 2017, Definition 1.1]. Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudoeffective curve class. Then the volume of α is defined to be

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ big and nef divisor class}} \left(\frac{A\cdot\alpha}{\mathrm{vol}(A)^{1/n}} \right)^{n/n-1}.$$

When α is a curve class that is not pseudoeffective, we set $\widehat{\mathrm{vol}}(\alpha) = 0$.

This is a polar transformation of the volume function on the ample cone of divisors. The definition is inspired by the realization that the volume of a divisor has a similar intersection-theoretic description against curves as in [Xiao 2017, Theorem 2.1]. It fits into a much broader picture relating positivity of divisors and curves via cone duality; see [Lehmann and Xiao 2016].

The second function captures the asymptotic geometry of curves. Recall that general points impose independent codimension 1 conditions on divisors in a linear series. Thus for a divisor L, one can interpret $\dim \mathbb{P}(H^0(X, L))$ as a measurement of how many general points are contained in sections of L. Using this interpretation, we define the mobility function for curves in an analogous way.

Definition 1.2 [Lehmann 2016, Definition 1.1]. Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a curve class with integer coefficients. The mobility of α is defined to be

$$\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\max\{b \in \mathbb{Z}_{\geq 0} \mid \text{any } b \text{ general points are contained in an effective curve of class } m\alpha\}}{m^{n/(n-1)}/n!}.$$

There is a closely related function known as the weighted mobility which counts singular points of the curve with a "higher weight". We first recall the definition of the weighted mobility count for a class

[1]For a nonbig line bundle, the higher asymptotic cohomological functions carry more significant information [Küronya 2006].

$\alpha \in N_1(X)$ with integer coefficients (see [Lehmann 2016, Definition 8.6]):

$$\mathrm{wmc}(\alpha) = \sup_{\mu} \max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \text{there is an effective cycle of class } \mu\alpha \text{ through any } b \\ \text{points of } X \text{ with multiplicity at least } \mu \text{ at each point} \end{array} \right\}.$$

The supremum is shown to exist in [Lehmann 2016] — it is then clear that the supremum is achieved by some positive integer μ. We define the weighted mobility to be

$$\mathrm{wmob}(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{wmc}(m\alpha)}{m^{n/n-1}}.$$

While the definition is slightly more complicated, the weighted mobility is easier to compute due to its close relationship with Seshadri constants. Lehmann [2016] showed that both the mobility and weighted mobility extend to continuous homogeneous functions on all of $N_1(X)$.

Main result. Our main theorem compares these functions. It continues a project begun in [Xiao 2017] (see especially Conjecture 3.1 and Theorem 3.2 there).

Theorem 1.3 (see Theorem 6.1). *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudoeffective curve class. Then*:

(1) $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.

(2) $\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha) \leq n!\,\widehat{\mathrm{vol}}(\alpha)$.

(3) *Assume Conjecture 1.4 below. Then* $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

This result is surprising: it suggests that the mobility count of *any* curve class is optimized by complete intersection curves; see the end of Section 2 (page 1256). Just as for curves on algebraic surfaces, the key to this result is the Zariski decomposition for curves on varieties of arbitrary dimension as constructed in [Lehmann and Xiao 2016; Fulger and Lehmann 2017b]. Part (3) of the theorem relies on the following conjectural description of the mobility of a complete intersection class:

Conjecture 1.4 [Lehmann 2016, Question 6.1]. *Let X be a smooth projective variety of dimension n and let A be an ample divisor on X. Then*

$$\mathrm{mob}(A^{n-1}) = A^n.$$

Example 1.5. Let α denote the class of a line on $\mathbb{P}^3$. The mobility count of α is determined by the following enumerative question: what is the minimal degree of a curve through b general points of $\mathbb{P}^3$? The answer is unknown, even in an asymptotic sense.

Perrin [1987] conjectured that the "optimal" curves (which maximize the number of points relative to their degree to the $\frac{3}{2}$) are complete intersections of two divisors of the same degree. Theorem 1.3 supports a vast generalization of Perrin's conjecture to all big curve classes on all smooth projective varieties.

Strict log concavity of the volume function for divisors. An important ingredient in the proof of Theorem 1.3 is the study of the volume function for divisors from the perspective of convexity theory. Since such results are of interest in their own right, we summarize the highlights below.

The first step is to analyze the strict log concavity of the volume function. It is well-known that the volume function for divisor classes is log concave (see e.g., [Lazarsfeld 2004, Theorem 11.4.9; Boucksom 2002b]). We show that it is strictly log concave on the big and movable cone of divisors (but on no larger cone), extending [Boucksom et al. 2009, Theorem D].

Theorem 1.6. *Let X be a smooth projective variety of dimension n. For any two big divisor classes L_1, L_2, the inequality*

$$\mathrm{vol}(L_1+L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

is an equality if and only if the (numerical) positive parts $P_\sigma(L_1)$, $P_\sigma(L_2)$ are proportional. Thus the function $L \mapsto \mathrm{vol}(L)$ is strictly log concave on the cone of big and movable divisors.

This result is proved in Section 3 (see Theorem 3.9). It shows that the volume function for divisors fits into the abstract convexity framework developed in [Lehmann and Xiao 2016]. A posteriori, this viewpoint motivates many of the well-known structure results for the volume function (such as the formula for the derivative, the Khovanskii–Teissier inequalities, the σ-decomposition, etc.).

Refined structure of the movable cone. The most important consequence is a refined version of a theorem of [Boucksom et al. 2013] describing the movable cone of curves. In [loc. cit.], it is proved that the movable cone $\mathrm{Mov}_1(X)$ is the closure of the cone *generated* by $(n-1)$-self positive products of big divisors. We show that $\mathrm{Mov}_1(X)$ is the *closure* of the set of $(n-1)$-self positive products of big divisors on the interior of $\mathrm{Mov}^1(X)$. (The definition of the positive product $\langle - \rangle$ is recalled in Section 2.)

Theorem 1.7. *Let X be a smooth projective variety of dimension n. The $(n-1)$-st positive product $\langle -^{n-1} \rangle$ defines a continuous bijection from the interior of the big and movable cone of divisors to the interior of* $\mathrm{Mov}_1(X)$.

In practice, Theorem 1.7 seems quite useful for working with the movable cone of curves. For example, it has an immediate corollary:

Corollary 1.8. *Let X be a projective variety of dimension n. Then the rays over classes of irreducible curves which deform to dominate X are dense in* $\mathrm{Mov}_1(X)$.

Polar transform of the volume function for divisors. Equipped with these results, we return to our discussion of positivity functions for curves.

First we review some facts about polar transforms. Let V be a real vector space of dimension n, and let V^* be its dual space. Let $\mathrm{Cvx}(V)$ be the space of lower semicontinuous convex functions on V. We denote the paring of $w^* \in V^*$ and $v \in V$ by $w^* \cdot v$. Recall that the classical Legendre–Fenchel transform

$$\mathcal{L} : \mathrm{Cvx}(V) \to \mathrm{Cvx}(V^*), \quad \mathcal{L}f(w^*) = \sup_{v \in V}\{w^* \cdot v - f(v)\}$$

is an order-reversing involution which relates the differentiability of a convex function with the strict convexity of its dual (see e.g., [Rockafellar 1970]).

When working with homogeneous functions on a cone, there is an analogue of the Legendre–Fenchel transform which plays a similar theoretical role. It is the concave homogeneous version of the well-known polar transform. Let $\mathfrak{C} \subset V$ be a proper closed convex cone of full dimension and let $\mathfrak{C}^* \subset V^*$ be its dual cone. We let $\mathrm{HConc}_s(\mathfrak{C})$ denote the collection of functions $f : \mathfrak{C} \to \mathbb{R}$, which are upper-semicontinuous, homogeneous of weight $s > 1$, strictly positive in the interior of $\mathfrak{C}$ and s-concave. The polar transform $\mathcal{H}$ associates to a function $f \in \mathrm{HConc}_s(\mathfrak{C})$ the function $\mathcal{H}f \in \mathrm{HConc}_{s/(s-1)}(\mathfrak{C}^*)$ defined as

$$\mathcal{H}f(w^*) := \inf_{v \in \mathfrak{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/(s-1)}.$$

By taking the logarithmic function of $\mathcal{H}f$, we get

$$\log \mathcal{H}f(w^*) = \frac{s}{s-1} \inf_{v \in \mathfrak{C}^\circ} \bigl(\log(w^* \cdot v) - \tfrac{1}{s} \log f(v) \bigr).$$

Thus the polar transform $\mathcal{H}$ can be considered as a variant of Legendre–Fenchel transform with a "coupling function" given by the logarithmic function. The papers [Xiao 2017; Lehmann and Xiao 2016] develop the theory of $\mathcal{H}$ in parallel with the classical Legendre–Fenchel transform $\mathcal{L}$ and demonstrate how it has fruitful applications in the positivity theory of curves.

In our geometric setting, polar duality yields two natural numerical positivity functions for curves. One is the function $\widehat{\mathrm{vol}}$ discussed above. If we instead take the polar transform of the volume on the pseudoeffective cone, then we obtain a polar function on the dual cone $\mathrm{Mov}_1(X)$.

Definition 1.9 [Xiao 2017, Definition 2.2]. Let X be a projective variety of dimension n. For any curve class $\alpha \in \mathrm{Mov}_1(X)$ define

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big divisor class}} \left(\frac{L \cdot \alpha}{\mathrm{vol}(L)^{1/n}} \right)^{n/(n-1)}.$$

When α is a curve class that is not movable, we set $\mathfrak{M}(\alpha) = 0$.

While the positivity functions $\widehat{\mathrm{vol}}$, mob, wmob are conjecturally the same, $\mathfrak{M}$ exhibits quite different behavior. It is best understood as a way of making Theorem 1.7 explicit (see Lemma 3.11, Theorem 3.14 and Corollary 3.23).

Theorem 1.10. *Let X be a smooth projective variety and let α be a curve class in $\mathrm{Mov}_1(X)$. Then exactly one of the following alternatives holds*:

- *$\alpha = \langle L^{n-1} \rangle$ for a big movable divisor class L.*
- *$\alpha \cdot M = 0$ for a nonzero movable divisor class M.*

In the first case, we have $\mathfrak{M}(\alpha) = \mathrm{vol}(L)$ and L achieves the infimum of Definition 1.9. In the second case we have $\mathfrak{M}(\alpha) = 0$.

A curve class α of the first type lies on the boundary of $\mathrm{Mov}_1(X)$ if and only if the corresponding big *divisor L lies on the boundary of* $\mathrm{Mov}^1(X)$. *Thus the homeomorphism between the interiors of* $\mathrm{Mov}^1(X)$ *and* $\mathrm{Mov}_1(X)$ *given by Theorem 1.7 extends to a homeomorphism from all big movable divisor classes to curve classes with* $\mathfrak{M} > 0$.

Conceptually, the function $\mathfrak{M}$ allows us to assign a movable divisor to a movable curve class by "taking an $(n-1)$-th root". For toric varieties, this coheres with a classical construction of Minkowski which assigns a polytope to a positive Minkowski weight.

Theorem 1.3 relies upon the following comparison between the two polar transforms $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ (see Section 5). Recall that the complete intersection cone $\mathrm{CI}_1(X)$ is the closure of the set of curve classes of the form A^{n-1} for an ample divisor class A. The set $\mathrm{CI}_1(X)$ is a closed cone but may fail to be convex (see [Lehmann and Xiao 2016]).

Theorem 1.11. *Let X be a smooth projective variety and let α be a big curve class in* $\mathrm{Mov}_1(X)$. *Then the following conditions are equivalent*:

- $\alpha \in \mathrm{CI}_1(X)$.
- $\widehat{\mathrm{vol}}(\alpha) = \mathfrak{M}(\alpha)$.
- $\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(\phi^*\alpha)$ *for every birational morphism $\phi : Y \to X$.*

While not strictly necessary for our main result, we also show that $\mathfrak{M}$ admits an enumerative interpretation. We define $\mathrm{mob}_{\mathrm{mov}}$ and $\mathrm{wmob}_{\mathrm{mov}}$ for curve classes analogously to mob and wmob, except that we only count contributions of families whose general member is a sum of irreducible movable curves (see paragraph after Definition 6.7 for more details).

Theorem 1.12. *Let X be a smooth projective variety of dimension n and let* $\alpha \in \mathrm{Mov}_1(X)^\circ$. *Then*:

(1) $\mathfrak{M}(\alpha) = \mathrm{wmob}_{\mathrm{mov}}(\alpha)$.

(2) *Assume Conjecture 1.4. Then* $\mathfrak{M}(\alpha) = \mathrm{mob}_{\mathrm{mov}}(\alpha)$.

Outline of the proof. We briefly outline the proof of Theorem 1.3(3), the most difficult part. As mentioned above, Zariski decompositions for positivity functions play an important role. Fix a function $f \in \{\widehat{\mathrm{vol}}, \mathrm{mob}\}$. A Zariski decomposition for a big curve class α with respect to f is an expression

$$\alpha = P + N$$

where N is pseudoeffective and P is a "positive part" satisfying $f(P) = f(\alpha)$.

The main distinction between the Zariski decompositions for mob and $\widehat{\mathrm{vol}}$ is where the positive part is required to lie. For $\widehat{\mathrm{vol}}$, the positive part $P_{\widehat{\mathrm{vol}}}$ constructed in [Lehmann and Xiao 2016] lies in the complete intersection cone $\mathrm{CI}_1(X)$. For mob, the positive part P_{mob} constructed in [Fulger and Lehmann 2017b] lies in $\mathrm{Mov}_1(X)$. In fact a stronger property is proved there: $\mathrm{mob}(P_{\mathrm{mob}}) = \mathrm{mob}(\phi^* P_{\mathrm{mob}})$ for any birational map ϕ. Using Conjecture 1.4 and a delicate comparison between mob and $\mathfrak{M}$, Theorem 1.11 allows us to conclude the stronger statement that $P_{\mathrm{mob}} \in \mathrm{CI}_1(X)$. Then the two positive parts should coincide, and one can again apply Conjecture 1.4 to deduce the equality of the two functions.

Examples.

Hyperkähler varieties. For hyperkähler varieties, positivity functions for curves admit interesting interpretations in terms of the Beauville–Bogomolov form. Let q denote the Beauville–Bogomolov quadratic form on $N^1(X)$ normalized so that $q(D)^{n/2} = D^n$ for ample D. The form induces an isomorphism $\psi : N^1(X) \to N_1(X)$. Then [Lehmann and Xiao 2016, Section 7] shows that the bijection of Theorem 1.7 can be understood using ψ:

- If D is a big movable divisor, then $\mathrm{vol}(D)^{(n-2)/n}\psi(D) = \langle D^{n-1}\rangle$. In other words, the bijection of Theorem 1.7 coincides with ψ up to a continuous rescaling factor. For a big movable divisor class D,

$$\mathfrak{M}(\psi(D)) = \mathrm{vol}(D)^{1/n-1}.$$

- In particular, ψ also induces a bijection between the big and nef cone of divisors and the complete intersection curve classes with positive $\widehat{\mathrm{vol}}$. For A big and nef we have

$$\widehat{\mathrm{vol}}(\psi(A)) = \mathrm{vol}(A)^{1/n-1}.$$

In general, the volume of a curve class is given by a Zariski decomposition projecting into the complete intersection cone. [Lehmann and Xiao 2016] furthermore shows how this decomposition is related via q-duality to the σ-decomposition of divisors.

Mori dream spaces. If X is a Mori dream space, then the movable cone of divisors admits a chamber structure defined via the ample cones on small $\mathbb{Q}$-factorial modifications. This chamber structure behaves compatibly with the σ-decomposition and the volume function for divisors.

For curves we obtain a complementary picture using the movable cone of curves. Note that $\mathrm{Mov}_1(X)$ is naturally preserved by small $\mathbb{Q}$-factorial modifications. We then have a chamber decomposition of $\mathrm{Mov}_1(X)$ induced by the decomposition for divisors via the bijection of Theorem 1.7. A good way to analyze the chambers is to compare the behavior of the two functions $\mathfrak{M}$ and $\widehat{\mathrm{vol}}$ restricted to $\mathrm{Mov}_1(X)$.

- By Theorem 1.7, a curve class in the interior of $\mathrm{Mov}_1(X)$ is the $(n-1)$-positive product of a big divisor class L and $\mathfrak{M}(\alpha) = \mathrm{vol}(L)$. Using the birational invariance of the volume for divisors, we see that $\mathfrak{M}$ is also invariant under small $\mathbb{Q}$-factorial modifications.
- Using the Zariski decomposition of [Lehmann and Xiao 2016], the movable cone of curves admits a "chamber structure" as a union of the complete intersection cones from small $\mathbb{Q}$-factorial modifications. However, $\widehat{\mathrm{vol}}$ is not invariant under small $\mathbb{Q}$-factorial modifications but changes to reflect the differing structure of the pseudoeffective cone of curves.

Theorem 1.11 shows that $\widehat{\mathrm{vol}}$ reaches its minimum value $\widehat{\mathrm{vol}}(\alpha) = \mathfrak{M}(\alpha)$ precisely on the complete intersection cone of X, and then increases on the chambers corresponding to birational models of X. In this way $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ are the right tools for understanding the birational geometry of curves on Mori dream spaces.

Toric varieties. Suppose that X is a simplicial projective toric variety of dimension n defined by a fan Σ. A class α in the interior of the movable cone of curves corresponds to a positive Minkowski weight on the rays of Σ. A fundamental theorem of Minkowski attaches to such a weight a polytope P_α whose facet normals are the rays of Σ and whose facet volumes are determined by the weights. In fact, Minkowski's construction exactly corresponds to the bijection of Theorem 1.7.

Lemma 1.13. *If L denotes the big and movable divisor class corresponding to the polytope P_α then $\langle L^{n-1}\rangle = \alpha$. Thus $\mathfrak{M}(\alpha) = n!\,\mathrm{vol}(P_\alpha)$.*

When α happens to be in the complete intersection cone, this quantity also agrees with $\widehat{\mathrm{vol}}(\alpha)$. In the toric setting, properties of $\mathfrak{M}$ can be interpreted via the classical theory of convex bodies, using constructions such as Blaschke addition and the Kneser–Süss inequality (see [Lehmann and Xiao 2017] for more details).

Further applications. The refined structure of the movable cone is not only important to study positivity functions for curves, but it should also have other applications. We briefly mention two areas for further study (which will not be addressed in the body of the paper).

The first is the study of moduli of vector bundles. Recently, the papers [Greb et al. 2016b; 2016c; 2016d; 2019] discussed some obstructions to generalizing the theory of slope-stability from surfaces to varieties of arbitrary dimension. Traditionally one uses stability conditions defined by H^{n-1} for an ample divisor H, but the walls are no longer linear in H. As discussed in [Greb et al. 2016d] the situation is improved by working in $\mathrm{CI}_1(X)$. Since this cone is not convex, it seems that a thorough understanding of Theorem 1.7 and of stability conditions constructed via movable curve classes (as in [Greb et al. 2016a]) will be helpful for filling out this picture. There are also some situations where one obtains a nice chamber structure of $\mathrm{Mov}_1(X)$ using stability conditions (see for example [Neumann 2010]), and it would be interesting to see the geometric input provided by the corresponding decomposition of $\mathrm{Mov}^1(X)$.

Another area is the geometry of curves on rationally connected varieties. The original proof of boundedness of smooth Fano varieties by [Campana 1992; Kollár et al. 1992] relied on constructing chains of rational curves and controlling the degree against an ample divisor. Such constructions also have interesting interaction with the volume function of curves (see for example Proposition 6.2). By considering the volume of connecting rational chains, one obtains a "birational" variant of boundedness problems which is interesting for arbitrary rationally connected varieties. See [Lehmann and Xiao 2016] for a more in-depth discussion.

Outline of the paper. In this paper we will work with projective varieties over $\mathbb{C}$, but related results can be also adjusted to arbitrary algebraically closed fields and compact Kähler manifolds. We give a general framework for this extension in Section 2.

In Section 2 we briefly recall the general convexity and duality framework in [Lehmann and Xiao 2016], and explain how the proofs can be adjusted to arbitrary algebraically closed fields and compact Kähler manifolds. In Section 3, we give a refined structure of the movable cone of curves and generalize several results on big and nef divisors to big and movable divisors. Section 4 discusses toric varieties, showing some relationships with convex geometry. Section 5 compares the complete intersection and movable cone of curves. In Section 6 we compare the (weighted) mobility functions and $\widehat{\mathrm{vol}}$, $\mathfrak{M}$, finishing the proof of the main results.

2. Preliminaries

Positivity. In this section, we first fix some notations over a projective variety X:

- $N^1(X)$: the real vector space of numerical classes of divisors.
- $N_1(X)$: the real vector space of numerical classes of curves.
- $\overline{\mathrm{Eff}}^1(X)$: the cone of pseudoeffective divisor classes.
- $\mathrm{Nef}^1(X)$: the cone of nef divisor classes.
- $\mathrm{Mov}^1(X)$: the cone of movable divisor classes.
- $\overline{\mathrm{Eff}}_1(X)$: the cone of pseudoeffective curve classes.
- $\mathrm{Mov}_1(X)$: the cone of movable curve classes, equivalently by [Boucksom et al. 2013] the dual of $\overline{\mathrm{Eff}}^1(X)$.
- $\mathrm{CI}_1(X)$: the closure of the set of all curve classes of the form A^{n-1} for an ample divisor A.

With only a few exceptions, capital letters A, B, D, L will denote $\mathbb{R}$-Cartier divisor classes and Greek letters α, β, γ will denote curve classes. For two curve classes α, β, we write $\alpha \succeq \beta$ and $\alpha \preceq \beta$ to denote that $\alpha - \beta$ and $\beta - \alpha$, respectively, belong to $\overline{\mathrm{Eff}}_1(X)$. We will do similarly for divisor classes, or two elements of a cone $\mathfrak{C}$ if the cone is understood.

We will use the notation $\langle - \rangle$ for the positive product as in [Boucksom 2002a; Boucksom et al. 2009; 2013]. Let us recall briefly recall its definition. Let X be a projective manifold (or compact Kähler manifold) of dimension n, and let $L_1, \dots, L_r$ be big $(1, 1)$ classes. Then

$$\langle L_1 \cdots L_r \rangle := \lim_{m \to \infty} \mu_{m*}(\hat{A}_1 \cdots \hat{A}_r),$$

where $\mu_m : X_m \to X$ is a suitable sequence of Fujita approximations such that the limit class has the most positivity (see [Boucksom et al. 2009; 2013] for more details). Note that μ_m satisfies $\mu_m^* L_i = \hat{A}_{i,m} + E_{i,m}$ for some effective divisor class $E_{i,m}$ and big nef class $\hat{A}_{i,m}$ such that $\hat{A}_{i,m}^n \to \mathrm{vol}(L_i)$. We make a few remarks on this construction for singular projective varieties. Suppose that X has dimension n. Then $N_{n-1}(X)$ denotes the vector space of $\mathbb{R}$-classes of Weil divisors up to numerical equivalence as in [Fulton 1984, Chapter 19]. In this setting, the first and $(n-1)$-st positive product should be interpreted respectively as maps $\overline{\mathrm{Eff}}^1(X) \to N_{n-1}(X)$ and $\overline{\mathrm{Eff}}^1(X)^{\times n-1} \to \mathrm{Mov}_1(X)$. We will also let $P_\sigma(L)$ denote the positive part in this sense — that is, pull back L to better and better Fujita approximations, take its positive part, and push the numerical class forward to X as a numerical Weil divisor class. With these conventions, we still have the crucial result of [Boucksom et al. 2009; Lazarsfeld and Mustaţă 2009] that the derivative of the volume is controlled by intersecting against the positive part.

We define the movable cone of divisors $\mathrm{Mov}^1(X)$ to be the subset of $\overline{\mathrm{Eff}}^1(X)$ consisting of divisor classes L such that $N_\sigma(L) = 0$ and $P_\sigma(L) = L \cap [X] \in N_{n-1}(X)$. On any projective variety, by [Fulton 1984, Example 19.3.3] capping with X defines an injective linear map $N^1(X) \to N_{n-1}(X)$. Thus if $D, L \in \mathrm{Mov}^1(X)$ have the same positive part in $N_{n-1}(X)$, then by the injectivity of the capping map we must have $D = L$.

To extend our results (especially the results in Section 3) to arbitrary compact Kähler manifolds, we need to deal with transcendental objects which are not given by divisors or curves. Let X be a compact Kähler manifold of dimension n. By analogue with the projective situation, we need to deal with the following spaces and positive cones:

- $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$: the real Bott–Chern cohomology group of bidegree $(1, 1)$.
- $H^{n-1,n-1}_{\mathrm{BC}}(X, \mathbb{R})$: the real Bott–Chern cohomology group of bidegree $(n-1, n-1)$.
- $\mathcal{N}(X)$: the cone of pseudoeffective $(n-1, n-1)$-classes.
- $\mathcal{M}(X)$: the cone of movable $(n-1, n-1)$-classes.
- $\overline{\mathcal{K}}(X)$: the cone of nef $(1, 1)$-classes, equivalently the closure of the Kähler cone.
- $\mathcal{E}(X)$: the cone of pseudoeffective $(1, 1)$-classes.

Recall that we call a Bott–Chern class pseudoeffective if it contains a d-closed positive current, and call an $(n-1, n-1)$-class movable if it is contained in the closure of the cone generated by the classes of the form $\mu_*(\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_{n-1})$ where $\mu : \tilde{X} \to X$ is a modification and $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}$ are Kähler metrics on $\tilde{X}$. For the basic theory of positive currents, we refer the reader to [Demailly 2012].

Fields of characteristic p. Almost all the results in the paper will hold for smooth varieties over an arbitrary algebraically closed field. The necessary technical generalizations are verified in the following references:

- The existence of Fujita approximations over an arbitrary algebraically closed field is proved in [Takagi 2007].
- The basic properties of the σ-decomposition in positive characteristic are considered in [Mustaţă 2013].
- The results of [Cutkosky 2015] lay the foundations of the theory of positive products and volumes over an arbitrary field.
- [Fulger and Lehmann 2017b] describes how to extend [Boucksom et al. 2013] and most of the results of [Boucksom et al. 2009] over an arbitrary algebraically closed field. In particular the description of the derivative of the volume function in [Boucksom et al. 2009, Theorem A] holds for smooth varieties in any characteristic.

Compact Kähler manifolds. The following results enable us to extend our results in Section 3 and Section 5 to arbitrary compact hyperkähler manifolds and projective manifolds.

- The theory of positive intersection products for pseudoeffective $(1,1)$-classes has been developed by [Boucksom 2002a; Boucksom et al. 2010; 2013].
- Divisorial Zariski decomposition for pseudoeffective $(1,1)$-classes has been studied in [Boucksom 2004; Boucksom et al. 2013].
- By [Boucksom et al. 2013, Theorem 10.12] and [Nyström and Boucksom 2016], the transcendental analogues of the results in [Boucksom et al. 2009; 2013] are true for compact hyperkähler manifolds and projective manifolds. In particular, we have the cone duality $\mathcal{E}^* = \mathcal{M}$ and the description of the derivative of the volume for pseudoeffective $(1,1)$-classes.

Polar transforms. As explained in the introduction, our results use convex analysis, and in particular a Legendre–Fenchel type transform for functions defined on a cone. We briefly recall some definitions and results from [Lehmann and Xiao 2016] which will be used to study the function $\mathfrak{M}$.

Duality transforms. Let V be a finite-dimensional $\mathbb{R}$-vector space of dimension n, and let V^* be its dual. We denote the pairing of $w^* \in V^*$ and $v \in V$ by $w^* \cdot v$. Let $\mathfrak{C} \subset V$ be a proper closed convex cone of full dimension and let $\mathfrak{C}^* \subset V^*$ denote the dual cone of $\mathfrak{C}$. We let $\mathrm{HConc}_s(\mathfrak{C})$ denote the collection of functions $f : \mathfrak{C} \to \mathbb{R}$ satisfying:

- f is upper-semicontinuous and homogeneous of weight $s > 1$.
- f is strictly positive in the interior of $\mathfrak{C}$ (and hence nonnegative on $\mathfrak{C}$).
- f is s-concave: for any $v, x \in \mathfrak{C}$ we have $f(v)^{1/s} + f(x)^{1/s} \le f(v+x)^{1/s}$.

The polar transform $\mathcal{H}$ associates to a function $f \in \mathrm{HConc}_s(\mathfrak{C})$ the function $\mathcal{H}f : \mathfrak{C}^* \to \mathbb{R}$ defined as

$$\mathcal{H}f(w^*) := \inf_{v \in \mathfrak{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/(s-1)}.$$

The definition is unchanged if we instead vary v over all elements of $\mathfrak{C}$ where f is positive. It is not hard to see that $\mathcal{H}^2 f = f$ for any $f \in \mathrm{HConc}_s(\mathfrak{C})$.

It will be crucial to understand which points obtain the infimum in the definition of $\mathcal{H}f$.

Definition 2.1. Let $f \in \mathrm{HConc}_s(\mathfrak{C})$. For any $w^* \in \mathfrak{C}^*$, we define G_{w^*} to be the set of all $v \in \mathfrak{C}$ which satisfy $f(v) > 0$ and which achieve the infimum in the definition of $\mathcal{H}f(w^*)$, so that

$$\mathcal{H}f(w^*) = \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/(s-1)}.$$

Remark 2.2. The set G_{w^*} is the analogue of super-gradients of concave functions. In particular, we know the differential of $\mathcal{H}f$ at w^* lies in G_{w^*} if $\mathcal{H}f$ is differentiable.

We next identify the collection of points where f is controlled by $\mathcal{H}$.

Definition 2.3. Let $f \in \mathrm{HConc}_s(\mathfrak{C})$. We define $\mathfrak{C}_f$ to be the set of all $v \in \mathfrak{C}$ such that $v \in G_{w^*}$ for some $w^* \in \mathfrak{C}$ satisfying $\mathcal{H}f(w^*) > 0$.

We say that $f \in \mathrm{HConc}_s(\mathfrak{C})$ is differentiable if it is $\mathcal{C}^1$ on $\mathfrak{C}^\circ$. In this case we define the function

$$D : \mathfrak{C}^\circ \to V^* \quad \text{by} \quad v \mapsto \frac{df(v)}{s}.$$

We will need to understand the behavior of the derivative along the boundary.

Definition 2.4. We say that $f \in \mathrm{HConc}_s(\mathfrak{C})$ is $+$-differentiable if f is $\mathcal{C}^1$ on $\mathfrak{C}^\circ$ and the derivative on $\mathfrak{C}^\circ$ extends to a continuous function on all of $\mathfrak{C}_f$.

Remark 2.5. For $+$-differentiable functions f, we define the function $D : \mathfrak{C}_f \to V^*$ by extending continuously from $\mathfrak{C}^\circ$.

Teissier proportionality and strict log concavity. In [Lehmann and Xiao 2016], we gave some conditions which are equivalent to the strict log concavity.

Definition 2.6. Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ be $+$-differentiable and let $\mathfrak{C}_T$ be a nonempty subcone of $\mathfrak{C}_f$. We say that f satisfies Teissier proportionality with respect to $\mathfrak{C}_T$ if for any $v, x \in \mathfrak{C}_T$ satisfying

$$D(v) \cdot x = f(v)^{s-1/s} f(x)^{1/s}$$

we have that v and x are proportional.

Note that we do not assume that $\mathfrak{C}_T$ is convex — indeed, in examples it is important to avoid this condition. However, since f is defined on the convex hull of $\mathfrak{C}_T$, we can (somewhat abusively) discuss the strict log concavity of $f|_{\mathfrak{C}_T}$:

Definition 2.7. Let $\mathfrak{C}' \subset \mathfrak{C}$ be a (possibly nonconvex) subcone. We say that f is strictly log concave on $\mathfrak{C}'$ if

$$f(v)^{1/s} + f(x)^{1/s} < f(v+x)^{1/s}$$

holds whenever $v, x \in \mathfrak{C}'$ are not proportional. Note that this definition makes sense even when $\mathfrak{C}'$ is not itself convex.

Theorem 2.8 [Lehmann and Xiao 2016, Theorem 4.12]. *Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ be $+$-differentiable. For any nonempty subcone $\mathfrak{C}_T$ of $\mathfrak{C}_f$, consider the following conditions*:

(1) *The restriction $f|_{\mathfrak{C}_T}$ is strictly log concave (in the sense defined above).*

(2) *f satisfies Teissier proportionality with respect to $\mathfrak{C}_T$.*

(3) *The restriction of D to $\mathfrak{C}_T$ is injective.*

Then we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *If $\mathfrak{C}_T$ is convex, then we have* (2) $\Longrightarrow$ (1). *If $\mathfrak{C}_T$ is an open subcone, then we have* (3) $\Longrightarrow$ (1).

Sublinear boundary conditions. Under certain conditions we can control the behavior of $\mathcal{H}f$ near the boundary, and thus obtain the continuity.

Definition 2.9. Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ and let $\alpha \in (0, 1)$. We say that f satisfies the sublinear boundary condition of order α if for any nonzero v on the boundary of $\mathfrak{C}$ and for any x in the interior of $\mathfrak{C}$, there exists a constant $C := C(v, x) > 0$ such that $f(v+\epsilon x)^{1/s} \geq C\epsilon^{\alpha}$.

Note that the condition is always satisfied at v if $f(v) > 0$. Furthermore, the condition is satisfied for any v, x with $\alpha = 1$ by homogeneity and log-concavity, so the crucial question is whether we can decrease α slightly.

Using this sublinear condition, we get the vanishing of $\mathcal{H}f$ along the boundary.

Proposition 2.10 [Lehmann and Xiao 2016, Proposition 4.21]. *Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ satisfy the sublinear boundary condition of order α. Then $\mathcal{H}f$ vanishes along the boundary. As a consequence, $\mathcal{H}f$ extends to a continuous function over V^* by setting $\mathcal{H}f = 0$ outside $\mathfrak{C}^*$.*

Remark 2.11. If f satisfies the sublinear condition, then $\mathfrak{C}^*_{\mathcal{H}f} = \mathfrak{C}^{*\circ}$.

Formal Zariski decompositions. The Legendre–Fenchel transform relates the strict concavity of a function to the differentiability of its transform. The transform $\mathcal{H}$ will play the same role in our situation; however, one needs to interpret the strict concavity slightly differently. We will encapsulate this property using the notion of a Zariski decomposition.

Definition 2.12. Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ and let $U \subset \mathfrak{C}$ be a nonempty subcone. We say that f admits a strong Zariski decomposition with respect to U if:

(1) For every $v \in \mathfrak{C}_f$ there are unique elements $p_v \in U$ and $n_v \in \mathfrak{C}$ satisfying

$$v = p_v + n_v \quad \text{and} \quad f(v) = f(p_v).$$

We call the expression $v = p_v + n_v$ the Zariski decomposition of v, and call p_v the positive part and n_v the negative part of v.

(2) For any $v, w \in \mathfrak{C}_f$ satisfying $v + w \in \mathfrak{C}_f$ we have

$$f(v)^{1/s} + f(w)^{1/s} \leq f(v+w)^{1/s}$$

with equality only if p_v and p_w are proportional.

In [Lehmann and Xiao 2016, Theorem 4.3], we proved the following theorem linking the existence of Zariski decomposition structure with differentiability.

Theorem 2.13. *Let $f \in \mathrm{HConc}_s(\mathfrak{C})$. Then we have the following results*:

- *If f is $+$-differentiable, then $\mathcal{H}f$ admits a strong Zariski decomposition with respect to the cone $D(\mathfrak{C}_f) \cup \{0\}$.*
- *If $\mathcal{H}f$ admits a strong Zariski decomposition with respect to a cone U, then f is differentiable.*

In the first situation, one can construct the positive part of w^* by choosing any $v \in G_{w^*}$ with $f(v) > 0$ and choosing p_{w^*} to be the unique element of the ray spanned by $D(v)$ with $\mathcal{H}f(p_{w^*}) = \mathcal{H}f(w^*)$.

Under some additional conditions, we can get the continuity of formal Zariski decompositions (see [Lehmann and Xiao 2016, Theorem 4.6]). Note that for the divisorial Zariski decomposition the continuity is already well known due to the concavity of taking positive parts (see e.g., [Boucksom et al. 2009; Küronya and Maclean 2013; Nakayama 2004]).

Theorem 2.14. *Let $f \in \mathrm{HConc}_s(\mathfrak{C})$ be $+$-differentiable. Then the function taking an element $w^* \in \mathfrak{C}^{*\circ}$ to its positive part p_{w^*} is continuous.*

If furthermore $G_v \cup \{0\}$ is a unique ray for every $v \in \mathfrak{C}_f$ and $\mathcal{H}f$ is continuous on all of $\mathfrak{C}^*_{\mathcal{H}f}$, then the Zariski decomposition is continuous on all of $\mathfrak{C}^*_{\mathcal{H}f}$.

Zariski decomposition for curves. In [Lehmann and Xiao 2016], as an application of the above formal Zariski decomposition to the situation

$$\mathfrak{C} = \mathrm{Nef}^1(X), \quad f = \mathrm{vol}, \quad \mathfrak{C}^* = \overline{\mathrm{Eff}}_1(X), \quad \mathcal{H}f = \widehat{\mathrm{vol}},$$

we obtain the Zariski decomposition for curves. The following result is important in the proof of Theorem 1.3.

Definition 2.15. Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ be a big curve class. Then a Zariski decomposition for α is a decomposition

$$\alpha = B^{n-1} + \gamma$$

where B is a big and nef $\mathbb{R}$-Cartier divisor class, γ is pseudoeffective, and $B \cdot \gamma = 0$. We call B^{n-1} the "positive part" and γ the "negative part" of the decomposition.

Theorem 2.16. *Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ be a big curve class. Then α admits a unique Zariski decomposition $\alpha = B_\alpha^{n-1} + \gamma$. Furthermore,*

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B_\alpha^{n-1}) = \mathrm{vol}(B_\alpha)$$

and B_α is the unique big and nef divisor class with this property satisfying $B_\alpha^{n-1} \preceq \alpha$. The class B_α depends continuously on α.

Remark 2.17. As explained in [Lehmann and Xiao 2016, Remark 5.1], the above result holds in the Kähler setting — we have a similar decomposition for any interior point of the pseudoeffective $(n-1, n-1)$-cone $\mathcal{N}$.

3. Refined structure of the movable cone

In this section, we study the movable cone of curves and its relationship to the positive product of divisors. A key tool in this study is the following function of [Xiao 2017, Definition 2.2]:

Definition 3.1. Let X be a projective variety of dimension n. For any curve class $\alpha \in \mathrm{Mov}_1(X)$ define

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big divisor class}} \left(\frac{L \cdot \alpha}{\mathrm{vol}(L)^{1/n}} \right)^{n/(n-1)}.$$

We say that a big class L computes $\mathfrak{M}(\alpha)$ if this infimum is achieved by L. When α is a curve class that is not movable, we set $\mathfrak{M}(\alpha) = 0$.

In other words, $\mathfrak{M}$ is the function on $\mathrm{Mov}_1(X)$ defined as the polar transform of the volume function on $\overline{\mathrm{Eff}}^1(X)$. Dually, we can think of the volume function on divisors as the polar transform of $\mathfrak{M}$; this viewpoint allows us to apply the general theory of convexity developed in [Lehmann and Xiao 2016] to vol.

In this section we first prove some new results concerning the volume function for divisors. We will then return to the study of $\mathfrak{M}$ below, where we show that it measures the volume of the "$(n-1)$-st root" of α.

The volume function on big and movable divisors. We first extend several well-known results on big and nef divisors to big and movable divisors. The key will be an extension of Teissier proportionality theorem for big and nef divisors (see [Lehmann and Xiao 2016; Boucksom et al. 2009]) to big and movable divisors.

Lemma 3.2. *Let X be a projective variety of dimension n. Let L_1 and L_2 be big movable divisor classes. Set s to be the largest real number such that $L_1 - sL_2$ is pseudoeffective. Then*

$$s^n \leq \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. We first prove the case when X is smooth. Certainly we have $\mathrm{vol}(L_1) \geq \mathrm{vol}(sL_2) = s^n \,\mathrm{vol}(L_2)$. If they are equal, then since sL_2 is movable and $L_1 - sL_2$ is pseudoeffective we get a Zariski decomposition of

$$L_1 = sL_2 + (L_1 - sL_2)$$

in the sense of [Fulger and Lehmann 2017b]. By [Fulger and Lehmann 2017b, Proposition 5.3], this decomposition coincides with the numerical version of the σ-decomposition of [Nakayama 2004] so that $P_\sigma(L_1) = sL_2$. Since L_1 is movable, we obtain equality $L_1 = sL_2$.

For arbitrary X, let $\phi : X' \to X$ be a resolution. The inequality follows by pulling back L_1 and L_2 and replacing them by their positive parts. Indeed using the numerical analogue of [Nakayama 2004, III.1.14 Proposition] we see that $\phi^* L_1 - sP_\sigma(\phi^* L_2)$ is pseudoeffective if and only if $P_\sigma(\phi^* L_1) - sP_\sigma(\phi^* L_2)$ is pseudoeffective, so that s can only go up under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = sP_\sigma(\phi^* L_2)$ as elements of $N_{n-1}(X)$, then $L_1 = sL_2$ as elements of $N^1(X)$ by the injectivity of the capping map. □

Next we prove the Diskant inequality for big and movable divisor classes, generalizing the version for big and nef divisors in [Boucksom et al. 2009].

Proposition 3.3. *Let X be a smooth projective variety of dimension n. Let L_1, L_2 be big and movable divisor classes. Set s_L to be the largest real number such that $L_1 - s_L L_2$ is pseudoeffective. Then*

$$(\langle L_1^{n-1}\rangle \cdot L_2)^{n/(n-1)} - \mathrm{vol}(L_1)\,\mathrm{vol}(L_2)^{1/n-1} \geq ((\langle L_1^{n-1}\rangle \cdot L_2)^{1/n-1} - s_L\,\mathrm{vol}(L_2)^{1/n-1})^n. \qquad (\dagger)$$

Proof. Fix an ample divisor H on X.

For any $\epsilon > 0$, by taking sufficiently good Fujita approximations we may find a birational map $\phi_\epsilon : Y_\epsilon \to X$ and ample divisor classes $A_{1,\epsilon}$ and $A_{2,\epsilon}$ such that

- $\phi_\epsilon^* L_i - A_{i,\epsilon}$ is pseudoeffective for $i = 1, 2$;
- $\mathrm{vol}(A_{i,\epsilon}) > \mathrm{vol}(L_i) - \epsilon$ for $i = 1, 2$;
- $\phi_{\epsilon *} A_{i,\epsilon}$ is in an ϵ-ball around L_i for $i = 1, 2$.

Furthermore:

- By applying the argument of [Fulger and Lehmann 2017b, Theorem 6.22], we may ensure
$$\phi_\epsilon^*(\langle L_1^{n-1}\rangle - \epsilon H^{n-1}) \preceq A_{1,\epsilon}^{n-1} \preceq \phi_\epsilon^*(\langle L_1^{n-1}\rangle + \epsilon H^{n-1}).$$
- Set s_ϵ to be the largest real number such that $A_{1,\epsilon} - s_\epsilon A_{2,\epsilon}$ is pseudoeffective. Then we may ensure that $s_\epsilon < s_L + \epsilon$.

By the Khovanskii–Teissier inequality for nef divisor classes, we have

$$(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/(n-1)} \geq \mathrm{vol}(A_{1,\epsilon})\,\mathrm{vol}(A_{2,\epsilon})^{1/n-1}.$$

Note that $\langle L^{n-1}\rangle \cdot L_2$ is approximated by $A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon}$ by the projection formula. Taking a limit as ϵ goes to 0, we see that

$$\langle L_1^{n-1}\rangle \cdot L_2 \geq \mathrm{vol}(L_1)^{n-1/n}\,\mathrm{vol}(L_2)^{1/n}. \qquad (\star)$$

On the other hand, the Diskant inequality for big and nef divisors in [Boucksom et al. 2009, Theorem F] implies that

$$\begin{aligned}(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/(n-1)} - \mathrm{vol}(A_{1,\epsilon})\,\mathrm{vol}(A_{2,\epsilon})^{1/n-1} &\geq ((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - s_\epsilon\,\mathrm{vol}(A_{2,\epsilon})^{1/n-1})^n \\ &\geq ((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - (s_L + \epsilon)\,\mathrm{vol}(A_{2,\epsilon})^{1/n-1})^n.\end{aligned}$$

Taking a limit as ϵ goes to 0 again, we see that

$$(\langle L_1^{n-1}\rangle \cdot L_2)^{n/(n-1)} - \mathrm{vol}(L_1)\,\mathrm{vol}(L_2)^{1/n-1} \geq ((\langle L_1^{n-1}\rangle \cdot L_2)^{1/n-1} - s_L\,\mathrm{vol}(L_2)^{1/n-1})^n.$$

This finishes the proof of the Diskant inequality for big and movable divisor classes. □

Remark 3.4. As shown in [Lehmann and Xiao 2017, Section 3] and implicitly proved in [Fulger and Lehmann 2017b] (which in turn follows from a result of [Boucksom et al. 2009]), for two big movable divisor classes L_1, L_2, we indeed have $\langle L_1^{n-1}\rangle \cdot L_2 = \langle L_1^{n-1} \cdot L_2\rangle$.

As a corollary of Proposition 3.3, we get:

Proposition 3.5. *Let X be a projective variety of dimension n. Let L_1, L_2 be big and movable divisor classes. Then*

$$\langle L_1^{n-1}\rangle \cdot L_2 \geq \mathrm{vol}(L_1)^{n-1/n}\,\mathrm{vol}(L_2)^{1/n}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. If X is smooth, then the result follows directly from Lemma 3.2, $\star$ and $\dagger$.

Now suppose X is singular. The inequality can be computed by passing to a resolution $\phi : X' \to X$ and replacing L_1 and L_2 by their positive parts, since the left-hand side can only decrease under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = s P_\sigma(\phi^* L_2)$ as elements of $N_{n-1}(X)$, then $L_1 = s L_2$ as elements of $N^1(X)$ by the injectivity of the capping map. □

Remark 3.6. In the analytic setting, applying Proposition 3.5 and the same method as [Lehmann and Xiao 2016], it is not hard to generalize Proposition 3.5 to any number of big and movable divisor classes provided we have sufficient regularity for degenerate Monge–Ampère equations in big classes:

- Let $L_1, \ldots, L_n$ be n big divisor classes over a smooth complex projective variety X, then we have

$$\langle L_1 \cdots L_n\rangle \geq \mathrm{vol}(L_1)^{1/n} \cdots \mathrm{vol}(L_n)^{1/n}$$

  where the equality is obtained if and only if $P_\sigma(L_1), \ldots, P_\sigma(L_n)$ are proportional.

We only need to characterize the equality situation. To see this, we need the fact that the above positive intersection $\langle L_1 \cdots L_n\rangle$ depends only on the positive parts $P_\sigma(L_i)$, which follows from the analytic construction of positive product [Boucksom 2002a, Proposition 3.2.10]. Then by the method in [Lehmann and Xiao 2016] where we apply [Boucksom et al. 2010] or [Demailly et al. 2014, Theorem D], we reduce it to the case of a pair of divisor classes, e.g., we get

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2)\rangle = \mathrm{vol}(L_1)^{n-1/n}\,\mathrm{vol}(L_2)^{1/n}.$$

By the definition of positive product we always have

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2)\rangle \geq \langle P_\sigma(L_1)^{n-1}\rangle \cdot P_\sigma(L_2) \geq \mathrm{vol}(L_1)^{n-1/n}\,\mathrm{vol}(L_2)^{1/n},$$

this then implies the equality

$$\langle P_\sigma(L_1)^{n-1}\rangle \cdot P_\sigma(L_2) = \mathrm{vol}(L_1)^{n-1/n}\,\mathrm{vol}(L_2)^{1/n}.$$

By Proposition 3.5, we immediately obtain the desired result. See also [Lehmann and Xiao 2017, Section 7] for an alternative approach.

Corollary 3.7. *Let X be a smooth projective variety of dimension n. Let $\alpha \in \mathrm{Mov}_1(X)$ be a big movable curve class. All big divisor classes L satisfying $\alpha = \langle L^{n-1}\rangle$ have the same positive part $P_\sigma(L)$.*

Proof. Suppose L_1 and L_2 have the same positive product. We have $\mathrm{vol}(L_1) = \langle L_2^{n-1}\rangle \cdot L_1$ so that $\mathrm{vol}(L_1) \geq \mathrm{vol}(L_2)$. By symmetry we obtain the reverse inequality, hence equality everywhere, and we conclude by Proposition 3.5 and the σ-decomposition for smooth varieties. □

As a consequence of Proposition 3.5, we show the strict log concavity of the volume function vol on the cone of big and movable divisors.

Proposition 3.8. *Let X be a projective variety of dimension n. Then the volume function* vol *is strictly log concave on the cone of big and movable divisor classes.*

Proof. Since the big and movable cone is convex and since the derivative of vol is continuous, this follows immediately from Proposition 3.5 and Theorem 2.8. □

As a consequence, we get:

Theorem 3.9. *Let X be a projective variety of dimension n. Then for any two big divisor classes L_1, L_2, the equality*

$$\mathrm{vol}(L_1+L_2)^{1/n} = \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

holds if and only if the positive parts $P(L_1)$, $P(L_2)$ are proportional.

It is well known that $\mathrm{vol}(L_1+L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$, thus the above result give a characterization on the equality case.

Proof. First, we assume the equality holds. Note that $\mathrm{vol}(L_i) = \mathrm{vol}(P(L_i))$ for $i = 1, 2$, then we get

$$\mathrm{vol}(L_1+L_2)^{1/n} \geq \mathrm{vol}(P(L_1)+P(L_2))^{1/n} \geq \mathrm{vol}(P(L_1))^{1/n} + \mathrm{vol}(P(L_2))^{1/n} = \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}.$$

The equality assumption implies that $\mathrm{vol}(P(L_1)+P(L_2))^{1/n} = \mathrm{vol}(P(L_1))^{1/n} + \mathrm{vol}(P(L_2))^{1/n}$, then by Proposition 3.8 the positive parts $P(L_1)$, $P(L_2)$ are proportional.

Next we assume that the positive parts $P(L_1)$, $P(L_2)$ are proportional. We claim that $P(L_1+L_2) = P(L_1)+P(L_2)$. With this claim, it is easy to see the equality for volumes holds. Next we prove the claim. By the divisorial Zariski decomposition, we have two decompositions for L_1+L_2:

$$L_1+L_2 = P(L_1+L_2)+N(L_1+L_2) = P(L_1)+P(L_2)+N(L_1)+N(L_2).$$

Since $P(L_1)$, $P(L_2)$ are proportional, the orthogonality estimate in the divisorial Zariski decomposition implies

$$\langle (P(L_1)+P(L_2))^{n-1}\rangle \cdot (N(L_1)+N(L_2)) = 0.$$

Multiplying by $\langle (P(L_1)+P(L_2))^{n-1}\rangle$ in the two decompositions of L_1+L_2, we get

$$\langle (P(L_1)+P(L_2))^{n}\rangle \geq P(L_1+L_2)\cdot \langle (P(L_1)+P(L_2))^{n-1}\rangle.$$

By the Khovanski–Teissier inequality, this yields that $\mathrm{vol}(P(L_1)+P(L_2)) \geq \mathrm{vol}(P(L_1+L_2))$. However, we always have $\mathrm{vol}(P(L_1+L_2)) \geq \mathrm{vol}(P(L_1)+P(L_2))$, thus the equality holds everywhere. In particular, Proposition 3.5 implies that $P(L_1+L_2) = P(L_1)+P(L_2)$, finishing the proof of our claim. □

The function $\mathfrak{M}$. We now return to the study of the function $\mathfrak{M}$. We are in the situation:

$$\mathfrak{C} = \overline{\mathrm{Eff}}^1(X), \quad f = \mathrm{vol}, \quad \mathfrak{C}^* = \mathrm{Mov}_1(X), \quad \mathcal{H}f = \mathfrak{M}.$$

Note that $\mathfrak{C}^* = \mathrm{Mov}_1(X)$ follows from the main result of [Boucksom et al. 2013].

As preparation for using the polar transform theory, we recall the analytic properties of the volume function for divisors on smooth varieties. By [Boucksom et al. 2009] the volume function on the pseudoeffective cone of divisors is differentiable on the big cone (with $D(L) = \langle L^{n-1}\rangle$). In the notation of Definition 2.3 the cone $\overline{\mathrm{Eff}}^1(X)_{\mathrm{vol}}$ coincides with the big cone, so that vol is $+$-differentiable. The volume function is n-concave, and is strictly n-concave on the big and movable cone by Proposition 3.8. Furthermore, it admits a strong Zariski decomposition with respect to the movable cone of divisors using the σ-decomposition of [Nakayama 2004] and Proposition 3.8.

Remark 3.10. Note that if X is not smooth (or at least $\mathbb{Q}$-factorial), then it is unclear whether vol admits a Zariski decomposition structure with respect to the cone of movable divisors. For this reason, we will focus on smooth varieties in this section. See Remark 3.24 for more details.

Our first task is to understand the behavior of $\mathfrak{M}$ on the boundary of the movable cone of curves. Note that vol does not satisfy a sublinear condition, so that $\mathfrak{M}$ may not vanish on the boundary of $\mathrm{Mov}_1(X)$.

Lemma 3.11. *Let X be a smooth projective variety of dimension n and let α be a movable curve class. Then $\mathfrak{M}(\alpha) = 0$ if and only if α has vanishing intersection against a nonzero movable divisor class L.*

Proof. We first show that if there exists some nonzero movable divisor class M such that $\alpha \cdot M = 0$ then $\mathfrak{M}(\alpha) = 0$. Fix an ample divisor class A. Note that $M + \epsilon A$ is big and movable for any $\epsilon > 0$. Thus there exists some modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class A_ϵ on Y_ϵ such that $M + \frac{\epsilon}{2} A = \mu_{\epsilon *} A_\epsilon$. So we can write

$$M + \epsilon A = \mu_{\epsilon *}\big(A_\epsilon + \tfrac{\epsilon}{2}\mu_\epsilon^* A\big),$$

which implies

$$\begin{aligned} \mathrm{vol}(M + \epsilon A) &= \mathrm{vol}\big(\mu_{\epsilon *}\big(A_\epsilon + \tfrac{\epsilon}{2}\mu_\epsilon^* A\big)\big) \\ &\geq \mathrm{vol}\big(A_\epsilon + \tfrac{\epsilon}{2}\mu_\epsilon^* A\big) \\ &\geq n\big(\tfrac{\epsilon}{2}\mu_\epsilon^* A\big)^{n-1} \cdot A_\epsilon \\ &\geq c\epsilon^{n-1} A^{n-1} \cdot M. \end{aligned}$$

This estimate shows that the intersection number

$$\rho_\epsilon = \alpha \cdot \frac{M + \epsilon A}{\mathrm{vol}(M + \epsilon A)^{1/n}}.$$

tends to zero as ϵ tends to zero, and so $\mathfrak{M}(\alpha) = 0$.

Conversely, suppose that $\mathfrak{M}(\alpha) = 0$. From the definition of $\mathfrak{M}(\alpha)$, we can take a sequence of big divisor classes L_k with $\mathrm{vol}(L_k) = 1$ such that

$$\lim_{k \to \infty} (\alpha \cdot L_k)^{n/(n-1)} = \mathfrak{M}(\alpha).$$

Moreover, let $P_\sigma(L_k)$ be the positive part of L_k. Then we have $\mathrm{vol}(P_\sigma(L_k)) = 1$ and

$$\alpha \cdot P_\sigma(L_k) \leq \alpha \cdot L_k$$

since α is movable. Thus we can assume the sequence of big divisor classes L_k is movable in the beginning.

Fix an ample divisor A of volume 1, and consider the classes $L_k/(A^{n-1} \cdot L_k)$. These lie in a compact slice of the movable cone, so they must have a nonzero movable accumulation point L, which without loss of generality we may assume is a limit.

Choose a modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class $A_{\epsilon,k}$ on Y such that

$$A_{\epsilon,k} \leq \mu_\epsilon^* L_k, \quad \mathrm{vol}(A_{\epsilon,k}) > \mathrm{vol}(L_k) - \epsilon$$

Then

$$L_k \cdot A^{n-1} \geq A_{\epsilon,k} \cdot \mu_\epsilon^* A^{n-1} \geq \mathrm{vol}(A_{\epsilon,k})^{1/n}$$

by the Khovanskii–Teissier inequality. Taking a limit over all ϵ, we find $L_k \cdot A^{n-1} \geq \mathrm{vol}(L_k)^{1/n}$. Thus

$$L \cdot \alpha = \lim_{k\to\infty} \frac{L_k \cdot \alpha}{L_k \cdot A^{n-1}} \leq \mathfrak{M}(\alpha)^{n-1/n} = 0. \qquad \square$$

Example 3.12. Note that a movable curve class α with positive $\mathfrak{M}$ need not lie in the interior of the movable cone of curves. A simple example is when X is the blow-up of $\mathbb{P}^2$ at one point, H denotes the pullback of the hyperplane class. For surfaces the functions $\mathfrak{M}$ and vol coincide, so $\mathfrak{M}(H) = 1$ even though H is on the boundary of $\mathrm{Mov}_1(X) = \mathrm{Nef}^1(X)$.

It is also possible for a big movable curve class α to have $\mathfrak{M}(\alpha) = 0$. This occurs for the projective bundle $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1))$. There are two natural divisor classes on X: the class f of the fibers of the projective bundle and the class ξ of the sheaf $\mathcal{O}_{X/\mathbb{P}^1}(1)$. Using for example [Fulger 2011, Theorem 1.1] and [Fulger and Lehmann 2017b, Proposition 7.1], one sees that f and ξ generate the algebraic cohomology classes with the relations $f^2 = 0$, $\xi^2 f = -\xi^3 = 1$ and that $\mathrm{Mov}^1(X) = \langle f, \xi \rangle$ and $\mathrm{Mov}_1(X) = \langle \xi f, \xi^2 + \xi f \rangle$. We see that the big and movable curve class $\xi^2 + \xi f$ has vanishing intersection against the movable divisor ξ so that $\mathfrak{M}(\xi^2 + \xi f) = 0$ by Lemma 3.11.

Remark 3.13. Another perspective on Lemma 3.11 is provided by the numerical dimension of [Nakayama 2004; Boucksom 2004]. On a smooth variety the following conditions are equivalent for a class $L \in \mathrm{Mov}^1(X)$. (They both correspond to the nonvanishing of the numerical dimension.)

- Fix an ample divisor class A. For any big class D, there is a positive constant C such that $Ct^{n-1} < \mathrm{vol}(L + tA)$ for all $t > 0$.
- $L \neq 0$.

In particular, this implies that vol satisfies the sublinear boundary condition of order $n - 1/n$ when restricted to the movable cone, and this fact can be used in the previous proof. A variant of this statement in characteristic p is proved by [Cascini et al. 2014].

In many ways it is most natural to define $\mathfrak{M}$ using the movable cone of divisors instead of the pseudoeffective cone of divisors. Conceptually, this coheres with the fact that the polar transform can be calculated using the positive part of a Zariski decomposition. Recall that the positive part $P_\sigma(L)$ of a pseudoeffective divisor L has $P_\sigma(L) \preceq L$ and $\mathrm{vol}(P_\sigma(L)) = \mathrm{vol}(L)$. Arguing as in Lemma 3.11 by taking positive parts, we see that for any $\alpha \in \mathrm{Mov}_1(X)$ we have

$$\mathfrak{M}(\alpha) = \inf_{D \text{ big and movable}} \left(\frac{D \cdot \alpha}{\mathrm{vol}(D)^{1/n}} \right)^{n/(n-1)}.$$

Thus for X smooth it is perhaps better to consider the following polar transform:

$$\mathfrak{C} = \mathrm{Mov}^1(X), \qquad f = \mathrm{vol}, \qquad \mathfrak{C}^* = \mathrm{Mov}^1(X)^*, \qquad \mathcal{H}f = \mathfrak{M}'.$$

Since vol satisfies a sublinear condition on $\mathrm{Mov}^1(X)$, the function $\mathfrak{M}'$ is strictly positive exactly in $\mathrm{Mov}^1(X)^{*\circ}$ and extends to a continuous function over $N_1(X)$. The relationship between the two functions is given by

$$\mathfrak{M}'|_{\mathrm{Mov}_1(X)} = \mathfrak{M};$$

this follows immediately from the description for $\mathfrak{M}$ earlier in this paragraph. In fact by Theorem 2.13 $\mathfrak{M}'$ admits a strong Zariski decomposition. Using the interpretation of positive parts via derivatives as in Theorem 2.13, the results of [Boucksom et al. 2009; Lazarsfeld and Mustaţă 2009] show that the positive parts for the Zariski decomposition of $\mathfrak{M}'$ lie in $\mathrm{Mov}_1(X)$. In this way one can think of $\mathfrak{M}$ as the "Zariski projection" of $\mathfrak{M}'$.

Note one important consequence of this perspective: Lemma 3.11 shows that the subcone of $\mathrm{Mov}_1(X)$ where $\mathfrak{M}$ is positive lies in the interior of $\mathrm{Mov}^1(X)^*$. Thus this region agrees with $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and $\mathfrak{M}$ extends to a differentiable function on an open set containing this cone by applying Theorem 2.13. In particular $\mathfrak{M}$ is $+$-differentiable and continuous on $\mathrm{Mov}_1(X)$.

We next prove a refined structure of the movable cone of curves. Recall that by [Boucksom et al. 2013] the movable cone of curves $\mathrm{Mov}_1(X)$ is generated by the $(n-1)$-self positive products of big divisors. In other words, any curve class in the interior of $\mathrm{Mov}_1(X)$ is a *convex combination* of such positive products. We show that $\mathrm{Mov}_1(X)$ actually coincides with the closure of such products (which naturally form a cone).

Theorem 3.14. *Let X be a smooth projective variety of dimension n. Then any movable curve class α with $\mathfrak{M}(\alpha) > 0$ has the form*

$$\alpha = \langle L_\alpha^{n-1} \rangle$$

for a unique big and movable divisor class L_α. We then have $\mathfrak{M}(\alpha) = \mathrm{vol}(L_\alpha)$ and any big and movable divisor computing $\mathfrak{M}(\alpha)$ is proportional to L_α.

Proof. Applying Theorem 2.13 to $\mathfrak{M}'$, we get

$$\alpha = D(L_\alpha) + n_\alpha$$

where L_α is a big movable class computing $\mathfrak{M}(\alpha)$ and $n_\alpha \in \mathrm{Mov}^1(X)^*$. As D is the differential of $\mathrm{vol}^{1/n}$ on big and movable divisor classes, we have $D(L_\alpha) = \langle L_\alpha^{n-1} \rangle$. Note that $\mathfrak{M}(\alpha) = \langle L_\alpha^{n-1} \rangle \cdot L_\alpha = \mathrm{vol}(L_\alpha)$.

To finish the proof, we observe that $n_\alpha \in \mathrm{Mov}_1(X)$. This follows since α is movable: by the definition of L_α, for any pseudoeffective divisor class E and $t \geq 0$ we have

$$\frac{\alpha \cdot L_\alpha}{\mathrm{vol}(L_\alpha)^{1/n}} \leq \frac{\alpha \cdot P_\sigma(L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}} \leq \frac{\alpha \cdot (L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}}$$

with equality at $t = 0$. This then implies

$$n_\alpha \cdot E \geq 0.$$

Thus $n_\alpha \in \mathrm{Mov}_1(X)$. Intersecting against L_α, we have $n_\alpha \cdot L_\alpha = 0$. This shows $n_\alpha = 0$ because L_α is an interior point of $\overline{\mathrm{Eff}}^1(X)$ and $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$. So we have $\alpha = D(L_\alpha) = \langle L_\alpha^{n-1} \rangle$.

Finally, uniqueness follows from Corollary 3.7. □

We note in passing an immediate consequence:

Corollary 3.15. *Let X be a projective variety of dimension n. Then the rays spanned by classes of irreducible curves which deform to cover X are dense in* $\mathrm{Mov}_1(X)$.

Proof. It suffices to prove this on a resolution of X. By Theorem 3.14 it suffices to show that any class of the form $\langle L^{n-1} \rangle$ for a big divisor L is a limit of rescalings of classes of irreducible curves which deform to cover X. Indeed, we may even assume that L is a $\mathbb{Q}$-Cartier divisor. Then the positive product is a limit of the pushforward of complete intersections of ample divisors on birational models, whence the result. □

We can also describe the boundary of $\mathrm{Mov}_1(X)$, in combination with Lemma 3.11.

Corollary 3.16. *Let X be a smooth projective variety of dimension n. Let α be a movable class with $\mathfrak{M}(\alpha) > 0$ and let L_α be the unique big movable divisor whose positive product is α. Then α is on the boundary of* $\mathrm{Mov}_1(X)$ *if and only if L_α is on the boundary of* $\mathrm{Mov}^1(X)$.

Proof. Note that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if it has vanishing intersection against a class D lying on an extremal ray of $\overline{\mathrm{Eff}}^1(X)$. Lemma 3.11 shows that in this case D is not movable, so by [Nakayama 2004, Chapter III.1] D is (after rescaling) the class of an integral divisor on X which we continue to call D. By [Boucksom et al. 2009, Proposition 4.8 and Theorem 4.9], the equation $\langle L_\alpha^{n-1} \rangle \cdot D = 0$ holds if and only if $D \in \mathbb{B}_+(L_\alpha)$. Altogether, we see that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if L_α is on the boundary of $\mathrm{Mov}^1(X)$. □

Arguing using abstract properties of polar transforms just as in [Lehmann and Xiao 2016], the good analytic properties of the volume function for divisors imply most of the other analytic properties of $\mathfrak{M}$.

Theorem 3.17 (see Theorem 2.14, and compare with [Lehmann and Xiao 2016, Theorem 5.6]). *Let X be a smooth projective variety of dimension n. For any movable curve class α with $\mathfrak{M}(\alpha) > 0$, let L_α denote the unique big and movable divisor class satisfying $\langle L_\alpha^{n-1} \rangle = \alpha$. As we vary α in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$, L_α depends continuously on α.*

Theorem 3.18 (compare with [Lehmann and Xiao 2016, Theorem 5.11]). *Let X be a smooth projective variety of dimension n. For a curve class $\alpha = \langle L_\alpha^{n-1}\rangle$ in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and for an arbitrary curve class $\beta \in N_1(X)$ we have*

$$\frac{d}{dt}\bigg|_{t=0} \mathfrak{M}(\alpha + t\beta) = \frac{n}{n-1} P_\sigma(L_\alpha)\cdot\beta.$$

Theorem 3.19 (see Theorem 2.13, and compare with [Lehmann and Xiao 2016, Theorem 5.10]). *Let X be a smooth projective variety of dimension n. Let α_1, α_2 be two big and movable curve classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then*

$$\mathfrak{M}(\alpha_1+\alpha_2)^{n-1/n} \geq \mathfrak{M}(\alpha_1)^{n-1/n} + \mathfrak{M}(\alpha_2)^{n-1/n}$$

with equality if and only if α_1 and α_2 are proportional.

Remark 3.20. Theorem 3.19 can be interpreted as an analogue of the Knesser–Süss inequality for polytopes. We clarify this relationship when discussing toric varieties in Section 4.

Another application of the results in this section is the Morse-type bigness criterion for movable curve classes, which is slightly different from [Lehmann and Xiao 2016, Theorem 5.18].

Theorem 3.21. *Let X be a smooth projective variety of dimension n. Let α, β be two curve classes lying in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Write $\alpha = \langle L_\alpha^{n-1}\rangle$ and $\beta = \langle L_\beta^{n-1}\rangle$ for the unique big and movable divisor classes L_α, L_β given by Theorem 3.14. Then we have*

$$\mathfrak{M}(\alpha-\beta)^{n-1/n} \geq (\mathfrak{M}(\alpha) - nL_\alpha\cdot\beta)\cdot\mathfrak{M}(\alpha)^{-1/n} = (\mathrm{vol}(L_\alpha) - nL_\alpha\cdot\beta)\cdot\mathrm{vol}(L_\alpha)^{-1/n}.$$

In particular, we have

$$\mathfrak{M}(\alpha-\beta) \geq \mathrm{vol}(L_\alpha) - \frac{n^2}{n-1}L_\alpha\cdot\beta$$

and the curve class $\alpha-\beta$ is big whenever $\mathfrak{M}(\alpha) - nL_\alpha\cdot\beta > 0$.

Proof. By [Lehmann and Xiao 2016, Section 4.2] it suffices to prove a Morse-type bigness criterion for the difference of two movable divisor classes. So we need to prove $L-M$ is big whenever

$$\langle L^n\rangle - n\langle L^{n-1}\rangle\cdot M > 0.$$

This is proved (in the Kähler setting) in [Xiao 2018, Theorem 1.1]. □

Remark 3.22. We remark that we cannot extend this Morse-type criterion from big and movable divisors to arbitrary pseudoeffective divisor classes. A very simple construction provides the counter-examples, e.g., the blow up of $\mathbb{P}^2$ (see [Trapani 1995, Example 3.8]).

Combining Theorem 3.14 and Theorem 3.17, we obtain:

Corollary 3.23. *Let X be a smooth projective variety of dimension n. Then*

$$\Phi : \mathrm{Mov}^1(X)_{\mathrm{vol}} \to \mathrm{Mov}_1(X)_{\mathfrak{M}}, \quad L \mapsto \langle L^{n-1}\rangle$$

is a homeomorphism.

Remark 3.24. Modified versions of many of the results in this section hold for singular varieties as well (see Remark 3.10). For example, by similar arguments we can see that any element in the interior of $\mathrm{Mov}_1(X)$ is the positive product of some big divisor class regardless of singularities. Conversely, whenever $\mathfrak{M}$ is $+$-differentiable we obtain a Zariski decomposition structure for vol by Theorem 2.13.

Remark 3.25. All the results above extend to smooth varieties over algebraically closed fields. However, for compact Kähler manifolds some results rely on Demailly's conjecture on the transcendental holomorphic Morse-type inequality, or equivalently, on the extension of the results of [Boucksom et al. 2009] to the Kähler setting. Since the results of [Boucksom et al. 2009] are used in an essential way in the proof of Theorems 3.14 and 3.2 (via the proof of [Fulger and Lehmann 2017b, Proposition 5.3]), the only statement in this section which extends unconditionally to the Kähler setting is Lemma 3.11. However, these conjectures are known if X is a compact hyperkähler manifold or projective manifold (see [Boucksom et al. 2013, Theorem 10.12; Nyström and Boucksom 2016]), so all of our results extend to compact hyperkähler manifolds.

4. Positivity functions on toric varieties

We study the function $\mathfrak{M}$ on toric varieties, showing that it can be interpreted by the underlying special structures. In this section, X will denote a simplicial projective toric variety of dimension n. In terms of notation, X will be defined by a fan Σ in a lattice N with dual lattice M. We let $\{v_i\}$ denote the primitive generators of the rays of Σ and $\{D_i\}$ denote the corresponding classes of T-divisors. Our goal is to interpret the properties of the function $\mathfrak{M}$ in terms of toric geometry.

Positive product on toric varieties. Suppose that L is a big movable divisor class on the toric variety X. Then L naturally defines a (nonlattice) polytope Q_L; if we choose an expression $L = \sum a_i D_i$, then

$$Q_L = \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle + a_i \geq 0\}$$

and changing the choice of representative corresponds to a translation of Q_L. Conversely, suppose that Q is a full-dimensional polytope such that the unit normals to the facets of Q form a subset of the rays of Σ. Then Q uniquely determines a big movable divisor class L_Q on X. The divisors in the interior of the movable cone correspond to those polytopes whose facet normals coincide with the rays of Σ.

Given polytopes $Q_1, \ldots, Q_n$, let $V(Q_1, \ldots, Q_n)$ denote the mixed volume of the polytopes. [Boucksom et al. 2009] explains that the positive product of big movable divisors $L_1, \ldots, L_n$ can be interpreted via the mixed volume of the corresponding polytopes:

$$\langle L_1 \cdots L_n \rangle = n! V(Q_1, \ldots, Q_n).$$

The function $\mathfrak{M}$. In this section we use a theorem of Minkowski to describe the function $\mathfrak{M}$. We thank J. Huh for a conversation working out this picture.

Recall that a class $\alpha \in \mathrm{Mov}_1(X)$ defines a nonnegative Minkowski weight on the rays of the fan Σ — that is, an assignment of a positive real number t_i to each vector v_i such that $\sum t_i v_i = 0$. From now on we will identify α with its Minkowski weight. We will need to identify which movable curve classes are positive along a set of rays which span $\mathbb{R}^n$.

Lemma 4.1. *Suppose $\alpha \in \mathrm{Mov}_1(X)$ satisfies $\mathfrak{M}(\alpha) > 0$. Then α is positive along a spanning set of rays of Σ.*

We will soon see that the converse is also true in Theorem 4.2.

Proof. Suppose that there is a hyperplane V which contains every ray of Σ along which α is positive. Since X is projective, Σ has rays on both sides of V. Let D be the effective divisor consisting of the sum over all the primitive generators of rays of Σ not contained in V. It is clear that the polytope defined by D has nonzero projection onto the subspace spanned by $V^{\perp}$, and in particular, that the polytope defined by D is nonzero. Thus the asymptotic growth of sections of mD is at least linear in m, so $P_\sigma(D) \neq 0$ and α has vanishing intersection against a nonzero movable divisor. Lemma 3.11 shows that $\mathfrak{M}(\alpha) = 0$. □

Minkowski's theorem asserts the following. Suppose that $u_1, \dots, u_s$ are unit vectors which span $\mathbb{R}^n$ and that $r_1, \dots, r_s$ are positive real numbers. Then there exists a polytope P with unit normals $u_1, \dots, u_s$ and with corresponding facet volumes $r_1, \dots, r_s$ if and only if the u_i satisfy

$$r_1 u_1 + \cdots + r_s u_s = 0.$$

Moreover, the resulting polytope is unique up to translation. (See [Klain 2004] for a proof which is compatible with the results below.) If a vector u is a unit normal to a facet of P, we will use the notation $\mathrm{vol}(P^u)$ to denote the volume of the facet corresponding to u.

If α is positive on a spanning set of rays, then it canonically defines a polytope (up to translation) via Minkowski's theorem by choosing the vectors u_i to be the unit vectors in the directions v_i and assigning to each the constant

$$r_i = \frac{t_i |v_i|}{(n-1)!}.$$

Note that this is the natural choice of volume for the corresponding facet, as it accounts for

- the discrepancy in length between u_i and v_i, and
- the factor $1/(n-1)!$ relating the volume of an $(n-1)$-simplex to the determinant of its edge vectors.

We denote the corresponding polytope in $M_{\mathbb{R}}$ defined by the theorem of Minkowski by P_α.

Theorem 4.2. *Suppose α is a movable curve class which is positive on a spanning set of rays and let P_α be the corresponding polytope. Then*

$$\mathfrak{M}(\alpha) = n!\,\mathrm{vol}(P_\alpha).$$

Furthermore, the big movable divisor L_α corresponding to the polytope P_α satisfies $\langle L_\alpha^{n-1} \rangle = \alpha$.

Proof. Let $L \in \mathrm{Mov}^1(X)$ be a big movable divisor class and denote the corresponding polytope by Q_L. We claim that the intersection number can be interpreted as a mixed volume:

$$L \cdot \alpha = n! V(P_\alpha^{n-1}, Q_L).$$

To see this, define for a compact convex set K the function $h_K(u) = \sup_{v \in K}\{v \cdot u\}$. Using [Klain 2004, Equation (5)]

$$V(P_\alpha^{n-1}, Q_L) = \frac{1}{n} \sum_{u \text{ a facet of } P_\alpha + Q_L} h_{Q_L}(u)\, \mathrm{vol}(P_\alpha^u) = \frac{1}{n} \sum_{\text{rays } v_i} \left(\frac{a_i}{|v_i|}\right)\left(\frac{t_i|v_i|}{(n-1)!}\right) = \frac{1}{n!} \sum_{\text{rays } v_i} a_i t_i = \frac{1}{n!} L \cdot \alpha.$$

Note that we actually have equality in the second line because L is big and movable. Recall that by the Brunn–Minkowski inequality

$$V(P_\alpha^{n-1}, Q_L) \geq \mathrm{vol}(P_\alpha)^{n-1/n}\, \mathrm{vol}(Q_L)^{1/n}$$

with equality only when P_α and Q_L are homothetic. Thus

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big movable class}} \left(\frac{L \cdot \alpha}{\mathrm{vol}(L)^{1/n}}\right)^{n/(n-1)} = \inf_{L \text{ big movable class}} \left(\frac{n! V(P_\alpha^{n-1}, Q_L)}{n!^{1/n}\, \mathrm{vol}(Q_L)^{1/n}}\right)^{n/(n-1)} \geq n!\, \mathrm{vol}(P_\alpha).$$

Furthermore, the equality is achieved for divisors L whose polytope is homothetic to P_α, showing the computation of $\mathfrak{M}(\alpha)$. Furthermore, since the divisor L_α defined by the polytope computes $\mathfrak{M}(\alpha)$ we see that $\langle L_\alpha^{n-1}\rangle$ is proportional to α. By computing $\mathfrak{M}$ we deduce the equality:

$$\mathfrak{M}(\langle L_\alpha^{n-1}\rangle) = \mathrm{vol}(L) = n!\, \mathrm{vol}(P_\alpha) = \mathfrak{M}(\alpha). \qquad \square$$

The previous result shows:

Corollary 4.3. *Let α be a curve class in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then $\alpha \in \mathrm{CI}_1(X)$ if and only if the normal fan to the corresponding polytope P_α is refined by Σ. In this case we have*

$$\widehat{\mathrm{vol}}(\alpha) = n!\, \mathrm{vol}(P_\alpha).$$

Proof. By the uniqueness in Theorem 3.14, $\alpha \in \mathrm{CI}_1(X)$ if and only if the corresponding divisor L_α as in Theorem 4.2 is big and nef. $\square$

For toric varieties, much of the theory developed in this paper reduces to results from the theory of convex bodies. For example, suppose that we have movable curve classes α_1, α_2. Then the polytope corresponding to $\alpha_1 + \alpha_2$ is (essentially by definition) the Blaschke sum of the polytopes P_{α_1} and P_{α_2}. Thus the inequality

$$\mathfrak{M}(\alpha_1 + \alpha_2)^{n-1/n} \geq \mathfrak{M}(\alpha_1)^{n-1/n} + \mathfrak{M}(\alpha_2)^{n-1/n}$$

of Theorem 3.19 is exactly the Kneser–Süss inequality when interpreted via toric geometry. Similarly, the derivative formula of Theorem 3.18 follows from the theory of mixed volumes. See [Lehmann and Xiao 2017] for more details.

5. Comparing the complete intersection cone and the movable cone

Consider the functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ on the movable cone of curves $\mathrm{Mov}_1(X)$. By their definitions we always have $\widehat{\mathrm{vol}} \geq \mathfrak{M}$ on the movable cone, and [Xiao 2017, Remark 3.1] asks whether one can characterize when equality holds. In this section we show:

Theorem 5.1. *Let X be a smooth projective variety of dimension n and let α be a big and movable class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathfrak{M}(\alpha)$ if and only if $\alpha \in \mathrm{CI}_1(X)$.*

Thus $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ can be used to distinguish whether a big movable curve class lies in $\mathrm{CI}_1(X)$ or not. This result is important in Section 6.

Proof. If $\alpha = B^{n-1}$ is a complete intersection class, then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B) = \mathfrak{M}(\alpha)$. By continuity the equality holds true for any big curve class in $\mathrm{CI}_1(X)$.

Conversely, suppose that α is not in the complete intersection cone. The claim is clearly true if $\mathfrak{M}(\alpha) = 0$, so by Theorem 3.14 it suffices to consider the case when there is a big and movable divisor class L such that $\alpha = \langle L^{n-1} \rangle$. Note that L can not be big and nef since $\alpha \notin \mathrm{CI}_1(X)$.

We prove $\widehat{\mathrm{vol}}(\alpha) > \mathfrak{M}(\alpha)$ by contradiction. First, by the definition of $\widehat{\mathrm{vol}}$ we always have

$$\widehat{\mathrm{vol}}(\langle L^{n-1} \rangle) \geq \mathfrak{M}(\langle L^{n-1} \rangle) = \mathrm{vol}(L).$$

Suppose $\widehat{\mathrm{vol}}(\langle L^{n-1} \rangle) = \mathrm{vol}(L)$. For convenience, we assume $\mathrm{vol}(L) = 1$. By rescaling the positive part of a Zariski decomposition, we find a big and nef divisor class B with $\mathrm{vol}(B) = 1$ such that $\widehat{\mathrm{vol}}(\langle L^{n-1} \rangle) = (\langle L^{n-1} \rangle \cdot B)^{n/(n-1)}$. For the divisor class B we get

$$\langle L^{n-1} \rangle \cdot B = 1 = \mathrm{vol}(L)^{n-1/n} \,\mathrm{vol}(B)^{1/n}.$$

By Proposition 3.5, this implies L and B are proportional which contradicts the nonnefness of L. Thus we must have $\widehat{\mathrm{vol}}(\langle L^{n-1} \rangle) > \mathrm{vol}(L) = \mathfrak{M}(\langle L^{n-1} \rangle)$. □

We also obtain:

Proposition 5.2. *Let X be a smooth projective variety of dimension n and let α be a big and movable curve class. Then $\alpha \in \mathrm{CI}_1(X)$ if and only if for any birational morphism $\phi : Y \to X$ we have $\widehat{\mathrm{vol}}(\phi^*\alpha) = \widehat{\mathrm{vol}}(\alpha)$.*

Proof. The forward implication is clear. For the reverse implication, we first consider the case when $\mathfrak{M}(\alpha) > 0$. Let L be a big movable divisor class satisfying $\langle L^{n-1} \rangle = \alpha$. Choose a sequence of birational maps $\phi_\epsilon : Y_\epsilon \to X$ and ample divisor classes A_ϵ on Y_ϵ defining an ϵ-Fujita approximation for L. Then $\mathrm{vol}(L) \geq \mathrm{vol}(A_\epsilon) > \mathrm{vol}(L) - \epsilon$ and the classes $\phi_{\epsilon *} A_\epsilon$ limit to L. Note that $A_\epsilon \cdot \phi_\epsilon^* \alpha = \phi_{\epsilon *} A_\epsilon \cdot \alpha$. This implies that for any $\epsilon > 0$ we have

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(\phi_\epsilon^* \alpha) \leq \frac{(\alpha \cdot \phi_{\epsilon *} A_\epsilon)^{n/(n-1)}}{\mathrm{vol}(L)^{1/n-1}}.$$

As ϵ shrinks the right-hand side approaches $\mathrm{vol}(L) = \mathfrak{M}(\alpha)$, and we conclude by Theorem 5.1.

Next we consider the case when $\mathfrak{M}(\alpha) = 0$. Choose a class ξ in the interior of $\mathrm{Mov}_1(X)$ and consider the classes $\alpha + \delta\xi$ for $\delta > 0$. The argument above shows that for any $\epsilon > 0$, there is a birational model $\phi_\epsilon : Y_\epsilon \to X$ such that

$$\widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha + \delta\xi)) < \mathfrak{M}(\alpha + \delta\xi) + \epsilon.$$

But we also have $\widehat{\mathrm{vol}}(\phi_\epsilon^*\alpha) \leq \widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha+\delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudoeffective. Taking limits as $\epsilon \to 0$, $\delta \to 0$, we see that we can make the volume of the pullback of α arbitrarily small, a contradiction to the assumption and the bigness of α. □

As an illustration of the comparison between $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$, we discuss Mori dream spaces.

Example 5.3. Let X be a Mori dream space. Recall that a small $\mathbb{Q}$-factorial modification (henceforth SQM) $\phi : X \dashrightarrow X'$ is a birational contraction (i.e., does not extract any divisors) defined in codimension 1 such that X' is projective $\mathbb{Q}$-factorial. Hu and Keel [2000] showed that for any SQM the strict transform defines an isomorphism $\phi_* : N^1(X) \to N^1(X')$ which preserves the pseudoeffective and movable cones of divisors. (More generally, any birational contraction induces an injective pullback $\phi^* : N^1(X') \to N^1(X)$ and dually a surjection $\phi_* : N_1(X) \to N_1(X')$.) The SQM structure induces a chamber decomposition of the pseudoeffective and movable cones of divisors.

One would like to see a "dual picture" in $N_1(X)$ of this chamber decomposition. However, it does not seem interesting to simply dualize the divisor decomposition: the resulting cones are no longer pseudo-effective and are described as intersections instead of unions. Motivated by the Zariski decomposition for curves, we define a chamber structure on the movable cone of curves as a union of the complete intersection cones on SQMs.

Note that for each SQM we obtain by duality an isomorphism $\phi_* : N_1(X) \to N_1(X')$ which preserves the movable cone of curves. We claim that the strict transforms of the various complete intersection cones define a chamber structure on $\mathrm{Mov}_1(X)$. More precisely, given any birational contraction $\phi : X \dashrightarrow X'$ with X' normal projective, define

$$\mathrm{CI}^\circ_\phi := \bigcup_{A \text{ ample on } X'} \langle \phi^* A^{n-1} \rangle.$$

Then:

- $\mathrm{Mov}_1(X)$ is the union over all SQMs $\phi : X \dashrightarrow X'$ of $\overline{\mathrm{CI}^\circ_\phi} = \phi_*^{-1}\,\mathrm{CI}_1(X')$, and the interiors of the $\overline{\mathrm{CI}^\circ_\phi}$ are disjoint.
- The set of classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ is the disjoint union over all birational contractions $\phi : X \dashrightarrow X'$ of the CI°_ϕ.

To see this, first recall that for a pseudoeffective divisor L the σ-decomposition of L and the volume are preserved by ϕ_*. We know that each $\alpha \in \mathrm{Mov}_1(X)_{\mathfrak{M}}$ has the form $\langle L^{n-1} \rangle$ for a unique big and movable divisor L. If $\phi : X \dashrightarrow X'$ denotes the birational canonical model obtained by running the L-MMP, and A denotes the corresponding ample divisor on X', then $\phi_*\alpha = A^{n-1}$ and $\alpha = \langle \phi^* A^{n-1} \rangle$. The various claims now can be deduced from the properties of divisors and the MMP for Mori dream spaces as in [Hu and Keel 2000, 1.11 Proposition].

Since the volume of divisors behaves compatibly with strict transforms of pseudoeffective divisors, the description of ϕ_* above shows that $\mathfrak{M}$ also behaves compatibly with strict transforms of movable curves under an SQM. However, the volume function can change: we may well have $\widehat{\mathrm{vol}}(\phi_*\alpha) \neq \widehat{\mathrm{vol}}(\alpha)$. The reason is that the pseudoeffective cone of curves is also changing as we vary ϕ. In particular, the set

$$C_{\alpha,\phi} := \{\phi_*\alpha - \gamma \mid \gamma \in \overline{\mathrm{Eff}}_1(X')\}$$

will look different as we vary ϕ. Since $\widehat{\mathrm{vol}}$ is the same as the maximum value of $\mathfrak{M}(\beta)$ for $\beta \in C_{\alpha,\phi}$, the volume and Zariski decomposition for a given model will depend on the exact shape of $C_{\alpha,\phi}$.

Remark 5.4. Theorem 5.1 also holds for smooth varieties over any algebraically closed field and for compact hyperkähler manifolds or projective manifolds as explained in Section 2.

6. Comparison between the positivity functions for curves

Asymptotic point counts and $\widehat{\mathbf{vol}}$. In this section we give the proof of the main result, comparing the volume function for pseudoeffective curves with its mobility function. Recall from the introduction what we are trying to show (slightly reordered):

Theorem 6.1. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudoeffective curve class. Then the following results hold*:

(1) $\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha) \leq n!\,\widehat{\mathrm{vol}}(\alpha)$.

(2) *Assume Conjecture 1.4. Then* $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

(3) $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.

The upper bound in the first part improves the related result [Xiao 2017, Theorem 3.2]. Before giving the proof, we repeat the following estimate of $\widehat{\mathrm{vol}}$ in [Lehmann and Xiao 2016].

Proposition 6.2. *Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \dots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geq k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{n-1/n} \geq \mathfrak{M}(\alpha)^{n-1/n} \geq \frac{\sum_i k_i}{r^{1/n}}.$$

This is just a rephrasing of well-known results in birational geometry; see for example [Kollár 1996, V.2.9 Proposition].

Proof. By continuity and rescaling invariance, it suffices to show that if L is a big and movable Cartier divisor class then

$$\left(\sum_{i=1}^r k_i\right)\frac{\mathrm{vol}(L)^{1/n}}{r^{1/n}} \leq L \cdot C.$$

A standard argument (see for example [Lehmann 2016, Example 8.19]) shows that for any $\epsilon > 0$ and any very general points $\{x_i\}_{i=1}^r$ of X there is a positive integer m and a Cartier divisor M numerically equivalent to mL and such that $\mathrm{mult}_{x_i} M \geq mr^{-1/n}\,\mathrm{vol}(L)^{1/n} - \epsilon$ for every i. By the assumption on the family of curves we may find an irreducible curve C with multiplicity $\geq k_i$ at each x_i that is not contained M. Then

$$m(L\cdot C)\geq \sum_{i=1}^{r} k_i \operatorname{mult}_{x_i} M \geq \left(\sum_{i=1}^{r} k_i\right)\left(\frac{m\operatorname{vol}(L)^{1/n}}{r^{1/n}}-\epsilon\right).$$

Divide by m and let ϵ go to 0 to conclude. □

Example 6.3. The most important special case is when α is the class of a family of irreducible curves such that for any two general points of X there is a curve in our family containing them. Proposition 6.2 then shows that $\widehat{\operatorname{vol}}(\alpha)\geq 1$ and $\mathfrak{M}(\alpha)\geq 1$.

We also need to give a formal definition of the mobility count. Its properties are studied in more depth in [Lehmann 2016].

Definition 6.4. Let X be an integral projective variety and let W be a reduced variety. Suppose that $U\subset W\times X$ is a subscheme and let $p:U\to W$ and $s:U\to X$ denote the projection maps. The mobility count $\operatorname{mc}(p)$ of the morphism p is the maximum nonnegative integer b such that the map

$$U\times_W U\times_W\cdots\times_W U\xrightarrow{s\times s\times\cdots\times s} X\times X\times\cdots\times X$$

is dominant, where we have b terms in the product on each side. (If the map is dominant for every positive integer b, we set $\operatorname{mc}(p)=\infty$.)

For $\alpha\in N_1(X)_{\mathbb{Z}}$, the mobility count of α, denoted $\operatorname{mc}(\alpha)$, is defined to be the largest mobility count of any family of effective curves representing α.

The mobility is then defined as

$$\operatorname{mob}(\alpha)=\limsup_{m\to\infty}\frac{\operatorname{mc}(m\alpha)}{m^{n/(n-1)}/n!}.$$

Proof of Theorem 6.1. (1) We compare mob and $\widehat{\operatorname{vol}}$. We first prove the upper bound. By continuity and homogeneity it suffices to prove the upper bound for a class α in the natural sublattice of integral classes $N_1(X)_{\mathbb{Z}}$. Suppose that $p:U\to W$ is a family of curves representing $m\alpha$ of maximal mobility count for a positive integer m. Suppose that a general member of p decomposes into irreducible components $\{C_i\}$; arguing as in [Lehmann 2016, Corollary 4.10], we must have $\operatorname{mc}(p)=\sum_i \operatorname{mc}(U_i)$, where U_i represents the closure of the family of deformations of C_i. We also let β_i denote the numerical class of C_i.

Suppose that $\operatorname{mc}(U_i)>1$. Then we may apply Proposition 6.2 with all $k_i=1$ and $r=\operatorname{mc}(U_i)-1$ to deduce that

$$\widehat{\operatorname{vol}}(\beta_i)\geq \operatorname{mc}(U_i)-1.$$

If $\operatorname{mc}(U_i)\leq 1$ then Proposition 6.2 does not apply but at least we still know that $\widehat{\operatorname{vol}}(\beta_i)\geq 0\geq \operatorname{mc}(U_i)-1$. Fix an ample Cartier divisor A, and note that the number of components C_i is at most $mA\cdot\alpha$. All told, we have

$$\widehat{\operatorname{vol}}(m\alpha)\geq\sum_i\widehat{\operatorname{vol}}(\beta_i)\geq\sum_i(\operatorname{mc}(U_i)-1)\geq \operatorname{mc}(m\alpha)-mA\cdot\alpha.$$

Thus,

$$\widehat{\mathrm{vol}}(\alpha) = \limsup_{m\to\infty} \frac{\widehat{\mathrm{vol}}(m\alpha)}{m^{n/(n-1)}} \geq \limsup_{m\to\infty} \frac{\mathrm{mc}(m\alpha) - mA\cdot\alpha}{m^{n/(n-1)}} = \frac{\mathrm{mob}(\alpha)}{n!}.$$

The lower bound relies on the Zariski decomposition of curves in Theorem 2.16. By [Lehmann 2016, Example 6.2] we have

$$B^n \leq \mathrm{mob}(B^{n-1})$$

for any nef divisor B. With Theorem 2.16, this implies

$$\widehat{\mathrm{vol}}(B^{n-1}) \leq \mathrm{mob}(B^{n-1}).$$

In general, for a big curve class α we have

$$\mathrm{mob}(\alpha) \geq \sup_{\substack{B \text{ nef},\\ \alpha\succeq B^{n-1}}} \mathrm{mob}(B^{n-1}) \geq \sup_{\substack{B \text{ nef},\\ \alpha\succeq B^{n-1}}} B^n = \widehat{\mathrm{vol}}(\alpha).$$

where the last equality again follows from Theorem 2.16. This finishes the proof.

(2) To prove the second part of Theorem 6.1, we need the following result:

Lemma 6.5 [Fulger and Lehmann 2017b, Corollary 6.16]. *Let X be a smooth projective variety of dimension n and let α be a big curve class. Then there is a big movable curve class β satisfying $\beta \preceq \alpha$ such that*

$$\mathrm{mob}(\alpha) = \mathrm{mob}(\beta) = \mathrm{mob}(\phi^*\beta)$$

for any birational map $\phi : Y \to X$ from a smooth variety Y.

We now prove the statement via a sequence of claims.

Claim. *Assume Conjecture 1.4. If β is a movable curve class with $\mathfrak{M}(\beta) > 0$, then for any $\epsilon > 0$ there is a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that*

$$\mathfrak{M}(\beta) - \epsilon \leq \mathrm{mob}(\phi_\epsilon^*\beta) \leq \mathfrak{M}(\beta) + \epsilon.$$

By Theorem 3.14, we may suppose that there is a big divisor L such that $\beta = \langle L^{n-1}\rangle$. Without loss of generality we may assume that L is effective. Fix an ample effective divisor G as in [Fulger and Lehmann 2017b, Proposition 6.24]; the proposition shows that for any sufficiently small ϵ there is a birational morphism $\phi_\epsilon : Y_\epsilon \to X$ and a big and nef divisor A_ϵ on Y_ϵ satisfying

$$A_\epsilon \leq P_\sigma(\phi_\epsilon^* L) \leq A_\epsilon + \epsilon\phi_\epsilon^* G.$$

Note that $\mathrm{vol}(A_\epsilon) \leq \mathrm{vol}(L) \leq \mathrm{vol}(A_\epsilon + \epsilon\phi_\epsilon^* G)$. Furthermore, we have

$$\mathrm{vol}(A_\epsilon + \epsilon\phi_\epsilon^* G) \leq \mathrm{vol}(\phi_{\epsilon *}A_\epsilon + \epsilon G) \leq \mathrm{vol}(L + \epsilon G).$$

Applying [Fulger and Lehmann 2017b, Lemma 6.21] and the invariance of the positive product under passing to positive parts, we have

$$A_\epsilon^{n-1} \preceq \phi_\epsilon^* \beta \preceq (A_\epsilon + \epsilon \phi_\epsilon^* G)^{n-1}.$$

Applying Conjecture 1.4 (which is only stated for ample divisors but applies to big and nef divisors by continuity of mob), we find

$$\mathrm{vol}(A_\epsilon) = \mathrm{mob}(A_\epsilon^{n-1}) \leq \mathrm{mob}(\phi_\epsilon^* \beta) \leq \mathrm{mob}((A_\epsilon + \epsilon \phi_\epsilon^*(G))^{n-1}) = \mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G).$$

As ϵ shrinks the two outer terms approach $\mathrm{vol}(L) = \mathfrak{M}(\beta)$.

Claim. *Assume Conjecture 1.4. If a big movable curve class β satisfies $\mathrm{mob}(\beta) = \mathrm{mob}(\phi^* \beta)$ for every birational ϕ then we must have $\beta \in \mathrm{CI}_1(X)$.*

When $\mathfrak{M}(\beta) > 0$, by the previous claim we see from taking a limit that $\mathrm{mob}(\beta) = \mathfrak{M}(\beta)$. By Theorem 6.1(1) and Theorem 5.1 we get

$$\widehat{\mathrm{vol}}(\beta) \leq \mathfrak{M}(\beta) \leq \widehat{\mathrm{vol}}(\beta)$$

and Theorem 5.1 implies the result. When $\mathfrak{M}(\beta) = 0$, fix a class ξ in the interior of the movable cone and consider $\beta + \delta\xi$ for $\delta > 0$. By the previous claim, for any $\epsilon > 0$ we can find a sufficiently small δ and a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that $\mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi)) < \epsilon$. We also have $\mathrm{mob}(\phi_\epsilon^* \beta) \leq \mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudoeffective. By the assumption on the birational invariance of $\mathrm{mob}(\beta)$, we can take a limit to obtain $\mathrm{mob}(\beta) = 0$, a contradiction to the bigness of β.

To finish the proof, recall that Lemma 6.5 implies that the mobility of α must coincide with the mobility of a movable class β lying below α and satisfying $\mathrm{mob}(\pi^* \beta) = \mathrm{mob}(\beta)$ for any birational map π. Thus we have shown

$$\mathrm{mob}(\alpha) = \sup_{\substack{B \text{ nef},\\ \alpha \succeq B^{n-1}}} \mathrm{mob}(B^{n-1}).$$

By Conjecture 1.4 again, we obtain

$$\mathrm{mob}(\alpha) = \sup_{\substack{B \text{ nef},\\ \alpha \succeq B^{n-1}}} B^n.$$

But the right-hand side agrees with $\widehat{\mathrm{vol}}(\alpha)$ by Theorem 2.16. This proves the equality $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$ under the Conjecture 1.4.

(3) We now prove the equality $\widehat{\mathrm{vol}} = \mathrm{wmob}$. The key advantage is that the analogue of Conjecture 1.4 is known for the weighted mobility: Example 8.19 of [Lehmann 2016] shows that for any big and nef divisor B we have $\mathrm{wmob}(B^{n-1}) = B^n$.

We first prove the inequality $\widehat{\mathrm{vol}} \geq \mathrm{wmob}$. The argument is essentially identical to the upper bound in Theorem 6.1(1); by continuity and homogeneity it suffices to prove it for classes in $N_1(X)_{\mathbb{Z}}$. Choose a positive integer μ and a family of curves of class $\mu m \alpha$ achieving $\mathrm{wmc}(m\alpha)$. By splitting up into components and applying Proposition 6.2 with equal weight μ at every point we see that for any component U_i with class β_i we have

$$\widehat{\mathrm{vol}}(\beta_i) \geq \mu^{n/(n-1)}(\mathrm{wmc}(U_i) - 1)$$

Arguing as in Theorem 6.1(1), we see that for any fixed ample Cartier divisor A we have

$$\widehat{\mathrm{vol}}(m\mu\alpha) \geq \mu^{n/(n-1)}(\mathrm{wmc}(m\alpha) - mA \cdot \alpha).$$

Rescaling by μ and taking a limit proves the statement.

We next prove the inequality $\widehat{\mathrm{vol}} \leq \mathrm{wmob}$. Again, the argument is identical to the lower bound in Theorem 6.1(1). It is clear that the weighted mobility can only increase upon adding an effective class. Using continuity and homogeneity, the same is true for any pseudoeffective class. Thus we have

$$\mathrm{wmob}(\alpha) \geq \sup_{\substack{B \text{ nef},\\ \alpha \succeq B^{n-1}}} \mathrm{wmob}(B^{n-1}) = \sup_{\substack{B \text{ nef},\\ \alpha \succeq B^{n-1}}} B^n = \widehat{\mathrm{vol}}(\alpha).$$

where the second equality follows from [Lehmann 2016, Example 8.19]. This finishes the proof of the equality $\widehat{\mathrm{vol}} = \mathrm{wmob}$. □

Remark 6.6. We expect Theorem 6.1 to also hold over any algebraically closed field, but we have not thoroughly checked the results on asymptotic multiplier ideals used in the proof of [Fulger and Lehmann 2017b, Proposition 6.24].

Theorem 6.1 yields two interesting consequences:

- The theorem indicates (loosely speaking) that if the mobility count of complete intersection classes is optimized by complete intersection curves, then the mobility count of *any* curve class is optimized by complete intersection curves lying below the class.

  This result is very surprising: it indicates that the "positivity" of a curve class is coming from ample divisors in a strong sense. For example, suppose that X and X' are isomorphic in codimension 1. If we take a complete intersection class α on X, we expect that complete intersections of ample divisors maximize the mobility count. However, the strict transform of these curves on X' should not maximize the mobility count. Instead, if we deform these curves so that they break off a piece contained in the exceptional locus, the part left over will lie in a family which deforms more than the original.
- The theorem suggests that the Zariski decomposition constructed in [Fulger and Lehmann 2017b] for curves is not optimal: instead of defining a positive part in the movable cone, if Conjecture 1.4 is true we should instead define a positive part in the complete intersection cone. It would be interesting to see an analogous improvement for higher dimension cycles.

Asymptotic point counts and $\mathfrak{M}$. Finally, we show that $\mathfrak{M}$ can be given an enumerative interpretation.

Definition 6.7. Let $p : U \to W$ be a family of curves on X with morphism $s : U \to X$. We say that U is strictly movable if:

(1) For each component U_i of U, the morphism $s|_{U_i}$ is dominant.

(2) For each component U_i of U, the morphism $p|_{U_i}$ has generically irreducible fibers.

We then define $\mathrm{mob}_{\mathrm{mov}}$ and $\mathrm{wmob}_{\mathrm{mov}}$ exactly analogously to mob and wmob, except that we only allow contributions of strictly movable families of curves. Note that $\mathrm{mob}_{\mathrm{mov}}$ and $\mathrm{wmob}_{\mathrm{mov}}$ vanish outside of $\mathrm{Mov}_1(X)$ since these classes are not represented by a sum of irreducible curves which deform to dominate X. Arguing just as in [Lehmann 2016, Section 5], one sees that $\mathrm{mob}_{\mathrm{mov}}$ and $\mathrm{wmob}_{\mathrm{mov}}$ are homogeneous of weight $n/(n-1)$, and are continuous in the interior of $\mathrm{Mov}_1(X)$.

Lemma 6.8. *Let $\phi : Y \to X$ be a birational morphism of smooth projective varieties. Let $p : U \to W$ be a family of irreducible curves admitting a dominant map $s : U \to X$. Let U_Y be the family of curves defined by strict transforms. Letting α, α_Y denote respectively the classes of the families on X, Y, we have that $\phi^*\alpha - \alpha_Y$ is the class of an effective $\mathbb{R}$-curve.*

Proof. Since α_Y is the class of a family of irreducible curves which dominates Y, it has nonnegative intersection against every effective divisor. Arguing as in the negativity of contraction lemma, we can find a basis $\{e_i\}$ of $\ker(\phi_* : N_1(Y) \to N_1(X))$ consisting of effective curves and a basis $\{f_j\}$ of $\ker(\phi_* : N^1(Y) \to N^1(X))$ consisting of effective divisors such that the intersection matrix is negative definite and the only negative entries are on the diagonal. Just as in [Bauer et al. 2012, Lemma 4.1], this shows that

$$\alpha_Y = \phi^*\phi_*\alpha_Y - \beta = \phi^*\alpha - \beta$$

for some effective curve class β supported on the exceptional divisors. □

Theorem 6.9. *Let X be a smooth projective variety of dimension n and let $\alpha \in \mathrm{Mov}_1(X)^\circ$. Then*:

(1) $\mathfrak{M}(\alpha) = \mathrm{wmob}_{\mathrm{mov}}(\alpha)$.

(2) *Assume Conjecture 1.4. Then $\mathfrak{M}(\alpha) = \mathrm{mob}_{\mathrm{mov}}(\alpha)$.*

Proof. (1) Suppose that $\phi : Y \to X$ is a birational model of X and that A is an ample Cartier divisor on X. By pushing-forward complete intersection families, we see that $\mathrm{wmob}_{\mathrm{mov}}(\phi_* A^{n-1}) \geq A^n$. By continuity we obtain the inequality $\mathfrak{M}(\alpha) \leq \mathrm{wmob}_{\mathrm{mov}}(\alpha)$ for any $\alpha \in \mathrm{Mov}_1(X)^\circ$.

To see the reverse inequality, by continuity and homogeneity it suffices to consider the case when $\alpha \in \mathrm{Mov}_1(X)^\circ_{\mathbb{Z}}$. Choose a positive integer μ and a strictly movable family of curves U of class $\mu m\alpha$ achieving $\mathrm{wmc}_{\mathrm{mov}}(m\alpha)$. Let $\phi : Y \to X$ be a birational model and let U_Y denote the strict transform class on Y with numerical class α'. By arguing as in the proof of Theorem 6.1, we find that

$$\mathfrak{M}(\alpha') \geq \mu^{n/(n-1)}(\mathrm{wmc}_{\mathrm{mov}}(m\alpha) - mA \cdot \alpha).$$

Furthermore by Lemma 6.8 we have $\widehat{\mathrm{vol}}(m\mu\phi^*\alpha) \geq \widehat{\mathrm{vol}}(\alpha')$. Dividing by $m^{n/(n-1)}$ and taking a limit as m increases, we see that $\mathfrak{M}(\alpha) \geq \mathrm{wmob}_{\mathrm{mov}}(\alpha)$.

(2) The proof of $\mathfrak{M}(\alpha) \leq \mathrm{mob}_{\mathrm{mov}}(\alpha)$ is the same as in (1). Conversely, suppose that U is a strictly movable family of curves achieving $\mathrm{mc}_{\mathrm{mov}}(m\alpha)$. Let $\phi : Y \to X$ be a birational morphism of smooth varieties; by combining Lemma 6.8 with [Fulger and Lehmann 2017b, Section 4], we see that $\mathrm{mc}_{\mathrm{mov}}(m\alpha) \leq \mathrm{mc}_{\mathcal{K}}(m\phi^*\alpha)$, where $\mathcal{K}$ is a cone chosen as in [Fulger and Lehmann 2017b, Definition 4.8] and includes a fixed effective basis of the kernel of $\phi_* : N_1(Y) \to N_1(X)$ chosen as in Lemma 6.8. Taking limits, we see that $\mathrm{mob}_{\mathrm{mov}}(\alpha) \leq \mathrm{mob}(\phi^*\alpha)$ for any birational map ϕ.

Choose a sequence of birational maps $\phi_i : Y_i \to X$ as in the proof of Proposition 5.2 so that $\widehat{\mathrm{vol}}(\phi_i^*\alpha)$ limits to $\mathfrak{M}(\alpha)$. By taking a limit over i and applying Theorem 6.1(2) we finish the proof. □

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