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The generalized Riemann hypothesis implies that at least 50% of the central values $L\left(\frac{1}{2}, \chi\right)$ are nonvanishing as χ ranges over primitive characters modulo q. We show that one may unconditionally go beyond GRH, in the sense that if one averages over primitive characters modulo q and averages q over an interval, then at least 50.073% of the central values are nonvanishing. The proof utilizes the mollification method with a three-piece mollifier, and relies on estimates for sums of Kloosterman sums due to Deshouillers and Iwaniec.

1. Introduction

It is widely believed that no primitive Dirichlet L-function $L(s, \chi)$ vanishes at the central point $s = \frac{1}{2}$. Most of the progress towards this conjecture has been made by working with various families of Dirichlet L-functions. Balasubramanian and Murty [1992] showed that, in the family of primitive characters modulo q, a positive proportion of the L-functions do not vanish at the central point. Iwaniec and Sarnak [1999] later improved this lower bound, showing that at least $\frac{1}{3}$ of the L-functions in this family do not vanish at the central point. Bui [2012] improved this further to 34.11%, and Khan and Ngo [2016] showed at least $\frac{3}{8}$ of the central values are nonvanishing[1] for prime moduli. Soundararajan [2000] worked with a family of quadratic Dirichlet characters, and showed that $\frac{7}{8}$ of the family do not vanish at $s = \frac{1}{2}$. These proofs all proceed through the mollification method, which we discuss in Section 2 below.

If one assumes the generalized Riemann hypothesis, one can show that at least half of the primitive characters $\chi \pmod q$ satisfy $L\left(\frac{1}{2}, \chi\right) \neq 0$ [Balasubramanian and Murty 1992; Sica 1998; Miller and Takloo-Bighash 2006, Exercise 18.2.8]. One uses the explicit formula, rather than mollification, and the proportion $\frac{1}{2}$ arises from the choice of a test function with certain positivity properties.

It seems plausible that one may obtain a larger proportion of nonvanishing by also averaging over moduli q. Indeed, Iwaniec and Sarnak [1999] already claimed that by averaging over moduli one can prove at least half of the central values are nonzero. This is striking, in that it is as strong, on average, as the proportion obtained via GRH.

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[1] A clear preference for “non-vanishing” or “nonvanishing” has not yet materialized in the literature. We exclusively use the latter term throughout this work.

A natural question is whether, by averaging over moduli, one can breach the 50% barrier, thereby going beyond the immediate reach of GRH. We answer this question in the affirmative.

Let $\sum^*_{\chi(q)}$ denote a sum over the primitive characters modulo q, and define $\varphi^*(q)$ to be the number of primitive characters modulo q.

Theorem 1.1. *Let Ψ be a fixed, nonnegative smooth function, compactly supported in $\left[\frac{1}{2}, 2\right]$, which satisfies*

$$\int_{\mathbb{R}} \Psi(x)\, dx > 0.$$

Then for Q sufficiently large we have

$$\sum_q \Psi\left(\frac{q}{Q}\right)\frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi(q)\\ L(\frac{1}{2},\chi)\neq 0}} 1 \geq 0.50073 \sum_q \Psi\left(\frac{q}{Q}\right)\frac{q}{\varphi(q)}\varphi^*(q).$$

Thus, roughly speaking, a randomly chosen central value $L\left(\frac{1}{2}, \chi\right)$ is more likely nonzero than zero. We remark also that the appearance of the arithmetic weight $q/\varphi(q)$ is technically convenient, but not essential.

2. Mollification and a sketch for Theorem 1.1

The proof of Theorem 1.1 relies on the powerful technique of mollification. For each character χ we associate a function $\psi(\chi)$, called a mollifier, that serves to dampen the large values of $L\left(\frac{1}{2}, \chi\right)$. By the Cauchy–Schwarz inequality we have

$$\frac{\left|\sum_{q\asymp Q}\sum^*_{\chi(q)} L\left(\frac{1}{2}, \chi\right)\psi(\chi)\right|^2}{\sum_{q\asymp Q}\sum^*_{\chi(q)}\left|L\left(\frac{1}{2}, \chi\right)\psi(\chi)\right|^2} \leq \sum_{q\asymp Q} \sideset{}{^*}\sum_{\substack{\chi(q)\\ L(\frac{1}{2},\chi)\neq 0}} 1. \tag{2-1}$$

The better the mollification by ψ, the larger proportion of nonvanishing one can deduce.

It is natural to choose $\psi(\chi)$ such that

$$\psi(\chi) \approx \frac{1}{L\left(\frac{1}{2}, \chi\right)}.$$

Since $L\left(\frac{1}{2}, \chi\right)$ can be written as a Dirichlet series

$$L\left(\tfrac{1}{2}, \chi\right) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{1/2}}, \tag{2-2}$$

this suggests the choice

$$\psi(\chi) \approx \sum_{\ell\leq y} \frac{\mu(\ell)\chi(\ell)}{\ell^{1/2}}. \tag{2-3}$$

We have introduced a truncation y in anticipation of the need to control various error terms that will arise. We write $y = Q^\theta$, where $\theta > 0$ is a real number. At least heuristically, larger values of θ yield better mollification by (2-3). Iwaniec and Sarnak [1999] made this choice (2-3) (up to some smoothing), and found that the proportion of nonvanishing attained was

$$\frac{\theta}{1+\theta}. \tag{2-4}$$

When $\theta = 1$ we see (2-4) is exactly $\frac{1}{2}$, so we need $\theta > 1$ in order to conclude Theorem 1.1. This seems beyond the range of present technology. Without averaging over moduli we may take $\theta = \frac{1}{2} - \varepsilon$, and the asymptotic large sieve of Conrey, Iwaniec, and Soundararajan [Conrey et al. 2011] allows one to take $\theta = 1 - \varepsilon$ if one averages over moduli. This just falls short of our goal.

Thus, a better mollifier than (2-3) is required. Part of the problem is that (2-2) is an inefficient representation of $L\left(\frac{1}{2}, \chi\right)$. A better representation of $L\left(\frac{1}{2}, \chi\right)$ may be obtained through the approximate functional equation, which states

$$L\left(\tfrac{1}{2}, \chi\right) \approx \sum_{n \le q^{1/2}} \frac{\chi(n)}{n^{1/2}} + \epsilon(\chi) \sum_{n \le q^{1/2}} \frac{\bar{\chi}(n)}{n^{1/2}}. \tag{2-5}$$

Here $\epsilon(\chi)$ is the root number, which is a complex number of modulus 1 defined by

$$\epsilon(\chi) = \frac{1}{q^{1/2}} \sum_{h \pmod q} \chi(h) e\left(\frac{h}{q}\right). \tag{2-6}$$

Inspired by (2-5), Michel and VanderKam [2000] chose a mollifier

$$\psi(\chi) \approx \sum_{\ell \le y} \frac{\mu(\ell)\chi(\ell)}{\ell^{1/2}} + \bar{\epsilon}(\chi) \sum_{\ell \le y} \frac{\mu(\ell)\bar{\chi}(\ell)}{\ell^{1/2}}. \tag{2-7}$$

We note that Soundararajan [1995] earlier used a mollifier of this shape in the context of the Riemann zeta function.

For $y = Q^{\theta}$, Michel and VanderKam found that (2-7) gives a nonvanishing proportion of

$$\frac{2\theta}{1+2\theta}. \tag{2-8}$$

Thus, we need $\theta = \frac{1}{2} + \varepsilon$ in order for (2-8) to imply a proportion of nonvanishing greater than $\frac{1}{2}$. However, the more complicated nature of the mollifier (2-7) means that, without averaging over moduli, only the choice $\theta = \frac{3}{10} - \varepsilon$ is acceptable [Khan and Ngo 2016].

As we allow ourselves to average over moduli, however, one might hope to obtain (2-8) for $\theta = \frac{1}{2} + \varepsilon$. Again we fall just short of our goal. Using a powerful result of Deshouillers and Iwaniec on cancellation in sums of Kloosterman sums (see Lemma 5.1 below) we shall show that $\theta = \frac{1}{2} - \varepsilon$ is acceptable, but increasing θ any further seems very difficult. It follows that we need any extra amount of mollification in order to obtain a proportion of nonvanishing strictly greater than $\frac{1}{2}$.

The solution is to attach yet another piece to the mollifier $\psi(\chi)$, but here we wish for the mollifier to have a very different shape from (2-7). Such a mollifier was utilized by Bui [2012], who showed that

$$\psi_{\mathrm{B}}(\chi) \approx \frac{1}{\log q} \sum\sum_{bc \leq y} \frac{\Lambda(b)\mu(c)\bar{\chi}(b)\chi(c)}{(bc)^{1/2}} \tag{2-9}$$

is a mollifier for $L\left(\frac{1}{2}, \chi\right)$. It turns out that adding (2-9) to (2-7) gives a sufficient mollifier to conclude Theorem 1.1.

One may roughly motivate a mollifier of the shape (2-9) as follows. Working formally,

$$\begin{aligned}\frac{1}{L\left(\frac{1}{2}, \chi\right)} = \frac{L\left(\frac{1}{2}, \bar{\chi}\right)}{L\left(\frac{1}{2}, \chi\right)L\left(\frac{1}{2}, \bar{\chi}\right)} &= \sum\sum\sum_{r,s,v} \frac{\bar{\chi}(r)\mu(s)\bar{\chi}(s)\mu(v)\chi(v)}{(rsv)^{1/2}} \\ &\approx \sum\sum\sum_{r,s,v} \frac{\log r}{\log q} \frac{\bar{\chi}(r)\mu(s)\bar{\chi}(s)\mu(v)\chi(v)}{(rsv)^{1/2}} \\ &= \frac{1}{\log q} \sum\sum_{u,v} \frac{(\mu \star \log)(u)\bar{\chi}(u)\mu(v)\chi(v)}{(uv)^{1/2}}.\end{aligned}$$

One might wonder what percentage of nonvanishing one can obtain using only a mollifier of the shape (2-9). The analysis for Bui's mollifier is more complicated, and it does not seem possible to write down simple expressions like (2-4) or (2-8) that give a percentage of nonvanishing for (2-9) in terms of θ. If one assumes, perhaps optimistically, that averaging over moduli allows one to take any $\theta < 1$ in (2-9), then some numerical computation indicates that the nonvanishing percentage does not exceed 27%, say.

We remark that, in the course of the proof, the main terms are easily extracted and we have no need here for the averaging over moduli. We require the averaging over moduli in order to estimate some of the error terms.

The structure of the remainder of the paper is as follows. In Section 3 we reduce the proof of Theorem 1.1 to two technical results, Lemmas 3.3 and 3.4, which give asymptotic evaluations of certain mollified sums. In Section 4 we extract the main term of Lemma 3.3, and in Section 5 we use estimates on sums of Kloosterman sums to complete the proof of this lemma. Section 6 similarly proves the main term of Lemma 3.4, but this derivation is longer than that given in Section 4 because the main terms are more complicated. In the final section, Section 7, we bound the error term in Lemma 3.4, again using results on sums of Kloosterman sums.

3. Proof of Theorem 1.1: first steps

Let us fix some notation and conventions that shall hold for the remainder of the paper.

The notation $a \equiv b(q)$ means $a \equiv b \pmod q$, and when $a(q)$ occurs beneath a sum it indicates a summation over residue classes modulo q.

We denote by ϵ an arbitrarily small positive quantity that may vary from one line to the next, or even within the same line. Thus, we may write $X^{2\epsilon} \leq X^{\epsilon}$ with no reservations.

We need to treat separately the even primitive characters and odd primitive characters. We focus exclusively on the even primitive characters, since the case of odd characters is nearly identical. We write $\sum_{\chi(q)}^{+}$ for a sum over even primitive characters modulo q, and we write $\varphi^{+}(q)$ for the number of such characters. Observe that $\varphi^{+}(q) = \frac{1}{2}\varphi^{*}(q) + O(1)$.

We shall encounter the Ramanujan sum $c_q(n)$ (see the proof of Proposition 5.2), defined by

$$c_q(n) = \sum_{\substack{a(q) \\ (a,q)=1}} e\left(\frac{an}{q}\right).$$

We shall only need to know that $c_q(1) = \mu(q)$ and $|c_q(n)| \le (q, n)$, where (q, n) is the greatest common divisor of q and n.

We now fix a smooth function Ψ as in the statement of Theorem 1.1, and allow all implied constants to depend on Ψ. We let Q be a large real number, and set $y_i = Q^{\theta_i}$ for $i \in \{1, 2, 3\}$, where $0 < \theta_i < \frac{1}{2}$ are fixed real numbers. We further define $L = \log Q$. The notation $o(1)$ denotes a quantity that goes to zero as Q goes to infinity.

Let us now begin the proof of Theorem 1.1 in earnest. As discussed in Section 2, we choose our mollifier $\psi(\chi)$ to have the form

$$\psi(\chi) = \psi_{\mathrm{IS}}(\chi) + \psi_{\mathrm{B}}(\chi) + \psi_{\mathrm{MV}}(\chi), \tag{3-1}$$

where

$$\begin{aligned}
\psi_{\mathrm{IS}}(\chi) &= \sum_{\ell \le y_1} \frac{\mu(\ell)}{\ell^{1/2}} P_1\left(\frac{\log(y_1/\ell)}{\log y_1}\right), \\
\psi_{\mathrm{B}}(\chi) &= \frac{1}{L} \sum\sum_{bc \le y_2} \frac{\Lambda(b)\mu(c)\overline{\chi}(b)\chi(c)}{(bc)^{1/2}} P_2\left(\frac{\log(y_2/bc)}{\log y_2}\right), \\
\psi_{\mathrm{MV}}(\chi) &= \epsilon(\overline{\chi}) \sum_{\ell \le y_3} \frac{\mu(\ell)\overline{\chi}(\ell)}{\ell^{1/2}} P_3\left(\frac{\log(y_3/\ell)}{\log y_3}\right).
\end{aligned} \tag{3-2}$$

The smoothing polynomials P_i are real and satisfy $P_i(0) = 0$. For notational convenience we write

$$P_i\left(\frac{\log(y_i/x)}{\log y_i}\right) = P_i[x].$$

There is some ambiguity in this notation because of the y_i-dependence in the polynomials, and this needs to be remembered in calculations.

Now define sums S_1 and S_2 by

$$\begin{aligned}
S_1 &= \sum_q \Psi\left(\frac{q}{Q}\right) \frac{q}{\varphi(q)} \sum_{\chi(q)}^{+} L\left(\tfrac{1}{2}, \chi\right)\psi(\chi), \\
S_2 &= \sum_q \Psi\left(\frac{q}{Q}\right) \frac{q}{\varphi(q)} \sum_{\chi(q)}^{+} \left|L\left(\tfrac{1}{2}, \chi\right)\psi(\chi)\right|^2.
\end{aligned} \tag{3-3}$$

We apply Cauchy–Schwarz as in (2-1) and get

$$\sum_q \Psi\left(\frac{q}{Q}\right)\frac{q}{\varphi(q)} \sideset{}{^+}\sum_{\substack{\chi(q)\\ L(\frac12,\chi)\neq 0}} 1 \geq \frac{S_1^2}{S_2}. \tag{3-4}$$

The proof of Theorem 1.1 therefore reduces to estimating S_1 and S_2. We obtain asymptotic formulas for these two sums.

Lemma 3.1. *Suppose* $0 < \theta_1, \theta_2 < 1$ *and* $0 < \theta_3 < \frac12$. *Then*

$$S_1 = \left(P_1(1) + P_3(1) + \frac{\theta_2}{2}\widetilde{P}_2(1) + o(1)\right)\sum_q \Psi\left(\frac{q}{Q}\right)\frac{q}{\varphi(q)}\varphi^+(q),$$

where

$$\widetilde{P}_2(x) = \int_0^x P_2(u)\,du.$$

Lemma 3.2. *Let* $0 < \theta_1, \theta_2, \theta_3 < \frac12$ *with* $\theta_2 < \theta_1, \theta_3$. *Then*

$$S_2 = \left(2P_1(1)P_3(1) + P_3(1)^2 + \frac{1}{\theta_3}\int_0^1 P_3'(x)^2\,dx + \kappa + \lambda + o(1)\right)\sum_q \Psi\left(\frac{q}{Q}\right)\frac{q}{\varphi(q)}\varphi^+(q),$$

where

$$\kappa = 3\theta_2 P_3(1)\widetilde{P}_2(1) - 2\theta_2\int_0^1 P_2(x)P_3(x)\,dx$$

and

$$\begin{aligned}\lambda = P_1(1)^2 + \frac{1}{\theta_1}\int_0^1 P_1'(x)^2\,dx - \theta_2 P_1(1)\widetilde{P}_2(1) + 2\theta_2\int_0^1 P_1\left(1-\frac{\theta_2(1-x)}{\theta_1}\right)P_2(x)\,dx \\ + \frac{\theta_2}{\theta_1}\int_0^1 P_1'\left(1-\frac{\theta_2(1-x)}{\theta_1}\right)P_2(x)\,dx + \theta_2^2\int_0^1 (1-x)P_2(x)^2\,dx \\ + \frac{\theta_2}{2}\int_0^1 (1-x)^2 P_2'(x)^2\,dx - \frac{\theta_2^2}{4}\widetilde{P}_2(1)^2 + \frac{\theta_2}{4}\int_0^1 P_2(x)^2\,dx.\end{aligned}$$

Proof of Theorem 1.1. Lemmas 3.1 and 3.2 give the evaluations of S_1 and S_2 for even characters. The identical formulas hold for odd characters. Theorem 1.1 then follows from (3-4) upon choosing $\theta_1 = \theta_3 = \frac12$, $\theta_2 = 0.163$, and

$$\begin{aligned}P_1(x) &= 4.86x + 0.29x^2 - 0.96x^3 + 0.974x^4 - 0.17x^5,\\ P_2(x) &= -3.11x - 0.3x^2 + 0.87x^3 - 0.18x^4 - 0.53x^5,\\ P_3(x) &= 4.86x + 0.06x^2.\end{aligned}$$

These choices actually yield a proportion[2]

$$\geq 0.50073004\ldots,$$

which allows us to state Theorem 1.1 with a clean inequality. □

[2] A Mathematica file with this computation is included with this paper on arXiv at https://arxiv.org/e-print/1804.01445v1.

We note without further comment the curiosity in the proof of Theorem 1.1 that the largest permissible value of θ_2 is not optimal.

We can dispense with S_1 quickly.

Proof of Lemma 3.1. Apply [Bui 2012, Theorem 2.1] and the argument of [Michel and VanderKam 2000, §3], using the facts $L = \log q + O(1)$ and $y_i = q^{\theta_i + o(1)}$. □

The analysis of S_2 is much more involved, and we devote the remainder of the paper to this task. We first observe that (3-1) yields

$$|\psi(\chi)|^2 = |\psi_{\mathrm{IS}}(\chi) + \psi_{\mathrm{B}}(\chi)|^2 + 2\,\mathrm{Re}\{\psi_{\mathrm{IS}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi}) + \psi_{\mathrm{B}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi})\} + |\psi_{\mathrm{MV}}(\chi)|^2.$$

By [Bui 2012, Theorem 2.2] we have

$$\sum_{\chi(q)}^{+} \bigl|L\bigl(\tfrac{1}{2}, \chi\bigr)\bigr|^2 |\psi_{\mathrm{IS}}(\chi) + \psi_{\mathrm{B}}(\chi)|^2 = \lambda \varphi^+(q) + O(qL^{-1+\epsilon}),$$

where λ is as in Lemma 3.2. We also have

$$\begin{aligned}\frac{1}{\varphi^+(q)} \sum_{\chi(q)}^{+} \bigl|L\bigl(\tfrac{1}{2}, \chi\bigr)\bigr|^2 |\psi_{\mathrm{MV}}(\chi)|^2 &= \frac{1}{\varphi^+(q)} \sum_{\chi(q)}^{+} \bigl|L\bigl(\tfrac{1}{2}, \chi\bigr)\bigr|^2 \biggl|\sum_{\ell \le y_3} \frac{\mu(\ell)\chi(\ell)P_3[\ell]}{\ell^{1/2}}\biggr|^2 \\ &= P_3(1)^2 + \frac{1}{\theta_3}\int_0^1 P_3'(x)^2\,dx + O(L^{-1+\epsilon}),\end{aligned}$$

by the analysis of the Iwaniec–Sarnak mollifier [Bui 2012, §2.3].

Therefore, in order to prove Lemma 3.2 it suffices to prove the following two results.

Lemma 3.3. *For* $0 < \theta_1, \theta_3 < \frac{1}{2}$ *we have*

$$\sum_q \Psi\Bigl(\frac{q}{Q}\Bigr)\frac{q}{\varphi(q)} \sum_{\chi(q)}^{+} \bigl|L\bigl(\tfrac{1}{2}, \chi\bigr)\bigr|^2 \psi_{\mathrm{IS}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi}) = (P_1(1)P_3(1) + o(1)) \sum_q \Psi\Bigl(\frac{q}{Q}\Bigr)\frac{q}{\varphi(q)}\varphi^+(q).$$

Lemma 3.4. *Let* $0 < \theta_2 < \theta_3 < \frac{1}{2}$. *Then*

$$\begin{aligned}&\sum_q \Psi\Bigl(\frac{q}{Q}\Bigr)\frac{q}{\varphi(q)} \sum_{\chi(q)}^{+} \bigl|L\bigl(\tfrac{1}{2}, \chi\bigr)\bigr|^2 \psi_{\mathrm{B}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi}) \\ &\qquad = \Bigl(\frac{3\theta_2}{2} P_3(1)\widetilde{P}_2(1) - \theta_2 \int_0^1 P_2(x)P_3(x)\,dx + o(1)\Bigr) \sum_q \Psi\Bigl(\frac{q}{Q}\Bigr)\frac{q}{\varphi(q)}\varphi^+(q).\end{aligned}$$

4. Lemma 3.3: main term

The goal of this section is to extract the main term in Lemma 3.3. The main term analysis is given in [Michel and VanderKam 2000, §6], but as the ideas also appear in the proof of Lemma 3.4 we give details here.

We begin with two lemmas.

Lemma 4.1. *Let χ be a primitive even character modulo q. Let $G(s)$ be an even polynomial satisfying $G(0)=1$, and which vanishes to second order at $\frac{1}{2}$. Then we have*

$$\left|L\left(\tfrac{1}{2},\chi\right)\right|^2 = 2\sum_{m,n}\sum \frac{\chi(m)\overline{\chi}(n)}{(mn)^{1/2}} V\left(\frac{mn}{q}\right),$$

where

$$V(x) = \frac{1}{2\pi i}\int_{(1)} \frac{\Gamma^2\left(\frac{1}{2}s+\frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \frac{G(s)}{s}\pi^{-s}x^{-s}\,ds. \tag{4-1}$$

Proof. See [Iwaniec and Sarnak 1999, (2.5)]. The result follows along the lines of [Iwaniec and Kowalski 2004, Theorem 5.3]. □

We remark that V satisfies $V(x) \ll_A (1+x)^{-A}$, as can be seen by moving the contour of integration to the right. We also note that the choice of $G(s)$ in Lemma 4.1 is almost completely free. In particular, we may choose G to vanish at whichever finite set of points is convenient for us (see (4-6) below for an application).

Lemma 4.2. *Let $(mn,q)=1$. Then*

$$\sideset{}{^+}\sum_{\chi(q)} \chi(m)\overline{\chi}(n) = \frac{1}{2}\sum_{\substack{vw=q\\ w\mid m\pm n}}\sum \mu(v)\varphi(w).$$

Proof. See [Bui and Milinovich 2011, Lemma 4.1], for instance. □

We do not need the averaging over q in order to extract the main term of Lemma 3.3. We insert the definitions of the mollifiers $\psi_{\mathrm{IS}}(\chi)$ and $\psi_{\mathrm{MV}}(\overline{\chi})$, then apply Lemma 4.1, and interchange orders of summation. We obtain

$$\begin{aligned}\sideset{}{^+}\sum_{\chi(q)} &\left|L\left(\tfrac{1}{2},\chi\right)\right|^2 \psi_{\mathrm{IS}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi}) \\ &= 2\sum_{\substack{\ell_1\le y_1\\ \ell_3\le y_3\\ (\ell_1\ell_3,q)=1}}\sum \frac{\mu(\ell_1)\mu(\ell_3)P_1[\ell_1]P_3[\ell_3]}{(\ell_1\ell_3)^{1/2}} \sum_{(mn,q)=1}\sum \frac{1}{(mn)^{1/2}} V\left(\frac{mn}{q}\right) \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi)\chi(m\ell_1\ell_3)\overline{\chi}(n).\end{aligned} \tag{4-2}$$

Opening $\epsilon(\chi)$ using (2-6) and applying Lemma 4.2, we find after some work [Iwaniec and Sarnak 1999, (3.4) and (3.7)] that

$$\sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi)\chi(m\ell_1\ell_3)\overline{\chi}(n) = \frac{1}{q^{1/2}}\sum_{\substack{vw=q\\ (v,w)=1}}\sum \mu^2(v)\varphi(w)\cos\frac{2\pi n\overline{m\ell_1\ell_3 v}}{w}. \tag{4-3}$$

The main term comes from $m\ell_1\ell_3=1$. With this constraint in place we apply character orthogonality in reverse, obtaining that the main term $M_{\mathrm{IS,MV}}$ of Lemma 3.3 is

$$M_{\mathrm{IS,MV}} = 2P_1(1)P_3(1)\sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi)\sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{n}{q}\right).$$

We have the following proposition.

Proposition 4.3. *Let χ be a primitive even character modulo q, and let $T > 0$ be a real number. Let V be defined as in (4-1). Then*

$$\sum_n \frac{\bar\chi(n)}{n^{1/2}} V\left(\frac{Tn}{q}\right) = L\left(\tfrac12, \bar\chi\right) - \epsilon(\bar\chi) \sum_n \frac{\chi(n)}{n^{1/2}} F\left(\frac{n}{T}\right),$$

where

$$F(x) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac12 s + \frac14\right)\Gamma\left(-\frac12 s + \frac14\right)}{\Gamma^2\left(\frac14\right)} \frac{G(s)}{s} x^{-s}\, ds. \tag{4-4}$$

Before proving Proposition 4.3, let us see how to use it to finish the evaluation of $M_{\mathrm{IS,MV}}$. Proposition 4.3 gives

$$M_{\mathrm{IS,MV}} = 2P_1(1)P_3(1) \sum_{\chi(q)}^{+} \epsilon(\chi) L\left(\tfrac12, \bar\chi\right) - 2P_1(1)P_3(1) \sum_{\chi(q)}^{+} \sum_n \frac{\chi(n)}{n^{1/2}} F(n),$$

and by the first moment analysis (see [Michel and VanderKam 2000, §3] and also Section 6 below) we have

$$2P_1(1)P_3(1) \sum_{\chi(q)}^{+} \epsilon(\chi) L\left(\tfrac12, \bar\chi\right) = (1+o(1)) 2P_1(1)P_3(1)\varphi^{+}(q). \tag{4-5}$$

For the other piece, we apply Lemma 4.2 to obtain

$$-2P_1(1)P_3(1) \sum_{\chi(q)}^{+} \sum_n \frac{\chi(n)}{n^{1/2}} F(n) = -P_1(1)P_3(1) \sum_{w|q} \varphi(w)\mu(q/w) \sum_{\substack{n \equiv \pm 1 (w) \\ (n,q)=1}} \frac{1}{n^{1/2}} F(n).$$

We choose G to vanish at all the poles of

$$\Gamma\left(\tfrac12 s + \tfrac14\right)\Gamma\left(-\tfrac12 s + \tfrac14\right)$$

in the disc $|s| \le A$, where $A > 0$ is large but fixed. By moving the contour of integration to the right we see

$$F(x) \ll \frac{1}{(1+x)^{100}}, \tag{4-6}$$

say, and therefore, the contribution from $n > q^{1/10}$ is negligible. By trivial estimation the contribution from $w \le q^{1/4}$ is also negligible. For $w > q^{1/4}$ and $n \le q^{1/10}$, we can only have $n \equiv \pm 1 \pmod{w}$ if $n = 1$. Adding back in the terms with $n \le q^{1/4}$, the contribution from these terms is therefore

$$-(1+o(1)) 2P_1(1)P_3(1)F(1)\varphi^{+}(q). \tag{4-7}$$

Since the integrand in $F(1)$ is odd, we may evaluate $F(1)$ through a residue at $s = 0$. We shift the line of integration in (4-4) to $\operatorname{Re} s = -1$, picking up a contribution from the simple pole at $s = 0$. In the integral on the line $\operatorname{Re} s = -1$ we change variables $s \to -s$. This yields the relation $F(1) = 1 - F(1)$, whence $F(1) = \frac12$. Combining (4-5) and (4-7), we obtain

$$M_{\mathrm{IS,MV}} = (1+o(1)) P_1(1)P_3(1)\varphi^{+}(q),$$

as desired. This yields the main term of Lemma 3.3.

Proof of Proposition 4.3. We write V using its definition and interchange orders of summation and integration to get

$$\sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{Tn}{q}\right) = \frac{1}{2\pi i}\int_{(1)} \frac{\Gamma^2\left(\frac{1}{2}s+\frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \frac{G(s)}{s}\left(\frac{q}{\pi}\right)^s T^{-s} L\left(\tfrac{1}{2}+s, \overline{\chi}\right) ds.$$

We move the line of integration to $\operatorname{Re} s=-1$, picking up a contribution of $L\left(\frac{1}{2}, \overline{\chi}\right)$ from the pole at $s=0$. Observe that we do not get any contribution from the double pole of $\Gamma^2\left(\frac{1}{2}s+\frac{1}{4}\right)$ at $s=-\frac{1}{2}$ because of our assumption that G vanishes at $s=\pm\frac{1}{2}$ to second order.

Now, for the integral on the line $\operatorname{Re} s=-1$, we apply the functional equation for $L\left(\frac{1}{2}+s, \overline{\chi}\right)$ and then change variables $s \to -s$ to obtain

$$-\epsilon(\overline{\chi})\frac{1}{2\pi i}\int_{(1)} \frac{\Gamma\left(\frac{1}{2}s+\frac{1}{4}\right)\Gamma\left(-\frac{1}{2}s+\frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \frac{G(s)}{s} T^s L\left(\tfrac{1}{2}+s, \chi\right) ds.$$

The desired result follows by expanding $L\left(\frac{1}{2}+s, \chi\right)$ in its Dirichlet series and interchanging the order of summation and integration. □

5. Lemma 3.3: error term

Here we show that the remainder of the terms in (4-2) (those with $m\ell_1\ell_3 \neq 1$) contribute only to the error term of Lemma 3.3. Here we must avail ourselves of the averaging over q.

Inserting (4-3) into (4-2) and averaging over moduli, we wish to show that

$$\mathscr{E}_1 = \sum\sum_{(v,w)=1} \mu^2(v)\frac{v}{\varphi(v)}\frac{w^{1/2}}{v^{1/2}}\Psi\left(\frac{vw}{Q}\right) \sum\sum_{\substack{\ell_1\le y_1\\ \ell_3\le y_3\\ (\ell_1\ell_3,vw)=1}} \frac{\mu(\ell_1)\mu(\ell_3)P_1[\ell_1]P_3[\ell_3]}{(\ell_1\ell_3)^{1/2}} \times \sum\sum_{(mn,vw)=1} \frac{1}{(mn)^{1/2}} Z\left(\frac{mn}{vw}\right) \cos\frac{2\pi n\overline{m\ell_1\ell_3 v}}{w} \ll Q^{2-\epsilon+o(1)}, \tag{5-1}$$

where $m\ell_1\ell_3 \neq 1$, but we do not indicate this in the notation. The function Z is actually just V in (4-1), but we do not wish to confuse the function V with the scale V that shall appear shortly.

Observe that the arithmetic weight $q/\varphi(q)$ has become $(v/\varphi(v))(w/\varphi(w))$ by multiplicativity, and that this factor of $\varphi(w)$ has canceled with $\varphi(w)$ in (4-3), making the sum on w smooth.

The main tool we use to bound $\mathscr{E}_1'$ is the following result, due to Deshouillers and Iwaniec, on cancellation in sums of Kloosterman sums.

Lemma 5.1. *Let C, D, N, R, S be positive numbers, and let $b_{n,r,s}$ be a complex sequence supported in $(0, N]\times(R, 2R]\times(S, 2S]\cap\mathbb{N}^3$. Let $g_0(\xi, \eta)$ be a smooth function having compact support in $\mathbb{R}^+\times\mathbb{R}^+$,* *and let $g(c,d)=g_0(c/C,d/D)$. Then*

$$\sum_c \sum_d \sum_n \sum_r \sum_s_{(rd,sc)=1} b_{n,r,s}\, g(c,d)\, e\Big(n\frac{\overline{rd}}{sc}\Big) \ll_{\epsilon,g_0} (CDNRS)^{\epsilon} K(C,D,N,R,S)\|b_{N,R,S}\|_2,$$

where

$$\|b_{N,R,S}\|_2=\Big(\sum_n\sum_r\sum_s |b_{n,r,s}|^2\Big)^{1/2}$$

and

$$K(C,D,N,R,S)^2 = CS(RS+N)(C+RD)+C^2DS\sqrt{(RS+N)R}+D^2NRS^{-1}.$$

Proof. This is essentially [Bombieri et al. 1986, Lemma 1], which is an easy consequence of [Deshouillers and Iwaniec 1982, Theorem 12]. □

We need to massage (5-1) before it is in a form where an application of Lemma 5.1 is appropriate. Let us briefly describe our plan of attack. We apply partitions of unity to localize the variables and then separate variables with integral transforms. By using the orthogonality of multiplicative characters we will be able to assume that v is quite small, which is advantageous when it comes time to remove coprimality conditions involving v. We next reduce to the case in which n is somewhat small. This is due to the fact that the sum on n is essentially a Ramanujan sum, and Ramanujan sums experience better than square-root cancellation on average. We next use Möbius inversion to remove the coprimality condition between n and w. This application of Möbius inversion introduces a new variable, call it f, and another application of character orthogonality allows us to assume f is small. We then remove the coprimality conditions on m. We finally apply Lemma 5.1 to get the desired cancellation, and it is crucial here that f and v are no larger than Q^{ϵ}.

Let us turn to the details in earnest. We apply smooth partitions of unity (see [Blomer et al. 2017, Lemma 1.6], for instance) in all variables, so that $\mathscr{E}_1$ can be written

$$\sum_{M,N,L_1,L_3,V,W}\cdots\sum \mathscr{E}_1(M,N,L_1,L_3,V,W), \tag{5-2}$$

where

$$\begin{aligned}\mathscr{E}_1(M,N,L_1,L_3,V,W) = \sum\sum_{(v,w)=1} \mu^2(v)\frac{v}{\varphi(v)}\frac{w^{1/2}}{v^{1/2}}\Psi\Big(\frac{vw}{Q}\Big)G\Big(\frac{v}{V}\Big)G\Big(\frac{w}{W}\Big) \\ \times \sum\sum_{\substack{\ell_1\le y_1\\ \ell_3\le y_3\\ (\ell_1\ell_3,vw)=1}} \frac{\mu(\ell_1)\mu(\ell_3)P_1[\ell_1]P_3[\ell_3]}{(\ell_1\ell_3)^{1/2}} G\Big(\frac{\ell_1}{L_1}\Big)G\Big(\frac{\ell_3}{L_3}\Big) \\ \times \sum\sum_{(mn,vw)=1}\frac{1}{(mn)^{1/2}} Z\Big(\frac{mn}{vw}\Big)G\Big(\frac{m}{M}\Big)G\Big(\frac{n}{N}\Big)\cos\frac{2\pi nm\overline{\ell_1\ell_3 v}}{w}.\end{aligned}$$

Here G is a smooth, nonnegative function supported in $\left[\frac{1}{2}, 2\right]$, and the numbers M, N, L_i, V, W in (5-2) range over powers of two. We may assume

$$M, N, L_1, L_3, V, W \gg 1, \qquad VW \asymp Q, \qquad L_i \ll y.$$

Furthermore, by the rapid decay of Z we may assume $MN \le Q^{1+\epsilon}$. Thus, the number of summands $\mathscr{E}_1(M, \dots, W)$ in (5-2) is $\ll Q^{o(1)}$.

Up to changing the definition of G, we may rewrite $\mathscr{E}_1(M, \dots, W)$ as

$$\begin{aligned}\mathscr{E}_1(M, N, L_1, L_3, V, W) = \frac{W^{1/2}}{(MNL_1L_3V)^{1/2}} &\sum\sum_{(v,w)=1} \alpha(v) G\left(\frac{v}{V}\right) G\left(\frac{w}{W}\right) \Psi\left(\frac{vw}{Q}\right) \\ &\times \sum\sum_{\substack{\ell_i \le y_i \\ (\ell_i, vw)=1}} \beta(\ell_1)\gamma(\ell_3) G\left(\frac{\ell_1}{L_1}\right) G\left(\frac{\ell_3}{L_3}\right) \\ &\times \sum\sum_{(mn,vw)=1} Z\left(\frac{mn}{vw}\right) G\left(\frac{m}{M}\right) G\left(\frac{n}{N}\right) \cos \frac{2\pi n m \overline{\ell_1 \ell_3 v}}{w},\end{aligned}$$

where α, β, γ are sequences satisfying $|\alpha(v)|, |\beta(\ell_1)|, |\gamma(\ell_3)| \ll Q^{o(1)}$.

We separate the variables in Z by writing Z using its definition as an integral (4-1) and moving the line of integration to $\operatorname{Re} s = L^{-1}$. By the rapid decay of the Γ function in vertical strips we may restrict to $|\operatorname{Im} s| \le Q^\epsilon$. We similarly separate the variables in Ψ using the inverse Mellin transform. Therefore, up to changing the definition of some of the functions G, it suffices to prove that

$$\begin{aligned}\mathscr{E}_1'(M, N, L_1, L_3, V, W) = \frac{W^{1/2}}{(MNL_1L_3V)^{1/2}} &\sum\sum_{(v,w)=1} \alpha(v) G\left(\frac{v}{V}\right) G\left(\frac{w}{W}\right) \\ &\times \sum\sum_{\substack{\ell_i \le y_i \\ (\ell_i, vw)=1}} \beta(\ell_1)\gamma(\ell_3) G\left(\frac{\ell_3}{L_3}\right) G\left(\frac{\ell_3}{L_3}\right) \\ &\times \sum\sum_{(mn,vw)=1} G\left(\frac{m}{M}\right) G\left(\frac{n}{N}\right) e\left(\frac{nm\overline{\ell_1 \ell_3 v}}{w}\right) \ll Q^{2-\epsilon+o(1)}. \qquad (5\text{-}3)\end{aligned}$$

Our smooth functions G all satisfy $G^{(j)}(x) \ll_j Q^{j\epsilon}$ for $j \ge 0$. To save on space we write the left side of (5-3) as simply $\mathscr{E}_1'$.

Observe that the trivial bound for $\mathscr{E}_1'$ is

$$\mathscr{E}_1' \ll V^{1/2} W^{3/2} (MN)^{1/2} (L_1L_3)^{1/2} Q^{o(1)} \ll \frac{Q^{2+\epsilon}(y_1y_3)^{1/2}}{V}. \qquad (5\text{-}4)$$

This bound is worst when V is small. Since y_i will be taken close to $Q^{1/2}$, we therefore need to save $\approx Q^{1/2}$ in order to obtain (5-1). The trivial bound does show, however, that the contribution from $V > Q^{1/2+2\epsilon}$ is acceptably small, and we may therefore assume that $V \le Q^{1/2+2\epsilon}$. Note this implies $W \gg Q^{1/2-\epsilon}$.

We now reduce to the case $V \ll Q^{\epsilon}$. We accomplish this by reintroducing multiplicative characters. The orthogonality of multiplicative characters yields

$$e\left(\frac{nm\overline{\ell_1\ell_3 v}}{w}\right) = \frac{1}{\varphi(w)} \sum_{\chi(w)} \tau(\overline{\chi})\chi(n)\overline{\chi}(m\ell_1\ell_3 v). \tag{5-5}$$

Using the Gauss sum bound $|\tau(\overline{\chi})| \ll w^{1/2}$ we then arrange $\mathscr{E}_1'$ as

$$\mathscr{E}_1' \ll \frac{W}{(MNL_1L_3V)^{1/2}} \sum_{w \asymp W} \frac{1}{\varphi(w)} \sum_{v \asymp V} \left| \sum\sum_{(mn,v)=1} \chi(n)\overline{\chi}(m) \right| \left| \sum\sum_{(\ell_1\ell_3,v)=1} \overline{\chi}(\ell_1\ell_3) \right|,$$

where we have suppressed some things in the notation for brevity. By Cauchy–Schwarz and character orthogonality we obtain

$$\sum_{\chi(w)} \left| \sum\sum_{m,n} \right| \left| \sum\sum_{\ell_1,\ell_3} \right| \ll Q^{o(1)}(MNL_1L_3)^{1/2}(MN+W)^{1/2}(L_1L_3+W)^{1/2},$$

which yields a bound of

$$Q^{-o(1)}\mathscr{E}_1' \ll \frac{Q(MN)^{1/2}(y_1y_3)^{1/2}}{V^{1/2}} + \frac{Q^{3/2}(MN)^{1/2}}{V} + \frac{Q^{3/2}(y_1y_3)^{1/2}}{V} + \frac{Q^2}{V^{3/2}}. \tag{5-6}$$

We observe that (5-6) is acceptable for $V \geq Q^{3\epsilon}$, say. We may therefore assume $V \leq Q^{\epsilon}$.

We next show that $\mathscr{E}_1'$ is small provided N is somewhat large.

Proposition 5.2. *Assume the hypotheses of Lemma 3.3. If $N \geq MQ^{-2\epsilon}$ and $m\ell_1\ell_3 \neq 1$, then $\mathscr{E}_1' \ll Q^{2-\epsilon+o(1)}$.*

Proof. We make use only of cancellation in the sum on n, say

$$\Sigma_N = \sum_{(n,vw)=1} G\left(\frac{n}{N}\right) e\left(\frac{nm\overline{\ell_1\ell_3 v}}{w}\right).$$

We use Möbius inversion to detect the condition $(n, v) = 1$, and then break n into primitive residue classes modulo w. Thus,

$$\Sigma_N = \sum_{d|v} \mu(d) \sum_{(a,w)=1} e\left(\frac{adm\overline{\ell_1\ell_3 v}}{w}\right) \sum_{n \equiv a(w)} G\left(\frac{dn}{N}\right).$$

We apply Poisson summation to each sum on n, and obtain

$$\Sigma_N = \sum_{d|v} \mu(d) \sum_{(a,w)=1} e\left(\frac{adm\overline{\ell_1\ell_3 v}}{w}\right) \frac{N}{dw} \sum_{|h| \leq W^{1+\epsilon}d/N} e\left(\frac{ah}{w}\right) \widehat{G}\left(\frac{hN}{dw}\right) + O_\epsilon(Q^{-100}),$$

say. The contribution of the error term is, of course, negligible. The contribution of the zero frequency $h = 0$ to Σ_N is

$$\widehat{G}(0)\frac{N}{w} \sum_{d|v} \frac{\mu(d)}{d} \sum_{(a,w)=1} e\left(\frac{adm\overline{\ell_1\ell_3 v}}{w}\right) = \widehat{G}(0)\mu(w)\frac{N}{w}\frac{\varphi(v)}{v},$$

and upon summing this contribution over the remaining variables, the zero frequency contributes

$$\ll V^{1/2}W^{1/2}(MN)^{1/2}(y_1y_3)^{1/2}Q^{o(1)} \ll Q^{3/2}$$

to $\mathscr{E}_1'$, and this contribution is sufficiently small.

It takes just a bit more work to bound the contribution of the nonzero frequencies $|h| > 0$. We rearrange the sum as

$$\sum_{d|v}\mu(d)\frac{N}{dw}\sum_{|h|\le W^{1+\epsilon}d/N}\widehat{G}\left(\frac{hN}{dw}\right)\sum_{(a,w)=1}e\left(\frac{adm\overline{\ell_1\ell_3v}}{w}+\frac{ah}{w}\right).$$

By a change of variables the inner sum is equal to the Ramanujan sum $c_w(hm\ell_1\ell_3v+d)$. Note that $hm\ell_1\ell_3v+d\neq 0$ because $m\ell_1\ell_3\neq 1$. The nonzero frequencies therefore contribute to $\mathscr{E}_1'$ an amount

$$\ll Q^{\epsilon}\frac{(VWL_1L_3M)^{1/2}}{N^{1/2}}\sup_{0<|k|\ll Q^{O(1)}}\sum_{w\asymp W}|c_w(k)|.$$

Since $|c_w(k)|\le (k,w)$ the sum on w is $\ll W^{1+o(1)}$. It follows that

$$\mathscr{E}_1'\ll Q^{3/2}+Q^{3/2+\epsilon}(y_1y_3)^{1/2}\frac{M^{1/2}}{N^{1/2}}.$$

Since $y_i=Q^{\theta_i}$ and $\theta_i<\frac{1}{2}-3\epsilon$, say, this bound for $\mathscr{E}_1'$ is acceptable provided $N\ge MQ^{-2\epsilon}$. □

By Proposition 5.2 we may assume $N\le MQ^{-2\epsilon}$. Since $MN\le Q^{1+\epsilon}$, the condition $N\le MQ^{-2\epsilon}$ implies $N\le Q^{1/2}$.

We now pause to make a comment on the condition $m\ell_1\ell_3\neq 1$, which we have assumed throughout this section but not indicated in the notation for $\mathscr{E}_1'$. Observe that this condition is automatic if $ML_1L_3>2019$ (say). If $ML_1L_3\ll 1$, then we may use the trivial bound (5-4) along with the bound $N\le Q^{1/2}\le Q^{1-\epsilon}$ to obtain

$$\mathscr{E}_1'\ll Q^{2-\epsilon}.$$

We may therefore assume $ML_1L_2\gg 1$, so that the condition $m\ell_1\ell_3\neq 1$ is satisfied.

We now remove the coprimality condition $(n,w)=1$. By Möbius inversion we have

$$\mathbf{1}_{(n,w)=1}=\sum_{\substack{f|n\\ f|w}}\mu(f).$$

We move the sum on f to be the outermost sum, and note $f\ll N$. We then change variables $n\to nf$ and $w\to wf$. If a_*, say, is any lift of the multiplicative inverse of $m\ell_1\ell_3v$ modulo wf, then $a_*\equiv\overline{m\ell_1\ell_3v}\pmod{w}$, and therefore,

$$\frac{nf\overline{m\ell_1\ell_3v}}{wf}\equiv\frac{n\overline{m\ell_1\ell_3v}}{w}\pmod{1}.$$

It follows that

$$\mathscr{E}_1' = \frac{W^{1/2}}{(MNL_1L_3V)^{1/2}} \sum_{f \ll N} \mu(f) \sum_{(v,wf)=1}\sum \alpha(v) G\Big(\frac{v}{V}\Big) G\Big(\frac{wf}{W}\Big) \times \sum_{\substack{\ell_i \le y_i \\ (\ell_i, fvw)=1}}\sum \beta(\ell_1)\gamma(\ell_3) G\Big(\frac{\ell_1}{L_1}\Big) G\Big(\frac{\ell_3}{L_3}\Big) \sum_{\substack{(m,fvw)=1 \\ (n,v)=1}}\sum G\Big(\frac{m}{M}\Big) G\Big(\frac{nf}{N}\Big) e\Big(\frac{n\overline{m\ell_1\ell_3 v}}{w}\Big).$$

We next reduce the size of f by a similar argument to the one that let us impose the condition $V \le Q^\epsilon$. We obtain by transitioning to multiplicative characters (recall (5-5)) that the sum over v, w, m, n, ℓ_1, ℓ_3 is bounded by

$$\ll \frac{W^{1/2+o(1)}V^{1/2}}{f^{1/2}} \sum_{w \asymp W/f} \frac{1}{w^{1/2}}\Big(\frac{(MN)^{1/2}}{f^{1/2}} + w^{1/2}\Big)((L_1L_3)^{1/2} + w^{1/2}) \ll \frac{Q^{2+\epsilon}}{f^{3/2}},$$

and therefore, the contribution from $f > Q^{4\epsilon}$ is negligible.

Now the only barrier to applying Lemma 5.1 is the conditions $(m, f) = 1$ and $(m, v) = 1$. We remove both of these conditions with Möbius inversion, obtaining

$$\sum_{f \ll \min(N, Q^\epsilon)} \mu(f) \sum_{h \mid f} \mu(h) \sum_{t \ll V} \mu(t) \frac{W^{1/2}}{(MNL_1L_3V)^{1/2}} \sum_{\substack{(v,wf)=1 \\ (w,ht)=1}}\sum \alpha(v) G\Big(\frac{vt}{V}\Big) G\Big(\frac{wf}{W}\Big) \times \sum_{\substack{\ell_i \le y_i \\ (\ell_i, fvw)=1}}\sum \beta(\ell_1)\gamma(\ell_3) G\Big(\frac{\ell_1}{L_1}\Big) G\Big(\frac{\ell_3}{L_3}\Big) \sum_{\substack{(m,w)=1 \\ (n,v)=1}}\sum G\Big(\frac{mht}{M}\Big) G\Big(\frac{nf}{N}\Big) e\Big(\frac{n\overline{mht^2\ell_1\ell_3 v}}{w}\Big).$$

We set

$$b_{n,ht^2k} = \mathbf{1}_{(n,v)=1} G\Big(\frac{nf}{N}\Big) \sum_{\substack{\ell_1 \\ \ell_1\ell_3 v = k \\ (\ell_1\ell_3, v)=1}} \sum_{\ell_3} \sum_{v} \beta(\ell_1)\gamma(\ell_3)\alpha(v) G\Big(\frac{vt}{V}\Big) G\Big(\frac{\ell_1}{L_1}\Big) G\Big(\frac{\ell_3}{L_3}\Big)$$

if $(k, f) = 1$, and for integers r not divisible by ht^2 we set $b_{n,r} = 0$. It follows that if $b_{n,r} \ne 0$, then $n \asymp N/f$ and $r \asymp htL_1L_3V$ with $r \equiv 0(ht^2)$. The sum over n, r, m, w is therefore a sum of the form to which Lemma 5.1 may be applied. We note that

$$\|b_{N,R}\|_2 \ll \frac{Q^{o(1)}}{(ft)^{1/2}}(NL_1L_3V)^{1/2},$$

and therefore, by Lemma 5.1 we have

$$\begin{aligned}\mathscr{E}_1' &\ll Q^\epsilon \sum_{f \ll Q^\epsilon} \frac{1}{f^{1/2}} \sum_{h \mid f} \sum_{t \ll Q^\epsilon} \frac{1}{t^{1/2}} \frac{W^{1/2}}{M^{1/2}} \\ &\qquad \times \bigg\{ \frac{W^{1/2}}{f^{1/2}} ((htL_1L_3V)^{1/2} + \frac{N^{1/2}}{f^{1/2}})(\frac{W^{1/2}}{f^{1/2}} + (ML_1L_3V)^{1/2}) \\ &\qquad\qquad + \frac{W}{f} \frac{M^{1/2}}{(ht)^{1/2}} ((htL_1L_3V)^{1/2} + (htL_1L_3NV)^{1/4}) + \frac{M}{ht}(htL_1L_3NV)^{1/2} \bigg\} \\ &\ll Q^\epsilon \bigg(\frac{W^{3/2}(y_1y_3)^{1/2}}{M^{1/2}} + Wy_1y_3 + W^{3/2}\frac{N^{1/2}}{M^{1/2}} + W(y_1y_3)^{1/2}N^{1/2} + W^{3/2}(y_1y_3)^{1/2} \\ &\qquad\qquad + W^{3/2}(y_1y_3)^{1/4}N^{1/4} + W^{1/2}Q^{1/2}(y_1y_3)^{1/2} \bigg) \ll Q^{2-\epsilon},\end{aligned}$$

upon recalling the bounds $W \ll Q$ and $y_i \le Q^{\theta_i}$ with $\theta_i < \frac{1}{2}$, and $N \le Q^{1-\epsilon}$. This completes the proof of Lemma 3.3.

6. Lemma 3.4: main term

In this section we obtain the main term of Lemma 3.4. We allow ourselves to recycle some notation from Sections 4 and 5.

Recall that we wish to asymptotically evaluate

$$\sum_{\chi(q)}^{+} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \psi_{\mathrm{B}}(\chi)\psi_{\mathrm{MV}}(\overline{\chi}).$$

We begin precisely as in Section 4. Inserting the definitions of $\psi_{\mathrm{B}}(\chi)$ and $\psi_{\mathrm{MV}}(\overline{\chi})$, we must asymptotically evaluate

$$\frac{2}{L} \sum_{\substack{bc \le y_2 \\ (bc,q)=1}} \sum \frac{\Lambda(b)\mu(c)P_2[bc]}{(bc)^{1/2}} \sum_{\substack{\ell \le y_3 \\ (\ell,q)=1}} \frac{\mu(\ell)P_3[\ell]}{\ell^{1/2}} \sum_{(mn,q)=1} \sum \frac{1}{(mn)^{1/2}} V\left(\frac{mn}{q}\right) \sum_{\chi(q)}^{+} \epsilon(\chi)\chi(c\ell m)\overline{\chi}(bn). \tag{6-1}$$

The main term of Lemma 3.3 arose from $m\ell_1\ell_3 = 1$. In the present case, the main term contains more than just $c\ell m = 1$; the main term arises from those $c\ell m$ which divide b. The support of the von Mangoldt function constrains b to be a prime power, so the condition $c\ell m \mid b$ is straightforward, but tedious, to handle.

There are three different cases to consider. The first case is $c\ell m = 1$. In the second case we have $c\ell m = p$ and $b = p$. Both of these cases contribute to the main term. The third case is everything else ($b = p^j$ with $j \ge 2$ and $c\ell m \mid b$ with $c\ell m \ge p$), and this case contributes only to the error term.

***First case*: $c\ell m = 1$.** If $c\ell m$ is equal to 1, then certainly $c\ell m$ divides b for every b. The contribution from $c\ell m = 1$ is equal to

$$M = \frac{2P_3(1)}{L} \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi) \sum_{b \le y_2} \frac{\Lambda(b)\overline{\chi}(b)P_2[b]}{b^{1/2}} \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{n}{q}\right).$$

By an application of Proposition 4.3,

$$M = M_1 + M_2, \tag{6-2}$$

where

$$M_1 = \frac{2P_3(1)}{L} \sum_{\substack{b \le y_2 \\ (b,q)=1}} \frac{\Lambda(b)P_2[b]}{b^{1/2}} \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi)\overline{\chi}(b) L\left(\tfrac{1}{2}, \overline{\chi}\right),$$

$$M_2 = -\frac{2P_3(1)}{L} \sum_{\substack{b \le y_2 \\ (bn,q)=1}}\sum \frac{\Lambda(b)P_2[b]}{(bn)^{1/2}} F(n) \sideset{}{^+}\sum_{\chi(q)} \chi(n)\overline{\chi}(b),$$

and F is the rapidly decaying function given by (4-4). A main term arises from M_1, and M_2 contributes only to the error term.

Let us first investigate M_2. By Lemma 4.2 we have

$$M_2 = -\frac{P_3(1)}{L} \sum_{w|q} \varphi(w)\mu(q/w) \sum_{\substack{b \le y_2 \\ b \equiv \pm n(w) \\ (bn,q)=1}}\sum \frac{\Lambda(b)P_2[b]}{(bn)^{1/2}} F(n).$$

By the rapid decay of F (recall (4-6)) we may restrict n to $n \le q^{1/10}$. The contribution from $w \le q^{1/2+\epsilon}$ is then trivially $\ll q^{1-\epsilon}$, since $y_2 \ll q^{1/2-\epsilon}$. For the remaining terms, the congruence condition $b \equiv \pm n(w)$ becomes $b = n$, and thus,

$$M_2 \ll q^{1-\epsilon} + \frac{1}{L} \sum_{\substack{w|q \\ w > q^{1/2+\epsilon}}} \varphi(w) \sum_{b \le q^{1/10}} \frac{\Lambda(b)P_2[b]}{b} F(b) \ll qL^{-1}.$$

Let us turn to M_1. We use the following lemma to represent the central value $L\left(\frac{1}{2}, \overline{\chi}\right)$.

Lemma 6.1. *Let $\overline{\chi}$ be a primitive even character modulo q. Then*

$$L\left(\tfrac{1}{2}, \overline{\chi}\right) = \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right) + \epsilon(\overline{\chi}) \sum_n \frac{\chi(n)}{n^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right),$$

where

$$V_1(x) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{1}{2}s + \frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \frac{G_1(s)}{s} \pi^{-s/2} x^{-s}\, ds$$

and $G_1(s)$ is an even polynomial satisfying $G_1(0) = 1$.

Proof. See [Iwaniec and Sarnak 1999, (2.2)]. □

Applying Lemma 6.1, the main term M_1 naturally splits as $M_1 = M_{1,1} + M_{1,2}$, where

$$M_{1,1} = \frac{2P_3(1)}{L} \sum_{\substack{b \le y_2 \\ (bn,q)=1}} \sum \frac{\Lambda(b) P_2[b]}{(bn)^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right) \sum_{\chi(q)}^{+} \epsilon(\chi)\overline{\chi}(bn),$$

$$M_{1,2} = \frac{2P_3(1)}{L} \sum_{\substack{b \le y_2 \\ (bn,q)=1}} \sum \frac{\Lambda(b) P_2[b]}{(bn)^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right) \sum_{\chi(q)}^{+} \chi(n)\overline{\chi}(b).$$

Applying character orthogonality to $M_{1,1}$ we arrive at

$$M_{1,1} = \frac{2P_3(1)}{Lq^{1/2}} \sum_{\substack{vw=q \\ (v,w)=1}} \sum \mu^2(v)\varphi(w) \sum_{\substack{b \le y_2 \\ (bn,q)=1}} \sum \frac{\Lambda(b) P_2[b]}{(bn)^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right) \cos \frac{2\pi bn\bar{v}}{w},$$

and a trivial estimation shows

$$M_{1,1} \ll q^{1-\epsilon}.$$

Let us lastly examine $M_{1,2}$, from which a main term arises. By character orthogonality we have

$$M_{1,2} = \frac{P_3(1)}{L} \sum_{w|q} \varphi(w)\mu(q/w) \sum_{\substack{b \le y_2 \\ b \equiv \pm n(w) \\ (b,q)=1}} \sum \frac{\Lambda(b) P_2[b]}{(bn)^{1/2}} V_1\left(\frac{n}{q^{1/2}}\right).$$

By trivial estimation, the contribution from $w \le q^{1/2+\epsilon}$ is

$$\ll \sum_{\substack{w|q \\ w \le q^{1/2+\epsilon}}} \varphi(w) \sum_{b \le y_2} \frac{1}{b^{1/2}} \sum_{\substack{n \le q^{1/2+\epsilon} \\ n \equiv \pm b(w)}} \frac{1}{n^{1/2}} \ll y_2^{1/2} \sum_{\substack{w|q \\ w \le q^{1/2+\epsilon}}} \varphi(w)\left(\frac{q^{1/4+\epsilon}}{w} + O(1)\right) \ll q^{3/4+\epsilon}.$$

By the rapid decay of V_1, for $w > q^{1/2+\epsilon}$ the congruence $b \equiv \pm n(w)$ becomes $b = n$. Adding back in the terms $w \le q^{1/2+\epsilon}$, we have

$$M_{1,2} = \frac{2P_3(1)}{L}\varphi^+(q) \sum_{\substack{b \le y_2 \\ (b,q)=1}} \frac{\Lambda(b) P_2[b]}{b} V_1\left(\frac{b}{q^{1/2}}\right) + O(q^{1-\epsilon}).$$

For $x \ll 1$ we see by a contour shift that

$$V_1(x) = 1 + O(x^{1/3}),$$

and we have $bq^{-1/2} \ll q^{-\epsilon}$. It follows that

$$M_{1,2} = O(q^{1-\epsilon}) + \frac{2P_3(1)}{L}\varphi^+(q) \sum_{\substack{b \le y_2 \\ (b,q)=1}} \frac{\Lambda(b) P_2[b]}{b}.$$

We have

$$\sum_{(b,q)>1} \frac{\Lambda(b)}{b} \ll 1 + \sum_{p|q} \frac{\log p}{p} \ll \log\log q,$$

and therefore, we may remove the condition $(b, q) = 1$ at the cost of an error $O(qL^{-1+\epsilon})$. From the estimate

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1),$$

summation by parts, and elementary manipulations, we obtain

$$\sum_{b \le y_2} \frac{\Lambda(b) P_2[b]}{b} = (\log y_2) \int_0^1 P_2(u)\, du + O(1).$$

Therefore, the contribution to the main term of Lemma 3.4 from $c\ell m = 1$ is

$$(2\theta_2 P_3(1) \widetilde{P}_2(1) + o(1)) \varphi^+(q). \tag{6-3}$$

***Second case*: $c\ell m = p$, $b = p$.** Another main term which contributes to Lemma 3.4 comes from $c\ell m = p$ and $b = p$. There are three subcases: $(c, \ell, m) = (p, 1, 1)$, $(1, p, 1)$, or $(1, 1, p)$. These three cases give (compare with (6-1))

$$N_1 = -\frac{2P_3(1)}{L} \sum_{\substack{p \le y_2^{1/2} \\ (p,q)=1}} \frac{(\log p) P_2(\log(y_2^{1/2}/p)/\log(y_2^{1/2}))}{p} \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{n}{q}\right),$$

$$N_2 = -\frac{2}{L} \sum_{\substack{p \le y_2 \\ (p,q)=1}} \frac{(\log p) P_2[p] P_3[p]}{p} \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{n}{q}\right),$$

$$N_3 = \frac{2P_3(1)}{L} \sum_{\substack{p \le y_2 \\ (p,q)=1}} \frac{(\log p) P_2[p]}{p} \sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{pn}{q}\right).$$

The first two are somewhat easier to handle than the last one. We apply Proposition 4.3 and then argue as in Section 4 and the $c\ell m = 1$ case to obtain

$$\sideset{}{^+}\sum_{\chi(q)} \epsilon(\chi) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{n}{q}\right) = \tfrac{1}{2}\varphi^+(q) + O(q^{1-\epsilon}).$$

It follows that

$$\begin{aligned} N_1 &= -\left(\frac{\theta_2}{2} P_3(1) \widetilde{P}_2(1) + o(1)\right) \varphi^+(q), \\ N_2 &= -\left(\theta_2 \int_0^1 P_2(u) P_3(u)\, du + o(1)\right) \varphi^+(q). \end{aligned} \tag{6-4}$$

Combining (6-3) and (6-4) gives the main term of Lemma 3.4.

The final term N_3 is more difficult because the inner sum now depends on p. However, M_3 contributes only to the error term. By Proposition 4.3 with $T = p$,

$$\sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V\left(\frac{pn}{q}\right) = L\left(\tfrac{1}{2}, \overline{\chi}\right) - \sum_n \frac{\chi(n)}{n^{1/2}} F\left(\frac{n}{p}\right). \tag{6-5}$$

The first term on the right side of (6-5) contributes to N_3 an amount

$$(2\theta_2 P_3(1)\widetilde{P}_2(1)+o(1))\varphi^+(q). \tag{6-6}$$

For the second term on the right side of (6-5) we use character orthogonality and get

$$-\frac{2P_3(1)}{L}\sum_{\substack{p\le y_2\\(p,q)=1}}\frac{(\log p)P_2[p]}{p}\frac{1}{2}\sum_{w\mid q}\varphi(w)\mu(q/w)\sum_{n\equiv\pm1(w)}\frac{1}{n^{1/2}}F\left(\frac{n}{p}\right).$$

By the rapid decay of F the contribution from $n>p^{11/10}$, say, is $O(qL^{-1})$. We next estimate trivially the contribution from $w\le q^{3/5}$, say. We have the bound

$$\sum_{\substack{n\equiv\pm1(w)\\n\le p^{11/10}}}\frac{1}{n^{1/2}}F\left(\frac{n}{p}\right)\ll q^{\epsilon}\left(\frac{p^{11/20}}{w}+1\right),$$

and this contributes to N_3 an amount

$$\ll q^{3/5+\epsilon}+q^{\epsilon}\sum_{p\le y_2}p^{-9/20}\ll q^{3/5+\epsilon},$$

since $y_2\ll q^{1/2}$. For $w>q^{3/5}$ and $n\le p^{11/10}$ the congruence $n\equiv\pm1(w)$ becomes $n=1$. By a contour shift we have

$$F\left(\frac{1}{p}\right)=1+O(p^{-1/2}).$$

Thus, the second term on the right side of (6-5) contributes to N_3 an amount

$$-(2\theta_2 P_3(1)\widetilde{P}_2(1)+o(1))\varphi^+(q), \tag{6-7}$$

and (6-6) and (6-7) together imply N_3 is negligible.

Third case*: *everything else. This case is the contribution from $b=p^j$ with $j\ge 2$ and $c\ell m\mid b$ with $c\ell m\ge p$. This case contributes an error of size $O(qL^{-1+\epsilon})$, essentially because the sum

$$\sum_{\substack{p^k\\k\ge2}}\frac{\log(p^k)}{p^k}$$

converges. There are four different subcases to consider, since the Möbius functions attached to c and ℓ imply $c,\ell\in\{1,p\}$. The same techniques we have already employed allow one to bound the resulting sums, so we leave the details for the interested reader. This completes the proof of Lemma 3.4.

7. Lemma 3.4: error term

After the results of the previous section, it remains to finish the proof of Lemma 3.4 by showing the error term of (6-1) is negligible. The argument is very similar to that given in Section 5, and, indeed, the arguments are identical after a point.

The error term has the form

$$\mathscr{E}_2 = \sum_{(v,w)=1}\sum \mu^2(v)\frac{v}{\varphi(v)}\frac{w^{1/2}}{v^{1/2}}\Psi\left(\frac{vw}{Q}\right)\sum_{\substack{\ell\le y_3\\(\ell,vw)=1}}\frac{\mu(\ell)P_3[\ell]}{\ell^{1/2}}$$
$$\times\sum_{\substack{bc\le y_2\\(bc,vw)=1}}\sum\frac{\Lambda(b)\mu(c)P_2[bc]}{(bc)^{1/2}}\sum_{(mn,vw)=1}\sum\frac{1}{(mn)^{1/2}}V\left(\frac{mn}{vw}\right)\cos\frac{2\pi bnc\overline{\ell m v}}{w},$$

where we also have the condition $c\ell m\nmid b$, which we do not indicate in the notation. This condition is awkward, but turns out to be harmless.

We note that we may separate the variables b and c from one another in $P_2[bc]$ by linearity, the additivity of the logarithm, and the binomial theorem. Thus, it suffices to study $\mathscr{E}_1$ with $P_2[bc]$ replaced by $(\log b)^{j_1}(\log c)^{j_2}$, for j_i some fixed nonnegative integers. Arguing as in the reduction to (5-3), we may bound $\mathscr{E}_2$ by $\ll Q^{o(1)}$ instances of $\mathscr{E}_2'=\mathscr{E}_2'(B,C,L,M,N,V,W)$, where

$$\mathscr{E}_2'=\frac{W^{1/2}}{(BCLMNV)^{1/2}}\sum_{(v,w)=1}\sum\alpha(v)G\left(\frac{v}{V}\right)G\left(\frac{w}{W}\right)\sum_{\substack{\ell\le y_3\\(\ell,vw)=1}}\beta(\ell)G\left(\frac{\ell}{L}\right)\sum_{\substack{bc\le y_2\\(bc,vw)=1}}\sum\gamma(b)\delta(c)G\left(\frac{b}{B}\right)G\left(\frac{c}{C}\right)$$
$$\times\sum_{(mn,vw)=1}\sum G\left(\frac{m}{M}\right)G\left(\frac{n}{N}\right)e\left(\frac{bnc\overline{\ell m v}}{w}\right),\tag{7-1}$$

the function G is smooth as before, and $\alpha,\beta,\gamma,\delta$ are sequences f satisfying $|f(z)|\ll Q^{o(1)}$. We also have the conditions

$$VW\asymp Q,\qquad MN\le Q^{1+\epsilon},\qquad BC\ll y_2,\qquad L\ll y_3,\qquad B,C,L,M,N,V,W\gg 1.$$

By the argument that gave (5-6) we may also assume $V\le Q^{\epsilon}$. Lastly, we may remove the condition $bc\le y_2$ by Mellin inversion, at the cost of changing γ and δ by b^{it_0} and c^{it_0}, respectively, where $t_0\in\mathbb{R}$ is arbitrary (see [Duke et al. 1997, Lemma 9], for instance).

Recall the condition $c\ell m\nmid b$. This condition is unnecessary if $CLM>2019\,B$, so it is only in the case $CLM\ll B$ where we need to deal with it. However, the case $CLM\ll B$ is exceptional, since B is bounded by $y_2\ll Q^{1/2}$, but generically we would expect CLM to be much larger than $Q^{1/2}$.

Indeed, we now show that when $CLM\ll B$ it suffices to get cancellation from the n variable alone. The proof is essentially Proposition 5.2, so we just remark upon the differences. By Möbius inversion and Poisson summation we have

$$\sum_{(n,vw)=1}G\left(\frac{n}{N}\right)e\left(\frac{bnc\overline{\ell m v}}{w}\right)=\mu(w)\frac{N}{w}\frac{\varphi(v)}{v}$$
$$+\sum_{d\mid v}\mu(d)\frac{N}{dw}\sum_{|h|\le W^{1+\epsilon}d/N}\widehat{G}\left(\frac{hN}{dw}\right)\sum_{(a,w)=1}e\left(\frac{abd\overline{c\ell m v}}{w}+\frac{ah}{w}\right)$$
$$+O(Q^{-100}).$$

The first and third terms contribute acceptable amounts, so consider the second term. The sum over a is the Ramanujan sum $c_w(hc\ell mv+bd)$, and since $c\ell m$ does not divide b the argument of the Ramanujan sum is nonzero. Following the proof of Proposition 5.2, we therefore obtain a bound of

$$\mathscr{E}_2' \ll \frac{Q^{3/2+\epsilon}(BCLM)^{1/2}}{N^{1/2}}. \tag{7-2}$$

By the reasoning immediately after Proposition 5.2, the bound (7-2) allows us to assume $N \leq MQ^{-2\epsilon}$, so that $N \leq Q^{1/2}$, regardless of whether $CLM \ll B$. In the case $CLM \ll B$, the bound (7-2) becomes

$$\mathscr{E}_2' \ll \frac{Q^{3/2+\epsilon}B}{N^{1/2}} \ll Q^{3/2+\epsilon}B \ll Q^{3/2+\theta_2+\epsilon} \ll Q^{2-\epsilon},$$

which of course is acceptable.

At this point we can follow the rest of the proof in Section 5. We change variables $bn \to n$, and the rest follows mutatis mutandis (it is important that with $N \ll Q^{1/2}$ we have $BN \ll Q^{1-\epsilon}$). This completes the proof of Lemma 3.4.

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