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Algebraic cycles on genus-2 modular fourfolds

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To the memory of my father

This paper studies universal families of stable genus-2 curves with level structure. Among other things, it is shown that the $(1,1)$-part is spanned by divisor classes, and that there are no cycles of type $(2,2)$ in the third cohomology of the first direct image of $\mathbb{C}$ under projection to the moduli space of curves. Using this, it shown that the Hodge and Tate conjectures hold for these varieties.

One of the goals of this article is to extend some results from Shioda's study [1972] of elliptic modular surfaces to families of genus-2 curves. We recall that elliptic modular surfaces $f : \bar{C}_{1,1}[n] \to \bar{M}_{1,1}[n]$ are the universal families of elliptic curves over modular curves. Among other things, Shioda showed that $\bar{C}_{1,1}[n]$ has maximal Picard number in the sense that $H^{1,1}(\bar{C}_{1,1}[n])$ is spanned by divisors. He also showed that the Mordell–Weil rank is zero. A related property, observed later by Viehweg and Zuo [2004], is that a certain Arakelov inequality becomes equality. As they observe, this is equivalent to the map

$$f_*\omega_{\bar{C}_{1,1}[n]/\bar{M}_{1,1}[n]} \to \Omega^1_{\bar{M}_{1,1}[n]}(\log D) \otimes R^1 f_* \mathcal{O}_{\bar{C}_{1,1}[n]}$$

induced by the Kodaira–Spencer class being an isomorphism. The divisor D is the discriminant of f.

In this paper, we study universal curves $f' : \bar{C}_2[\Gamma] \to \bar{M}_2[\Gamma]$ over the moduli space of stable genus-2 curves with generalized level structure. The level Γ is a finite-index subgroup of the mapping class group Γ_2. The classical level n-structures correspond to the case where Γ is the preimage $\widetilde{\Gamma}(n)$ of the principal congruence subgroup $\Gamma(n) \subset \mathrm{Sp}_4(\mathbb{Z})$. We fix a suitable nonsingular birational model $f : X \to Y$ for f'. Let $D \subset Y$ be the discriminant, and $U = Y - D$. We show that, as before, for a classical level, the Mordell–Weil rank of $\mathrm{Pic}^0(X) \to Y$ is zero and $H^{1,1}(X)$ is spanned by divisors. These results are deduced from Raghunathan's vanishing theorem [1967]. We also prove an analogue of Viehweg–Zuo that the map

$$\Omega^1_Y(\log D) \otimes f_*\omega_{X/Y} \to \Omega^2_Y(\log D) \otimes R^1 f_* \mathcal{O}_X$$

is an isomorphism. We will see that this implies that there are no cycles of type $(2,2)$ in the mixed Hodge structure $H^3(U, R^1 f_*\mathbb{C})$. As an application, we deduce that the Hodge conjecture holds for X. We also show that the Tate conjecture holds for X for a classical level using, in addition, Faltings' p-adic Hodge theorem [1988] and Weissauer's work [1988] on Siegel modular threefolds.

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If X is a complex variety, then unless indicated otherwise, sheaves should be understood as sheaves on the associated analytic space X^{an}.

1. Hodge theory of semistable maps

We start with some generalities. By a log pair $\mathcal{X} = (X, E)$, we mean a smooth variety X together with a divisor with simple normal crossings E. We usually denote log pairs by the symbols $\mathcal{X}, \mathcal{Y}, \ldots$ with corresponding varieties $X, Y, \ldots$. Given $\mathcal{X}$, set

$$\Omega^1_{\mathcal{X}} = \Omega^1_X(\log E) \quad \text{and} \quad T_{\mathcal{X}} = (\Omega^1_{\mathcal{X}})^\vee.$$

Recall that a semistable map $f : (X, E) \to (Y, D)$ of log pairs is a morphism $f : X \to Y$ such that $f^{-1}D = E$ and étale locally it is given by

$$\begin{aligned} y_1 &= x_1 \cdots x_{r_1}, \\ &\vdots \\ y_k &= x_{r_{k-1}+1} \cdots x_{r_k}, \\ y_{k+1} &= x_{r_k+1}, \\ &\vdots \end{aligned}$$

where $y_1 \cdots y_k = 0$ and $x_1 \cdots x_{r_k} = 0$ are local equations for D and E respectively. We will say f is log étale if it is semistable of relative dimension zero. (This is a bit more restrictive than the usual definition).

Fix a projective semistable map $f : (X, E) \to (Y, D)$. The map restricts to a smooth projective map f^o from $\widetilde{U} = X - E$ to $U = Y - D$. Let

$$\Omega^i_{\mathcal{X}/\mathcal{Y}} = \Omega^i_{X/Y}(\log E).$$

The sheaf $\mathcal{L}^m = R^m f^o_* \mathbb{Q}$ is a local system, which is part of a variation of Hodge structure. Let $V^m = R^m f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$ with filtration F induced by the stupid filtration $R^m f_* \Omega^{\geq p}_{\mathcal{X}/\mathcal{Y}}$. It carries an integrable logarithmic connection

$$\nabla : V^m \to \Omega^1_{\mathcal{Y}} \otimes V^m$$

such that $\ker \nabla|_U = \mathbb{C}_U \otimes \mathcal{L}^m$. Griffiths transversality

$$\nabla(F^p) \subseteq \Omega^1_{\mathcal{Y}} \otimes F^{p-1}$$

holds. The relative de Rham to Hodge spectral sequence

$$E_1 = R^j f_* \Omega^i_{\mathcal{X}/\mathcal{Y}} \Rightarrow R^{i+j} f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$$

degenerates at E_1 by [Illusie 1990, Corollaire 2.6; Fujisawa 1999, Theorem 6.10]. Therefore

$$\mathrm{Gr}^p_F V^m \cong R^{m-p} f_* \Omega^p_{\mathcal{X}/\mathcal{Y}}.$$

The Kodaira–Spencer class

$$\kappa : \mathcal{O}_Y \to \Omega^1_{\mathcal{Y}} \otimes R^1 f_* T_{\mathcal{X}/\mathcal{Y}} \tag{1}$$

is given as the transpose of the map

$$T_{\mathcal{Y}} \to R^1 f_* T_{\mathcal{X}/\mathcal{Y}}$$

induced by the sequence

$$0 \to T_{\mathcal{X}/\mathcal{Y}} \to T_{\mathcal{X}} \to f^* T_{\mathcal{Y}} \to 0.$$

Proposition 1.1. *The associated graded*

$$\mathrm{Gr}(\nabla) : R^{m-p} f_* \Omega^p_{\mathcal{X}/\mathcal{Y}} \to \Omega^1_{\mathcal{Y}} \otimes R^{m-p+1} f_* \Omega^{p-1}_{\mathcal{X}/\mathcal{Y}}$$

coincides with cup product and contraction with κ.

Proof. In the nonlog setting, this is stated in [Katz 1970, Theorem 3.5], and the argument indicated there extends to the general case. □

By [Arapura 2005], we can give $H^i(U, R^j f_* \mathbb{Q})$ a mixed Hodge structure by identifying it with the associated graded of $H^{i+j}(\widetilde{U}, \mathbb{Q})$ with respect to the Leray filtration. It can also be defined intrinsically using mixed Hodge module theory, but the first description is more convenient for us. We will need a more precise description of the Hodge filtration. We define a complex

$$K_{\mathcal{X}/\mathcal{Y}}(m, p) = [R^{m-p} f_* \Omega^p_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\kappa} \Omega^1_{\mathcal{Y}} \otimes R^{m-p+1} f_* \Omega^{p-1}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\kappa} \Omega^2_{\mathcal{Y}} \otimes R^{m-p+2} f_* \Omega^{p-2}_{\mathcal{X}/\mathcal{Y}} \cdots].$$

Proposition 1.2. $\mathrm{Gr}^p_F H^i(U, R^j f_* \mathbb{C}) \cong H^i(K_{\mathcal{X}/\mathcal{Y}}(j, p))$.

Proof. (Compare with [Zucker 1979, 2.16].) Define a filtration

$$L^i \Omega^\bullet_{\mathcal{X}} = \mathrm{im}\, f^* \Omega^i_{\mathcal{Y}} \otimes \Omega^\bullet_{\mathcal{X}}.$$

Then

$$\mathrm{Gr}^i_L \, \Omega^\bullet_{\mathcal{X}} = f^* \Omega^i_{\mathcal{Y}} \otimes \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}[-i]$$

from which we deduce that

$$\mathrm{Gr}^i_L \, R f_* \Omega^\bullet_{\mathcal{X}} \cong \Omega^i_{\mathcal{Y}} \otimes R f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}[-i]. \tag{2}$$

Therefore, we obtain a spectral sequence

$${}_L E_1^{i,j} = \mathcal{H}^{i+j}(\mathrm{Gr}^i_L \, R f_* \Omega^\bullet_{\mathcal{X}}) \cong \Omega^i_{\mathcal{Y}} \otimes R^j f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}} = \Omega^i_{\mathcal{Y}} \otimes V^j \Rightarrow R^{i+j} f_* \Omega^\bullet_{\mathcal{X}}$$

Recall that to L we can associate a new filtration $\mathrm{Dec}(L)$ [Deligne 1971] such that

$${}_{\mathrm{Dec}(L)} E_0^{i,j} \cong {}_L E_1^{2i+j,-i}.$$

Therefore we obtain a quasiisomorphism

$$\mathrm{Gr}^i_{\mathrm{Dec}(L)} \, R f_* \Omega^\bullet_{\mathcal{X}} \xrightarrow{\sim} \Omega^\bullet_{\mathcal{Y}} \otimes V^{-i}[i]. \tag{3}$$

This becomes a map of filtered complexes with respect to the filtration induced by Hodge filtration $F^p = \Omega^{\geq p}_{\mathcal{X}}$. On the right of (3), it becomes

$$F^p \Omega^\bullet_{\mathcal{Y}} \otimes V^{-i} = F^p V^{-i} \to \Omega^1_{\mathcal{Y}} \otimes F^{p-1} V^{-i} \to \cdots.$$

The relative de Rham to Hodge spectral sequence

$$_F E_1 = R^j f_* \Omega^i_{\mathcal{X}} \Rightarrow R^{i+j} f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$$

degenerates at E_1 [Illusie 1990, Corollaire 2.6; Fujisawa 1999, Theorem 6.10]. Therefore by [Deligne 1971, 1.3.15], we can conclude that (3) is a filtered quasiisomorphism.

The spectral sequence associated to the filtration induced by $\mathrm{Dec}(L)$ on $R\Gamma(Rf_*\Omega^\bullet_{\mathcal{X}})$

$$_{\mathrm{Dec}(L)} E_1^{i,j} = H^{2i+j}(Y, \Omega^\bullet_{\mathcal{Y}} \otimes V^{-i}) = H^{2i+j}(U, R^{-i} f_* \mathbb{C})$$

coincides with Leray after reindexing. Therefore this degenerates at the first page by [Deligne 1968]. The above arguments plus [Deligne 1971, 1.3.17] show that F-filtration on the $H^{2i+j}(Y, \Omega^\bullet_{\mathcal{Y}} \otimes V^{-i})$ coincides with the filtration on $_{\mathrm{Dec}(L)} E_\infty$, which is the Hodge filtration on $H^{2i+j}(U, R^{-i} f_* \mathbb{C})$. The proposition follows immediately from this. □

One limitation of the notion of semistability is that it is not stable under base change. In order to handle this, we need to work in the broader setting of log schemes [Kato 1989]. We recall that a log scheme consists of a scheme X and a sheaf of monoids M on $X_{\text{ét}}$ together with a multiplicative homomorphism $\alpha : M \to \mathcal{O}_X$ such that α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$. A log pair (X, E) gives rise to a log scheme where M is the sheaf of functions invertible outside of E. If $f : (X, E) \to (Y, D)$ is semistable, and $\pi : (Y', D') \to (Y, D)$ is log étale in our sense, then $X' = X \times_Y Y'$ can be given the log structure pulled back from Y'. Then $X' \to Y'$ becomes a morphism $\mathcal{X}' \to \mathcal{Y}'$ of log schemes, which is log smooth and exact. Logarithmic differentials can be defined for log schemes [Kato 1989], so the complexes $K_{\mathcal{X}'/\mathcal{Y}'}(m, p)$ can be constructed exactly as above. Since π is log étale, we easily obtain:

Lemma 1.3. *With the above notation,* $\pi^* K_{\mathcal{X}/\mathcal{Y}}(m, p) \cong K_{\mathcal{X}'/\mathcal{Y}'}(m, p)$.

Let us spell things out for curves. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is semistable with relative dimension 1. From

$$0 \to \Omega^1_{\mathcal{Y}} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\mathcal{Y}} \to 0$$

we get an isomorphism

$$\Omega^1_{\mathcal{X}/\mathcal{Y}} = \det \Omega^1_{\mathcal{X}} \otimes (\det \Omega^1_{\mathcal{Y}})^{-1} \cong \omega_{X/Y}.$$

The complex $K_{\mathcal{X}/\mathcal{Y}}(1, i)$ satisfies

$$K_{\mathcal{X}/\mathcal{Y}}(1, i) = [\Omega^{i-1}_{\mathcal{Y}} \otimes f_* \omega_{X/Y} \to \Omega^i_{\mathcal{Y}} \otimes R^1 f_* \mathcal{O}_X],$$

where the first term sits in degree $i - 1$. We note that this complex, which we now denote by $K_{X/Y}(1, i)$, can be defined when X/Y is a semistable curve in the usual sense (a proper flat map of relative dimension 1 with reduced connected nodal geometric fibres). In general, any such curve carries a natural log structure [Kato 2000], and the differential of this complex can be interpreted as a cup product with the associated Kodaira–Spencer class. Consequently, given a map $\pi : X' \to X$ of curves over Y, we get an induced map of complexes $\pi^* : K_{X/Y}(1, i) \to K_{X'/Y'}(1, i)$. Finally, we note that these constructions can be extended to Deligne–Mumford stacks, such as the moduli stack of stable curves $\overline{\mathbb{M}}_g$, without difficulty.

2. Consequences of Raghunathan's vanishing

Let M_g be the moduli space of smooth projective curves of genus g, let $M_{g,n}$ be the moduli space of smooth genus-g curves with n marked points, and let A_2 be the moduli space of principally polarized g-dimensional abelian varieties. The symbols $\mathbb{M}_g$, $\mathbb{A}_g$ etc. will be reserved for the corresponding moduli stacks. We note that $\dim M_2 = 3$. The Torelli map $\tau : M_2 \to A_2$ is injective and the image of M_2 is the complement of the divisor parametrizing products of two elliptic curves. As an analytic space, M_2^{an} is a quotient of the Teichmüller space T_2 by the mapping class group Γ_2. Given a finite-index subgroup $\Gamma \subset \Gamma_2$, let $M_2[\Gamma] = T_2/\Gamma$. We view this as the moduli space of curves with generalized level structure. When $\Gamma = \widetilde{\Gamma}(n)$ is the preimage of the principal congruence subgroup $\Gamma(n) \subseteq \mathrm{Sp}_4(\mathbb{Z})$ under the canonical map $\Gamma_2 \to \mathrm{Sp}_4(\mathbb{Z})$, the space $M_2[n] := M_2[\widetilde{\Gamma}(n)]$ is the moduli space of curves with classical (or abelian or Jacobi) level n-structure. It is smooth and fine as soon as $n \geq 3$, and defined over the cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$. More generally $M_2[\Gamma]$ is smooth, and defined over a number field, as soon as $\Gamma \subseteq \widetilde{\Gamma}(n)$ with $n \geq 3$. We refer to Γ as *fine*, when the last condition holds. Torelli extends to a map $M_2[n] \to A_2[n]$ to the moduli space of abelian varieties with level n-structure.

Let $\overline{M}_2$ denote the Deligne–Mumford compactification of M_2. The boundary divisor Δ consists of a union of two components $\Delta_0 \cup \Delta_1$. The generic point of Δ_0 corresponds to an irreducible curve with a single node, and the generic point of Δ_1 corresponds to a union of two elliptic curves meeting transversally. Let $\pi : \overline{M}_2[\Gamma] \to \overline{M}_2$ denote the normalization of $\overline{M}_2$ in the function field of $M_2[\Gamma]$. When $\Gamma = \widetilde{\Gamma}(n)$, we denote this by $\overline{M}_2[n]$. On the other side $A_2[n]$ has a unique smooth toroidal compactification, first constructed by Igusa, and τ extends to an isomorphism between $\overline{M}_2[n]$ and the Igusa compactification [Namikawa 1980, §9]. The space $\overline{M}_2[\Gamma]$ is smooth, when $\Gamma = \widetilde{\Gamma}(n)$, $n \geq 3$, and in some other cases [Pikaart and de Jong 1995]. Suppose that $n \geq 3$. The boundary $D = \overline{M}_2[n] - M_2[n]$ is a divisor with normal crossings. Let $D_i = \pi^{-1}\Delta_i$. Since D_1 parametrizes unordered pairs of (generalized) elliptic curves with level structure, its irreducible components are isomorphic to symmetric products $\overline{M}_{1,1}[n] \times \overline{M}_{1,1}[n]/S_2$ of the modular curve of full level n. Let $\overline{C}_{1,m}[n]/\overline{M}_{1,m}[n]$ denote the pullback of the universal elliptic curve under the canonical map $\overline{M}_{1,m}[n] \to \overline{\mathbb{M}}_{1,m}$. The components of D_0 are birational to the elliptic modular surfaces $\overline{C}_{1,1}[n]$ [Oda and Schwermer 1990, §1.4].

Given a fine level structure Γ, let $\overline{C}_2[\Gamma] \to \overline{M}_2[\Gamma]$ be the pullback of the universal curve from $\overline{\mathbb{M}}_2$. The space $\overline{C}_2[\Gamma]$ will be singular [Boggi and Pikaart 2000, Proposition 1.4], so we will replace it with a suitable birational model $f : X \to Y$ whose construction we now explain. If $\Gamma = \widetilde{\Gamma}[n]$, we set $Y = \overline{M}_2[n]$. As noted above, Y is smooth. For other Γ's, we choose a desingularization $Y \to \overline{M}_2[\Gamma]$ which is an isomorphism over $M_2[\Gamma]$ and such that boundary divisor D has simple normal crossings. We have a morphism $Y \to \overline{\mathbb{M}}_2$ to the moduli stack, which is log étale. It follows in particular that $\Omega^1_{\overline{\mathbb{M}}_2}(\log \Delta)$ pulls back to $\Omega^1_Y(\log D)$. The space Y will carry a stable curve $X' \to Y$ obtained by pulling back the universal family over $\overline{\mathbb{M}}_2$. The space X' will be singular, however:

Lemma 2.1. (a) *X' will have rational singularities.*

(b) *There exists a desingularization $\pi : X \to X'$ such that $X \to Y$ is semistable.*

(c) *The map $\pi : X \to X'$ can be chosen so as to have the following additional property. After extending scalars to $\overline{\mathbb{Q}}$, let E be a component of an exceptional divisor of π. Then*:

(i) *If $\pi(E)$ is a point, E is a rational variety.*

(ii) *If $\pi(E) = C$ is a curve, there is a map $E \to C$ such that the pullback under a finite map $\widetilde{C} \to C$ is birational to $\mathbb{P}^2 \times \widetilde{C}$.*

(iii) *If $\pi(E)$ is a surface, there is a map $E \to D_i$, for some i, such that the pullback of E to an étale cover $\widetilde{D}_i \to D_i$ is birational to $\mathbb{P}^1 \times \widetilde{D}_i$.*

(iv) *If $\dim \pi(E) = 3$, then $E \to \pi(E)$ is birational.*

(d) $K_{X/Y}(1, i) \cong K_{X'/Y}(1, i)$.

Proof. The singularities of X' are analytically of the form $xy = t_1^a t_2^b t_3^c$. These are toroidal singularities, in the sense that is local analytically, or étale locally, isomorphic to a toric variety. (This is a bit weaker than the notion of toroidal embedding in [Kempf et al. 1973], but it is sufficient for our needs.) Such singularities are well known to be rational; see [Kempf et al. 1973; Viehweg 1977]. Item (b) follows from [de Jong 1996, Proposition 3.6].

To prove (c), we need to recall some details of the construction of X from [de Jong 1996]. First, as explained in the proof of [loc. cit., Lemma 3.2], one blows up a codimension-2 component $T \subset X'_{\text{sing}}$. The locus T is an étale cover of some component D_i. Furthermore, from the description in [loc. cit.] we can see that T is compatible with the toroidal structure. Consequently, we can find a toric variety V with torus fixed point 0, and an étale local isomorphism between X' and $V \times T$, over the generic point of T, which takes T to $\{0\} \times T$. This shows that, over the generic point, the exceptional divisor E to T is étale locally a product of T with a toric curve. So we get case (iii). Note that this step is repeated until the X'_{sing} has codimension at least 3. One does further blow ups to obtain X. An examination of the proof of [loc. cit., Proposition 3.6] shows that the required blow ups are also compatible with the toroidal structure in the previous sense. If the centre of the blow up is a point, then the exceptional divisor is toric and we have case (i). If the centre is a smooth curve C, we obtain case (ii). The last item (iv) is automatic for blow ups.

By the remarks at the end of the last section, there is a commutative diagram marked with solid arrows

$$\begin{array}{ccc} \Omega_{\mathcal{Y}}^{i-1} \otimes f_* \omega_{X/Y} & \longrightarrow & \Omega_{\mathcal{Y}}^{i} \otimes R^1 f_* \mathcal{O}_X \\ {\scriptstyle \pi^*} \uparrow \downarrow {\scriptstyle \pi_*} & & {\scriptstyle \pi^*} \uparrow \downarrow {\scriptstyle \pi_*} \\ \Omega_{\mathcal{Y}}^{i-1} \otimes f_* \omega_{X'/Y} & \longrightarrow & \Omega_{\mathcal{Y}}^{i} \otimes R^1 f_* \mathcal{O}_{X'} \end{array}$$

Since X' has rational singularities, the dotted arrows labelled with π_* are isomorphisms, and these are left inverse to the arrows labelled with π^*. Therefore π^* are also isomorphisms, and this proves (d). □

We refer to $f : X \to Y$ constructed in Lemma 2.1 as *a good model* of $\overline{C}_2[\Gamma] \to \overline{M}_2[\Gamma]$. We let $U = Y - D$, $E = f^{-1}D$, and $\widetilde{U} = X - E$ as above.

Corollary 2.2. *After extending scalars to $\overline{\mathbb{Q}}$, let E_1 be an irreducible component of E for a classical fine level $\widetilde{\Gamma}(n)$. Then there exists a dominant rational map $\widetilde{E}_1 \dashrightarrow E_1$, where $\widetilde{E}_1$ is one of*

(1) $\overline{C}_{1,1}[n] \times \overline{M}_{1,1}[n]$,

(2) $\overline{C}_{1,2}[n]$,

(3) $\overline{C}_{1,1}[m] \times \mathbb{P}^1$ *for some* $n \mid m$,

(4) $\overline{M}_{1,1}[m] \times \overline{M}_{1,1}[m] \times \mathbb{P}^1$ *for some* $n \mid m$,

(5) *a product of* $\mathbb{P}^2$ *with a curve*,

(6) $\mathbb{P}^3$.

Proof. The preimage of D_1 in $\overline{C}_2[n]$ parametrizes a union of pairs of (generalized) elliptic curves with level structure together with a point on the union. It follows that a component of E dominating D_1 is dominated by $\overline{C}_{1,1}[n] \times \overline{M}_{1,1}[n]$. The preimage of D_0 in $\overline{C}_2[n]$ is a family of nodal curves over D_0; its normalization is $\overline{C}_{1,2}[n]$. Therefore a component of E dominating D_0 is birational to $\overline{C}_{1,2}[n]$. Case (3) follows from Lemma 2.1(ciii) once we observe that an étale cover of $\overline{C}_{1,1}[n]$ is dominated by $\overline{C}_{1,1}[m]$ for some $n \mid m$. This is because we have a surjection of étale fundamental groups

$$\pi_1^{\text{ét}}(M_{1,1}[n]) \cong \pi_1^{\text{ét}}(C_{1,1}[n]) \to \pi_1^{\text{ét}}(\overline{C}_{1,1}[n])$$

[Cox and Zucker 1979, Theorem 1.36], and $\{M_{1,1}[m]\}_{n \mid m}$ is cofinal in the set of étale covers of $M_{1,1}[n]$. Case (4) is similar. The remaining cases follow immediately from the lemma. □

Proposition 2.3. *When $\Gamma = \widetilde{\Gamma}(n)$, with $n \geq 3$, $H^1(U, R^1 f_* \mathbb{C}) = 0$.*

Proof. As explained above, $Y = \overline{M}_2[n] = \bar{A}_2[n]$ and $U = A_2[n] - D_1^o$, where $D_1^o = D_1 - D_0$. Let $g : \mathrm{Pic}^0(X/Y) \to Y$ denote the relative Picard scheme. Then $R^1 f_* \mathbb{C} = R^1 g_* \mathbb{C}|_U$. We have an exact sequence

$$H^1(A_2[n], R^1 g_* \mathbb{C}) \to H^1(U, R^1 f_* \mathbb{C}) \to H^0(D_1^o, R^1 g_* \mathbb{C}).$$

The group on the left vanishes by Raghunathan [1967, p. 423, Corollary 1]. The local system $R^1 g_* \mathbb{C}|_{D_1^o}$ decomposes into a sum of two copies of the standard representation of the congruence group $\Gamma(n) \subset \mathrm{SL}_2(\mathbb{Z})$. Therefore it has no invariants. Consequently, $H^1(U, R^1 f_* \mathbb{C}) = 0$ as claimed. □

Lemma 2.4. *Let η denote the generic point of Y. Then we have an exact sequence*

$$0 \to \mathrm{Pic}(U) \xrightarrow{s} \mathrm{Pic}(\widetilde{U}) \xrightarrow{r} \mathrm{Pic}(X_\eta) \to 0,$$

where r and s are the natural maps.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}(U)^* & \longrightarrow & \mathbb{C}(U)^* \oplus \mathbb{C}(\widetilde{U})^* & \longrightarrow & \mathbb{C}(\widetilde{U})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{div}} & & \downarrow{\scriptstyle \mathrm{div+div}} & & \downarrow{\scriptstyle \mathrm{div}} & & \\
0 & \longrightarrow & \mathrm{Div}(U) & \xrightarrow{s'} & \mathrm{Div}(\widetilde{U}) & \xrightarrow{r'} & \mathrm{Div}(X_\eta) & \longrightarrow & 0
\end{array}$$

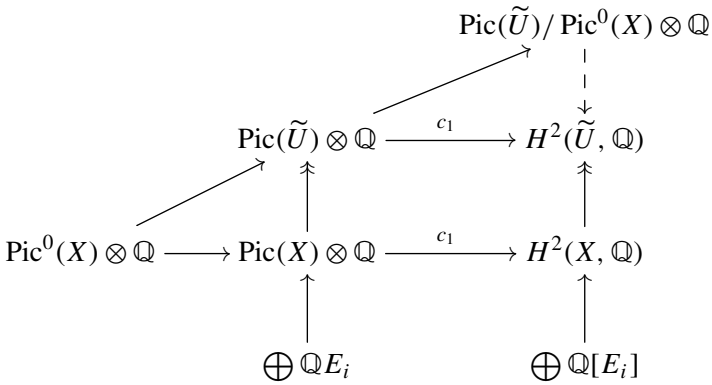
The map r' is surjective because any codimension-1 point of X_η is the restriction of its scheme-theoretic closure. A straightforward argument also shows that s' is injective and $\ker r' = \operatorname{im} s'$. The lemma now follows from the snake lemma. □

Lemma 2.5. *The first Chern class map induces injections*

$$\operatorname{Pic}(\widetilde{U})/\operatorname{Pic}^0(X) \otimes \mathbb{Q} \hookrightarrow H^2(\widetilde{U}, \mathbb{Q}), \tag{4}$$

$$\operatorname{Pic}(U)/\operatorname{Pic}^0(Y) \otimes \mathbb{Q} \hookrightarrow H^2(U, \mathbb{Q}). \tag{5}$$

Proof. To prove (4), we observe that there is a commutative diagram with exact lines



The existence and injectivity of the dotted arrow follows from this diagram. Existence and injectivity of the map of (5) is proved similarly. □

We refer to the group of $\mathbb{C}(Y)$ rational points of $\operatorname{Pic}^0(X_\eta)$ as the Mordell–Weil group of X/Y.

Theorem 2.6. *Let $f : X \to Y$ be a good model of $\overline{C}_2[n] \to \overline{M}_2[n]$, where $n \geq 3$:*

(a) *The space $H^{1,1}(X)$ is spanned by divisors.*

(b) *The rank of the Mordell–Weil group of X/Y is zero.*

Proof. We have an sequence

$$\bigoplus \mathbb{Q}[E_i] \to H^2(X) \to \operatorname{Gr}_2^W H^2(\widetilde{U}) \to 0$$

of mixed Hodge structures. So for (a), it suffices to show that the (1, 1)-part of rightmost Hodge structure is spanned by divisors. The Leray spectral sequence together with Proposition 2.3 gives an exact sequence

$$0 \to H^2(U, f_*\mathbb{Q}) \to H^2(\widetilde{U}) \to H^0(U, R^2 f_*\mathbb{Q}) \to 0$$

of mixed Hodge structures. Therefore, we get an exact sequence

$$0 \to \operatorname{Gr}_2^W H^2(U, f_*\mathbb{Q}) \to \operatorname{Gr}_2^W H^2(\widetilde{U}) \to \operatorname{Gr}_2^W H^0(U, R^2 f_*\mathbb{Q}) \to 0.$$

The space on the right is 1-dimensional and spanned by the class of any horizontal divisor. We can identify

$$\mathrm{Gr}_2^W H^2(U, f_*\mathbb{Q}) = \mathrm{Gr}_2^W H^2(U, \mathbb{Q})$$

with a quotient of $H^2(Y)$. Weissauer [1988, p. 101] showed that $H^{1,1}(Y)$ is spanned by divisors. This proves (a).

By Lemma 2.4, we have isomorphisms

$$\mathrm{Pic}(X_\eta) \otimes \mathbb{Q} \cong \frac{\mathrm{Pic}(\widetilde{U})}{\mathrm{Pic}(U)} \otimes \mathbb{Q} \cong \frac{\mathrm{Pic}(\widetilde{U})/\mathrm{Pic}^0(X)}{\mathrm{Pic}(U)/\mathrm{Pic}^0(Y)} \otimes \mathbb{Q}$$

and, by Lemma 2.5, the last group embeds into $H^2(\widetilde{U}, \mathbb{Q})/H^2(U, \mathbb{Q})$. Therefore, $\mathrm{Pic}^0(X_\eta) \otimes \mathbb{Q}$ embeds into

$$\frac{\ker[H^2(\widetilde{U}, \mathbb{Q}) \to H^2(X_t, \mathbb{Q})]}{H^2(U, \mathbb{Q})} \cong H^1(U, R^1 f_*\mathbb{Q}) = 0,$$

where $t \in U$. For the first isomorphism, we use the fact the Leray spectral sequence over U degenerates by [Deligne 1968]; the second is Proposition 2.3. □

3. Key vanishing

Let us fix a fine level structure $\Gamma \subseteq \Gamma_2$. We do not assume that it is classical. Choose a good model $f : X \to Y$ for $\bar{C}_2[\Gamma] \to \overline{M}_2[\Gamma]$, with $U, E, \widetilde{U}$ as above. Our goal in this section is to establish the vanishing of $\mathrm{Gr}_F^2 H^3(U, R^1 f_*\mathbb{C})$. This is the key fact which, when combined with Lemma 4.3 proved later on, will allow us to prove the Hodge conjecture for X.

Theorem 3.1. *$K_{X/Y}(1, 2)$ is quasiisomorphic to 0.*

Proof. The moduli stack $\overline{\mathbb{M}}_2$ is smooth and proper, the boundary divisor has normal crossings, and the universal curve is semistable. So we can define an analogue of $K(1, 2)$ on it. Since the canonical map $Y \to \overline{\mathbb{M}}_2$ is log étale, $K_{X/Y}(1, 2)$ is the pullback of the corresponding complex on the moduli stack. So we replace Y by $\overline{\mathbb{M}}_2$ and X by the universal curve $\overline{\mathbb{M}}_{2,1}$.

Set

$$H = f_*\omega_{X/Y}.$$

By duality, we have an isomorphism

$$H \cong R^1 f_* \mathcal{O}_X^\vee.$$

Thus the Kodaira–Spencer map

$$H \to \Omega^1_{\mathcal{Y}} \otimes H^\vee$$

induces an adjoint map

$$(H)^{\otimes 2} \to \Omega^1_{\mathcal{Y}}.$$

This factors through the symmetric power to yield a map

$$S^2 H \to \Omega^1_{\mathcal{Y}}. \tag{6}$$

After identifying $\overline{\mathbb{M}}_2 \cong \overline{\mathbb{A}}_2$, and $\mathrm{Pic}^0(X/Y)$ with the universal semiabelian variety, we see that (6) is an isomorphism by [Faltings and Chai 1990, Chapter IV, Theorem 5.7].

With the above notation $K(1,2)$ can be written as

$$\Omega^1_{\mathcal{Y}} \otimes H \to \Omega^2_{\mathcal{Y}} \otimes H^\vee.$$

We need to show that the map in this complex is an isomorphism. It is enough to prove that the map is surjective, because both sides are locally free of the same rank. To do this, it suffices to prove that the adjoint map

$$\kappa' : \Omega^1_{\mathcal{Y}} \otimes (H)^{\otimes 2} \to \Omega^2_{\mathcal{Y}}$$

is surjective. Let κ'' denote the restriction of κ' to $\Omega^1_{\mathcal{Y}} \otimes S^2 H$. We can see that we have a commutative diagram

$$\begin{array}{ccc} \Omega^1_{\mathcal{Y}} \otimes S^2 H & \xrightarrow{\kappa''} & \Omega^2_{\mathcal{Y}} \\ \downarrow \cong & & \downarrow = \\ \Omega^1_{\mathcal{Y}} \otimes \Omega^1_{\mathcal{Y}} & \xrightarrow{\wedge} & \Omega^2_{\mathcal{Y}} \end{array}$$

This implies that κ'', and therefore κ', is surjective. □

From Proposition 1.2, we obtain:

Corollary 3.2. $\mathrm{Gr}^2_F H^*(U, R^1 f_* \mathbb{C}) = 0.$

Remark 3.3. The referee has pointed out that for a classical level, a short alternative proof of the corollary can be deduced using Faltings' BGG resolution as follows. It suffices to prove $\mathrm{Gr}^2_F H^*(A_2[\Gamma], R^1 f'_* \mathbb{C}) = 0$, where f' is the universal abelian variety, because the restriction map to $\mathrm{Gr}^2_F H^*(U, R^1 f_* \mathbb{C})$ can be seen to be surjective. By [Faltings and Chai 1990, Chapter VI, Theorem 5.5] (see also [Petersen 2015, Theorem 2.4] for a more explicit statement) $\mathrm{Gr}^a_F H^*(A_2[\Gamma], R^1 f_* \mathbb{C})$ is zero unless $a \in \{0, 1, 3, 4\}$.

4. Hodge and Tate

Given a smooth projective variety X defined over $\mathbb{C}$, a Hodge cycle of degree $2p$ is an element of $\mathrm{Hom}_{\mathrm{HS}}(\mathbb{Q}(-p), H^{2p}(X, \mathbb{Q}))$, and given a smooth projective variety X defined over a finitely generated field K, an ℓ-adic Tate cycle of degree $2p$ is an element of $\sum H^{2p}_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(p))^{\mathrm{Gal}(\overline{K}/L)}$ as L/K runs over finite extensions. The image of the cycle maps from $CH^p(X) \otimes \mathbb{Q}$ or $CH^p(X \otimes \overline{K}) \otimes \mathbb{Q}_\ell$ lands in these spaces. We say that the Hodge or Tate conjecture holds for X (in a given degree) if the space of Hodge or Tate cycles (of the given degree) are spanned by algebraic cycles. Here is the main result of the paper:

Theorem 4.1. *Let $f : X \to Y$ be a good model of $\bar{C}_2[\Gamma] \to \overline{M}_2[\Gamma]$, where $\Gamma \subseteq \Gamma_2$ is a fine level*:

(A) *The Hodge conjecture holds for X.*

(B) *When* $\Gamma = \widetilde{\Gamma}(n)$ *is a classical level, the Tate conjecture holds for* X.

We deduce this with the help of the following lemmas.

Lemma 4.2. *Let* X_1 *and* X_2 *be smooth projective varieties defined over a finitely generated field*:

(1) *If* X_1 *and* X_2 *are birational, then the Tate conjecture holds in degree* 2 *for* X_1 *if and only if it holds for* X_2.

(2) *If the Tate conjecture holds in degree* 2 *for* X_1 *and there is a dominant rational map* $X_1 \dashrightarrow X_2$, *then the Tate conjecture holds in degree* 2 *for* X_2; *if the Tate conjecture holds in degree* $2d$ *for* X_1 *and there is a surjective regular map* $X_1 \dashrightarrow X_2$, *then the Tate conjecture holds in degree* $2d$ *for* X_2.

(3) *If the Tate conjecture holds in degree* 2 *for* X_i, *then the Tate conjecture holds in degree* 2 *for* $X_1 \times X_2$.

Proof. See [Tate 1994, Theorem 5.2]. □

Lemma 4.3. *Let* $f : (X, E) \to (Y, D)$ *be a semistable map of smooth projective varieties with* $\dim Y = 3$ *and* $\dim X = 4$. *Suppose that*

$$\mathrm{Gr}_F^2 H^3(U, R^1 f_* \mathbb{C}) = 0,$$

where $U = Y - D$. *Then the Hodge conjecture holds for* X.

Proof. Also let $\widetilde{U} = X - E$. Since X is a fourfold, it is enough to prove that Hodge cycles in $H^4(X)$ are algebraic. The other cases follow from the Lefschetz (1, 1) and hard Lefschetz theorems. Using the main theorems of [Deligne 1968; Arapura 2005], and the semisimplicity of the category of polarizable Hodge structures, we have a noncanonical isomorphism of Hodge structures

$$\mathrm{Gr}_4^W H^4(\widetilde{U}) \cong \underbrace{\mathrm{Gr}_4^W H^4(U, f_* \mathbb{Q})}_{I} \oplus \underbrace{\mathrm{Gr}_4^W H^3(U, R^1 f_* \mathbb{Q})}_{II} \oplus \underbrace{\mathrm{Gr}_4^W H^2(U, R^2 f_* \mathbb{Q})}_{III}. \tag{7}$$

The first summand I can be identified with

$$\mathrm{im}[H^4(Y) \to H^4(U)] \cong \frac{H^4(Y)}{\sum \mathrm{im}\, H^2(D_i)(-1)} \cong L\left(\frac{H^2(Y)}{\sum L^{-1} \mathrm{im}\, H^2(D_i)(-1)}\right),$$

where L is the Lefschetz operator with respect to an ample divisor on Y. The Lefschetz (1, 1) theorem shows that the Hodge cycles in I are algebraic.

We have an isomorphism $\mathbb{Q}_U \cong R^2 f_*^o \mathbb{Q}$, under which $1 \in H^0(U, \mathbb{Q})$ maps to the class of a multisection $[\sigma] \in H^0(U, R^2 f_*^o \mathbb{Q})$. Thus the summand III can be identified with

$$[\sigma] \cup \mathrm{im}[H^2(Y) \to H^2(U)] \cong \frac{[\sigma] \cup H^2(Y)}{\sum [\sigma] \cup [D_i]}.$$

It follows again, by the Lefschetz (1, 1) theorem, that any Hodge cycle in the summand III is algebraic. This is also vacuously true for II because, by assumption, there are no Hodge cycles in $\mathrm{Gr}_4^W H^3(U, R^1 f_* \mathbb{Q})$.

From the sequence

$$\bigoplus H^2(E_i)(-1) \to H^4(X) \to \mathrm{Gr}_4^W H^4(\widetilde{U}) \to 0$$

we deduce that

$$H^4(X) \cong \bigoplus \operatorname{im} H^2(E_i)(-1) \oplus \operatorname{Gr}_4^W H^4(\widetilde{U}) \quad \text{(noncanonically)}.$$

Therefore all the Hodge cycles in $H^4(X)$ are algebraic. □

Lemma 4.4. *Let $f : (X, E) \to (Y, D)$ be a semistable map of smooth projective varieties defined over a finitely generated subfield $K \subset \mathbb{C}$ with $\dim Y = 3$ and $\dim X = 4$. Let $U = Y - D$. Suppose that*

$$\operatorname{Gr}_F^2 H^3(U, R^1 f_* \mathbb{C}) = 0,$$

that $H^{1,1}(Y)$ is spanned by algebraic cycles, and that the Tate conjecture holds in degree 2 for the components E_i of E. Then the Tate conjecture holds for X in degree 4.

Proof. By the Hodge index theorem

$$\langle \alpha, \beta \rangle = \pm \operatorname{tr}(\alpha \cup \beta)$$

gives a positive definite pairing on the primitive part of $H^4(X)$, and this can be extended to the whole of H^4 by hard Lefschetz. Let

$$\begin{aligned} S_B &= \sum \operatorname{im} H^2(E_i(\mathbb{C}), \mathbb{C})(1) \subseteq H^4(X(\mathbb{C}), \mathbb{C})(2), \\ S_{\mathrm{Hdg}} &= \sum \operatorname{im} H^1(E_i, \Omega^1_{E_i}) \subseteq H^2(X, \Omega^2_X), \\ S_\ell &= \sum \operatorname{im} H^2_{\text{ét}}(E_i \otimes \overline{K}, \mathbb{Q}_\ell(1)) \subseteq H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)), \end{aligned}$$

where the images above are with respect to the Gysin maps. Set

$$\begin{aligned} V_B &= H^4(X(\mathbb{C}), \mathbb{C})(2)/S_B, \\ V_{\mathrm{Hdg}} &= H^2(X, \Omega^2_X)/S_{\mathrm{Hdg}}, \\ V_\ell &= H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2))/S_\ell. \end{aligned}$$

Observe that V_B is a Hodge structure and V_ℓ is a Galois module. Let us say that a class in any one of these spaces is algebraic if it lifts to an algebraic cycle in $H^4(X)$ or $H^2(X, \Omega^2_X)$. Let us write

$$\operatorname{Tate}(-) = \sum_{[L:K]<\infty} (-)^{\operatorname{Gal}(\overline{K}/L)},$$

where $(-)$ can stand for V_ℓ or any other Galois module. Clearly

$$\dim(\text{space of algebraic classes in } V_\ell) \leq \dim \operatorname{Tate}(V_\ell). \tag{8}$$

We also claim that

$$\dim \operatorname{Tate}(V_\ell) \leq \dim V_{\mathrm{Hdg}}. \tag{9}$$

This will follow from the Hodge–Tate decomposition. After passing to a finite extension, we can assume that all elements of $\operatorname{Tate}(H^4(X, \mathbb{Q}_\ell(2)))$ and $\operatorname{Tate}(V_\ell)$ are fixed by $\operatorname{Gal}(\overline{K}/K)$. Let K_ℓ denote

the completion of K at a prime lying over ℓ, and let $\mathbb{C}_\ell = \widehat{\overline{K}}_\ell$. By [Faltings 1988] there is a Hodge–Tate decomposition, i.e., a functorial isomorphism of $\mathrm{Gal}(\overline{K}_\ell/K_\ell)$-modules

$$H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)) \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong \bigoplus_{a+b=4} H^a(X, \Omega^b_X) \otimes_K \mathbb{C}_\ell(2-b).$$

This is compatible with products, Poincaré/Serre duality, and cycle maps. Since we can decompose $H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)) = S_\ell \oplus S_\ell^\perp$ as an orthogonal direct sum, and this is a decomposition of $\mathrm{Gal}(\overline{K}/K)$-modules, an element of $\gamma \in \mathrm{Tate}(V_\ell)$ can be lifted to $\gamma_1 \in \mathrm{Tate}(H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)))$. This gives a $\mathrm{Gal}(\overline{K}_\ell/K_\ell)$-invariant element of $H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)) \otimes \mathbb{C}_\ell$, and thus an element of $\gamma_2 \in H^2(X, \Omega^2_X) \otimes K_\ell$. Let $\gamma_3 \in V_{\mathrm{Hdg}} \otimes K_\ell$ denote the image. One can check that $\gamma \mapsto \gamma_3$ is a well-defined injection of $\mathrm{Tate}(V_\ell) \otimes K_\ell \to V_{\mathrm{Hdg}} \otimes K_\ell$. This proves that (9) holds.

As in the proof of Lemma 4.3, we can split $V_B(-2)$ as

$$V_B(-2) = I \oplus II \oplus III,$$

where the summands are defined as in (7). Arguing as above, but with the stronger assumption that $H^{1,1}(Y)$ is algebraic, we can see that the (not necessarily rational) $(2,2)$-classes in I and III are algebraic, and that II has no such classes. Therefore V_{Hdg} is spanned by algebraic classes. Combined with inequalities (8) and (9), we find that every element of $\mathrm{Tate}(V_\ell)$ is an algebraic class. Therefore given a Tate cycle $\gamma \in \mathrm{Tate}(H^4_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(2)))$, there is an algebraic cycle γ' so that $\gamma - \gamma' \in S_\ell$. This means that $\gamma - \gamma'$ is the sum of images of Tate cycles in $H^2(E_i)$. By assumption, this is again algebraic. □

Lemma 4.5. *Let X be a smooth projective variety defined over a finitely generated subfield $K \subset \mathbb{C}$. If $H^{1,1}(X)$ is spanned by divisors, the Tate conjecture holds for X in degree 2.*

Proof. This is similar to the previous proof. We have inequalities

$$\operatorname{rank} \mathrm{NS}(X) \le \dim \mathrm{Tate}(H^2_{\text{ét}}(X \otimes \overline{K}, \mathbb{Q}_\ell(1))) \le h^{1,1}(X),$$

where the second follows from Hodge–Tate. Since $H^{1,1}(X)$ is spanned by divisors, we must have equality above. □

Proof of Theorem 4.1. The statement (A) about the Hodge conjecture follows immediately from Corollary 3.2 and Lemma 4.3.

We now turn to part (B) on the Tate conjecture. We break the analysis into cases. Tate in degree 2 follows from Theorem 2.6 and Lemma 4.5. Hard Lefschetz then implies Tate in degree 6. In degree 4, by Lemmas 4.2 and 4.4, it is enough to verify that $H^{1,1}(Y)$ is spanned by divisors and that the Tate conjecture holds in degree 2 for varieties rationally dominating components of the divisor E. The first condition for Y is due to [Weissauer 1988, p. 101]. By Corollary 2.2, irreducible components of E are dominated by $\overline{C}_{1,1}[n] \times \overline{M}_{1,1}[n]$, $\overline{C}_{1,2}[n]$, $\overline{C}_{1,1}[m] \times \mathbb{P}^1$, $\overline{M}_{1,1}[m] \times \overline{M}_{1,1}[m]$, $\mathbb{P}^2$ times a curve, or $\mathbb{P}^3$. The Tate conjecture in degree 2 is trivially true for the last two cases. The Tate conjecture in degree 2 for the other cases follows from [Gordon 1993, Theorem 5] and Lemma 4.2. □

Part (B) of the previous theorem can be extended slightly. Suppose that $\Gamma \subseteq \Gamma_2$ is the preimage of a finite-index subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ such that $\overline{M}_2[\Gamma]$ is smooth. With this assumption, we may choose a good model $X \to Y$ of $\bar{C}_2[\Gamma] \to \overline{M}_2[\Gamma]$, with $Y = \overline{M}_2[\Gamma]$.

Corollary 4.6. *The Tate conjecture holds for X as above.*

Proof. We first note that Γ contains some $\widetilde{\Gamma}(n)$, because the congruence subgroup problem has a positive solution for $\mathrm{Sp}_4(\mathbb{Z})$ [Bass et al. 1967]. Therefore the good model $X[n]$ for $\widetilde{\Gamma}(n)$ surjects onto X. Since we know that Tate holds for $X[n]$, it holds for X by Lemma 4.2. □

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