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Effective generation and twisted weak positivity of direct images

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We study pushforwards of log pluricanonical bundles on projective log canonical pairs (Y, Δ) over the complex numbers, partially answering a Fujita-type conjecture due to Popa and Schnell in the log canonical setting. We show two effective global generation results. First, when Y surjects onto a projective variety, we show a quadratic bound for generic generation for twists by big and nef line bundles. Second, when Y is fibered over a smooth projective variety, we show a linear bound for twists by ample line bundles. These results additionally give effective nonvanishing statements. We also prove an effective weak positivity statement for log pluricanonical bundles in this setting, which may be of independent interest. In each context we indicate over which loci positivity holds. Finally, using the description of such loci, we show an effective vanishing theorem for pushforwards of certain log-sheaves under smooth morphisms.

1. Introduction

Throughout this paper, all varieties will be over the complex numbers.

Popa and Schnell proposed the following relative version of Fujita's conjecture:

Conjecture 1.1 [Popa and Schnell 2014, Conjecture 1.3]. *Let $f : Y \to X$ be a morphism of smooth projective varieties, with* $\dim X = n$, *and let $\mathcal{L}$ be an ample line bundle on X. For each $k \geq 1$, the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated for all $\ell \geq k(n+1)$.

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Additionally assuming that $\mathcal{L}$ is globally generated, Popa and Schnell proved Conjecture 1.1 more generally for log canonical pairs (Y, Δ). Previously, Deng [2017, Theorem C] and the first author [Dutta 2017, Proposition 1.2] studied this conjecture for klt $\mathbb{Q}$-pairs, and were able to remove the global generation assumption on $\mathcal{L}$ to obtain generic effective generation statements. In this paper, we obtain similar generic generation results, more generally for log canonical pairs (Y, Δ).

First, when X is arbitrarily singular and $\mathcal{L}$ is only big and nef, we obtain the following quadratic bound on ℓ. The case when (Y, Δ) is klt and $k = 1$ is due to de Cataldo [1998, Theorem 2.2].

Theorem A. *Let $f : Y \to X$ be a surjective morphism of projective varieties, where X is of dimension n. Let (Y, Δ) be a log canonical $\mathbb{R}$-pair and let $\mathcal{L}$ be a big and nef line bundle on X. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$. Then, the sheaf*

$$f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$$

is generated by global sections on an open set U for every integer $\ell \geq k(n^2 + 1)$.

On the other hand, we have the following linear bound when X is smooth and $\mathcal{L}$ is ample. The statement in (i) extends [Deng 2017, Theorem C] to log canonical pairs. As we were writing this, we learned that a statement similar to (ii) was also obtained by Iwai [2017, Theorem 1.5].

Theorem B. *Let $f : Y \to X$ be a fibration of projective varieties, where X is smooth of dimension n. Let (Y, Δ) be a log canonical $\mathbb{R}$-pair and let $\mathcal{L}$ be an ample line bundle on X. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$. Then, the sheaf*

$$f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$$

is globally generated on an open set U for

(i) *every integer $\ell \geq k(n + 1) + n^2 - n$; and*

(ii) *every integer $\ell > k(n + 1) + \frac{1}{2}(n^2 - n)$ when (Y, Δ) is a klt $\mathbb{Q}$-pair.*

Here, a *fibration* is a morphism whose generic fiber is irreducible.

In both Theorems A and B, when Y is smooth and Δ has simple normal crossing support, we have explicit descriptions of the open set U. See Remark 5.1. Thus, we have descriptions of the loci where global generation holds up to a log resolution.

When X is smooth of dimension ≤ 3 and $\mathcal{L}$ is ample, the bound on ℓ can be improved. This gives the predicted bound in Conjecture 1.1 for surfaces; see Remark 5.2.

Remark 1.2 (effective nonvanishing). Theorems A and B can be interpreted as effective nonvanishing statements. With notation as in the theorems, it follows that $f_*\mathcal{O}_Y(P) \otimes \mathcal{L}^{\otimes \ell}$ admits global sections for all $\ell \geq k(n^2 + 1)$ when $\mathcal{L}$ is big and nef, and for all $\ell \geq k(n + 1) + n^2 - n$ when $\mathcal{L}$ is ample and X is smooth. Moreover, just as in Theorem B(ii), the effective bound of the second nonvanishing statement can be improved in the case when (Y, Δ) is a klt $\mathbb{Q}$-pair.

We now state the technical results used in proving Theorems A and B.

An extension theorem. Recall that if $\mu : X' \to X$ is the blow-up of a projective variety X at x with exceptional divisor E, then the *Seshadri constant* of a nef Cartier divisor L at x is

$$\varepsilon(L; x) := \sup\{t \in \mathbb{R}_{\geq 0} \mid \mu^* L - tE \text{ is nef}\}.$$

The following replaces the role of Deng's extension theorem [2017, Theorem 2.11] in our proofs.

Theorem C. *Let $f : Y \to X$ be a surjective morphism of projective varieties, where X is of dimension n and Y is smooth. Let Δ be an $\mathbb{R}$-divisor on Y with simple normal crossing support and coefficients in* $(0, 1]$, *and let L be a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Suppose there exists a closed point $x \in U(f, \Delta)$ and a real number $\ell > n/\varepsilon(L; x)$ such that*

$$P_\ell \sim_{\mathbb{R}} K_Y + \Delta + \ell f^* L$$

for some Cartier divisor P_ℓ on Y. Then, the restriction map

$$H^0(Y, \mathcal{O}_Y(P_\ell)) \longrightarrow H^0(Y_x, \mathcal{O}_{Y_x}(P_\ell)) \tag{1}$$

is surjective, and the sheaf $f_* \mathcal{O}_Y(P_\ell)$ is globally generated at x.

See Notation 2.1(a) for the definition of the open set $U(f, \Delta)$.

Remark 1.3 (comments on the proofs). The proofs of Theorems A and B(i) are in a way an algebraization of Deng's techniques, exploiting a generic lower bound for Seshadri constants due to Ein, Küchle, and Lazarsfeld (Theorem 2.20). In the algebraic setting, this lower bound was first used by de Cataldo to prove a version of Theorem A for klt pairs when $k = 1$. One of our main challenges was to extend de Cataldo's theorem to the log canonical case (see Theorem C above).

To obtain the better bound in Theorem B(ii) for klt $\mathbb{Q}$-pairs, we use [Dutta 2017, Proposition 1.2] instead of Seshadri constants.

In Theorems A and C, in order to work with line bundles $\mathcal{L}$ that are big and nef instead of ample, we needed to study the augmented base locus $\boldsymbol{B}_+(\mathcal{L})$ of $\mathcal{L}$ (see Definition 2.22). We used Birkar's generalization of Nakamaye's theorem [Birkar 2017, Theorem 1.4] and [Küronya 2013, Proposition 2.7], which capture how $\mathcal{L}$ fails to be ample.

The proof of Theorem C relies on a cohomological injectivity theorem due to Fujino [2017a, Theorem 5.4.1]. If (Y, Δ) is replaced by an arbitrary log canonical $\mathbb{R}$-pair, then the global generation statement in Theorem C still holds over some open set (Corollary 3.2).

Remark 1.4 (effective vanishing). With the new input of weak positivity, which is discussed next, we give some effective vanishing statements for certain cases of such pushforwards under smooth morphisms (see Theorem 5.3). This improves similar statements in [Dutta 2017] and is in the spirit of [Popa and Schnell 2014, Proposition 3.1], where they showed a similar statement with the assumption that $\mathcal{L}$ is ample and globally generated.

Effective twisted weak positivity. In order to prove Theorem B, we also use the following weak positivity result for log canonical pairs. This may be of independent interest.

In this setting, weak positivity was partially known due to Campana [2004, Theorem 4.13], and later more generally due to Fujino [2017b, Theorem 1.1], but using a slightly weaker notion of weak positivity (see [loc. cit., Definition 7.3] and the comments thereafter). Our result extends these results.

Theorem D (twisted weak positivity). *Let $f : Y \to X$ be a fibration of normal projective varieties such that X is Gorenstein of dimension n. Let Δ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y such that (Y, Δ) is log canonical and $k(K_Y + \Delta)$ is $\mathbb{R}$-linearly equivalent to a Cartier divisor for some integer $k \geq 1$. Then, the sheaf*

$$f_*\mathcal{O}_Y(k(K_{Y/X} + \Delta))$$

is weakly positive.

Recall that a torsion-free coherent sheaf $\mathscr{F}$ is *weakly positive* if there exists a nonempty open set U such that for every integer a, there is an integer $b \geq 1$ such that

$$\mathrm{Sym}^{[ab]} \mathscr{F} \otimes H^{\otimes b}$$

is generated by global sections on U for all ample line bundles H. Here, $\cdot^{[s]}$ is the reflexive hull of $\cdot^{s}$ (see Notation 2.6).

Popa and Schnell [2014, Theorem 4.2] showed that if $\Delta = 0$, the morphism f has generically reduced fibers in codimension 1, and $H = \omega_X \otimes \mathcal{L}^{\otimes n+1}$ with $\mathcal{L}$ ample and globally generated, then weak positivity in Theorem D holds over $U(f, 0)$ for all $b \geq k$. In a similar spirit, we prove the following "effective" version of twisted weak positivity when Y is smooth and Δ has simple normal crossing support. Moreover, Theorem D is deduced from this result and therefore we also obtain an explicit description, up to a log resolution, of the locus over which weak positivity holds. This extends [Popa and Schnell 2014, Theorem 4.2] to arbitrary fibrations.

Theorem E (effective weak positivity). *Let $f : Y \to X$ be a fibration of projective varieties, where Y is smooth and X is normal and Gorenstein of dimension n. Let Δ be an $\mathbb{R}$-divisor on Y with simple normal crossing support and with coefficients of Δ^h in $(0, 1]$. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$. Let U be the intersection of $U(f, \Delta)$ with the largest open set over which $f_*\mathcal{O}_Y(P)$ is locally free, and let $H = \omega_X \otimes \mathcal{L}^{\otimes n+1}$ for $\mathcal{L}$ an ample and globally generated line bundle on X. Then, the sheaf*

$$(f_*\mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]} \otimes H^{\otimes \ell}$$

is generated by global sections on U for all integers $\ell \geq k$ and $s \geq 1$.

Here, Δ^h is the *horizontal part* of Δ; see Notation 2.1(b).

When $\lfloor \Delta \rfloor = 0$, one can, in a way, get rid of the assumption that $f_*\mathcal{O}_Y(P)$ is locally free on U using invariance of log plurigenera [Hacon et al. 2018, Theorem 4.2]; see Remark 4.2.

The proof of Theorem E relies on Viehweg's fiber product trick; see [Viehweg 1983, §3], [Popa and Schnell 2014, Theorem 4.2], or [Höring 2010, §3] for an exposition.

2. Definitions and preliminary results

Throughout this paper, a *variety* is an integral separated scheme of finite type over the complex numbers. We will also fix the following notation:

Notation 2.1. Let $f : Y \to X$ be a morphism of projective varieties, where Y is smooth, and let Δ be an $\mathbb{R}$-divisor with simple normal crossing support on Y.

(a) We denote by $U(f, \Delta)$ the largest open subset of X such that

- $U(f, \Delta)$ is contained in the smooth locus X_{reg} of X;
- $f : f^{-1}(U(f, \Delta)) \to U(f, \Delta)$ is smooth; and
- the fibers $Y_x := f^{-1}(x)$ intersect each component of Δ transversely for all closed points $x \in U(f, \Delta)$.

This open set $U(f, \Delta)$ is nonempty by generic smoothness; see [Hartshorne 1977, Corollary III.10.7] and [Lazarsfeld 2004a, Lemma 4.1.11].

(b) We write

$$\Delta = \Delta^v + \Delta^h,$$

where Δ^v and Δ^h do not share any components, such that

- every component of Δ^h is *horizontal* over X, i.e., surjects onto X; and
- Δ^v is *vertical* over X, i.e., $f(\mathrm{Supp}(\Delta^v)) \subsetneq X$.

Note that $U(f, \Delta)$ satisfies $U(f, \Delta) \cap f(\Delta^v) = \varnothing$.

Reflexive sheaves and weak positivity. In this section, fix an integral noetherian scheme X. To prove Theorem E, we need some basic results on reflexive sheaves, which we collect here.

Definition 2.2. A coherent sheaf $\mathscr{F}$ on X is *reflexive* if the natural morphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism, where $\mathscr{G}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathcal{O}_X)$. In particular, locally free sheaves are reflexive.

A coherent sheaf $\mathscr{F}$ on X is *normal* if the restriction map

$$\Gamma(U, \mathscr{F}) \longrightarrow \Gamma(U \smallsetminus Z, \mathscr{F})$$

is bijective for every open set $U \subseteq X$ and every closed subset Z of U of codimension at least 2.

Proposition 2.3 (see [Hartshorne 1994, Proposition 1.11]). *If X is normal, then every reflexive coherent sheaf $\mathscr{F}$ is normal.*

Lemma 2.4 [Stacks 2018, Tag 0AY4]. *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X, and assume that $\mathscr{F}$ is reflexive. Then, $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$ is also reflexive.*

We will often use these facts to extend morphisms from the complement of codimension at least 2, as recorded in the following:

Corollary 2.5. *Suppose X is normal, and let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X such that $\mathscr{F}$ is reflexive. If $U \subseteq X$ is an open subset such that* $\operatorname{codim}(X \smallsetminus U) \geq 2$, *then every morphism $\varphi : \mathscr{G}|_U \to \mathscr{F}|_U$ extends uniquely to a morphism $\tilde{\varphi} : \mathscr{G} \to \mathscr{F}$.*

Proof. The morphism φ corresponds to a section of the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$ over U. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$ is reflexive by Lemma 2.4; hence the section φ extends uniquely to a section $\tilde{\varphi}$ of $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$ over X by Proposition 2.3. ☐

We will use the following notation throughout this paper:

Notation 2.6 [Höring 2010, Notation 3.3]. Let $\mathscr{F}$ be a torsion-free coherent sheaf on a normal variety X. Let $i : X^* \hookrightarrow X$ be the largest open set such that $\mathscr{F}|_{X^*}$ is locally free. We define

$$\operatorname{Sym}^{[b]} \mathscr{F} := i_* \operatorname{Sym}^b(\mathscr{F}|_{X^*}) \quad \text{and} \quad \mathscr{F}^{[b]} := i_*((\mathscr{F}|_{X^*})^{\otimes b}).$$

We can also describe these sheaves as follows:

$$\operatorname{Sym}^{[b]} \mathscr{F} \simeq (\operatorname{Sym}^b(\mathscr{F}))^{\vee\vee} \quad \text{and} \quad \mathscr{F}^{[b]} \simeq (\mathscr{F}^{\otimes b})^{\vee\vee}.$$

Indeed, these pairs of reflexive sheaves coincide in codimension 1 and hence are isomorphic (see [Hartshorne 1994, Theorem 1.12]).

We can now define the positivity notion appearing in Theorem D.

Definition 2.7 (weak positivity [Viehweg 1983, Definition 1.2]). Let X be a normal variety, and let $U \subseteq X$ be an open set. A torsion-free coherent sheaf $\mathscr{F}$ on X is said to be *weakly positive on U* if for every positive integer a and every ample line bundle $\mathcal{L}$ on X, there exists an integer $b \geq 1$ such that $\operatorname{Sym}^{[ab]} \mathscr{F} \otimes \mathcal{L}^{\otimes b}$ is globally generated on U. We say $\mathscr{F}$ is *weakly positive* if $\mathscr{F}$ is weakly positive on some open set U.

Dualizing complexes and canonical sheaves. The main reference for this section is [Hartshorne 1966]. We define the following:

Definition 2.8. Let $h : X \to \operatorname{Spec} k$ be an equidimensional scheme of finite type over a field k. Then the *normalized dualizing complex* for X is $\omega_X^\bullet := h^! k$, where $h^!$ is the exceptional pullback of Grothendieck duality [loc. cit., Corollary VII.3.4]. One defines the *canonical sheaf* on X to be the coherent sheaf

$$\omega_X := \boldsymbol{H}^{-\dim X} \omega_X^\bullet.$$

When X is smooth and equidimensional over a field, the canonical sheaf ω_X is isomorphic to the invertible sheaf of volume forms $\Omega_X^{\dim X}$ [loc. cit., III.2].

We will need the explicit description of the exceptional pullback functor for finite morphisms. Let $\nu : Y \to X$ be a finite morphism of equidimensional schemes of finite type over a field. Consider the functor

$$\bar{\nu}^* : \mathsf{Mod}(\nu_* \mathcal{O}_Y) \longrightarrow \mathsf{Mod}(\mathcal{O}_Y)$$

obtained from the morphism $\bar{\nu} : (Y, \mathcal{O}_Y) \to (X, \nu_* \mathcal{O}_Y)$ of ringed spaces. This functor $\bar{\nu}^*$ satisfies the following properties (see [loc. cit., III.6]):

(a) The functor $\bar{\nu}^*$ is exact since the morphism $\bar{\nu}$ of ringed spaces is flat. We define the functor

$$\begin{aligned} \nu^! : \mathsf{D}^+(\mathsf{Mod}(\mathcal{O}_X)) &\longrightarrow \mathsf{D}^+(\mathsf{Mod}(\mathcal{O}_Y)), \\ \mathscr{F} &\longmapsto \bar{\nu}^* \, \boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_Y, \mathscr{F}). \end{aligned}$$

(b) For every $\mathcal{O}_X$-module $\mathscr{G}$, we have $\nu^* \mathscr{G} \simeq \bar{\nu}^*(\mathscr{G} \otimes_{\mathcal{O}_X} \nu_* \mathcal{O}_Y)$.

(c) If $\omega_X^\bullet$ is the normalized dualizing complex for X, then $\nu^! \omega_Y^\bullet$ is the normalized dualizing complex for Y.

Using the above description, we construct the following *pluri-trace map* for integral schemes over fields, which we will use in the proof of Theorem E. We presume that this construction is already known to the experts, but we could not find a reference.

Lemma 2.9. *Let $d : Y' \to Y$ be a dominant proper birational morphism of integral schemes of finite type over a field, where Y' is normal and Y is Gorenstein. Then, there is a map of pluricanonical sheaves*

$$d_* \omega_{Y'}^{\otimes k} \longrightarrow \omega_Y^{\otimes k}$$

which is an isomorphism where d is an isomorphism.

Proof. By the universal property of normalization [Stacks 2018, Tag 035Q], we can factor d as

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[rr, bend right, "d"'] & \overline{Y} \arrow[r, "\nu"] & Y \end{tikzcd}$$

where ν is the normalization. Note that d' is proper and birational since d is.

We first construct a similar morphism for ν. Let $n = \dim Y$. Since Y is Gorenstein, the canonical sheaf ω_Y is invertible and the normalized dualizing complex is $\omega_Y[n]$ [Hartshorne 1966, Proposition V.9.3]. Using property (c) above we have

$$\begin{aligned} \omega_{\overline{Y}} = \boldsymbol{H}^{-n}(\nu^! \omega_Y^\bullet) &\simeq \bar{\nu}^*(\boldsymbol{R}^{-n}\, \mathcal{H}om_{\mathcal{O}_Y}(\nu_* \mathcal{O}_{\overline{Y}}, \mathcal{O}_Y[n]) \otimes_{\mathcal{O}_Y} \omega_Y) \\ &\simeq \bar{\nu}^*(\mathcal{H}om_{\mathcal{O}_Y}(\nu_* \mathcal{O}_{\overline{Y}}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_Y), \end{aligned}$$

where we get the first isomorphism since $\bar{\nu}^*$ is exact by (a) and since ω_Y is invertible.

Now $\mathcal{H}om_{\mathcal{O}_Y}(\nu_* \mathcal{O}_{\overline{Y}}, \mathcal{O}_Y)$ admits a morphism to $\nu_* \mathcal{O}_{\overline{Y}}$, which makes it the largest ideal in $\nu_* \mathcal{O}_{\overline{Y}}$ that is also an ideal in $\mathcal{O}_Y$. It is the so-called *conductor ideal* of the normalization map [Kollár 2013, (5.2)]. Thus, we get a morphism

$$\omega_{\overline{Y}} \hookrightarrow \bar{\nu}^*(\nu_* \mathcal{O}_{\overline{Y}} \otimes \omega_Y) \simeq \nu^* \omega_Y.$$

The last isomorphism follows from (b) above. By taking the $(k-1)$-fold tensor product of the above morphism we have

$$\omega_{\overline{Y}}^{\otimes(k-1)} \hookrightarrow \nu^* \omega_Y^{\otimes(k-1)}. \tag{2}$$

Finally, we use (2) to construct a map

$$d_*\omega_{Y'}^{\otimes k} \longrightarrow \nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}.$$

First, we construct the above morphism over U, where d' is an isomorphism. Define $V := d'^{-1}(U)$. The identity map

$$\mathrm{id} : d'_*\omega_V^{\otimes k} \longrightarrow \omega_U^{\otimes k}$$

composed with map obtained from (2) gives the map

$$\tau : d'_*\omega_V^{\otimes k} \longrightarrow \nu^*\omega_Y^{\otimes(k-1)}|_U \otimes_{\mathcal{O}_U} \omega_U.$$

Since $\nu^*\omega_Y^{\otimes(k-1)}$ is invertible and $\omega_{\overline{Y}}$ is reflexive, the sheaf $\nu^*\omega_Y^{\otimes(k-1)} \otimes \omega_{\overline{Y}}$ is also reflexive. Now $\operatorname{codim}(Y \smallsetminus U) \geq 2$ by Zariski's main theorem; see [Hartshorne 1977, Theorem V.5.2]. Therefore by Corollary 2.5 we obtain

$$\tilde{\tau} : d'_*\omega_{Y'}^{\otimes k} \longrightarrow \nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}.$$

Composing $\nu_*\tilde{\tau}$ with one copy of the trace morphism $\nu_*\omega_{\overline{Y}} \to \omega_Y$ [Hartshorne 1966, Proposition III.6.5], we get

$$d_*\omega_{Y'}^{\otimes k} \xrightarrow{\nu_*\tilde{\tau}} \nu_*(\nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}) \simeq \omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_Y} \nu_*\omega_{\overline{Y}} \xrightarrow{\mathrm{id}\otimes\mathrm{Tr}} \omega_Y^{\otimes k}. \tag{3}$$

The statement about the isomorphism locus of the above morphism holds by construction of the maps above. Indeed, in (3) the trace morphism is compatible with flat base change [Hartshorne 1966, Proposition III.6.6(2)], and hence compatible with restriction to the open set where d is an isomorphism. □

Singularities of pairs. We follow the conventions of [Fujino 2017a, §2.3]; see also [Kollár 2013, §1.1,2.1]. Recall that X_{reg} denotes the regular locus of a scheme X; see Notation 2.1(a).

Definition 2.10 (canonical divisor). Let X be a normal variety of dimension n. A *canonical divisor* K_X on X is a Weil divisor such that

$$\mathcal{O}_{X_{\mathrm{reg}}}(K_X) \simeq \Omega^n_{X_{\mathrm{reg}}}.$$

The choice of a canonical divisor K_X is unique up to linear equivalence. Then one defines $\mathcal{O}_X(K_X)$ to be the reflexive sheaf of rank 1 associated to K_X.

The following lemma allows us to freely pass between divisor and sheaf notation on normal varieties:

Lemma 2.11. *Let X be a normal variety of dimension n. Then, $\mathcal{O}_X(K_X)$ is isomorphic to ω_X.*

Proof. The sheaf $\mathcal{O}_X(K_X)$ is reflexive by definition and the canonical sheaf ω_X is S_2, by [Stacks 2018, Tag 0AWE], and hence reflexive, by [Hartshorne 1994, Theorem 1.9]. Since they are both isomorphic to $\Omega^n_{X_{\mathrm{reg}}}$ on X_{reg} and $\operatorname{codim}(X \smallsetminus X_{\mathrm{reg}}) \geq 2$, we have $\mathcal{O}_X(K_X) \simeq \omega_X$ by [loc. cit., Theorem 1.12]. □

Definition 2.12 (discrepancy). Let (X, Δ) be a pair consisting of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose $f : Y \to X$ is a proper birational morphism from a normal variety Y, and choose canonical divisors K_Y and K_X such that $f_* K_Y = K_X$. In this case, we may write

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta) E_i,$$

where the E_i are irreducible Weil divisors. The real number $a(E_i, X, \Delta)$ is called the *discrepancy of E_i* with respect to (X, Δ), and the *discrepancy* of (X, Δ) is

$$\operatorname{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\},$$

where the infimum runs over all irreducible exceptional divisors of all proper birational morphisms $f : Y \to X$.

Definition 2.13 (singularities of pairs). Let (X, Δ) be a pair consisting of a normal variety X and an effective $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is *klt* if $\operatorname{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor = 0$. We say that (X, Δ) is *log canonical* if $\operatorname{discrep}(X, \Delta) \geq -1$.

We will repeatedly use the following results about log resolutions of log canonical $\mathbb{R}$-pairs.

Lemma 2.14. *Let (Y, Δ) be a log canonical (resp. klt) $\mathbb{R}$-pair, and consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta + H)$ for some integer $k \geq 1$ and some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H. Then, for every proper birational morphism $\mu : \widetilde{Y} \to Y$ such that $\widetilde{Y}$ is smooth and $\mu^{-1}(\Delta) + \operatorname{exc}(\mu)$ has simple normal crossing support, there exists a divisor $\widetilde{P}$ on $\widetilde{Y}$ and an $\mathbb{R}$-divisor $\tilde{\Delta}$ such that*

(i) *$\tilde{\Delta}$ has coefficients in $(0, 1]$ (resp. $(0, 1)$) and simple normal crossing support;*

(ii) *the divisor $\widetilde{P} - \mu^* P$ is an effective divisor with support in $\operatorname{Supp}(\operatorname{exc}(\mu))$;*

(iii) *the divisor $\widetilde{P}$ satisfies $\widetilde{P} \sim_{\mathbb{R}} k(K_{\widetilde{Y}} + \tilde{\Delta} + \mu^* H)$; and*

(iv) *there is an isomorphism $\mu_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}) \simeq \mathcal{O}_Y(P)$.*

Proof. On $\widetilde{Y}$, we can write

$$K_{\widetilde{Y}} - \mu^*(K_Y + \Delta) = Q - N,$$

where Q and N are effective $\mathbb{R}$-divisors without common components such that $Q - N$ has simple normal crossing support and Q is μ-exceptional. Note that since (Y, Δ) is log canonical (resp. klt), all coefficients in N are less than or equal to 1 (resp. less than 1). Let

$$\tilde{\Delta} := N + \lceil Q \rceil - Q$$

so that by definition $\tilde{\Delta}$ has simple normal crossing support and coefficients in $(0, 1]$ (resp. $(0, 1)$). Now setting $\widetilde{P} := \mu^* P + k\lceil Q \rceil$, we have

$$\begin{aligned} \widetilde{P} &\sim_{\mathbb{R}} k\mu^*(K_Y + \Delta + H) + k\lceil Q \rceil \\ &\sim_{\mathbb{R}} kK_{\widetilde{Y}} + k(N + \lceil Q \rceil - Q) + \mu^* H = k(K_{\widetilde{Y}} + \tilde{\Delta} + \mu^* H). \end{aligned}$$

Since $\lceil Q \rceil$ is μ-exceptional, we get $\mu_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}) \simeq \mathcal{O}_Y(P)$ by using the projection formula. $\square$

We also use the following stronger notion of log resolution due to Szabó:

Theorem 2.15 [Kollár 2013, Theorem 10.45.2]. *Let X be a variety, and let D be a Weil divisor on X. Then, there is a log resolution $\mu : \widetilde{X} \to X$ of (X, D) such that μ is an isomorphism over the locus where X is smooth and D has simple normal crossing support.*

A few tools from Popa–Schnell. The following result is a slight generalization of [Popa and Schnell 2014, Variant 1.6]. This will be instrumental in proving Theorems D and E.

Theorem 2.16. *Let $f : Y \to X$ be a morphism of projective varieties, where Y is normal and X is of dimension n. Let Δ be an $\mathbb{R}$-divisor on Y and H a semiample $\mathbb{Q}$-divisor on X such that for some integer $k \geq 1$, there is a Cartier divisor P on Y satisfying*

$$P \sim_{\mathbb{R}} k(K_Y + \Delta + f^*H).$$

Suppose, moreover, that Δ can be written as $\Delta = \Delta' + \Delta^v$, where (Y, Δ') is log canonical and Δ^v is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is vertical over X. Let $\mathcal{L}$ be an ample and globally generated line bundle on X. Then, the sheaf

$$f_*\mathcal{O}_Y(P) \otimes \mathcal{L}^{\otimes \ell}$$

is generated by global sections on some open set U for all $\ell \geq k(n+1)$. Moreover, when Δ' has simple normal crossing support, we have $U = X \smallsetminus f(\operatorname{Supp}(\Delta^v))$.

Proof. Possibly after a log resolution of (Y, Δ), we may assume that $\Delta = \Delta^h + \Delta^v$ in the sense of Notation 2.1(b) such that (Y, Δ^h) is log canonical and Δ has simple normal crossing support. Indeed, let $\mu : \widetilde{Y} \to Y$ be a log resolution of (Y, Δ). Then, by Lemma 2.14 applied to the pair (Y, Δ') and $H = \Delta^v$, we obtain a log canonical $\mathbb{R}$-divisor $\tilde{\Delta}$ with simple normal crossing support on $\widetilde{Y}$ satisfying

$$K_{\widetilde{Y}} + \tilde{\Delta} + \mu^*\Delta^v \sim_{\mathbb{R}} \mu^*(K_Y + \Delta) + N,$$

where N is an effective μ-exceptional divisor. We rename $\widetilde{Y}$ and $\tilde{\Delta} + \mu^*\Delta^v$ as Y and Δ respectively.

Now Δ has simple normal crossing support and Δ^h is log canonical. Moreover, since f^*H is semiample, by Bertini's theorem we can pick a $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} f^*H$ with smooth support and satisfying the conditions that $D + \Delta$ has simple normal crossing support and D does not share any components with Δ. Letting $\Delta'' := \Delta^v - \lfloor \Delta^v \rfloor$, we have

$$\Delta = \Delta^h + \Delta'' + \lfloor \Delta^v \rfloor$$

and $(Y, \Delta^h + \Delta'' + D)$ is log canonical. Since $\mathcal{L}$ is ample and globally generated, we therefore obtain that

$$f_*\mathcal{O}_Y(k(K_Y + \Delta^h + \Delta'' + f^*H)) \otimes \mathcal{L}^{\otimes \ell}$$

is generated by global sections for all $\ell \geq k(n+1)$ by [Popa and Schnell 2014, Variant 1.6]. But

$$f_*\mathcal{O}_Y(k(K_Y + \Delta^h + \Delta'' + f^*H)) \otimes \mathcal{L}^{\otimes \ell} \longrightarrow f_*\mathcal{O}_Y(P) \otimes \mathcal{L}^{\otimes \ell},$$

and they have the same stalks at every point $x \in U$. Thus, the sheaf on the right-hand side is generated by global sections at x for all $x \in U$ and for all $\ell \geq k(n+1)$. ☐

We will also need the following result, which is used in the proof of [loc. cit., Variant 1.6]:

Lemma 2.17 (cf. [Popa and Schnell 2014, p. 2280]). *Let $f : Y \to X$ be a morphism of projective varieties, and let $\mathscr{F}$ be a coherent sheaf on Y such that the image of the counit map*

$$f^* f_* \mathscr{F} \longrightarrow \mathscr{F}$$

of the adjunction $f^* \dashv f_*$ is of the form $\mathscr{F}(-E)$ for some effective Cartier divisor E on Y. Then, for every effective Cartier divisor $E' \preceq E$, we have $f_*(\mathscr{F}(-E')) \simeq f_*\mathscr{F}$.

Proof. We have the factorization

$$f^* f_* \mathscr{F} \longrightarrow \mathscr{F}(-E') \hookrightarrow \mathscr{F}$$

and by applying the adjunction $f^* \dashv f_*$, we have a factorization

$$\begin{tikzcd} f_*\mathscr{F} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & f_*(\mathscr{F}(-E')) \arrow[r, hook] & f_*\mathscr{F} \end{tikzcd}$$

of the identity. ☐

Finally, we record the following numerical argument that will appear in the proofs of Theorems A and B.

Lemma 2.18 (cf. [Popa and Schnell 2014, Theorem 1.7, Step 2]). *Let X be a smooth projective variety. Let Δ be an effective $\mathbb{R}$-Cartier divisor and E an effective $\mathbb{Z}$-divisor with simple normal crossing support such that $\Delta + E$ also has simple normal crossing support and Δ has coefficients in $(0, 1]$. Let $0 \leq c < 1$ be a real number. Then, there exists an effective Cartier divisor $E' \preceq E$ such that $\Delta + cE - E'$ has simple normal crossing support and coefficients in $(0, 1]$.*

Seshadri constants. The effectivity of our results in Theorems A and B relies on Seshadri constants. These were originally introduced by Demailly to measure local positivity of line bundles and thereby study Fujita-type conjectures. See [Lazarsfeld 2004a, Chapter 5] for more on these invariants.

Definition 2.19. Let X be a projective variety, and let $x \in X$ be a closed point. Let L be a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Denote by $\mu : X' \to X$ the blow-up of X at x with exceptional divisor E. The *Seshadri constant* of L at x is

$$\varepsilon(L; x) := \sup\{t \in \mathbb{R}_{\geq 0} \mid \mu^* L - tE \text{ is nef}\}.$$

If $\mathcal{L}$ is a nef line bundle, then we denote by $\varepsilon(\mathcal{L}; x)$ the Seshadri constant of the associated Cartier divisor L at x.

The following result is crucial in making our results effective.

Theorem 2.20 [Ein et al. 1995, Theorem 1]. *Let X be a projective variety of dimension n. Let L be a big and nef Cartier divisor on X. Then, for every $\delta > 0$, the locus*

$$\left\{ x \in X \;\middle|\; \varepsilon(L;x) > \frac{1}{n+\delta} \right\}$$

contains an open dense set.

Remark 2.21. If in the notation of Theorem 2.20 we also assume that X is smooth and L is ample, then better lower bounds are known if $n = 2, 3$. Under these additional assumptions, the locus

$$\left\{ x \in X \;\middle|\; \varepsilon(L;x) > \frac{1}{(n-1)+\delta} \right\}$$

contains an open dense set if $n = 2$ [Ein and Lazarsfeld 1993, Theorem] or $n = 3$ [Cascini and Nakamaye 2014, Theorem 1.2]. Here, we use [Ein et al. 1995, Lemma 1.4] to obtain results for general points from the cited results, which are stated for very general points. In general, it is conjectured that in the situation of Theorem 2.20, the locus

$$\left\{ x \in X \;\middle|\; \varepsilon(L;x) > \frac{1}{1+\delta} \right\}$$

contains an open dense set [Lazarsfeld 2004a, Conjecture 5.2.5].

The stable and augmented base locus. In order to deal with big and nef line bundles in Theorems A and C, we will need some facts about base loci, following [Ein et al. 2009].

Definition 2.22. Let X be a projective variety. If L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then the *stable base locus* of L is the closed set

$$\boldsymbol{B}(L) := \bigcap_m \mathrm{Bs}|mL|_{\mathrm{red}},$$

where m runs over all integers such that mL is Cartier. If L is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, the *augmented base locus* of L is the closed set

$$\boldsymbol{B}_+(L) := \bigcap_A \boldsymbol{B}(L-A),$$

where A runs over all ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors A such that $L - A$ is $\mathbb{Q}$-Cartier. By definition, if L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then

$$\boldsymbol{B}(L) \subseteq \boldsymbol{B}_+(L).$$

Note that $\boldsymbol{B}_+(L) \neq X$ if and only if L is big by Kodaira's lemma [Lazarsfeld 2004a, Proposition 2.2.22].

We will also need the following result, which shows how augmented base loci and Seshadri constants are related. The result follows from [Ein et al. 2009, §6] if the scheme X is a smooth variety, but we will need it more generally for singular varieties.

Corollary 2.23. *Let X be a projective variety, and let $x \in X$ be a closed point. Suppose L is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If $\varepsilon(L;x) > 0$, then $x \notin \boldsymbol{B}_+(L)$.*

Proof. If $x \in \boldsymbol{B}_+(L)$, then by [Birkar 2017, Theorem 1.4] there exists a closed subvariety $V \subseteq X$ containing x such that $L^{\dim V} \cdot V = 0$, in which case $\varepsilon(L; x) = 0$ by [Lazarsfeld 2004a, Proposition 5.1.9]. □

3. An extension theorem

We now turn to the proof of Theorem C. The proof relies on the following application of cohomology and base change.

Lemma 3.1. *Let $f : Y \to X$ be a proper morphism of separated noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on Y. Let $x \in X$ be a point that has an open neighborhood $U \subseteq X$, where $\mathscr{F}|_{f^{-1}(U)}$ is flat over U. Consider the following cartesian square:*

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec}(\kappa(x)) \arrow[r] & X \end{tikzcd}$$

If the restriction map

$$H^0(Y, \mathscr{F}) \longrightarrow H^0(Y_x, \mathscr{F}|_{Y_x})$$

is surjective, then the restriction map

$$H^0(X, f_*\mathscr{F}) \longrightarrow f_*\mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x)$$

is also surjective.

Proof. Let $f_U := f|_{f^{-1}(U)}$ and $\mathscr{F}_U := \mathscr{F}|_{f^{-1}(U)}$. We have the commutative diagram

$$\begin{tikzcd} H^0(X, f_*\mathscr{F}) \arrow[r] \arrow[dd, equal] & f_*\mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x) \arrow[d, "\beta", "\wr"'] \\ & f_{U*}\mathscr{F}_U \otimes_{\mathcal{O}_U} \kappa(x) \arrow[d, "\alpha^0(x)"] \\ H^0(Y, \mathscr{F}) \arrow[r, two heads] & H^0(Y_x, \mathscr{F}|_{Y_x}) \end{tikzcd}$$

where the bottom arrow is surjective by assumption, β is an isomorphism by computing affine-locally, and $\alpha^0(x)$ is the natural base change map [Illusie 2005, (8.3.2.3)]. By the commutativity of the diagram, this map $\alpha^0(x)$ is surjective, and hence is an isomorphism by cohomology and base change [loc. cit., Corollary 8.3.11]. Thus, the top horizontal arrow is also surjective. □

Before proving Theorem C, we first explain how to deduce a generic global generation statement for arbitrary log canonical $\mathbb{R}$-pairs (Y, Δ) from Theorem C by passing to a log resolution.

Corollary 3.2. *Let $f : Y \to X$ be a surjective morphism of projective varieties, where X is of dimension n. Let (Y, Δ) be a log canonical $\mathbb{R}$-pair, and let L be an big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let ℓ be a* *real number for which there exists a Cartier divisor P_ℓ on Y such that*

$$P_\ell \sim_{\mathbb{R}} K_Y + \Delta + \ell f^* L.$$

If $\ell > n/\varepsilon(L;x)$ for general $x \in X$, then the sheaf $f_*\mathcal{O}_Y(P_\ell)$ is generically globally generated.

Proof. Applying Lemma 2.14 for $H = \ell f^* L$ to a log resolution $\mu : \widetilde{Y} \to Y$ of (Y, Δ), we have the following commutative diagram:

$$\begin{array}{ccc} H^0(X, (f \circ \mu)_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell)) & \longrightarrow & (f \circ \mu)_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell) \otimes \kappa(x) \\ \uparrow\wr & & \uparrow\wr \\ H^0(X, f_* \mathcal{O}_Y(P_\ell)) & \longrightarrow & f_* \mathcal{O}_Y(P_\ell) \otimes \kappa(x) \end{array}$$

where $\widetilde{P}_\ell$ is the divisor on $\widetilde{Y}$ satisfying the properties in Lemma 2.14. Then, Theorem C for $(\widetilde{Y}, \tilde{\Delta})$ implies that for some open subset $U \subseteq X$, the top horizontal arrow is surjective for all closed points $x \in U$ such that $\ell > n/\varepsilon(L;x)$; hence the bottom horizontal arrow is also surjective at these closed points x. We therefore conclude that $f_* \mathcal{O}_Y(P_\ell)$ is generically globally generated. □

To prove Theorem C, we need the following result on augmented base loci.

Lemma 3.3. *Let X be a projective variety of dimension n, and let L be a big and nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $x \in X$ be a closed point, and suppose $\varepsilon(L;x) > 0$. Let $\mu : X' \to X$ be the blow-up of X at x with exceptional divisor E. For every positive real number $\delta < \varepsilon(L;x)$, we have*

$$\boldsymbol{B}_+(\mu^* L - \delta E) \cap E = \varnothing.$$

In particular, if $\mu^* L - \delta E$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then

$$\mathrm{Bs}|m(\mu^* L - \delta E)| \cap E = \varnothing$$

for all sufficiently large and divisible integers m.

Proof. First, the $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\mu^* L - \delta E$ is big and nef since

$$\mu^* L - \delta E \sim_{\mathbb{R}} \frac{\delta}{\varepsilon(L;x)}(\mu^* L - \varepsilon(L;x)E) + \left(1 - \frac{\delta}{\varepsilon(L;x)}\right)\mu^* L \tag{4}$$

is the sum of a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and a big and nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Thus, by [Birkar 2017, Theorem 1.4], we know that $\boldsymbol{B}_+(\mu^* L - \delta E)$ is the union of positive-dimensional closed subvarieties V of X' such that $(\mu^* L - \delta E)^{\dim V} \cdot V = 0$.

It suffices to show such a V cannot contain any point $y \in E$. First, if $V \subseteq E$, then

$$(\mu^* L - \delta E)^{\dim V} \cdot V = (-\delta E)^{\dim V} \cdot V = \delta^{\dim V} (-E|_E)^{\dim V} \cdot V > 0,$$

since $\mathcal{O}_E(-E) \simeq \mathcal{O}_E(1)$ is very ample. On the other hand, if $V \not\subseteq E$, then V is the strict transform of some closed subvariety $V_0 \subseteq X$ containing x, and by (4), we have

$$\begin{aligned}(\mu^*L - \delta E)^{\dim V} \cdot V &= \left(\frac{\delta}{\varepsilon(L;x)}(\mu^*L - \varepsilon(L;x)E) + \left(1 - \frac{\delta}{\varepsilon(L;x)}\right)\mu^*L\right)^{\dim V} \cdot V \\ &\geq \left(1 - \frac{\delta}{\varepsilon(L;x)}\right)^{\dim V} (\mu^*L)^{\dim V} \cdot V \\ &= \left(1 - \frac{\delta}{\varepsilon(L;x)}\right)^{\dim V} L^{\dim V} \cdot V_0 > 0,\end{aligned}$$

where the first inequality is by nefness of $\mu^*L - \varepsilon(L;x)E$, and the last inequality is by [Lazarsfeld 2004a, Proposition 5.1.9] and the condition $\varepsilon(L;x) > 0$.

The last statement about base loci follows from the fact that

$$\boldsymbol{B}_+(\mu^*L - \delta E) \supseteq \boldsymbol{B}(\mu^*L - \delta E) = \mathrm{Bs}|m(\mu^*L - \delta E)|_{\mathrm{red}}$$

for all sufficiently large and divisible integers m, where the last equality holds by [loc. cit., Proposition 2.1.21] since $\mu^*L - \delta E$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. □

Finally, we need the following cohomological injectivity theorem due to Fujino.

Theorem 3.4 [Fujino 2017a, Theorem 5.4.1]. *Let Y be a smooth complete variety and let Δ be an $\mathbb{R}$-divisor on Y with coefficients in $(0, 1]$ and simple normal crossing support. Let L be a Cartier divisor on Y and let D be an effective Weil divisor on Y whose support is contained in* Supp Δ. *Assume that $L \sim_{\mathbb{R}} K_Y + \Delta$. Then, the natural homomorphism*

$$H^i(Y, \mathcal{O}_Y(L)) \longrightarrow H^i(Y, \mathcal{O}_Y(L + D))$$

induced by the inclusion $\mathcal{O}_Y \to \mathcal{O}_Y(D)$ is injective for every i.

We can now prove Theorem C.

Proof of Theorem C. Fix $x \in U$, and consider the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{B} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{b} & X \end{array}$$

where b is the blow-up of X at x. Since f is flat in a neighborhood of x, the morphism B can be identified with the blow-up of Y along Y_x, which is a smooth subvariety of codimension n [Stacks 2018, Tag 0805]. Moreover, if E is the exceptional divisor of b and D is the exceptional divisor of B, then $f'^*E = D$. By Lemma 3.1, the surjectivity of (1) in the statement of Theorem C implies the generic global generation statement, so it suffices to show that the map in (1) is surjective.

First, we note that $(Y', B^*\Delta)$ is log canonical: since Y_x intersects every component of Δ transversely, the pullback $B^*\Delta$ of Δ is equal to the strict transform Δ' of Δ [Fulton 1984, Corollary 6.7.2], and so in particular, (Y', Δ') is log canonical.

Since $\varepsilon(L;x) > n/\ell$, we can choose a sufficiently small $\delta > 0$ such that $(n+\delta)/\ell \in \mathbb{Q}$ and $\varepsilon(L;x) > (n+\delta)/\ell$. Thus, using the fact that L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, for real numbers m of the form m_0/ℓ for sufficiently large and divisible integers m_0, we have that $m(\ell b^*L - (n+\delta)E)$ is Cartier. Lemma 3.3 then implies

$$S := \mathrm{Bs}|m(\ell b^*L - (n+\delta)E)|_{\mathrm{red}}$$

does not intersect E, i.e., $m(\ell b^*L - (n+\delta)E)$ is globally generated away from S, and in particular, is globally generated on an open set containing E. Thus, the pullback $m(\ell B^*f^*L - (n+\delta)D)$ of this divisor is globally generated away from $S' := f'^{-1}(S)$, and in particular is globally generated on an open set containing D. Choose

$$\mathfrak{D}_x \in |m(\ell B^*f^*L - (n+\delta)D)|$$

which is smooth and irreducible away from $f'^{-1}(S)$, and is such that the component of $\mathfrak{D}_x$ not contained in S' intersects each component of the support of Δ' transversely away from S'. Note that such a choice is possible by applying Bertini's theorem [Hartshorne 1977, Corollary III.10.9 and Remark III.10.9.3]. Since $\mathfrak{D}_x$ may have singularities along S', however, we will need to pass to a log resolution before applying Theorem 3.4.

By Theorem 2.15, there exists a common log resolution $\mu : \widetilde{Y} \to Y'$ for $\mathfrak{D}_x$ and (Y', Δ') that is an isomorphism away from $f'^{-1}(S) \subsetneq Y'$. We then write

$$\mu^*\mathfrak{D}_x = D' + F, \quad \mu^*\Delta' = \mu_*^{-1}\Delta' + F_1,$$

where D' is a smooth divisor intersecting Y_x transversely and F, F_1 are supported on $\mu^{-1}(S')$. Define

$$F' := \left\lfloor \frac{1}{m}F + F_1 \right\rfloor, \quad \tilde{\Delta} := \mu^*\Delta' + \frac{1}{m}\mu^*\mathfrak{D}_x - F' + \delta\mu^*D, \quad \widetilde{P}_\ell := \mu^*B^*P_\ell + K_{\widetilde{Y}/Y'}.$$

Note that $\tilde{\Delta}$ has simple normal crossing support containing μ^*D, and has coefficients in $(0,1]$ by assumption on the log resolution and by definition of F'. Note also that

$$\begin{aligned}
\widetilde{P}_\ell - F' &\sim_{\mathbb{R}} \mu^*B^*(K_Y + \Delta + \ell f^*L) + K_{\widetilde{Y}/Y'} - F' \\
&\sim_{\mathbb{R}} K_{\widetilde{Y}} + \mu^*\Delta' - F' + \mu^*(\ell B^*f^*L - (n-1)D) \\
&\sim_{\mathbb{R}} K_{\widetilde{Y}} + \mu^*\Delta' + \frac{1}{m}\mu^*\mathfrak{D}_x - F' + (1+\delta)\mu^*D \\
&\sim_{\mathbb{R}} K_{\widetilde{Y}} + \tilde{\Delta} + \mu^*D,
\end{aligned}$$

where the second equivalence follows from the fact that B is the blow-up of the smooth subvariety Y_x, which is of codimension n; hence

$$K_{\widetilde{Y}} = \mu^*K_{Y'} + K_{\widetilde{Y}/Y'} = \mu^*B^*K_Y + (n-1)\mu^*D + K_{\widetilde{Y}/Y'}.$$

We can now apply the injectivity result Theorem 3.4 to $\widetilde{P}_\ell - F' - \mu^* D \sim_{\mathbb{R}} K_{\widetilde{Y}} + \tilde{\Delta}$ to see that

$$H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell - F' - \mu^* D)) \longrightarrow H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell - F')) \tag{5}$$

is injective. Next, consider the following commutative diagram:

$$\begin{tikzcd}
H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell - F')) \arrow[r, two heads] \arrow[d, hook] & H^0(\mu^*(D), \mathcal{O}_{\mu^*(D)}(\widetilde{P}_\ell - F')) \arrow[d, hook, "\wr"] \\
H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\widetilde{P}_\ell)) \arrow[r] & H^0(\mu^*(D), \mathcal{O}_{\mu^*(D)}(\widetilde{P}_\ell)) \\
H^0(Y', \mathcal{O}_{Y'}(B^* P_\ell)) \arrow[r] \arrow[u, "\wr"] & H^0(D, \mathcal{O}_D(B^* P_\ell)) \arrow[u, "\wr"] \\
H^0(Y, \mathcal{O}_Y(P_\ell)) \arrow[r] \arrow[u, "\wr"] & H^0(Y_x, \mathcal{O}_{Y_x}(P_\ell)) \arrow[u, "\wr"]
\end{tikzcd}$$

The top right vertical arrow is an isomorphism since F' is disjoint from $\mu^*(D)$. The bottom right vertical arrow is an isomorphism since $B|_D$ realizes D as a projective bundle over Y_x; hence $(B|_D)_* \mathcal{O}_D \simeq \mathcal{O}_{Y_x}$. The other vertical isomorphisms follow from the projection formula and the fact that μ and B are birational. Finally, the top horizontal arrow is surjective by the long exact sequence on cohomology and the injectivity of (5). The commutativity of the diagram implies the bottom row is surjective, which is exactly the map in (1). □

4. Effective twisted weak positivity

We now prove Theorem E using Viehweg's fiber product trick. This trick enables us to reduce the global generation of the reflexivized s-fold tensor product $f_* \mathcal{O}_Y(k(K_Y + \Delta))^{[s]}$ to $s = 1$ with Y replaced by a suitable $\widetilde{Y}^s$. The main obstacle is picking a suitable boundary divisor on $\widetilde{Y}^s$. We tackle this using Theorem 2.16. Readers are encouraged to consult [Popa and Schnell 2014, §4], [Viehweg 1983, §3], or [Höring 2010, §3].

Throughout the proof we use $\mathcal{O}_X(K_X)$ and ω_X interchangeably whenever X is a normal variety. We can do so by Lemma 2.11.

Proof of Theorem E. For every positive integer s, let Y^s denote the reduction of the unique irreducible component of

$$\underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{s \text{ times}}$$

that surjects onto X; note that it is unique since f has irreducible generic fiber. Setting $V := f^{-1}(U)$, we define V^s similarly.

Let $d : Y^{(s)} \to Y^s$ be a desingularization of Y^s, and note that d is an isomorphism over V^s. We will also denote by V^s the image of V^s under any birational modification of Y^s which is an isomorphism along V^s. Define $d_i = \pi_i \circ d$ for $i \in \{1, 2, \ldots, s\}$, where $\pi_i : Y^s \to Y$ is the i-th projection. Since d_i is a surjective morphism between integral varieties, the pullback $d_i^* \Delta_j$ of the Cartier divisor Δ_j is well-defined for every component Δ_j of Δ; see [Stacks 2018, Tag 02OO(1)].

Let $\mu : \widetilde{Y}^s \to Y^{(s)}$ be a log resolution as in Theorem 2.15 of the pair $\left(Y^{(s)}, \sum_i d_i^* \Delta\right)$ so that μ is an isomorphism over V^s. Define

$$\tilde{\Delta} = \mu^* \sum_i d_i^* \Delta.$$

Claim 4.1. *There exists a map*

$$\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}(k(K_{\widetilde{Y}^s/X} + \tilde{\Delta})) \longrightarrow (f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]} \tag{6}$$

which is an isomorphism over U.

Let X_0 be the open set in X such that

- the map f is flat over X_0;
- the regular locus of X contains X_0; and
- the sheaf $f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta))$ is locally free over X_0.

Then, $\operatorname{codim}(X \smallsetminus X_0) \geq 2$. Indeed, X is normal and both $f_* \mathcal{O}_Y$ and $f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta))$ are torsion-free. Now by construction, we have $U \subseteq X_0$. Since $(f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]}$ is reflexive and is isomorphic to $(f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{\otimes s}$ on X_0, a map

$$\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}(k(K_{\widetilde{Y}^s/X} + \tilde{\Delta})) \longrightarrow (f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{\otimes s}$$

over X_0 will extend to a map of the form in (6) on X by Corollary 2.5. This, together with flat base change [Hartshorne 1977, Proposition III.9.3], implies that it suffices to construct a map

$$\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s_0}(k(K_{\widetilde{Y}^s_0/X_0} + \tilde{\Delta}|_{\widetilde{Y}^s_0})) \longrightarrow (f_* \mathcal{O}_{Y_0}(k(K_{Y_0/X_0} + \Delta|_{Y_0})))^{\otimes s}$$

which is an isomorphism over U.

Define $Y_0 := f^{-1}(X_0)$. In this case, by [Höring 2010, Corollary 5.24] we know that

$$Y_0^s := \underbrace{Y_0 \times_X Y_0 \times_X \cdots \times_X Y_0}_{s \text{ times}} \simeq \underbrace{Y_0 \times_{X_0} Y_0 \times_{X_0} \cdots \times_{X_0} Y_0}_{s \text{ times}}$$

and that Y_0^s is Gorenstein. We can therefore apply Lemma 2.9 to $d \circ \mu$, to obtain a morphism

$$(d \circ \mu)_* \omega^{\otimes k}_{\widetilde{Y}^s_0/X_0} \longrightarrow \omega^{\otimes k}_{Y^s_0/X_0}$$

which is an isomorphism over V^s. Here $\omega_{Y^s_0/X_0} := \omega_{Y_0} \otimes f^{s*} \omega^{-1}_{X_0}$ and we define $\omega_{\widetilde{Y}^s_0/X_0}$ similarly. This induces a map

$$\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s_0}(k(K_{\widetilde{Y}^s_0/X_0} + \tilde{\Delta}|_{\widetilde{Y}^s_0})) \longrightarrow f^s_* \left(\omega^{\otimes k}_{Y^s_0/X_0} \otimes \bigotimes_i \pi_i^* \mathscr{M}|_{Y^s_0} \right) \tag{7}$$

which is an isomorphism over U, where $\mathscr{M} := \mathcal{O}_Y(P - kK_Y)$ is the line bundle associated to the Cartier divisor $P - kK_Y \sim_{\mathbb{R}} k\Delta$.

We will now show that the sheaf on the right-hand side of (7) admits an isomorphism to

$$(f_*\mathcal{O}_{Y_0}(k(K_{Y_0/X_0} + \Delta|_{Y_0})))^{\otimes s}.$$

Note that this would show Claim 4.1, since (7) is an isomorphism over U. We proceed by induction, adapting the argument in [Höring 2010, Lemma 3.15] to our twisted setting. Note that the case $s = 1$ is clear, since in this case $Y^s = Y$ and the sheaves in question are equal.

By [loc. cit., Corollary 5.24] we have

$$\omega^{\otimes k}_{Y_0^s/X_0} \otimes \bigotimes_i \pi_i^*(\mathscr{M}|_{Y_0}) \simeq \pi_s^*(\omega^{\otimes k}_{Y_0/X_0} \otimes \mathscr{M}|_{Y_0}) \otimes \pi'^*(\omega^{\otimes k}_{Y_0^{s-1}/X_0} \otimes \mathscr{M}^{s-1}|_{Y_0^{s-1}}),$$

where $\pi' : Y^s \to Y^{s-1}$ and $\mathscr{M}^{s-1} := \bigotimes_{i=1}^{s-1} \pi_i^*\mathscr{M}$. Since $\omega^{\otimes k}_{Y_0^{s-1}/X_0} \otimes \mathscr{M}^{s-1}|_{Y_0^{s-1}}$ is locally free, by the projection formula we obtain

$$f_*^s\Bigl(\omega^{\otimes k}_{Y_0^s/X_0} \otimes \bigotimes_{i=1}^{s} \pi_i^*\mathscr{M}|_{Y_0}\Bigr) \simeq f_*\bigl((\omega^{\otimes k}_{Y_0/X_0} \otimes \mathscr{M}|_{Y_0}) \otimes \pi_{s*}\pi'^*(\omega^{\otimes k}_{Y_0^{s-1}/X_0} \otimes \mathscr{M}^{s-1}|_{Y_0^{s-1}})\bigr).$$

Now by flat base change [Hartshorne 1977, Proposition III.9.3],

$$\pi_{s*}\pi'^*(\omega^{\otimes k}_{Y_0^{s-1}/X_0} \otimes \mathscr{M}^{s-1}|_{Y_0^{s-1}}) \simeq f^* f_*^{s-1}(\omega^{\otimes k}_{Y_0^{s-1}/X_0} \otimes \mathscr{M}^{s-1}|_{Y_0^{s-1}}).$$

By induction the latter is isomorphic to

$$f^*(f_*\mathcal{O}_{Y_0}(k(K_{Y_0/X_0} + \Delta|_{Y_0}))^{\otimes s-1}).$$

Therefore

$$f_*^s\Bigl(\omega^{\otimes k}_{Y_0^s/X_0} \otimes \bigotimes_i \pi_i^*\mathscr{M}|_{Y_0}\Bigr) \simeq f_*\bigl(\omega^{\otimes k}_{Y_0/X_0} \otimes \mathscr{M}|_{Y_0} \otimes f^*\bigl(f_*\mathcal{O}_{Y_0}(k(K_{Y_0/X_0} + \Delta|_{Y_0}))^{\otimes s-1}\bigr)\bigr).$$

Since $f_*\mathcal{O}_Y(k(K_{Y/X} + \Delta))$ is locally free over X_0, we can apply the projection formula to obtain

$$f_*^s\Bigl(\omega^{\otimes k}_{Y_0^s/X_0} \otimes \bigotimes_i \pi_i^*\mathscr{M}|_{Y_0}\Bigr) \simeq (f_*\mathcal{O}_{Y_0}(k(K_{Y_0/X_0} + \Delta|_{Y_0})))^{\otimes s}.$$

This concludes the proof of Claim 4.1.

We now use Theorem 2.16 to finish the proof of Theorem E.

We first claim $\tilde{\Delta}$ satisfies the hypothesis of Theorem 2.16. To do so, first note that on π_i is flat over Y_0, and therefore by flat pullback of cycles we have

$$\pi_i^*(\Delta_j)|_{Y_0^s} = \pi_i^{-1}(\Delta_j|_{Y_0}) = Y_0 \times_{X_0} \cdots \times_{X_0} \underbrace{\Delta_j}_{i\text{-th position}} \times_{X_0} \cdots \times_{X_0} Y_0.$$

Since $Y_0 \supseteq V$ and both d and μ are isomorphisms over V^s, the pullbacks $\mu^*(\pi_i \circ d)^*\Delta_j^h|_{V^s}$ of the horizontal components of Δ are smooth above U for all $i \in \{1, 2, \ldots, s\}$. In other words, the components of $\tilde{\Delta}$ either do not intersect V^s, or intersect the fiber over x transversely for all $x \in U$. Thus,

$$\tilde{\Delta}|_{V^s} = \mu^{-1} d^{-1} \sum_i \pi_i^{-1}(\Delta^h|_V).$$

In particular, using Notation 2.1(b), we have that the horizontal part $\tilde{\Delta}^h$ equals the closure $\overline{\tilde{\Delta}|_{V^s}}$ of $\tilde{\Delta}|_{V^s}$ in $\widetilde{Y}^s$. We can therefore write

$$\tilde{\Delta} = \tilde{\Delta}^h + \tilde{\Delta}^v,$$

where by construction, the coefficients of $\tilde{\Delta}^h$ are in $(0, 1]$ and $\tilde{f}^s(\tilde{\Delta}^v) \cap U = \varnothing$.

Finally, we note from Mori's cone theorem [Kollár and Mori 1998, Theorem 1.24] that $H = \omega_X \otimes \mathcal{L}^{\otimes n+1}$ is nef and hence semiample by the base point free theorem [loc. cit., Theorem 3.3]. Therefore $f^* H^{\otimes(\ell-k)}$ is also semiample for all $\ell \geq k$. Using H again to denote a divisor class of H, we argue that since

$$\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}(k(K_{\widetilde{Y}^s/X} + \tilde{\Delta})) \otimes H^{\otimes \ell} \simeq \tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}\bigl(k(K_{\widetilde{Y}^s} + \tilde{\Delta} + (\ell-k)\tilde{f}^{s*}H)\bigr) \otimes \mathcal{L}^{\otimes k(n+1)}, \tag{8}$$

with $\mathcal{L}$ ample and globally generated, we can apply Theorem 2.16 to conclude that the sheaf above in (8) is generated by global sections over U for all $\ell \geq k$. Now fix a closed point $x \in U$. We have the commutative diagram

$$\begin{tikzcd}
H^0\bigl(X, \tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}(k(K_{\widetilde{Y}^s/X} + \tilde{\Delta})) \otimes H^{\otimes \ell}\bigr) \arrow[r, two heads] \arrow[d] & \bigl(\tilde{f}^s_* \mathcal{O}_{\widetilde{Y}^s}(k(K_{\widetilde{Y}^s/X} + \tilde{\Delta})) \otimes H^{\otimes \ell}\bigr) \otimes \kappa(x) \arrow[d, "\wr"] \\
H^0\bigl(X, (f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]} \otimes H^{\otimes \ell}\bigr) \arrow[r] & \bigl((f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]} \otimes H^{\otimes \ell}\bigr) \otimes \kappa(x)
\end{tikzcd}$$

where the vertical arrows are induced by the map (6) from Claim 4.1, and the top horizontal arrow is surjective by the global generation of the sheaves in (8) over U. Since (6) is an isomorphism over U, the right vertical arrow is an isomorphism; hence by the commutativity of the diagram, the bottom horizontal arrow is surjective. We therefore conclude that

$$(f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]} \otimes H^{\otimes \ell}$$

is generated by global sections over U for all $\ell \geq k$. □

Remark 4.2. When $\lfloor \Delta \rfloor = 0$, if we moreover take $U(f, \Delta)$ to be an open set over which every stratum of (Y, Δ) is smooth, then applying invariance of log plurigenera [Hacon et al. 2018, Theorem 4.2], we can assert that $f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta))|_{U(f,\Delta)}$ is locally free. In this case we can take X_0 to be simply the locus inside X_{reg} over which f is flat. Moreover, the isomorphism

$$(f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{\otimes s} \simeq (f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta)))^{[s]}$$

automatically holds over $U(f, \Delta)$. Thus, Theorem E holds more generally over $U(f, \Delta)$.

We now deduce Theorem D from Theorem E.

Proof of Theorem D. Using Lemma 2.14, we assume that Y is smooth and Δ has simple normal crossing support. Then, Theorem E implies

$$(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))^{[s]} \otimes H^{\otimes \ell}$$

is generated by global sections for all $\ell \geq k$ on an open set $U \subseteq X$. Since $f_*\mathcal{O}_Y(k(K_{Y/X}+))$ is locally free over U, the map

$$(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))^{[s]} \longrightarrow \mathrm{Sym}^{[s]}(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))$$

is surjective over U; hence

$$\mathrm{Sym}^{[s]}(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta))) \otimes H^{\otimes \ell}$$

is also generated by global sections for all $\ell \geq k$ on U.

Note that for any ample line bundle $\mathcal{L}$, there is an integer $b \geq 1$ such that $H^{\otimes -k} \otimes \mathcal{L}^{\otimes b}$ is globally generated. For such a b, the sheaf

$$\mathrm{Sym}^{[s]}(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta))) \otimes \mathcal{L}^{\otimes b}$$

is also generated by global sections on U. Since b depends only on k and H and is independent of s, we can set $s = ab$. This implies weak positivity of $f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta))$ over U. □

Remark 4.3. The proof of Theorem D shows that when Y is smooth and Δ has simple normal crossing support, the sheaf $f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta))$ is weakly positive over the open set in the statement of Theorem E.

5. Generic generation for pluricanonical sheaves

Proof of Theorem A. We now prove Theorem A, following the strategy in [Popa and Schnell 2014, Theorem 1.7] and [Dutta 2017, Theorem A]. The idea is to reduce to the case where Y is smooth and Δ has simple normal crossing support, and then maneuver into a situation to which Theorem C applies.

Proof of Theorem A. We start with some preliminary reductions.

Step 0: We may assume that the image of the counit morphism

$$f^* f_*\mathcal{O}_Y(P) \longrightarrow \mathcal{O}_Y(P) \tag{9}$$

for the adjunction $f^* \dashv f_*$ is nonzero.

Suppose the image of (9) is the zero sheaf. Then, the natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^* f_*\mathcal{O}_Y(P), \mathcal{O}_Y(P)) \simeq \mathrm{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y(P), f_*\mathcal{O}_Y(P))$$

from the adjunction $f^* \dashv f_*$ implies that the identity morphism $\mathrm{id}: f_*\mathcal{O}_Y(P) \to f_*\mathcal{O}_Y(P)$ is the zero morphism. This implies $f_*\mathcal{O}_Y(P) = 0$; hence the conclusion of Theorem A trivially holds.

Step 1 (cf. [Popa and Schnell 2014, Theorem 1.7, Step 1]): We can reduce to the case where

(a) Y is smooth;

(b) Δ has simple normal crossing support and coefficients in $(0, 1]$; and

(c) the image of (9) is of the form $\mathcal{O}_Y(P - E)$ for a divisor E such that $\Delta + E$ has simple normal crossing support.

A priori, the image of the counit (9) is of the form $\mathfrak{b} \cdot \mathcal{O}_Y(P)$, where $\mathfrak{b} \subseteq \mathcal{O}_Y$ is the *relative base ideal* of $\mathcal{O}_Y(P)$. By Step 0, this ideal is nonzero, and so consider a simultaneous log resolution $\mu : \widetilde{Y} \to Y$ of $\mathfrak{b}$ and (Y, Δ). The image of the counit morphism

$$\mu^* f^* f_* \mathcal{O}_Y(P) \longrightarrow \mu^* \mathcal{O}_Y(P) = \mathcal{O}_{Y'}(\mu^* P) \tag{10}$$

is the sheaf $\mathcal{O}_{Y'}(\mu^* P - E')$ [Lazarsfeld 2004b, Generalization 9.1.17].

We then apply Lemma 2.14 to μ. With the notation of the lemma we note that on $\widetilde{Y}$ the counit morphism (10) becomes the surjective morphism

$$(f \circ \mu)^* (f \circ \mu)_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}) \twoheadrightarrow \mathcal{O}_{\widetilde{Y}}(\mu^* P - E') = \mathcal{O}_{\widetilde{Y}}(\widetilde{P} - (\widetilde{P} - \mu^* P) - E').$$

Setting $E := (\widetilde{P} - \mu^* P) + E'$, we see that (c) holds for $\widetilde{P}$.

Finally, Theorem A for $(\widetilde{Y}, \tilde{\Delta})$ and $\widetilde{P}$ implies that

$$(f \circ \mu)_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell} \simeq f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$$

is generated by global sections on some open set U for $\ell \geq k(n^2 + 1)$. This concludes Step 1.

Henceforth, we work in the situation of Step 1. Before moving on to Step 2, we fix some notation. Let L denote the divisor class of $\mathcal{L}$. Let U be the subset of $U(f, \Delta + E)$ where

$$\varepsilon(\mathcal{L}; x) > \frac{1}{n + \frac{1}{kn}}$$

for every $x \in U$, which is nonempty by Notation 2.1(a) and Theorem 2.20.

We set m to be the smallest positive integer such that $f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$ is globally generated on U. This integer m exists by [Küronya 2013, Proposition 2.7] since $U \cap \boldsymbol{B}_+(L) = \varnothing$ by Corollary 2.23.

Finally, we set $B := \mathrm{Bs}|P - E + mf^* L|_{\mathrm{red}} \subsetneq Y$ and note that $B \cap f^{-1}(U) = \varnothing$.

Step 2: Reducing the problem to $k = 1$ and a suitable pair.

From now on, fix a closed point $x \in U$.

The surjection

$$f^* f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_Y} f^* \mathcal{L}^{\otimes m} \twoheadrightarrow \mathcal{O}_Y(P - E) \otimes_{\mathcal{O}_Y} f^* \mathcal{L}^{\otimes m}$$

implies that $\mathcal{O}_Y(P - E) \otimes_{\mathcal{O}_Y} f^* \mathcal{L}^{\otimes m}$ is globally generated on $f^{-1}(U)$. Choose a general member

$$\mathfrak{D}_x \in |P - E + mf^* L|.$$

By Bertini's theorem [Hartshorne 1977, Corollary III.10.9 and Remark III.10.9.3], we may assume that $\mathfrak{D}_x$ is smooth away from the base locus B of the linear system $|P - E + mf^* L|$. We may also assume that

$\mathfrak{D}_x$ intersects the fiber Y_x transversely, and the support of Δ and E transversely away from B [Lazarsfeld 2004a, Lemma 4.1.11]. We then have

$$k(K_Y+\Delta)\sim_{\mathbb{R}} K_Y+\Delta+\frac{k-1}{k}\mathfrak{D}_x+\frac{k-1}{k}E-\frac{k-1}{k}mf^*L;$$

hence for every integer ℓ,

$$k(K_Y+\Delta)+\ell f^*L\sim_{\mathbb{R}} K_Y+\Delta+\frac{k-1}{k}\mathfrak{D}_x+\frac{k-1}{k}E+\left(\ell-\frac{k-1}{k}m\right)f^*L.$$

We now adjust the coefficients of Δ and E so they do not share any components. Applying Lemma 2.18 to $c=(k-1)/k$, we see that there exists an effective divisor $E'\preceq E$ such that

$$\Delta':=\Delta+\frac{k-1}{k}E-E'$$

is effective with simple normal crossing support, with components intersecting Y_x transversely, and with coefficients in $(0,1]$. We can then write

$$P-E'+\ell f^*L\sim_{\mathbb{R}} K_Y+\Delta'+\frac{k-1}{k}\mathfrak{D}+\left(\ell-\frac{k-1}{k}m\right)f^*L. \tag{11}$$

Step 3: Applying Theorem C to obtain global generation.

By Lemma 2.17, we have $f_*\mathcal{O}_X(P-E')\simeq f_*\mathcal{O}_X(P)$. It therefore suffices to show that

$$f_*\mathcal{O}_Y(P-E')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell} \tag{12}$$

is globally generated at x. We first modify $\mathfrak{D}_x$ to allow us to apply Theorem C. By Theorem 2.15, there exists a common log resolution $\mu:\widetilde{Y}\to Y$ for $\mathfrak{D}_x$ and (Y,Δ) that is an isomorphism away from $B\subsetneq Y$. We then write

$$\mu^*\mathfrak{D}_x=D+F,\quad \mu^*\Delta'=\mu_*^{-1}\Delta'+F_1,$$

where D is a smooth prime divisor intersecting the fiber over x transversely and F, F_1 are supported on $\mu^{-1}(B)$. Define

$$F':=\left\lfloor\frac{k-1}{k}F+F_1\right\rfloor,\quad \tilde{\Delta}:=\mu^*\Delta'+\frac{k-1}{k}\mu^*\mathfrak{D}_x-F',\quad \widetilde{P}:=\mu^*P+K_{\widetilde{Y}/Y}.$$

Note that $\tilde{\Delta}$ has simple normal crossing support and coefficients in $(0,1]$ by assumption on the log resolution and by definition of F'. Moreover, the support of $\tilde{\Delta}$ intersects the fiber over x transversely. Pulling back the decomposition in (11) and adding $K_{\widetilde{Y}/Y}-F'$ yields

$$\begin{aligned}\widetilde{P}-\mu^*E'-F'+\ell(f\circ\mu)^*L&\sim_{\mathbb{R}} K_{\widetilde{Y}}+\mu^*\Delta'+\frac{k-1}{k}\mu^*\mathfrak{D}_x-F'+\left(\ell-\frac{k-1}{k}m\right)(f\circ\mu)^*L\\&\sim_{\mathbb{R}} K_{\widetilde{Y}}+\tilde{\Delta}+\left(\ell-\frac{k-1}{k}m\right)(f\circ\mu)^*L.\end{aligned} \tag{13}$$

We now claim that it suffices to show

$$(f \circ \mu)_* \mathcal{O}_{\widetilde{Y}}(\widetilde{P} - \mu^* E' - F') \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell} \tag{14}$$

is globally generated at x. Consider the commutative diagram

$$\begin{tikzcd}
H^0\big(X, (f\circ\mu)_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P}-\mu^*E'-F')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big) \arrow[r] \arrow[d, hook] & \big((f\circ\mu)_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P}-\mu^*E'-F')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big)\otimes\kappa(x) \arrow[d, hook, "\wr"] \\
H^0\big(X, (f\circ\mu)_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P}-\mu^*E')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big) \arrow[r] \arrow[d, "\wr"] & \big((f\circ\mu)_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P}-\mu^*E')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big)\otimes\kappa(x) \arrow[d, "\wr"] \\
H^0\big(X, f_*(\mu_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P})\otimes\mathcal{O}_Y(-E'))\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big) \arrow[r] \arrow[d, "\wr"] & \big(f_*(\mu_*\mathcal{O}_{\widetilde{Y}}(\widetilde{P})\otimes\mathcal{O}_Y(-E'))\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big)\otimes\kappa(x) \arrow[d, "\wr"] \\
H^0\big(X, f_*\mathcal{O}_Y(P-E')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\big) \arrow[r] & f_*\mathcal{O}_Y(P-E')\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}\otimes\kappa(x)
\end{tikzcd}$$

where the top right isomorphism holds since F' is supported away from $(f \circ \mu)^{-1}(U)$; hence the stalks of the two sheaves are isomorphic, and the other isomorphisms follow from the projection formula and the fact that $K_{\widetilde{Y}/Y}$ is μ-exceptional. If the top horizontal arrow is surjective, then the commutativity of the diagram implies that the bottom horizontal arrow is also surjective, i.e., the sheaf in (12) is globally generated at x.

We now apply Theorem C to the decomposition (13) to see that the sheaf in (14) is globally generated at x for all

$$\ell - \frac{k-1}{k} m > \frac{n}{\varepsilon(\mathcal{L}; x)}.$$

By choice of U, we know that

$$\varepsilon(\mathcal{L}; x) > \frac{1}{n + \frac{1}{kn}}$$

at all $x \in U$, and so by applying the same argument used so far to all $x \in U$, we see $f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$ is globally generated on U for all

$$\ell > n\left(n + \frac{1}{kn}\right) + \frac{k-1}{k} m = n^2 + \frac{1}{k} + \frac{k-1}{k} m.$$

By the minimality of m, we know

$$m \le \left\lfloor n^2 + \frac{1}{k} + \frac{k-1}{k} m \right\rfloor + 1 \le n^2 + \frac{k-1}{k} m + 1.$$

The inequality between the leftmost and rightmost quantities is equivalent to $m \le k(n^2+1)$; that is, $f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$ is globally generated on U for $\ell \ge k(n^2+1)$. $\square$

Proof of Theorem B. Restricting to X smooth and $\mathcal{L}$ ample, we now show a slightly better bound. The strategy of Theorem B is the same as that for Theorem A: We first reduce to the case when Y is smooth and Δ has simple normal crossing support. Then, using twisted weak positivity this time, we maneuver to a situation in which we can apply Theorem C or [Dutta 2017, Proposition 1.2].

Proof of Theorem B. We begin with Steps 0 and 1 of the proof of Theorem A to reduce to a situation where Y is smooth and Δ has simple normal crossing support. Following Step 1, we also assume that there exists an effective divisor E with simple normal crossing support such that

$$f^* f_* \mathcal{O}_Y(P) \longrightarrow \mathcal{O}_Y(P-E) \tag{15}$$

is surjective.

Step 2: Reducing the problem to $k = 1$ and a suitable pair.

Unless otherwise mentioned, throughout this proof we fix U to denote the intersection of $U(f, \Delta + E)$ with the open set over which $f_* \mathcal{O}_Y(P)$ is locally free.

In the diagram

$$\begin{tikzcd}
f^*\big((f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))^{\otimes b}\big) \arrow[r, two heads] \arrow[d] & \mathcal{O}_Y(bk(K_{Y/X}+\Delta)-bE) \arrow[d, equal] \\
f^*\big((f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))^{[b]}\big) \arrow[r, dashed] & \mathcal{O}_Y(bk(K_{Y/X}+\Delta)-bE)
\end{tikzcd}$$

the dashed map exists making the diagram commute. Indeed, the map exists over the locus X_1 where $f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta))$ is locally free. Since X_1 has a complement of codimension ≥ 2, and the bottom right sheaf is locally free, we can extend the dashed map to all of X (Corollary 2.5).

Now the top arrow is the surjective map obtained by taking the b-th tensor power of (15). Then the commutativity of the diagram implies that the bottom arrow is also surjective. By Theorem E we know that over U,

$$f_* \mathcal{O}_Y(k(K_{Y/X} + \Delta))^{[b]} \otimes \mathcal{L}^{\otimes b}$$

is generated by global sections for $b \gg 1$. Therefore so is $\mathcal{O}_Y(bk(K_{Y/X} + \Delta) - bE) \otimes f^* \mathcal{L}^{\otimes b}$ over $f^{-1}(U)$.

We now fix a point $x \in U$.

Letting L denote a Cartier divisor class of $\mathcal{L}$, we can apply Bertini's theorem to choose a divisor

$$D \in |bk(K_{Y/X} + \Delta) - bE + bf^* L|$$

such that on $f^{-1}(U)$, D is smooth, $D + \Delta + E$ has simple normal crossing support, D is not contained in the support of $\Delta + E$, and D intersects the fiber over x transversely. Then write

$$\frac{1}{b} D \sim_{\mathbb{R}} k(K_{Y/X} + \Delta) - E + f^* L.$$

Multiplying both sides by $\frac{k-1}{k}$, and then adding $K_{Y/X} + \Delta + \frac{k-1}{k}E$, we have

$$K_{Y/X} + \Delta + \frac{k-1}{kb}D + \frac{k-1}{k}E \sim_{\mathbb{R}} k(K_{Y/X} + \Delta) + \frac{k-1}{k}f^*L. \tag{16}$$

Now applying Lemma 2.18 for $c = \frac{k-1}{k}$, there exists an effective divisor $E' \preceq E$ such that

$$\Delta' := \Delta + \frac{k-1}{k}E - E'$$

has coefficients in $(0, 1]$. Subtracting $E' + \frac{k-1}{k}f^*L$ from both sides in (16), we can therefore write

$$K_{Y/X} + \frac{k-1}{kb}D + \Delta' - \frac{k-1}{k}f^*L \sim_{\mathbb{R}} k(K_{Y/X} + \Delta) - E'.$$

Let us now denote by H the line bundle $\omega_X \otimes \mathcal{L}^{\otimes n+1}$ and a divisor class in it at the same time. For a positive integer ℓ, we add $f^*K_X + (k-1)f^*H + (\ell - (k-1)(n+1))f^*L$ to both sides to obtain

$$K_Y + \frac{k-1}{kb}D + \Delta' + (k-1)f^*H + \left(\ell - \frac{k-1}{k} - (k-1)(n+1)\right)f^*L \sim_{\mathbb{R}} P - E' + \ell f^*L. \tag{17}$$

As noted earlier $E' \preceq E$ is an effective Cartier divisor and therefore

$$f_*\mathcal{O}_Y(P - E') \simeq f_*\mathcal{O}_Y(P)$$

by Lemma 2.17. Moreover since the right-hand side of (17) is a Cartier divisor, it is enough to tackle the generation of the left side.

Step 3: Applying Theorem C to obtain global generation.

First, we need to modify D to be able to apply Theorem C.

Let $\mu : Y' \to Y$ be a log resolution of $\frac{k-1}{kb}D + \Delta'$ as in Theorem 2.15. Such a modification is an isomorphism over $f^{-1}(U)$ by choice of D. Write

$$\mu^*D = \widetilde{D} + F, \quad \mu^*\Delta' = \tilde{\Delta}' + F_1,$$

where $\widetilde{D}$ is the strict transform of the components of D that lie above U and $\tilde{\Delta}'$ is the strict transform of Δ'. Note that both F and F_1 has support outside of $f^{-1}(U)$.

Define

$$F' := \left\lfloor \frac{k-1}{kb}F + F_1 \right\rfloor, \quad \tilde{\Delta} := \mu^*D + \mu^*\Delta' - F', \quad \widetilde{P} := \mu^*P + K_{Y'/Y}.$$

By definition $\tilde{\Delta}$ has coefficients in $(0, 1]$. Now pulling back (17) and adding $K_{Y'/Y} - F'$ we and rewrite (17) as

$$K_{Y'} + \tilde{\Delta} + (k-1)\mu^*f^*H + \left(\ell - \frac{k-1}{k} - (k-1)(n+1)\right)\mu^*f^*L \sim_{\mathbb{R}} \widetilde{P} - \mu^*E' + \ell\mu^*f^*L - F'.$$

This can be compared to (13). By the arguments following (13) we can say that it is enough to show global generation for the pushforward of the left side under $f \circ \mu$ to deduce desired global generation for $f_*\mathcal{O}_Y(P) \otimes \mathcal{L}^{\otimes \ell}$ for suitable ℓ.

To do so, we note once again that from Mori theory it follows that $H = \omega_X \otimes \mathcal{L}^{\otimes n+1}$ is semiample. Therefore $(k-1)\mu^* f^* H$ is also semiample. Applying Bertini's theorem one more time we can pick an effective fractional $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} (k-1)\mu^* f^* H$ with smooth support and its support intersects components of $\tilde{\Delta} + D'$ and the fiber over x transversely. We can now rewrite the linear equivalence as

$$K_{Y'} + \tilde{\Delta} + D' + \left(\ell - \frac{k-1}{k} - (k-1)(n+1)\right)\mu^* f^* L \sim_{\mathbb{R}} \widetilde{P} - \mu^* E' + \ell\mu^* f^* L - F'. \tag{18}$$

Note that $\tilde{\Delta} + D'$ on the left-hand side of (18) has simple normal crossing support with coefficients in $(0, 1]$ and $\operatorname{Supp}(\tilde{\Delta} + D')$ intersects the fiber over x transversely. Thus, we can apply Theorem C on the left-hand side to conclude that

$$f_*\mathcal{O}_Y(P) \otimes \mathcal{L}^{\otimes \ell}$$

is generated by global sections over U for all

$$\ell > \frac{n}{\varepsilon(L; x)} + k(n+1) - n - \frac{1}{k}.$$

After possibly shrinking U we assume that

$$\varepsilon(\mathcal{L}; x) > \frac{1}{n + \frac{1}{n(k+1)}}$$

for all points $x \in U$, and hence

$$\ell > n\left(n + \frac{1}{n(k+1)}\right) + k(n+1) - n - \frac{1}{k} = k(n+1) + n^2 - n - \frac{1}{k(k+1)}.$$

Therefore, $\ell \geq k(n+1) + n^2 - n$. This proves (i).

Step 4: The case of klt $\mathbb{Q}$-pairs.

When Δ is a klt $\mathbb{Q}$-pair, we apply [Dutta 2017, Proposition 1.2] on the left-hand side of (18). To do so, we first trace the construction of $\tilde{\Delta} + D'$ to note that its coefficients lie in $(0, 1)$. We then apply the proposition with

$$H = \frac{1}{k}\mu^* f^* L, \quad A = (\ell - k(n+1) + n)\mu^* f^* L$$

to obtain global generation on U for all $\ell > k(n+1) + \frac{1}{2}(n^2 - n)$. This proves (ii). □

We summarize below the locus of global generation for Theorems A and B:

Remark 5.1. When Y is smooth and the relative base locus of P is an effective divisor E such that $\Delta + E$ has simple normal crossing support, the open set U for Theorem A contains the largest open subset of $U(f, \Delta+E)$ such that $\varepsilon(\mathcal{L};x) > \left(n+\frac{1}{kn}\right)^{-1}$ and for Theorem B(i), U contains the intersection of $U(f, \Delta+E)$, the locus where $f_*\mathcal{O}_Y(P)$ is locally free, and the open set where

$$\varepsilon(\mathcal{L};x) > \left(n+\frac{1}{n(k+1)}\right)^{-1}.$$

Finally, for Theorem B(ii), U contains the intersection of $U(f, \Delta+E)$ and the locus where $f_*\mathcal{O}_Y(P)$ is locally free.

Remark 5.2. Using the better bounds in Remark 2.21 for low dimensions ($n=2,3$), one can show that the lower bounds $\ell \geq k(n^2-n+1)$ in Theorem A and $\ell \geq k(n+1)+n^2-2n$ in Theorem B suffice when X is smooth and $\mathcal{L}$ is ample. In particular, Conjecture 1.1 for generic global generation holds for $n=2$. In the klt case, the conjectured lower bound in fact holds when $n \leq 4$ as was observed in [Dutta 2017].

If the conjectured lower bound for Seshadri constants in Remark 2.21 holds, then Theorem A would hold for the lower bound $\ell \geq k(n+1)$, thereby proving this generic version of Conjecture 1.1 in higher dimensions for big and nef line bundles.

An effective vanishing theorem. With the help of our effective twisted weak positivity, we improve the effective vanishing statement in [Dutta 2017, Theorem 3.1]:

Theorem 5.3. *Let $f: Y \to X$ be a smooth fibration of smooth projective varieties with* $\dim X = n$. *Let Δ be a $\mathbb{Q}$-divisor with simple normal crossing support with coefficients in* $[0,1)$ *such that every stratum of (Y,Δ) is smooth and dominant over X, and let $\mathcal{L}$ be an ample line bundle on X. Assume also that for some fixed integer $k \geq 1$, $k(K_Y+\Delta)$ is Cartier and $\mathcal{O}_Y(k(K_Y+\Delta))$ is relatively base point free. Then, for every $i > 0$ and all $\ell \geq k(n+1)-n$, we have*

$$H^i\big(X, f_*\mathcal{O}_Y(k(K_Y+\Delta)) \otimes \mathcal{L}^{\otimes \ell}\big) = 0.$$

Moreover, if K_X is semiample, for every $i > 0$ and every ample line bundle $\mathcal{L}$ we have

$$H^i\big(X, f_*\mathcal{O}_Y(k(K_Y+\Delta)) \otimes \mathcal{L}\big) = 0.$$

Proof. The hypothesis on f and Δ ensures invariance of log plurigenera, as noted in Remark 4.2; hence $f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta))$ is locally free. This means $U(f,\Delta) = X$. Furthermore, by the description of the open set in the proof of Theorem D, we have that there exists a positive integer b such that

$$(f_*\mathcal{O}_Y(k(K_{Y/X}+\Delta)))^{[b]} \otimes \mathcal{L}^{\otimes b}$$

is globally generated everywhere on X. Now since $\mathcal{O}_Y(k(K_Y+\Delta))$ is relatively base point free, we can choose a divisor $\frac{1}{b}D \sim_{\mathbb{R}} k(K_{Y/X}+\Delta) + f^*L$ satisfying the Bertini-type properties as in Step 2 of Theorem B. Define $H := K_X + (n+1)L$, which is semiample by Mori's cone theorem and the base point free theorem. As before, we then write

$$K_Y + \Delta + \frac{k-1}{kb}D + (k-1)f^*H + \left(\ell - \frac{k-1}{k} - (k-1)(n+1)\right)f^*L \sim_{\mathbb{R}} k(K_Y+\Delta) + \ell f^*L.$$

Since the divisor $\Delta + \frac{k-1}{kb} D$ is klt and $(k-1)H + \big(\ell - \frac{k-1}{k} - (k-1)(n+1)\big)L$ is ample for all $\ell \geq k(n+1) - n$, by Kollár's vanishing theorem [1995, Theorem 10.19] we obtain

$$H^i\big(X, f_*\mathcal{O}_Y(k(K_Y + \Delta)) \otimes \mathcal{L}^{\otimes \ell}\big) = 0$$

for all $\ell \geq k(n+1) - n$ and for all $i > 0$.

Moreover, when K_X is already semiample, we take $H = K_X$. In this case, the linear equivalence above looks as follows:

$$K_Y + \Delta + \frac{k-1}{kb} D + (k-1) f^*H + \left(\ell - \frac{k-1}{k}\right) f^*L \sim_{\mathbb{R}} k(K_Y + \Delta) + \ell f^*L.$$

Then, we obtain the desired vanishing for all $\ell \geq 1$ and $i > 0$. □

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