Algebra & Number Theory

Volume 7

2013

No. 3

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ALGEBRA AND NUMBER THEORY 7:3 (2013)
dx.doi.org/10.2140/ant.2013.7.595

The biHecke monoid of a finite Coxeter group and its representations

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For any finite Coxeter group W, we introduce two new objects: its cutting poset and its biHecke monoid. The cutting poset, constructed using a generalization of the notion of blocks in permutation matrices, almost forms a lattice on W. The construction of the biHecke monoid relies on the usual combinatorial model for the 0-Hecke algebra $H_0(W)$, that is, for the symmetric group, the algebra (or monoid) generated by the elementary bubble sort operators. The authors previously introduced the Hecke group algebra, constructed as the algebra generated simultaneously by the bubble sort and antisort operators, and described its representation theory. In this paper, we consider instead the *monoid* generated by these operators. We prove that it admits $|W|$ simple and projective modules. In order to construct the simple modules, we introduce for each $w \in W$ a combinatorial module T_w whose support is the interval $[1, w]_R$ in right weak order. This module yields an algebra, whose representation theory generalizes that of the Hecke group algebra, with the combinatorics of descents replaced by that of blocks and of the cutting poset.

1. Introduction

In this paper we introduce two novel objects for any finite Coxeter group W: its *cutting poset* and its *biHecke monoid*. The cutting poset is constructed using a generalization of blocks in permutation matrices to any Coxeter group and is almost a lattice. The biHecke monoid is generated simultaneously by the sorting and antisorting operators associated to the combinatorial model of the 0-Hecke algebra $H_0(W)$. It turns out that the representation theory of the biHecke monoid, and in particular the construction of its simple modules, is closely tied to the cutting poset.

Hivert was partly supported by ANR grant 06-BLAN-0380. Schilling was in part supported by NSF grants DMS–0652641, DMS–0652652, and DMS–1001256. Thiérly was in part supported by NSF grants DMS–0652641 and DMS–0652652.

MSC2010: primary 20M30, 20F55; secondary 06D75, 16G99, 20C08.

Keywords: Coxeter groups, Hecke algebras, representation theory, blocks of permutation matrices.

The study of these objects combines methods from and impacts several areas of mathematics: Coxeter group theory, monoid theory, representation theory, combinatorics (posets, permutations, descent sets), as well as computer algebra. The guiding principle is the use of representation theory, combined with computer exploration, to extract combinatorial structures from an algebra, and in particular a monoid algebra, often in the form of posets or lattices. This includes the structures associated to monoid theory (such as for example Green's relations), but also goes beyond. For example, we find connections between the classical orders of Coxeter groups (left, right, and left-right weak order and Bruhat order) and Green's relations on our monoids ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{H}$-order and ordered monoids), and these orders play a crucial role in the combinatorics and representation theory of the biHecke monoid.

The usual combinatorial model for the 0-Hecke algebra $H_0(\mathfrak{S}_n)$ of the symmetric group is the algebra (or monoid) generated by the (anti) bubble sort operators $\pi_1, \dots, \pi_{n-1}$, where π_i acts on words of length n and sorts the letters in positions i and $i+1$ decreasingly. By symmetry, one can also construct the bubble sort operators $\bar{\pi}_1, \dots, \bar{\pi}_{n-1}$, where $\bar{\pi}_i$ acts by sorting increasingly, and this gives an isomorphic construction $\overline{H}_0$ of the 0-Hecke algebra. This construction generalizes naturally to any finite Coxeter group W. Furthermore, when W is a Weyl group, and hence can be affinized, there is an additional operator π_0 projecting along the highest root.

In [Hivert and Thiéry 2009] the first and last author constructed the *Hecke group algebra* $\mathcal{H}W$ by gluing together the 0-Hecke algebra and the group algebra of W along their right regular representation. Alternatively, $\mathcal{H}W$ can be constructed as the *biHecke algebra* of W, by gluing together the two realizations $H_0(W)$ and $\overline{H}_0(W)$ of the 0-Hecke algebra. $\mathcal{H}W$ admits a more conceptual description as the algebra of all operators on $\mathbb{K}W$ preserving left antisymmetries; the representation theory of $\mathcal{H}W$ follows, governed by the combinatorics of descents. In [Hivert et al. 2009], the authors further proved that, when W is a Weyl group, $\mathcal{H}W$ is a natural quotient of the affine Hecke algebra.

In this paper, following a suggestion of Alain Lascoux, we study the *biHecke monoid* $M(W)$, obtained by gluing together the two 0-Hecke *monoids*. This involves the combinatorics of the usual poset structures on W (left, right, left-right, Bruhat order), as well as the new cutting poset. Building upon the extensive study of the representation theory of the 0-Hecke algebra [Norton 1979; Carter 1986; Denton 2010; 2011], we explore the representation theory of the biHecke monoid. In the process, we prove that the biHecke monoid is aperiodic and its Borel submonoid fixing the identity is $\mathcal{J}$-trivial. This sparked our interest in the representation theory of $\mathcal{J}$-trivial and aperiodic monoids, and the general results we found along the way are presented in [Denton et al. 2010/11].

We further prove that the simple and projective modules of M are indexed by the elements of W. In order to construct the simple modules, we introduce for each $w \in W$ a combinatorial module T_w whose support is the interval $[1, w]_R$ in right weak order. This module yields an algebra, whose representation theory generalizes that of the Hecke group algebra, with the combinatorics of descents replaced by that of blocks and of the cutting poset.

Let us finish by giving some additional motivation for the study of the biHecke monoid. In type A, the tower of algebras $(\mathbb{K}M(\mathfrak{S}_n))_{n\in\mathbb{N}}$ possesses long sought-after properties. Indeed, it is well known that several combinatorial Hopf algebras arise as Grothendieck rings of towers of algebras. The prototypical example is the tower of algebras of the symmetric groups that gives rise to the Hopf algebra Sym of symmetric functions, on the Schur basis [Macdonald 1995; Zelevinsky 1981]. Another example, due to Krob and Thibon [1997], is the tower of the 0-Hecke algebras of the symmetric groups that gives rise to the Hopf algebra QSym of quasisymmetric functions of [Gessel 1984], on the F_I basis. The product rule on the F_I is naturally lifted through the descent map to a product on permutations, leading to the Hopf algebra FQSym of free quasisymmetric functions [Duchamp et al. 2002]. This calls for the existence of a tower of algebras $(A_n)_{n\in\mathbb{N}}$, such that each A_n contains $H_0(\mathfrak{S}_n)$ and has its simple modules indexed by the elements of $\mathfrak{S}_n$. The biHecke monoids $M(\mathfrak{S}_n)$, and their Borel submonoids $M_1(\mathfrak{S}_n)$ and $M_{w_0}(\mathfrak{S}_n)$, satisfy these properties, and are therefore expected to yield new representation theoretical interpretations of the bases of FQSym.

In the remainder of this introduction, we briefly review Coxeter groups and their 0-Hecke monoids, introduce the biHecke monoid, which is our main object of study, and outline the rest of the paper.

1a. ***Coxeter groups.*** Let (W, S) be a Coxeter group, that is, a group W with a presentation

$$W = \langle S \mid (ss')^{m(s,s')} \quad \text{for all } s, s' \in S\rangle, \tag{1-1}$$

with $m(s, s') \in \{1, 2, \ldots, \infty\}$ and $m(s, s) = 1$. The elements $s \in S$ are called *simple reflections*, and the relations can be rewritten as

$$\begin{aligned} s^2 &= 1 && \text{for all } s \in S,\\ \underbrace{ss'ss's\cdots}_{m(s,s')} &= \underbrace{s'ss'ss'\cdots}_{m(s,s')} && \text{for all } s, s' \in S, \end{aligned} \tag{1-2}$$

where 1 denotes the identity in W.

Most of the time, we just write W for (W, S). In general, we follow the notation of [Björner and Brenti 2005], and we refer to this and to [Humphreys 1990] for details on Coxeter groups and their Hecke algebras. Unless stated otherwise, we always assume that W is finite, and denote its generators by $S = (s_i)_{i \in I}$, where $I = \{1, 2, \ldots, n\}$ is the *index set* of W.

The prototypical example is the Coxeter group of type A_{n-1} which is the n-th symmetric group $(W, S) := (\mathfrak{S}_n, \{s_1, \ldots, s_{n-1}\})$, where s_i denotes the *elementary transposition* which exchanges i and $i+1$. The relations are given by

$$\begin{aligned} s_i^2 &= 1 && \text{for } 1 \le i \le n-1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \ge 2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \le i \le n-2\,; \end{aligned} \tag{1-3}$$

the last two relations are called the *braid relations*. When writing a permutation $\mu \in \mathfrak{S}_n$ explicitly, we use *one-line notation*, that is the sequence $\mu_1 \mu_2 \ldots \mu_n$, where $\mu_i := \mu(i)$.

A *reduced word* $i_1 \ldots i_k$ for an element $w \in W$ corresponds to a decomposition $w = s_{i_1} \cdots s_{i_k}$ of w into a product of generators in S of minimal length $k = \ell(w)$. A *(right) descent* of w is an element $i \in I$ such that $\ell(ws_i) < \ell(w)$. If w is a permutation, this translates into $w_i > w_{i+1}$. *Left descents* are defined analogously. The sets of left and right descents of w are denoted by $\mathrm{D}_L(w)$ and $\mathrm{D}_R(w)$, respectively.

For $J \subseteq I$, we denote by $W_J = \langle s_j \mid j \in J \rangle$ the subgroup of W generated by s_j with $j \in J$. Furthermore, the longest element in W_J and W are denoted by s_J and w_0, respectively. Any finite Coxeter group $W := \langle s_i \mid i \in I \rangle$ can be realized as a finite reflection group; see for example [Humphreys 1990, Chapter 5.6] and [Björner and Brenti 2005, Chapter 4]. The generators s_i of W can be interpreted as reflections on hyperplanes in some $|I|$-dimensional vector space V. The simple roots α_i for $i \in I$ form a basis for V; the set of all roots is given by $\Phi := \{w(\alpha_i) \mid i \in I, w \in W\}$. One can associate reflections s_α to all roots $\alpha \in \Phi$. If $\alpha, \beta \in \Phi$ and $w \in W$, then $w(\alpha) = \beta$ if and only if $ws_\alpha w^{-1} = s_\beta$; see [Humphreys 1990, Chapter 5.7].

1b. ***The 0-Hecke monoid.*** The 0-*Hecke monoid* $H_0(W) = \langle \pi_i \mid i \in I \rangle$ of a Coxeter group W is generated by the *simple projections* π_i with relations

$$\begin{aligned} \pi_i^2 &= \pi_i && \text{for all } i \in I, \\ \underbrace{\pi_i \pi_j \pi_i \pi_j \cdots}_{m(s_i, s_j)} &= \underbrace{\pi_j \pi_i \pi_j \pi_i \cdots}_{m(s_i, s_j)} && \text{for all } i, j \in I\ . \end{aligned} \tag{1-4}$$

Thanks to these relations, the elements of $H_0(W)$ are canonically indexed by the elements of W by setting $\pi_w := \pi_{i_1} \cdots \pi_{i_k}$ for any reduced word $i_1 \ldots i_k$ of w. We further denote by π_J the longest element of the *parabolic submonoid* $H_0(W_J) := \langle \pi_i \mid i \in J \rangle$.

As mentioned before, any finite Coxeter group W can be realized as a finite reflection group, each generator s_i of W acting by reflection along an hyperplane.

The corresponding generator π_i of the 0-Hecke monoid acts as a *folding*, reflecting away from the fundamental chamber on one side of the hyperplane and as the identity on the other side. Both the action of W and of $H_0(W)$ stabilize the set of reflecting hyperplanes and therefore induce an action on chambers.

The right regular representation of $H_0(W)$, or equivalently the action on chambers, induce a concrete realization of $H_0(W)$ as a monoid of operators acting on W, with generators $\pi_1, \dots, \pi_n$ defined by

$$w\,.\,\pi_i := \begin{cases} w & \text{if } i \in \mathrm{D}_R(w), \\ ws_i & \text{otherwise.} \end{cases} \tag{1-5}$$

In type A, π_i sorts the letters at positions i and $i+1$ decreasingly, and $w\,.\,\pi_{w_0} = n \cdots 21$ for any permutation w. This justifies naming π_i an *elementary bubble antisorting operator*.

Another concrete realization of $H_0(W)$ can be obtained by considering instead the *elementary bubble sorting operators* $\bar{\pi}_1, \dots, \bar{\pi}_n$, whose action on W are defined by

$$w\,.\,\bar{\pi}_i := \begin{cases} ws_i & \text{if } i \in \mathrm{D}_R(w), \\ w & \text{otherwise.} \end{cases} \tag{1-6}$$

In geometric terms, this is folding toward the fundamental chamber. In type A, and for any permutation w, one has $w\,.\,\bar{\pi}_{w_0} = 12 \cdots n$.

Remark 1.1. For a given $w \in W$, define v by $wv = w_0$, where w_0 is the longest element of W. Then

$$i \in \mathrm{D}_R(w) \iff i \notin \mathrm{D}_L(v) \iff i \notin \mathrm{D}_R(v^{-1}) = \mathrm{D}_R(w_0 w).$$

Hence, the action of $\bar{\pi}_i$ on W can be expressed from the action of π_i on W using w_0:

$$w\,.\,\bar{\pi}_i = w_0[(w_0 w)\,.\,\pi_i].$$

1c. *The biHecke monoid $M(W)$.* We now introduce our main object of study.

Definition 1.2. Let W be a finite Coxeter group. The *biHecke monoid* is the submonoid of functions from W to W generated simultaneously by the elementary bubble sorting and antisorting operators of (1-5) and (1-6):

$$M := M(W) := \langle \pi_1, \pi_2, \dots, \pi_n, \bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_n \rangle.$$

As mentioned in [Hivert and Thiéry 2009; Hivert et al. 2009] this monoid admits several natural variants, depending on the choice of the generators:

$$\langle \pi_1, \pi_2, \dots, \pi_n, s_1, s_2, \dots, s_n \rangle,$$
$$\langle \pi_0, \pi_1, \pi_2, \dots, \pi_n \rangle,$$

where π_0 is defined when W is a Weyl group and hence can be affinized. Unlike the algebras they generate, which all coincide with the biHecke algebra (in particular due to the linear relation $1+s_i=\pi_i+\overline{\pi}_i$ which expresses how to recover a reflection by gluing together the two corresponding foldings), these monoids are all distinct as soon as W is large enough. Another close variant is the monoid of all strictly order-preserving functions on the Boolean lattice [Gaucher 2010]. All of these monoids, and their representation theory, remain to be studied.

1d. ***Outline.*** The remainder of this paper consists of two parts: We first introduce and study the new cutting poset structure on finite Coxeter groups, and then proceed to the biHecke monoid and its representation theory.

In Section 2, we recall some needed basic facts, definitions, and properties about posets, Coxeter groups, monoids, and representation theory.

In Section 3, we generalize the notion of blocks of permutation matrices to any Coxeter group, and use it to define a new poset structure on W, which we call the *cutting poset*; we prove that it is (almost) a lattice, and derive that its Möbius function is essentially that of the hypercube.

In Section 4, we study the combinatorial properties of $M(W)$. In particular, we prove that it preserves left and Bruhat order, derive consequences on the fibers and image sets of its elements, prove that it is aperiodic, and study Green's relations and idempotents.

In Section 5, our strategy is to consider a "Borel" triangular submonoid of $M(W)$ whose representation theory is simpler, but with the same number of simple modules, to later induce back information about the representation theory of $M(W)$. Namely, we study the submonoid $M_1(W)$ of the elements fixing 1 in $M(W)$. This monoid not only preserves Bruhat order, but furthermore is regressive. It follows that it is $\mathcal{J}$-trivial (in fact $\mathcal{B}$-trivial) which is the desired triangularity property. It is for example easily derived that $M_1(W)$ has $|W|$ simple modules, all of dimension 1. In fact most of our results about M_1 generalize to any $\mathcal{J}$-trivial monoid, which is the topic of a separate paper on the representation theory of $\mathcal{J}$-trivial monoids [Denton et al. 2010/11]. We also provide properties of the Cartan matrix and a combinatorial description of the quiver of M_1.

In Section 6, we construct, for each $w \in W$, the *translation module* T_w by induction of the corresponding simple $\mathbb{K}M_1(W)$-module. It is a quotient of the indecomposable projective module P_w of $\mathbb{K}M(W)$, and therefore admits the simple module S_w of $\mathbb{K}M(W)$ as top. It further admits a simple combinatorial model using the right classes with the interval $[1, w]_R$ as support, and which passes down to S_w. We derive a formula for the dimension of S_w, using an inclusion-exclusion on the sizes of intervals in $(W, \leq_R)$ along the cutting poset. On the way, we study the algebra $\mathcal{H}W^{(w)}$ induced by the action of $M(W)$ on T_w. It turns out to be a natural

w-analogue of the Hecke group algebra, acting not anymore on the full Coxeter group, but on the interval $[1, w]_R$ in right order. All the properties of the Hecke group algebra pass through this generalization, with the combinatorics of descents being replaced by that of blocks and of the cutting poset. In particular, $\mathscr{H}W^{(w)}$ is Morita equivalent to the incidence algebra of the sublattice induced by the cutting poset on the interval $[1, w]_\sqsubseteq$.

In Section 7, we apply the findings of Sections 4, 5, and 6 to derive results on the representation theory of $M(W)$. We conclude in Section 8 with discussions on further research in progress.

There are two appendices. Appendix A summarizes some results on colored graphs which are used in Section 4 to prove properties of the fibers and image sets of elements in the biHecke monoid. Appendix B we present tables of q-Cartan invariant and decomposition matrices for $M(\mathfrak{S}_n)$ for $n = 2, 3, 4$.

2. Background

We review some basic facts about partial orders and finite posets in Section 2a, finite lattices and Birkhoff's theorem in Section 2b, order-preserving functions in Section 2c, the usual partial orders on Coxeter groups (left and right weak order, Bruhat order) in Section 2d, and the notion of $\mathscr{J}$-order (and related orders) and aperiodic monoids in Section 2e. We also prove a result in Proposition 2.4 about the image sets of order-preserving and regressive idempotents on a poset that will be used later in the study of idempotents of the biHecke monoid. Sections 2f and 2g contain reviews of some representation theory of algebras and monoids that will be relevant in our study of translation modules.

2a. *Finite posets.* For a general introduction to posets and lattices, we refer the reader to for example [Pouzet 2013; Stanley 1997] or [Wikipedia 2010, Poset, Lattice]. Throughout this paper, all posets are finite.

A partially ordered set (or *poset* for short) $(P, \preceq)$ is a set P with a binary relation $\preceq$ such that for all $x, y, z \in P$:

(i) $x \preceq x$ (reflexivity);

(ii) if $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry);

(iii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity).

When we exclude the possibility that $x = y$, we write $x \prec y$.

If $x \preceq y$ in P, we define the *interval*

$$[x, y]_P := \{z \in P \mid x \preceq z \preceq y\}.$$

A pair (x, y) such that $x \prec y$ and there is no $z \in P$ such that $x \prec z \prec y$ is called a *covering*. We denote coverings by $x \to y$. The *Hasse diagram* of $(P, \preceq)$ is the diagram where the vertices are the elements $x \in P$, and there is an upward-directed edge between x and y if $x \to y$.

Definition 2.1. Let $(P, \preceq)$ be a poset and $X \subseteq P$.

(i) X is *convex* if for any $x, y \in X$ with $x \preceq y$ we have $[x, y] \subseteq X$.

(ii) X is *connected* if for any $x, y \in X$ with $x \prec y$ there is a path in the Hasse diagram $x = x_0 \to x_1 \to \cdots \to x_k = y$ such that $x_i \in X$ for $0 \leq i \leq k$.

The Möbius inversion formula [Stanley 1997, Proposition 3.7.1] generalizes the inclusion-exclusion principle to any poset. Namely, there exists a unique function μ, called the *Möbius function* of P, which assigns an integer to each ordered pair $x \preceq y$ and enjoys the following property: For any two functions $f, g : P \to G$ taking values in an additive group G,

$$g(x) = \sum_{y \preceq x} f(y) \quad \text{if and only if} \quad f(y) = \sum_{x \preceq y} \mu(x, y)\, g(x). \tag{2-1}$$

The Möbius function can be computed thanks to the following recursion:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \preceq z \prec y} \mu(x, z) & \text{for } x \prec y. \end{cases}$$

2b. ***Finite lattices and Birkhoff's theorem.*** Let $(P, \preceq)$ be a poset. The *meet* $z = \bigwedge A$ of a subset $A \subseteq P$ is an element such that, first, $z \preceq x$ for all $x \in A$ and, second, $u \preceq x$ for all $x \in A$ implies that $u \preceq z$. When the meet exists, it is unique and is denoted by $\bigwedge A$. The meet of the empty set $A = \{\}$ is the largest element of the poset, if it exists. The meet of two elements $x, y \in P$ is denoted by $x \wedge y$. A poset $(P, \preceq)$ for which every pair of elements has a meet is called a *meet-semilattice*. In that case, P endowed with the meet operation is a commutative $\mathcal{J}$-trivial semigroup, and in fact a monoid with unit the maximal element of P, if the latter exists.

Reversing all comparisons, one can similarly define the *join* $\bigvee A$ of a subset $A \subseteq P$ or $x \vee y$ of two elements $x, y \in P$, and *join-semilattices*. A *lattice* is a poset for which both meets and joins exist for pair of elements. Recall that we only consider finite posets, so we do not have to worry about the distinction between lattices and complete lattices.

A lattice $(L, \vee, \wedge)$ is *distributive* if the following additional identity holds for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

This condition is equivalent to its dual,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Birkhoff's representation theorem (see [Wikipedia 2010, Birkhoff's representation theorem], or [Stanley 1997, Theorem 3.4.1]) states that any finite distributive lattice can be represented as a sublattice of a Boolean lattice, that is, a collection of sets stable under union and intersection. Furthermore, there is a canonical such representation, which we construct now.

An element z in a lattice L is called *join-irreducible* if z is not the smallest element in L and $z = x \vee y$ implies $z = x$ or $z = y$ for any $x, y \in L$ (and similarly for meet-irreducible). Equivalently, since L is finite, z is join-irreducible if and only if it covers exactly one element in L. We denote by $I(L)$ the *poset of join-irreducible elements* of L, that is the restriction of L to its join-irreducible elements. Note that this definition still makes sense for nonlattices. From a monoid point of view, $I(L)$ is the minimal generating set of L.

A *lower set* of a poset P is a subset Y of P such that, for any pair $x \leq y$ of comparable elements of P, x is in Y whenever y is. *Upper sets* are defined dually. The family of lower sets of P ordered by inclusion is a distributive lattice, the *lower sets lattice* $O(P)$. Birkhoff's representation theorem [Birkhoff 1937] states that any finite distributive lattice L is isomorphic to the lattice $O(I(L))$ of lower sets of the poset $I(L)$ of its join-irreducible elements, via the reciprocal isomorphisms:

$$\begin{cases} L & \to O(I(L)), \\ x & \mapsto \{y \in I(L) \mid y \leq x\} \end{cases} \quad \text{and} \quad \bigvee : \begin{cases} O(I(L)) & \to L, \\ I & \mapsto \bigvee I\,. \end{cases}$$

Following Edelman [1986], a meet-semilattice L is *meet-distributive* if for every $y \in L$, if $x \in L$ is the meet of elements covered by y then $[x, y]$ is a Boolean algebra. A stronger condition is that any interval of L is a distributive lattice. A straightforward application of Birkhoff's representation theorem yields that L is then isomorphic to a lower set of $O(I(L))$.

2c. *Order-preserving functions.*

Definition 2.2. Let $(P, \preceq)$ be a poset and $f : P \to P$ a function.

(i) f is called *order-preserving* if $x \preceq y$ implies $f(x) \preceq f(y)$. We also say f preserves the order $\preceq$.

(ii) f is called *regressive* if $f(x) \preceq x$ for all $x \in P$.

(iii) f is called *extensive* if $x \preceq f(x)$ for all $x \in P$.

Lemma 2.3. *Let $(P, \preceq)$ be a poset and $f : P \to P$ an order-preserving map. Then, the preimage $f^{-1}(C)$ of a convex subset $C \subseteq P$ is convex. In particular, the preimage of a point is convex.*

Proof. Let $x, y \in f^{-1}(C)$ with $x \preceq y$. Since f is order-preserving, for any $z \in [x, y]$, we have $f(x) \preceq f(z) \preceq f(y)$, and therefore $f(z) \in C$. □

Proposition 2.4. *Let $(P, \preceq)$ be a poset and $f : P \to P$ be an order-preserving and regressive idempotent. Then, f is determined by its image set. Namely, for $u \in P$ we have*

$$f(u) = \sup_{\preceq}(\downarrow u \cap \mathrm{im}(f)),$$

the supremum being always well-defined. Here $\downarrow u = \{x \in P \mid x \preceq u\}$.

An equivalent statement is that, for $v \in \mathrm{im}(f)$,

$$f^{-1}(v) = \uparrow v \setminus \bigcup_{\substack{v' \in \mathrm{im}(f) \\ v' \succ v}} \uparrow v', \quad \textit{where } \uparrow v = \{x \in P \mid x \succeq v\}.$$

Proof. We first prove that $\downarrow u \cap \mathrm{im}(f) = f(\downarrow u)$. The inclusion $\supseteq$ follows from the fact that f is regressive: Taking $v \in \downarrow u$, we have $f(v) \preceq v \preceq u$ and therefore $f(v) \in \downarrow u \cap \mathrm{im}(f)$. The inclusion $\subseteq$ follows from the assumption that f is an idempotent: For $v \in \mathrm{im}(f)$ with $v \preceq u$, one has $v = f(v)$, so $v \in f(\downarrow u)$.

Since f is order-preserving, $f(\downarrow u)$ has a unique maximal element, namely $f(u)$. The first statement of the proposition follows. The second statement is a straightforward reformulation of the first one. □

An *interior operator* (sometimes also called a kernel operator) is a function $L \to L$ on a lattice L that is order-preserving, regressive and idempotent; see for example [Wikipedia 2010, Moore Family]. A subset $A \subseteq L$ is a *dual Moore family* if it contains the smallest element $\perp_L$ of L and is stable under joins. The image set of an interior operator is a *dual Moore family*. Reciprocally, any dual Moore family A defines an interior operator by

$$L \to L, \quad x \mapsto \mathrm{red}(x) := \bigvee_{a \in A, a \preceq x} a, \tag{2-2}$$

where $\bigvee_{\{\}} = \perp_L$ by convention.

A (dual) Moore family is itself a lattice with the order and join inherited from L. The meet operation usually differs from that of L and is given by $x \wedge_A y = \mathrm{red}(x \wedge_L y)$.

2d. ***Classical partial orders on Coxeter groups.*** A Coxeter group $W = \langle s_i \mid i \in I \rangle$ comes endowed with several natural partial orders: left (weak) order, right (weak) order, left-right (weak) order, and Bruhat order. All of these play an important role for the representation theory of the biHecke monoid $M(W)$.

Fix $u, w \in W$. Then, in *right* (*weak*) *order*,

$$u \leq_R w \quad \text{if } w = us_{i_1} \cdots s_{i_k} \text{ for some } i_j \in I \text{ and } \ell(w) = \ell(u) + k.$$

Similarly, in *left* (*weak*) *order*,

$$u \leq_L w \quad \text{if } w = s_{i_1} \cdots s_{i_k} u \text{ for some } i_j \in I \text{ and } \ell(w) = \ell(u) + k,$$

and in *left-right* (*weak*) *order*,

$$u \leq_{LR} w \quad \text{if } w = s_{i_1} \cdots s_{i_k} u s_{i'_1} \cdots s_{i'_\ell} \text{ for some } i_j, i'_j \in I \text{ and } \ell(w) = \ell(u) + k + \ell.$$

Note that left-right order is the transitive closure of the union of left and right order. Thanks to associativity, this is equivalent to the existence of a $v \in W$ such that $u \leq_L v$ and $v \leq_R w$.

Let $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced expression for w. Then, in *Bruhat order*,

$$u \leq_B w \quad \begin{array}{l}\text{if there exists a reduced expression } u = s_{j_1} \cdots s_{j_k} \\ \text{where } j_1 \ldots j_k \text{ is a subword of } i_1 \ldots i_\ell.\end{array}$$

For any finite Coxeter group W, the posets $(W, \leq_R)$ and $(W, \leq_L)$ are graded lattices [Björner and Brenti 2005, Section 3.2]. The following proposition states that any interval is isomorphic to some interval starting at 1:

Proposition 2.5 [Björner and Brenti 2005, Proposition 3.1.6]. *Let* $\mathbb{O} \in \{L, R\}$ *and* $u \leq_{\mathbb{O}} w \in W$. *Then* $[u, w]_{\mathbb{O}} \cong [1, t]_{\mathbb{O}}$ *where* $t = wu^{-1}$.

Definition 2.6. The *type* of an interval in left and right order are defined to be $\operatorname{type}([u, w]_L) := wu^{-1}$ and $\operatorname{type}([u, w]_R) := u^{-1}w$, respectively.

It is easily shown that, if $\mathbb{O}$ is considered as a colored poset, then the converse of Proposition 2.5 holds as well:

Remark 2.7. Fix a type t. Then, the collection of all intervals in left weak order of type t is in bijection with $[1, t^{-1}w_0]_R$, and the operators π_i and $\bar{\pi}_i$ act transitively on the right on this collection. More precisely: π_a induces an isomorphism from $[1, ba^{-1}]_L$ to $[a, b]_L$, and $\bar{\pi}_{a^{-1}}$ induces an isomorphism from $[a, b]_L$ to $[1, ba^{-1}]_L$.

Proof. Take $u \in [a, b]_L$, and let $s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of a. Let $s_{j_1} \cdots s_{j_\ell}$ be a reduced decomposition of $ua^{-1} = us_{i_k} \cdots s_{i_1}$. Then

$$u = (s_{j_1} \cdots s_{j_\ell})(s_{i_1} \cdots s_{i_k})$$

is a reduced decomposition of u and $u.\bar{\pi}_{a^{-1}} = s_{j_1} \cdots s_{j_\ell} = ua^{-1}$. Reciprocially, applying π_a to an element $u \in [1, ba^{-1}]_L$ progressively builds up a reduced word for a. The result follows. □

2e. ***Preorders on monoids.*** J. A. Green [1951] introduced several preorders on monoids, which are essential for the study of their structures; see for example

[Pin 2012, Chapter V]. *Throughout this paper, we only consider finite monoids.* Define $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}, \leq_{\mathcal{H}}$ for $x, y \in M$ as follows:

$$
\begin{aligned}
x \leq_{\mathcal{R}} y &\quad \text{if and only if } x = yu \text{ for some } u \in M,\\
x \leq_{\mathcal{L}} y &\quad \text{if and only if } x = uy \text{ for some } u \in M,\\
x \leq_{\mathcal{J}} y &\quad \text{if and only if } x = uyv \text{ for some } u, v \in M,\\
x \leq_{\mathcal{H}} y &\quad \text{if and only if } x \leq_{\mathcal{R}} y \text{ and } x \leq_{\mathcal{L}} y.
\end{aligned}
$$

These preorders give rise to equivalence relations:

$$
\begin{aligned}
x \,\mathcal{R}\, y &\quad \text{if and only if } xM = yM,\\
x \,\mathcal{L}\, y &\quad \text{if and only if } Mx = My,\\
x \,\mathcal{J}\, y &\quad \text{if and only if } MxM = MyM,\\
x \,\mathcal{H}\, y &\quad \text{if and only if } x\mathcal{R}y \text{ and } x\mathcal{L}y.
\end{aligned}
$$

Strict comparisons are defined by $x <_{\mathcal{R}} y$ if $x \leq_{\mathcal{R}} y$ but $x \notin \mathcal{R}(y)$, or equivalently $\mathcal{R}(x) \subset \mathcal{R}(y)$, and similarly for $<_{\mathcal{L}}, <_{\mathcal{J}}, <_{\mathcal{H}}$.

We further add the relation $\leq_{\mathcal{B}}$ (and its associated equivalence relation $\mathcal{B}$) defined as the finest preorder such that $x \leq_{\mathcal{B}} 1$, and

$$x \leq_{\mathcal{B}} y \text{ implies that } uxv \leq_{\mathcal{B}} uyv \text{ for all } x, y, u, v \in M.$$

(One can view $\leq_{\mathcal{B}}$ as the intersection of all preorders with the property above. There exists at least one such preorder, namely $x \leq y$ for all $x, y \in M$). In the semigroup community, this order is sometimes colloquially referred to as the *multiplicative $\mathcal{J}$-order*.

Beware that 1 is the largest element of those (pre)-orders. This is the usual convention in the semigroup community, but is the converse convention from the closely related notions of left/right/left-right/Bruhat order in Coxeter groups as introduced in Section 2d.

Example 2.8. For the 0-Hecke monoid of Section 1b, $\mathcal{K}$-order for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{B}\}$ corresponds to the reverse of right, left, left-right and Bruhat order of Section 2d. More precisely for $x, y \in H_0(W)$, $x \leq_{\mathcal{K}} y$ if and only if $x \geq_K y$ for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{B}\}$ and $K \in \{R, L, LR, B\}$ the corresponding letter.

Definition 2.9. Elements of a monoid M in the same $\mathcal{K}$-equivalence class are called $\mathcal{K}$-classes, where $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{B}\}$. The $\mathcal{K}$-class of $x \in M$ is denoted by $\mathcal{K}(x)$.

A monoid M is called *$\mathcal{K}$-trivial* if all $\mathcal{K}$-classes are of cardinality one.

An element $x \in M$ is called *regular* if it is $\mathcal{J}$-equivalent to an idempotent.

An equivalent formulation of $\mathcal{K}$-triviality is given in terms of *ordered* monoids. A monoid M is called

right-ordered	if $xy \le x$ for all $x, y \in M$,
left-ordered	if $xy \le y$ for all $x, y \in M$,
left-right-ordered	if $xy \le x$ and $xy \le y$ for all $x, y \in M$,
two-sided-ordered	if $xy = yz \le y$ for all $x, y, z \in M$ with $xy = yz$,
ordered with 1 *on top*	if $x \le 1$, and $x \le y$ implies $uxv \le uyv$ for all $x, y, u, v \in M$

for some partial order $\le$ on M.

Proposition 2.10. *M is right-ordered (respectively left-ordered, left-right-ordered, two-sided-ordered, ordered with* 1 *on top) if and only if M is $\mathcal{R}$-trivial (respectively $\mathcal{L}$-trivial, $\mathcal{J}$-trivial, $\mathcal{H}$-trivial, $\mathcal{B}$-trivial).*

When M is $\mathcal{K}$-trivial for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{B}\}$, the partial order $\le$ is finer than $\le_{\mathcal{K}}$; that is, for any $x, y \in M$, $x \le_{\mathcal{K}} y$ implies $x \le y$.

Proof. We give the proof for right-order as the other cases can be proved in a similar fashion.

Suppose M is right-ordered and that $x, y \in M$ are in the same $\mathcal{R}$-class. Then $x = ya$ and $y = xb$ for some $a, b \in M$. This implies that $x \le y$ and $y \le x$, so that $x = y$. Conversely, suppose that all $\mathcal{R}$-classes are singletons. Then $x \le_{\mathcal{R}} y$ and $y \le_{\mathcal{R}} x$ imply that $x = y$, so that the $\mathcal{R}$-preorder turns into a partial order. Hence M is right-ordered using $xy \le_{\mathcal{R}} x$. □

Definition 2.11. A monoid M is *aperiodic* if there is an integer $N > 0$ such that $x^N = x^{N+1}$ for each $x \in M$.

Since we are only dealing with finite monoids, it is enough to find such an $N = N_x$ depending on the element x. Indeed, taking $N := \max\{N_x\}$ gives a uniform bound. From this definition it is clear that, for an aperiodic monoid M, the sequence $(x^n)_{n\in\mathbb{N}}$ eventually stabilizes for every $x \in M$. We write x^ω for the stable element, which is idempotent, and $E(M) := \{x^\omega \mid x \in M\}$ for the set of idempotents.

Equivalent characterizations of (finite) aperiodic monoids M are that they are $\mathcal{H}$-trivial, or that the sub-semigroup S of M (the identity of S is not necessarily the one of M), which are also groups, are trivial; see for example [Pin 2012, VII, 4.2, Aperiodic monoids]. In this sense, the notion of aperiodic monoids is orthogonal to that of groups as they contain no group-like structure. By the same token, their representation theory is orthogonal to that of groups.

As we will see in Section 4d, the biHecke monoid $M(W)$ of Definition 1.2 is aperiodic. Its Borel submonoid $M_1(W)$ of functions fixing the identity is $\mathcal{J}$-trivial (see Section 5).

2f. ***Representation theory of algebras.*** We refer to [Curtis and Reiner 1962] for an introduction to representation theory, and to [Benson 1991] for more advanced notions such as Cartan matrices and quivers. Here we mostly review composition series and characters.

Let A be a finite-dimensional algebra. Given an A-module X, any strictly increasing sequence $(X_i)_{i\leq k}$ of submodules

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

is called a *filtration* of X. A filtration $(Y_j)_{i\leq \ell}$ such that, for any i, $Y_i = X_j$ for some j is called a *refinement of* $(X_i)_{i\leq k}$. A filtration $(X_i)_{i\leq k}$ without a nontrivial refinement is called a *composition series*. For a composition series, each quotient module X_j/X_{j-1} is simple and is called a *composition factor*. The multiplicity of a simple module S in the composition series is the number of indices j such that X_j/X_{j-1} is isomorphic to S. The Jordan–Hölder theorem states that this multiplicity does not depend on the choice of the composition series. Hence, we may define the *generalized character* (or *character* for short) of a module X as the formal sum

$$[X] := \sum_{i\in I} c_i [S_i],$$

where I indexes the simple modules of A and c_i is the multiplicity of the simple module S_i in any composition series for X.

The additive group of formal sums $\sum_{i\in I} m_i[S_i]$, with $m_i \in \mathbb{Z}$, is called the *Grothendieck group of the category of A-modules* and is denoted by $G_0(A)$. By definition, the character satisfies that, for any exact sequence

$$0 \to X \to Y \to Z \to 0,$$

the equality

$$[Y] = [X] + [Z]$$

holds in the Grothendieck group. See [Serre 1977] for more information about Grothendieck groups.

Suppose that B is a subalgebra of A. Any A-module X naturally inherits an action from B. The constructed B-module thereby is called the *restriction of X to B* and its B-character $[X]_B$ depends only on its A-character $[X]_A$. Indeed, any A-composition series can be refined to a B-composition series and the resulting multiplicities depend only on those in the A-composition series and in the composition series of the simple modules of A restricted to B. This defines a $\mathbb{Z}$-linear map $[X]_A \mapsto [X]_B$, called the *decomposition map*. Let $(S_i^A)_{i\in I}$ and $(S_j^B)_{j\in J}$ be complete families of simple module representatives for A and B, respectively. The matrix of the decomposition map is called the *decomposition matrix* of A over B;

its coefficient (i, j) is the multiplicity of S_j^B as a composition factor of S_i^A, viewed as a B-module.

The adjoint construction of restriction is called *induction*: For any right B-module X the space

$$X{\uparrow}_B^A := X \otimes_B A$$

is naturally endowed with a right A-module structure by right multiplication by elements of A, and is called the *module induced by X from B to A*.

The next subsection, and in particular the statement of Theorem 2.13, requires a slightly more general setting, where the identity e of B does not coincide with that of A. More precisely, let B be a subalgebra of eAe for some idempotent e of A. Then, for any A-module Y, the *restriction* of Y to B is defined as Ye, whereas, for any B-module X, the *induction* of X to A is defined as $X{\uparrow}_B^A := X \otimes_B eA$.

2g. ***Representation theory of monoids.*** Although representation theory started at the beginning of the 20th century with groups before being extended to more general algebraic structures such as algebras, one has to wait until [Clifford 1942] for the first results on the representation theory of semigroups and monoids. Renewed interest in this subject was sparked more recently by the emergence of connections with probability theory and combinatorics; see for example [Brown 2000; Saliola 2007]. Compared to groups, only a few general results are known, the most important one being the construction of the simple modules. It is originally due to Clifford, Munn, and Ponizovskiĭ, and we recall here the construction of [Ganyushkin et al. 2009] (see also the historical references therein) from the regular $\mathcal{J}$-classes and corresponding right class modules.

In principle, one should be specific about the ground field $\mathbb{K}$; in other words, one should consider the representation theory of the *monoid algebra* $\mathbb{K}M$ of a monoid M, and not of the monoid itself. However, the monoids under study in this paper are aperiodic, and their representation theory only depends on the characteristic. We focus on the case where $\mathbb{K}$ is of characteristic 0. Note that the general statements mentioned in this section may further require $\mathbb{K}$ to be large enough (e.g., $\mathbb{K} = \mathbb{C}$) for nonaperiodic monoids.

Let M be a finite monoid. Fix a *regular* $\mathcal{J}$-class J, that is, a $\mathcal{J}$-class containing an idempotent. Consider the sets

$$M_{\geq J} := \bigcup_{K \in \mathcal{J}(M),\ K \geq_{\mathcal{J}} J} K \quad \text{and} \quad I_J := M - M_{\geq J}.$$

Then, I_J is an ideal of M, so that the vector space $\mathbb{K}M_{\geq J}$ can be endowed with an algebra structure by identifying it with the quotient $\mathbb{K}M/\mathbb{K}I_J$. Note that any $\mathbb{K}M_{\geq J}$-module is then a $\mathbb{K}M$-module.

Definition 2.12. Let $f \in M$. Set $\mathbb{K}\mathcal{R}_<(f) := \mathbb{K}\{b \in fM \mid b <_{\mathcal{R}} f\}$. The *right class module* of f (also known as *right Schützenberger representation*) is the $\mathbb{K}M$-module

$$\mathbb{K}\mathcal{R}(f) := \mathbb{K}fM/\mathbb{K}\mathcal{R}_<(f).$$

$\mathbb{K}\mathcal{R}(f)$ is clearly a right module since $\mathbb{K}\mathcal{R}_<(f)$ is a submodule of $\mathbb{K}fM$. Also, as suggested by the notation, $\mathcal{R}(f)$ forms a basis of $\mathbb{K}\mathcal{R}(f)$. Moreover, for a fixed $\mathcal{J}$-class J and thanks to associativity and finiteness, the right class module $\mathbb{K}\mathcal{R}(f)$ does not depend on the choice of $f \in J$ (up to isomorphism). Our main tool for studying the representation theory of the biHecke monoid will be a combinatorial model for its right class modules, which we will call *translation modules* (see Section 6a).

We now choose a $\mathcal{J}$-class J, fix an idempotent e_J in J, and set $\mathbb{K}\mathcal{R}_J := \mathbb{K}\mathcal{R}(e_J)$. Recall that

$$\mathcal{R}(e_J) = e_J M \cap J = e_J M_{\geq J} \cap J.$$

Define similarly

$$G_J := G_{e_J} := e_J M e_J \cap J = e_J M_{\geq J} e_J \cap J.$$

Then, G_J is a group that does not depend on the choice of e_J. More precisely, if e and f are two idempotents in J, the ideals MeM and MfM are equal and the groups G_e and G_f are conjugate and isomorphic. Note that when working with the quotient algebra $\mathbb{K}M_{\geq J}$, the equations above simplify to

$$\mathbb{K}\mathcal{R}_J = e_J \mathbb{K}M_{\geq J} \quad \text{and} \quad \mathbb{K}G_J = e_J \mathbb{K}M_{\geq J} e_J.$$

With these notations, the simple $\mathbb{K}M$-modules can be constructed as follows:

Theorem 2.13 (Clifford, Munn, and Ponizovskiĭ; see [Ganyushkin et al. 2009, Theorem 7]). *Let M be a monoid, and $\mathcal{U}(M)$ be the set of its regular $\mathcal{J}$-classes. For any $J \in \mathcal{U}(M)$, define the right class module $\mathbb{K}\mathcal{R}_J$ and groups G_J as above, let $S_1^J, \dots, S_{n_J}^J$ be a complete family of simple $\mathbb{K}G_J$-modules, and set*

$$X_i^J := \operatorname{top}(S_i^J \uparrow_{\mathbb{K}G_J}^{\mathbb{K}M_{\geq J}}) = \operatorname{top}(S_i^J \otimes_{\mathbb{K}G_J} e_J \mathbb{K}M_{\geq J}) = \operatorname{top}(S_i^J \otimes_{\mathbb{K}G_J} \mathbb{K}\mathcal{R}_J), \quad (2\text{-}3)$$

where $\operatorname{top}(X) := X/\operatorname{rad} X$ is the semisimple quotient of the module X. Then, (X_i^J for $J \in \mathcal{U}(M)$ and $i = 1, \dots, n_J$) is a complete family of simple $\mathbb{K}M$-modules.

In the present paper we only need the very particular case of aperiodic monoids. The key point is that a monoid is aperiodic if and only if all the groups G_J are trivial [Pin 2012, Proposition 4.9]: $G_J = \{e_J\}$. As a consequence, the only $\mathbb{K}G_J$-module is the trivial one, 1, so that the previous construction boils down to the following theorem:

Theorem 2.14. *Let M be an aperiodic monoid. Choose an idempotent transversal $E = \{e_J \mid J \in \mathcal{U}(M)\}$ of the regular $\mathcal{J}$-classes. Further set*

$$X^J := \operatorname{top}(1\uparrow_{\mathbb{K}e_J}^{\mathbb{K}M_{\geq J}}) = \operatorname{top}(e_J \mathbb{K}M_{\geq J}) = \operatorname{top}(\mathbb{K}\mathcal{R}_J). \tag{2-4}$$

Then, the family $(X^J)_{J\in\mathcal{U}(M)}$ is a complete family of representatives of simple $\mathbb{K}M$-modules. In particular, there are as many isomorphic types of simple modules as regular $\mathcal{J}$-classes.

Since the top of $\mathbb{K}\mathcal{R}_J$ is simple, one obtains immediately the following corollary; see [Curtis and Reiner 1962, Corollary 54.14].

Corollary 2.15. *Each regular right class module $\mathbb{K}\mathcal{R}_J$ is indecomposable and a quotient of the projective module P_J corresponding to S_J.*

For a nonaperiodic finite monoid, each right class module remains indecomposable even if its top is not necessarily simple; see [Zalcstein 1971, Corollary 1.10].

The top of a right class module $\mathbb{K}\mathcal{R}_J$ is easy to compute; indeed, the radical of this module is nothing but the annihilator of J acting on it. This in turn boils down to the calculation of the kernel of a matrix as we see below.

Rees matrix monoids [Rees 1940] play an important role in the representation theory of monoids, because any $\mathcal{J}$-class J of any monoid M is, roughly speaking, isomorphic to such a monoid. We give here the definition of aperiodic Rees matrix monoids, which we use in a couple of examples (see Examples 7.8 and 7.9).

Definition 2.16 (aperiodic Rees matrix monoid). Let $P = (p_{ij})$ be an $n \times m$ 0-1-matrix. The *aperiodic Rees matrix monoid $M(P)$* is obtained by endowing the disjoint union

$$\{1\} \cup \{1, \dots, m\} \times \{1, \dots, n\} \cup \{0\}$$

with the product

$$(i, j)(i', j') := \begin{cases} (i, j') & \text{if } p_{ji'} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

1 being neutral and 0 being the zero element.

Note that (i, j) is an idempotent if and only if $p_{j,i} = 1$; hence $M(P)$ can be alternatively described by specifying which elements (i, j) are idempotent.

Without entering into the details, we note that the radical of the unique (up to isomorphism) nontrivial right class modules of $\mathbb{K}M(P)$ is given by the kernel of the matrix P, and thus the dimension of the nontrivial simple module of $\mathbb{K}M(P)$ is given by the rank of P [Clifford and Preston 1961; Lallement and Petrich 1969; Rhodes and Zalcstein 1991; Margolis and Steinberg 2011].

3. Blocks of Coxeter group elements and the cutting poset

In this section, we develop the combinatorics underlying the representation theory of the translation modules studied in Section 6. The key question is, Given $w \in W$, for which subsets $J \subseteq I$ does the canonical bijection between a Coxeter group W and the Cartesian product $W_J \times {}^J W$ of a parabolic subgroup W_J by its set of coset representatives ${}^J W$ in W restrict properly to an interval $[1, w]_R$ in right order (see Figure 1)? In type A, the answer is given by the so-called blocks in the permutation matrix of w, and we generalize this notion to any Coxeter group.

We start with some results on parabolic subgroups and quotients in Section 3a, which are used to define *blocks* and *cutting points* of Coxeter group elements in Section 3b. Then, we illustrate the notion of blocks in type A in Section 3c, recovering the usual blocks in permutation matrices. In Section 3d it is shown that $(W, \sqsubseteq)$ with the cutting order $\sqsubseteq$ is a poset (see Theorem 3.19). In Section 3e we show that blocks are closed under unions and intersections, and relate these to meets and joins in left and right order, thereby endowing the set of cutting points of a Coxeter group element with the structure of a distributive lattice (see Theorem 3.26). In Section 3f, we discuss various indexing sets for cutting points, which leads to the notion of w-analogues of descent sets in Section 3g. Properties of the *cutting poset* are studied in Section 3h (see Theorem 3.41, which also recapitulates the previous theorems).

Throughout this section $W := \langle s_i \mid i \in I \rangle$ denotes a finite Coxeter group.

3a. *Parabolic subgroups and cosets representatives.* For a subset $J \subseteq I$, the *parabolic subgroup* W_J of W is the Coxeter subgroup of W generated by s_j for $j \in J$. A complete system of minimal length representatives of the right cosets $W_J w$ and of the left cosets $w W_J$ are given respectively by

$$
\begin{aligned}
{}^J W &:= \{x \in W \mid \mathrm{D}_L(x) \cap J = \varnothing\},\\
W^J &:= \{x \in W \mid \mathrm{D}_R(x) \cap J = \varnothing\}.
\end{aligned}
$$

Every $w \in W$ has a unique decomposition $w = w_J\, {}^J w$ with $w_J \in W_J$ and ${}^J w \in {}^J W$. Similarly, there is a unique decomposition $w = w^K\, {}_K w$ with ${}_K w \in {}_K W = W_K$ and $w^K \in W^K$.

Lemma 3.1. *Take $w \in W$.*

(i) *For $J \subseteq I$ consider the unique decomposition $w = uv$, where $u = w_J$ and $v = {}^J w$. Then, the unique decomposition of $w s_k$ is $w s_k = (u s_j) v$ if $v s_k v^{-1}$ is a simple reflection s_j with $j \in J$ and $w s_k = u (v s_k)$ otherwise.*

(ii) *For $K \subseteq I$ consider the unique decomposition $w = vu$, where $u = {}_K w$ and $v = w^K$. Then, the unique decomposition of $s_j w$ is $s_j w = v (s_k u)$ if $v^{-1} s_j v$ is a simple reflection s_k with $k \in K$ and $s_j w = (s_j v) u$ otherwise.*

Proof. This follows directly from [Björner and Brenti 2005, Lemma 2.4.3 and Proposition 2.4.4]. □

Note in particular that, if we are in case (i) of Lemma 3.1, we have the following:

- If k is a right descent of w, then $(ws_k)_J \in [1, w_J]_R$ and ${}^J(ws_k) \in [1, {}^Jws_k]_R$.
- If k is not a right descent of w, then either s_k skew commutes with ${}^Jw$ (that is, there exists an i such that $s_i{}^Jw = {}^Jws_k$), or ${}^J(ws_k) = {}^Jws_k$. In particular, ${}^J(ws_k) \leq_R {}^Jws_k$.

Definition 3.2. A subset $J \subseteq I$ is *left reduced* with respect to w if $J' \subset J$ implies ${}^Jw <_L {}^{J'}w$ (or equivalently, if for any $j \in J$, s_j appears in some and hence all reduced words for w_J).

We say $K \subseteq I$ is *right reduced* with respect to w if $K' \subset K$ implies $w^K <_R w^{K'}$.

Lemma 3.3. *Let $w \in W$ and $J \subseteq I$ be left reduced with respect to w. Then*

(i) *$v = {}^Jw \leq_R w$ if and only if there exists $K \subseteq I$ and a bijection $\phi_R : J \to K$ such that $s_jv = vs_{\phi_R(j)}$ for all $j \in J$.*

For $K \subseteq I$ right reduced with respect to w, we have

(i) *$v = w^K \leq_L w$ if and only if there exists $J \subseteq I$ and a bijection $\phi_L : K \to J$ such that $vs_k = s_{\phi_L(k)}v$ for all $k \in K$.*

Proof. Assume first that the bijection ϕ_R exists, and write $w = s_{j_1} \cdots s_{j_\ell}v$, where the product is reduced and $j_i \in J$. Then,

$$w = s_{j_1} \cdots s_{j_\ell}v = s_{j_1} \cdots s_{j_{\ell-1}}vs_{\phi_R(j_\ell)} = vs_{\phi_R(j_1)} \cdots s_{\phi_R(j_\ell)},$$

where the last product is reduced. Therefore $v \leq_R w$.

Assume conversely that $v = {}^Jw \leq_R w$, write the reduced expression $w = vs_{k_1} \cdots s_{k_\ell} \geq_R v$, and set $K = \{k_1, \dots, k_\ell\}$. By Lemma 3.1, the sequence

$$v = {}^Jv,\ {}^J(vs_{k_1}), \dots, {}^J(vs_{k_1} \cdots s_{k_\ell}) = {}^Jw = v$$

preserves right order, and therefore is constant. Hence, at each step i

$${}^J(vs_{k_1} \cdots s_{k_i}) = {}^J({}^J(vs_{k_1} \cdots s_{k_{i-1}})s_{k_i}) = {}^J(vs_{k_i}) = v.$$

Applying Lemma 3.1 again, it follows that there is a subset $J' \subseteq J$, and a bijective map $\phi_R : J' \to K$ such that $s_jv = vs_{\phi_R(j)}$ for all $j \in J'$. Then, $w = s_{\phi_R^{-1}(k_1)} \cdots s_{\phi_R^{-1}(k_\ell)}v$, and, since J is left reduced, $J = J'$.

The second part is the symmetric statement. □

By Lemma 3.1, for any $w \in W$ and $J \subseteq I$ we have $[1, w]_R \subseteq [1, w_J]_R[1, {}^Jw]_R$ and similarly for any $K \subseteq I$ we have $[1, w]_L \subseteq [1, w^K]_L[1, {}_Kw]_L$.

Lemma 3.4. *Take $w \in W$, $K \subseteq I$, and assume that $s_i w = w s_k$ for $i \in I$ and $k \in K$, where the products are reduced. Then, there exists $k' \in K$ such that $s_i w^K = w^K s_{k'}$, where the products are again reduced.*

Proof. We have $w^K = (ws_k)^K = (s_i w)^K = (s_i w^K)^K$. Hence, by Lemma 3.1(ii) there exists $k' \in K$ such that $w^K s_{k'} = s_i w^K$, as desired. □

3b. *Definition and characterizations of blocks and cutting points.* We now come to the definition of *blocks* of Coxeter group elements and associated *cutting points*. They will lead to a new poset on the Coxeter group W, which we coin the *cutting poset* in Section 3d.

Definition 3.5 (blocks and cutting points). Let $w \in W$. We call $K \subseteq I$ a *right block* (or $J \subseteq I$ a *left block*) of w, if there exists $J \subseteq I$ (respectively $K \subseteq I$) such that

$$W_J w = w W_K.$$

In that case, $v := w^K$ is called a *cutting point* of w, which we denote by $v \sqsubseteq w$. Furthermore, K is *proper* if $K \neq \varnothing$ and $K \neq I$; it is *nontrivial* if $w^K \neq w$ (or equivalently ${}_K w \neq 1$); analogous definitions are made for left blocks.

We denote by $\mathcal{B}_{\mathcal{R}}(w)$ the set of all right blocks for w, and by $\mathcal{R}\mathcal{B}_{\mathcal{R}}(w)$ the set of all (right) reduced (see Definition 3.2) right blocks for w. The sets $\mathcal{B}_{\mathcal{L}}(w)$ and $\mathcal{R}\mathcal{B}_{\mathcal{L}}(w)$ are similarly defined on the left.

Here is an equivalent characterization of blocks, which also shows that cutting points can be equivalently defined using ${}^J w$ instead of w^K.

Proposition 3.6. *Let $w \in W$ and $J, K \subseteq I$. Then, the following are equivalent:*

(i) $W_J w = w W_K$.

(ii) *There exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} w^K$ (or equivalently $w^K(\alpha_k) = \alpha_{\phi(k)}$) for all $k \in K$.*

Furthermore, when any, and therefore all, of the above hold then,

(iii) $w^K = {}^J w$.

Proof. Suppose (i) holds. Then $W_J {}^J w = w^K W_K$. Since ${}^J w$ has no left descents in J and w^K has no right descents in K, we know that on both sides ${}^J w$ and w^K are the shortest elements and hence have to be equal: ${}^J w = w^K$; this proves (iii). Furthermore, every reduced expression $w^K s_k$ with $k \in K$ must correspond to some reduced expression $s_j {}^J w$ for some $j \in J$, and vice versa. Hence there exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} {}^J w = s_{\phi(k)} w^K$. Therefore point (ii) holds.

Suppose now that point (ii) holds. Then, for any expression $s_{k_1} \cdots s_{k_\ell} \in W_K$, we have

$$w^K s_{k_1} \cdots s_{k_\ell} = s_{\phi(k_1)} w^K s_{k_2} \cdots s_{k_\ell} = \cdots = s_{\phi(k_1)} \cdots s_{\phi(k_\ell)} w^K.$$

It follows that

$$w^K W_K = W_J w^K.$$

In particular $w \in W_J w^K$ and therefore

$$W_J w = W_J w^K = w^K W_K = w W_K. \qquad \square$$

In general, condition (iii) of Proposition 3.6 is only a necessary, but not sufficient condition for K to be a block. See Example 3.12.

Proposition 3.7. *If K is a right block of w (or more generally if $w^K = w^{K'}$ with K' a right block), then the bijection*

$$W^K \times {}_K W \to W, \quad (v, u) \mapsto vu$$

restricts to a bijection $[1, w^K]_L \times [1, {}_K w]_L \to [1, w]_L$.

Similarly, if J is a left block (or more generally if ${}^J w = {}^{J'} w$ with J' a left block), then the bijection

$$W_J \times {}^J W \to W, \quad (u, v) \mapsto uv$$

restricts to a bijection $[1, w_J]_R \times [1, {}^J w]_R \to [1, w]_R$ (see Figure 1).

Proof. By Proposition 3.6 we know that, if K is a right block, then there exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} w^K$. Hence the map $y \mapsto w^K y$ induces a skew-isomorphism between $[1, {}_K w]_L$ and $[w^K, w]_L$, where an edge k is mapped to edge $\phi(k)$. It follows in particular that $uv \leq_L w^K\ v \leq_L w^K\ {}_K w = w$ for any $u \in [1, w^K]_L$ and $v \in [1, {}_K w]_L$, as desired.

Assume now that K is not a block, but $w^K = w^{K'}$ with K' a block. Then, $[1, w^K]_L = [1, w^{K'}]_L$ and $[1, {}_K w]_L = [1, {}_{K'} w]_L$ and we are reduced to the previous case.

The second statement can be proved in the same fashion. $\square$

Due to Proposition 3.7, we also say that $[1, v]_R$ *tiles* $[1, w]_R$ if $v = {}^J w$ for some left block J (or equivalently $v = w^K$ for some right block K).

Proposition 3.8. *Let $w \in W$ and K be right reduced with respect to w. Then, the following are equivalent:*

(i) *K is a reduced right block of w.*

(ii) *$w^K \leq_L w$.*

The analogous statement can be made for left blocks.

See also Proposition 6.7 for yet another equivalent condition of reduced blocks.

Proof of Proposition 3.8. If K is a right block, then by Proposition 3.6 we have $w^K = {}^J w$, where J is the associated left block. In particular, $w^K = {}^J w \leq_L w$.

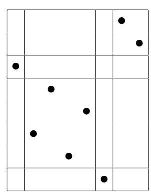
The converse statement follows from Lemma 3.3 and Proposition 3.6. $\square$

Example 3.9. For $w = w_0$, any $K \subseteq I$ is a reduced right block; of course $w_0^K \leq_L w_0$ and ${}_K w_0$ is the maximal element of the parabolic subgroup $W_K = {}_K W$. The cutting point $w^K \sqsubseteq w$ is the maximal element of the right descent class for the complement of K.

The associated left block is given by $J = \phi(K)$, where ϕ is the automorphism of the Dynkin diagram induced by conjugation by w_0 on the simple reflections. The tiling corresponds to the usual decomposition of W into right W_K cosets, or of W into left W_J cosets.

3c. *Blocks of permutations.* In this section we illustrate the notion of blocks and cutting points introduced in the previous section for type A. We show that, for a permutation $w \in \mathfrak{S}_n$, the blocks of Definition 3.5 correspond to the usual notion of blocks of the permutation matrix of w (or unions thereof), and the cutting points w^K for right blocks K correspond to putting the identity in those blocks.

A *matrix-block* of a permutation w is an interval $[k', k'+1, \dots, k]$ that is mapped to another interval. Pictorially, this corresponds to a square submatrix of the matrix of w that is again a permutation matrix (that of the *associated permutation*). For example, the interval $[2, 3, 4, 5]$ is mapped to the interval $[4, 5, 6, 7]$ by the permutation $w = 36475812 \in \mathfrak{S}_8$, and is therefore a matrix-block of w with associated permutation 3142. Similarly, $[7, 8]$ is a matrix-block with associated permutation 12:



For any permutation w, the singletons $[i]$ and the full set $[1, 2, \dots, n]$ are always matrix-blocks; the other matrix-blocks of w are called *proper*. A permutation with no proper matrix-block, such as 58317462, is called *simple*. See [Nozaki et al. 1995; Albert et al. 2003; Albert and Atkinson 2005] for a review of simple permutations. Simple permutations are also strongly related to dimension 2 posets.

A permutation $w \in \mathfrak{S}_n$ is *connected* if it does not stabilize any subinterval $[1, \dots, k]$ with $1 \leq k < n$, that is, if w is not in any proper parabolic subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. Pictorially, this means that there are no diagonal matrix-blocks. A matrix-block is *connected* if the corresponding induced permutation is connected. In the example above, the matrix-block $[2, 3, 4]$ is connected, but the matrix-block $[7, 8]$ is not.

Proposition 3.10. *Let $w \in \mathfrak{S}_n$. The right blocks of w are in bijection with disjoint unions of* (*nonsingleton*) *matrix-blocks for w*; *each matrix-block with column set*

$[i, i+1, \ldots, k]$ contributes $\{i, i+1, \ldots, k-1\}$ to the right block; each matrix-block with row set $[i, i+1, \ldots, k]$ contributes $\{i, i+1, \ldots, k-1\}$ to the left block.

In addition, trivial right blocks correspond to unions of identity matrix-blocks. Also, reduced right blocks correspond to unions of connected matrix-blocks.

Proof. Suppose $w \in \mathfrak{S}_n$ with a disjoint union of matrix-blocks with consecutive column sets $[i_1, \ldots, k_1]$ up to $[i_\ell, \ldots, k_\ell]$. Set $K_j = \{i_j, \ldots, k_j - 1\}$ for $1 \le j \le \ell$ and $K = K_1 \cup \cdots \cup K_\ell$. Define similarly J according to the rows of the blocks.

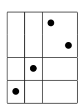
Then multiplying w on the right by some element of W_K permutes some columns of w, while stabilizing each block. Therefore, the same transformation can be achieved by some permutation of the rows stabilizing each block, that is, by multiplication of w on the left by some element of W_J. Hence, using symmetry, $W_J w = w W_K$, that is, J and K are corresponding left and right blocks for w.

Conversely, if K is a right block of w, then w^K maps each α_k with $k \in K$ to another simple root by Proposition 3.6. But then, splitting $K = K_1 \cup \cdots \cup K_\ell$ into consecutive subsets with $K_j = \{i_j, \ldots, k_j - 1\}$, the permutation w^K must contain the identity permutation in each matrix-block with column indices $[i_j, \ldots, k_j]$. This implies that w itself has matrix-blocks with column indices $[i_j, \ldots, k_j]$ for $1 \le j \le \ell$.

Note that, in the described correspondence, $w^K = w$ if and only if all matrix-blocks contain the identity. This proves the statement about trivial right blocks.

A reduced right block K has the property that $w^{K'} \neq w^K$ for every $K' \subset K$. This implies that no matrix-block is in a proper parabolic subgroup, and hence they are all connected. $\square$

Example 3.11. As in Figure 1, consider the permutation 4312, whose permutation matrix is



The reduced (right)-blocks are $K = \{\}$, $\{1\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. The cutting points are 4312, 3412, 4123, and 1234, respectively. The corresponding left blocks are $J = \{\}$, $\{3\}$, $\{1, 2\}$ and $\{1, 2, 3\}$, respectively. The nonreduced (right) blocks are $\{3\}$ and $\{1, 3\}$, as they are respectively equivalent to the blocks $\{\}$ and $\{1\}$. The trivial blocks are $\{\}$ and $\{3\}$.

Example 3.12. In general, condition (iii) of Proposition 3.6 is only a necessary, but not sufficient condition for K to be a block. For example, for $w = 43125$ (similar to 4312 of Example 3.11, but embedded in $\mathfrak{S}_5$), $J = \{3, 4\}$, and $K = \{1, 4\}$, one has ${}^J w = w^K$ yet neither J nor K are blocks. On the other hand (iii) of Proposition 3.6 becomes both necessary and sufficient for reduced blocks.

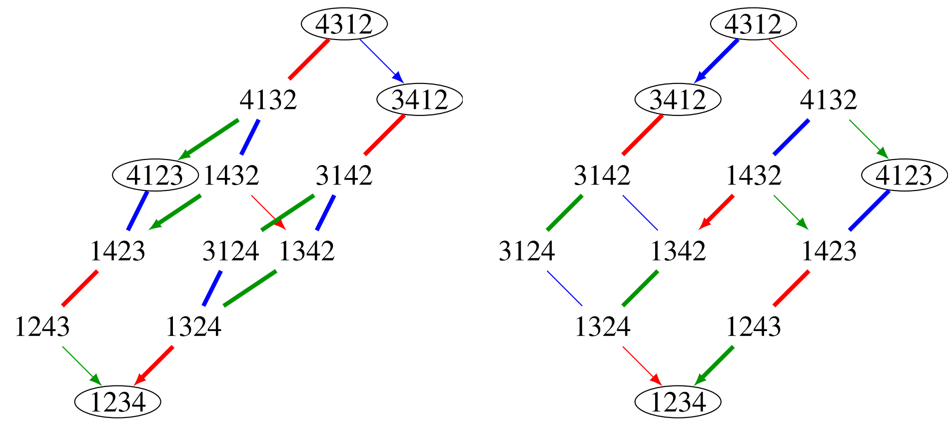


Figure 1. Two pictures of the interval $[1234, 4312]_R$ in right order in $\mathfrak{S}_4$ illustrating its proper tilings, for $J := \{3\}$ and $J := \{1, 2\}$, respectively. The thick edges highlight the tiling. The circled permutations are the cutting points, which are at the top of the tiling intervals. Blue, red, green lines correspond to s_1, s_2, s_3, respectively. See Section 6d for the definition of the orientation of the edges (this is $G^{(4312)}$); edges with no arrow tips point in both directions.

Remark 3.13. It is obvious that the union and intersection of overlapping (possibly with a trivial overlap) matrix-blocks in $\mathfrak{S}_n$ are again matrix-blocks; we will see in Proposition 3.22 that this property generalizes to all types.

Problem 3.14. Fix $J \subseteq \{1, 2, \ldots, n-1\}$ and enumerate the permutations $w \in \mathfrak{S}_n$ for which J is a left block.

3d. ***The cutting poset.*** In this section, we show that $(W, \sqsubseteq)$ indeed forms a poset. We start by showing that for a fixed $u \in W$, the set of elements w such that $u \sqsubseteq w$ admits a simple description. Recall that for $J \subseteq I$, we denote by s_J the longest element of W_J. Proposition 3.6 suggests the following definition.

Definition 3.15. Let $u \in W$. We call $k \in I$ a *short right nondescent* (or $j \in I$ a *short left nondescent*) of u if there exists $j \in I$ (respectively $k \in I$) such that

$$s_j u = u s_k,$$

where the product is reduced (that is, j and k are nondescents). An equivalent condition is that u maps the simple root α_k to a simple root (respectively the preimage of α_j is a simple root).

Set further

$$U_u := u W_K = [u, u s_K]_R = W_J u = [u, s_J u]_L,$$

where $K := K(u)$ and $J := J(u)$ are the sets of short right and left, respectively, nondescents of u.

Pictorially, one takes left and right order on W and associates to each vertex u the translate U_u above u of the parabolic subgroup generated by the short nondescents of u, which correspond to the simultaneous covers of u in both left and right order.

Example 3.16. In type A, i is short for $u \in \mathfrak{S}_n$ if $u(i+1) = u(i)+1$, that is, there is a 2×2 identity block in columns $(i, i+1)$ of the permutation matrix of u. Furthermore U_u is obtained by looking at all identity blocks in u and replacing each by any permutation matrix.

The permutation 4312 of Example 3.11 has a single nondescent 3 that is short, and $U_{4312} = \{4312, 4321\}$.

Proposition 3.17. *U_u is the set of all w such that $u \sqsubseteq w$.*
In particular, it follows that

- *if $u \leq_R v \leq_R w$ and $u \sqsubseteq w$, then $u \sqsubseteq v$; and*
- *if $u \sqsubseteq w$ and $u \sqsubseteq w'$, then $u \sqsubseteq w \vee_R w'$.*

Proof. Note that w is in U_u if and only if there exists K such that $K \subseteq K(u)$ and $w^K = u$. By Proposition 3.6, this is equivalent to the existence of a block K such that $w^K = u$, that is, $u \sqsubseteq w$. □

The following related lemma is used to prove that $(W, \sqsubseteq)$ is a poset.

Lemma 3.18. *If $u \sqsubseteq w$, then the set of short nondescents of w is a subset of the short nondescents of u, namely $K(w) \subseteq K(u)$.*

Proof. Let $k \in K(w)$, so that $ws_k = s_j w$ for some $j \in I$ and both sides are reduced. It follows from Lemma 3.4 that there exists $k' \in K(w)$ such that $s_j u = us_{k'}$ and both sides are reduced. Hence $k' \in K(u)$. Since the map $k \mapsto k'$ is injective it follows that $K(w) \subseteq K(u)$. □

Theorem 3.19. *$(W, \sqsubseteq)$ is a subposet of both left and right order.*

Proof. The relation $\sqsubseteq$ is reflexive since v is a cutting point of v with right block $\varnothing$; hence $v \sqsubseteq v$. Applying Proposition 3.6, it is a subrelation of left and right order: If $v \sqsubseteq w$ then $v = w^K \leq_R w$ for some K and $v = {}^J w \leq_L w$ for some J. Antisymmetry follows from the antisymmetry of left (or right) order.

For transitivity, let $v \sqsubseteq w$ and $w \sqsubseteq z$. Then $v = w^K$ and $w = z^{K'}$ for some right block K of w and K' of z. We claim that $v = z^{K \cup K'}$ with $K \cup K'$ a right block of z. Certainly $k \notin \mathrm{D}_R(v)$ for $k \in K$ since $v = w^K$. Since $w = z^{K'}$ with K' a block of z, all $k' \in K'$ are short nondescents of w and hence by Lemma 3.18 also short nondescents of v. This proves the claim. Therefore $v \sqsubseteq z$. □

Example 3.20. The cutting poset for $\mathfrak{S}_3$ and $\mathfrak{S}_4$ is given in Figure 2. As we can see on those figures, the cutting poset is not the intersection of the right and left order since w_0 is maximal for left and right order but not for cutting poset.

3e. ***Lattice properties of intervals.*** In this section we show that the set of blocks and the set of cutting points $\{u \mid u \sqsubseteq w\}$ of a fixed $w \in W$ are endowed with the structure of distributive lattices (see Theorem 3.26).

We begin with a lemma that gives some properties of blocks that are contained in each other.

Lemma 3.21. *Fix $w \in W$. Let $K \subseteq K'$ be two right blocks of w and $J \subseteq J'$ be the corresponding left blocks, so that*

$$W_J w = w W_K, \quad W_{J'} w = w W_{K'}, \quad {}^J w = w^K \sqsubseteq w, \quad \text{and} \quad {}^{J'} w = w^{K'} \sqsubseteq w.$$

Then,

(i) $w^{K'} \leq_R w^K$ *and* $w^{K'} \leq_L w^K$,

(ii) K' *is a right block of* w^K *and* $w^{K'} \sqsubseteq w^K$,

(iii) K *is a right block of* ${}_{K'}w$ *and* ${}_{K'}w^K \sqsubseteq {}_{K'}w$.
Furthermore K is reduced for ${}_{K'}w$ if and only if it is reduced for w.

The same statements hold for left blocks.

Proof. Part (i) holds because $w^{K'} = (w^K)^{K'} \leq_R w^K \leq_R w$, and similarly on the left.

Part (ii) is a trivial consequence of (i) and Proposition 3.17.

For (iii), first note that $({}_{K'}w)^K = {}_{K'}(w^K)$, so that the notation ${}_{K'}w^K$ is unambiguous. Consider the bijection ϕ from K' to J' of Proposition 3.6, and note that $W_J w^{K'} = w^{K'} W_{\phi^{-1}(J)}$. Therefore,

$$w^{K'}\, {}_{K'}w W_K = w W_K = W_J\, w = W_J\, w^{K'}\, {}_{K'}w = w^{K'} W_{\phi^{-1}(J)}\, {}_{K'}w.$$

Simplifying by $w^{K'}$ on the left, one obtains that

$${}_{K'}w\; W_K = W_{\phi^{-1}(J)}\; {}_{K'}w,$$

proving that K is also a block of ${}_{K'}w$. The reduction statement is trivial. □

We saw in Remark 3.13 that the set of blocks is closed under unions and intersections in type A. This holds for general type.

Proposition 3.22. *The set $\mathcal{B}_{\mathcal{R}}(w)$ (or $\mathcal{B}_{\mathcal{L}}(w)$) of right (respectively left) blocks is stable under union and intersection. Hence, it forms a distributive sublattice of the Boolean lattice $\mathcal{P}(I)$.*

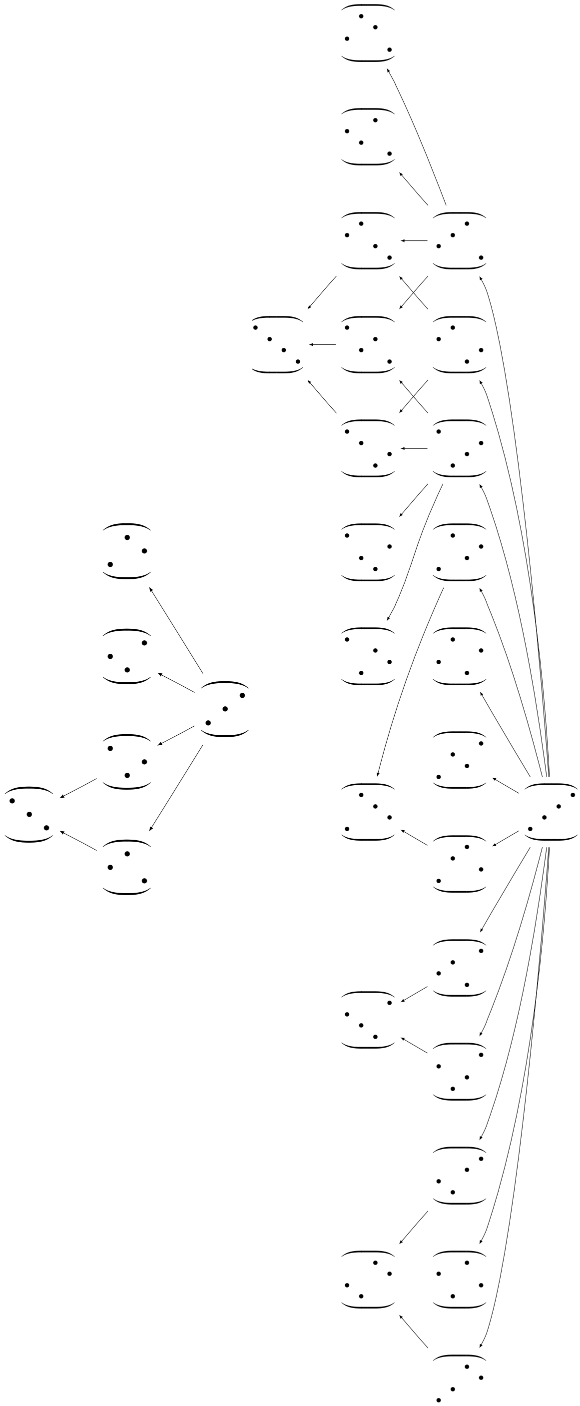


Figure 2. The cutting posets for the symmetric groups $\mathfrak{S}_3$ and $\mathfrak{S}_4$. Each permutation is represented by its permutation matrix, with the bullets marking the positions of the ones. Notice the Boolean sublattice appearing as the interval between the identity permutation at the bottom and the maximal permutation at the top; its elements are the minimal elements of the descent classes.

Proof. Let K and K' be right blocks for $w \in W$, and J and J' be the corresponding left blocks, so that

$$wW_K = W_J w \quad \text{and} \quad wW_{K'} = W_{J'} w.$$

Take $u \in W_{K\cap K'} = W_K \cap W_{K'}$. Then, wuw^{-1} is both in W_J and $W_{J'}$ and therefore in $W_J \cap W_{J'} = W_{J\cap J'}$. This implies $wW_{K\cap K'}w^{-1} \subseteq W_{J\cap J'}$. By symmetry, the inclusion $w^{-1}W_{J\cap J'}w \subseteq W_{K\cap K'}$ holds as well, and therefore $W_{J\cap J'}w = wW_{K\cap K'}$. In conclusion, $K \cap K'$ is a right block, with $J \cap J'$ as corresponding left block.

Now take $u \in W_{K\cup K'} = \langle W_K, W_{K'}\rangle$, and write u as a product $u_1u_1'u_2u_2' \cdots u_\ell u_\ell'$, where $u_i \in W_K$ and $u_i' \in W_{K'}$ for all $1 \leq i \leq \ell$. Then, for each i, $wu_iw^{-1} \in W_J$ and $wu_i'w^{-1} \in W_{J'}$. By composition, $wuw^{-1} \in W_J W_{J'} W_J W_{J'} \cdots W_J W_{J'} \subseteq W_{J\cup J'}$. Using symmetry as above, we conclude that $wW_{K\cup K'} = W_{J\cup J'}w$. In summary, $K \cup K'$ is a right block, with $J \cup J'$ as corresponding left block.

Finally, since blocks are stable under union and intersection, they form a sublattice of the Boolean lattice. Any sublattice of a distributive lattice is distributive. □

Next we relate the union and intersection operation on blocks with the meet and join operations in right and left order. We start with the following general statement which must be classical, though we have not found it in the literature.

Lemma 3.23. *Take $w \in W$ and $J, J', K, K' \subseteq I$. Then*

$$w^{K\cap K'} = w^K \vee_R w^{K'} \quad \text{and} \quad {}^{J\cap J'}w = {}^J w \vee_L {}^{J'}w.$$

Proof. We include a proof for the sake of completeness. By Lemma 3.21(i), $w^K, w^{K'} \leq_R w^{K\cap K'}$, and therefore $v \leq_R w^{K\cap K'}$, where $v = w^K \vee_R w^{K'}$. Suppose that v has a right descent $k \in K \cap K'$. Then vs_k is still bigger than w^K and $w^{K'}$ in right order, a contradiction to the definition of v. Hence $w^{K\cap K'} = w^K \vee_R w^{K'}$, as desired. The statement on the left follows by symmetry. □

Corollary 3.24. *Take $w \in W$. Let $K, K' \subseteq I$ be two right blocks of w and $J, J' \subseteq I$ the corresponding left blocks. Then, for the right block $K \cap K'$ and left block $J \cap J'$,*

$$w^{K\cap K'} = {}^{J\cap J'}w = w^K \vee_R w^{K'} = {}^J w \vee_L {}^{J'}w.$$

The analogous statement of Lemma 3.23 for unions fails in general: Take for example $w = 4231$ and $K = \{3\}$ and $K' = \{1, 2\}$, so that $w^K = 4213$ and $w^{K'} = 2341$; then $w^{K\cup K'} = 1234$, but $w^K \wedge_R w^{K'} = 2134$. However, it holds for blocks:

Lemma 3.25. *Take $w \in W$. Let $K, K' \subseteq I$ be two right blocks of w and $J, J' \subseteq I$ the corresponding left blocks. Then, for the right block $K \cup K'$ and left block $J \cup J'$,*

$$w^{K\cup K'} = {}^{J\cup J'}w = w^K \wedge_R w^{K'} = {}^J w \wedge_L {}^{J'}w.$$

Furthermore, $K \cup K'$ is reduced whenever K and K' are reduced, and similarly for the left blocks.

Proof. By symmetry, it is enough to prove the statements for right blocks.

By Lemma 3.21(i), $w^{K\cup K'} \leq_R w^K, w^{K'}$, and therefore $w^{K\cup K'} \leq_R w^K \wedge_R w^{K'}$.

Note that the interval $[w^{K\cup K'}, w]_R$ contains all the relevant points: w^K, $w^{K'}$, and $w^K \wedge_R w^{K'}$. Consider the translate of this interval obtained by dividing on the left by $w^{K\cup K'}$, or equivalently by using the map $u \mapsto {}_{K\cup K'}u$. By Lemma 3.21(iii), K and K' are still blocks of ${}_{K\cup K'}w$. From now on, we may therefore assume without loss of generality that $w^{K\cup K'} = 1$. It follows at once that $[1, w]_R$ lies in the parabolic subgroup $W_{K\cup K'}$ and that $J \cup J' = K \cup K'$.

If $w^K \wedge_R w^{K'} = 1 = w^{K\cup K'}$, then we are done. Otherwise, let $i \in K \cup K' = J \cup J'$ be the first letter of some reduced word for $w^K \wedge_R w^{K'}$. Since $w^K \wedge_R w^{K'}$ is in the interval $[1, w^K]_R$, i cannot be in J; by symmetry i cannot be in J' either, a contradiction.

Assume further that K and K' are reduced. Then, any $k \in K$ appears in any reduced word for ${}_K w$, and therefore in any reduced word for ${}_{K\cup K'}w$ since ${}_K w \leq_L {}_{K\cup K'}w$. By symmetry, the same holds for $k' \in K'$. Hence $K \cup K'$ is reduced. □

Theorem 3.26. *The map $K \mapsto w^K$ (or $J \mapsto {}^J w$) defines a lattice antimorphism from the lattice $\mathcal{B}_{\mathcal{R}}(w)$ (respectively $\mathcal{B}_{\mathcal{L}}(w)$) of right (respectively left) blocks of w to both right and left order on W.*

The set of cutting points for w, which is the image set

$$\{w^K \mid K \in \mathcal{B}_{\mathcal{R}}(w)\} = \{{}^J w \mid J \in \mathcal{B}_{\mathcal{L}}(w)\}$$

of the previous map, is a distributive sublattice of right (respectively left) order.

Proof. The first statement is the combination of Lemmas 3.23 and 3.25. The second statement follows from Proposition 3.22, since the quotient of a distributive sublattice by a lattice morphism is a distributive lattice. □

Corollary 3.27. *Every interval of $(W, \sqsubseteq)$ is a distributive sublattice and an induced subposet of both left and right order.*

Proof. Take an interval in $(W, \sqsubseteq)$; without loss of generality, we may assume that it is of the form $[1, w]_\sqsubseteq = \{w^K \mid K \in \mathcal{R}\mathcal{B}_{\mathcal{R}}(w)\}$. The interval $[1, w]_\sqsubseteq$ is not only a subposet of left (respectively right) order, but actually the induced subposet; indeed for K and K' right reduced blocks, and J and J' the corresponding left blocks,

$$w^K \leq_L w^{K'} \iff w^K \leq_R w^{K'} \iff J' \subseteq J \iff K' \subseteq K \iff w^K \leq_\sqsubseteq w^{K'}.$$

Therefore, using Theorem 3.26, it is a distributive sublattice of left (respectively right) order. □

Let us now consider the lower covers in the cutting poset for a fixed $w \in W$. They correspond to nontrivial blocks J that are minimal for inclusion, and in particular reduced.

Lemma 3.28. *Each minimal nontrivial* (*left*) *block J for $w \in W$ contains at least one element which is in no other minimal nontrivial block for w.*

Proof. Assume otherwise. Then, J is the union of its intersections with the other nontrivial blocks. Each such intersection is necessarily a trivial block, and a union of trivial blocks is a trivial block. Therefore, J is a trivial block, a contradiction. □

Corollary 3.29. *The semilattice of unions of minimal nontrivial blocks for a fixed $w \in W$ is free.*

Proof. This is a straightforward consequence of Lemma 3.28. Alternatively, this property is also a direct consequence of Corollary 3.27, since it holds in general for any distributive lattice. □

3f. ***Index sets for cutting points.*** Recall that by Theorem 3.26 the cutting points of w form a distributive lattice. Hence, by Birkhoff's representation theorem, they can be indexed by some collection of subsets closed under unions and intersections. We therefore now aim at finding a suitable choice of indexing scheme for the cutting points of w. More precisely, for each w, we are looking for a pair $(\mathcal{K}^{(w)}, \phi^{(w)})$, where $\mathcal{K}^{(w)}$ is a subset of some Boolean lattice (typically $\mathcal{P}(I)$) such that $\mathcal{K}^{(w)}$ ordered by inclusion is a lattice, and

$$\phi^{(w)} : \mathcal{K}^{(w)} \to [1, w]_{\sqsubseteq}$$

is an isomorphism (or antimorphism) of lattices.

Here are some of the desirable properties of this indexing:

(1) The indexing gives a Birkhoff representation of the lattice of cutting points of w. Namely, $\mathcal{K}^{(w)}$ is a sublattice of the chosen Boolean lattice, and unions and intersections of indices correspond to joins and meets of cutting points.

(2) The isomorphism $\phi^{(w)}$ is given by the map $J \mapsto {}^{J}w$. In that case the choice amounts to defining a section of those maps.

(3) The indexing generalizes the usual combinatorics of descents.

(4) The indices are blocks: $\mathcal{K}^{(w)} \subseteq \mathcal{B}_{\mathcal{L}}(w)$.

(5) We may actually want to have two indexing sets $\mathcal{K}^{(w)}$ and $\mathcal{K}^{(w)}$, one on the left and one on the right, with a natural isomorphism between them.

(6) The index of u in $\mathcal{K}^{(w)}$ does not depend on w (as long as u is a cutting point of w). One may further ask for this index to not depend on W, so that the indexing does not change through embedding of parabolic subgroups.

Unfortunately, there does not seem to be an ideal choice satisfying all of these properties at once, and we therefore propose several imperfect alternatives.

3f1. *Indexing by reduced blocks.* The first natural choice is to take reduced blocks as indices; then, $\mathcal{K}^{(w)} = \mathcal{RB}_{\mathcal{R}}(w)$ (and similarly $\mathcal{J}^{(w)} = \mathcal{RB}_{\mathcal{L}}(w)$ on the left). This indexing scheme satisfies most of the desired properties, except that it does not provide a Birkhoff representation, and depends on w.

Remark 3.30. By Lemma 3.25, if $K, K' \subseteq I$ are reduced right blocks for w, then $K \cup K'$ is also reduced. However, this is not necessarily the case for $K \cap K'$: consider for example the permutation $w = 4231$, $K = \{1, 2\}$ and $K' = \{2, 3\}$; then $K \cap K' = \{2\}$ is a block which is equivalent to the reduced block $\{\}$: $4231^{\{2\}} = 4231 = 4231^{\{\}}$.

The union $K \cup K'$ of two blocks may be reduced even when the blocks are not both reduced. Consider for example the permutation $w = 4312$ as in Figure 1. Then $K = \{1, 3\}$ and $K' = \{2, 3\}$ are blocks and their union $K \cup K' = \{1, 2, 3\}$ is reduced, yet K is not reduced.

Proposition 3.31. *The poset $(\mathcal{RB}_{\mathcal{R}}(w), \subseteq)$ of reduced right blocks is a distributive lattice, with the meet and join operation given respectively by*

$$K \vee K' = K \cup K' \quad \textit{and} \quad K \wedge K' = \operatorname{red}(K \cap K') ,$$

where, for a block K, $\operatorname{red}(K)$ is the unique largest reduced block contained in K.

The map $\phi^{(w)} : K \mapsto w^K$ restricts to a lattice antiisomorphism from the lattice $\mathcal{B}_{\mathcal{R}}(w)$ of reduced right blocks of w to $[1, w]_{\sqsubseteq}$.

The same statements hold on the left.

Proof. By Proposition 3.22 and Lemma 3.25, $\mathcal{RB}_{\mathcal{R}}(w)$ is a dual Moore family of the Boolean lattice of I, or even of $\mathcal{B}_{\mathcal{R}}(w)$. Therefore, using Section 2a, it is a lattice, with the given join and meet operations.

The lattice antiisomorphism of property follows from Lemma 3.25 and the coincidence of right order and $\sqsubseteq$ on $[1, w]_{\sqsubseteq}$ (Theorem 3.26). $\square$

3f2. *Indexing by largest blocks.* The indexing by reduced blocks corresponds to the section of the lattice morphism $K \mapsto w^K$ by choosing the smallest block K in the fiber of a cutting point u. Instead, one could choose the largest block in the fiber of u, which is given by the set of short nondescents of u. This indexing scheme is independent of w. Also, by the same reasoning as above, the indexing sets $\mathcal{J}^{(w)}$ come endowed with a natural lattice structure. However, it does not give a Birkhoff representation: The meet is given by intersection, but the join is not given by union (take $w = 2143$; its cutting points are 1234, 1243, 2134, and 2143, indexed respectively by $\{1, 2, 3\}$, $\{1\}$, $\{3\}$, and $\{\}$).

3f3. *Birkhoff's representation using nonblocks.* We now relax the condition for the indices to be blocks. That is, we consider $K \mapsto w^K$ as a function from the full Boolean lattice $\mathcal{P}(I)$ to the minimal coset representatives of w. Beware that this map is no longer a lattice antimorphism; yet, the fiber of any u still admits a largest set $K = \overline{\mathrm{D}}_R(u) \subseteq I$, which is the complement of the right descent set of u. One can define a similar indexing on the left by $J = \overline{\mathrm{D}}_L(u)$. These indexings are independent of w and provide a Birkhoff representation for the lattice of cutting points (see Proposition 3.34). Define

$$\mathfrak{DB}_{\mathfrak{L}}(w) = \{\overline{\mathrm{D}}_L(u) \mid u \sqsubseteq w\} \quad \text{and} \quad \mathfrak{DB}_{\mathfrak{R}}(w) = \{\overline{\mathrm{D}}_R(u) \mid u \sqsubseteq w\}\,. \tag{3-1}$$

Remark 3.32. Since $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ are not necessarily blocks anymore, the bijection between $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ is no longer induced by a bijection at the level of descents: For example, for $u = 3142$, one has $\overline{\mathrm{D}}_L(u) = \{1, 3\}$ and $\overline{\mathrm{D}}_R(u) = \{2\}$.

Remark 3.33. Using $\mathrm{D}_R(u)$ instead of $\overline{\mathrm{D}}_R(u)$ would give an isomorphism instead of an antiisomorphism, and make the indexing further independent of W, at the price of slightly cluttering the notation w^K for cutting points.

Proposition 3.34 (Birkhoff representation for the lattice of cutting points). *The set $\mathfrak{DB}_{\mathfrak{R}}(w)$ of Equation (3-1) is a sublattice of the Boolean lattice, and the maps $K \mapsto w^K$ and $u \mapsto \overline{\mathrm{D}}_R(u)$ form a pair of reciprocal lattice antiisomorphisms with the lattice of cutting points of w. The same statement holds on the left.*

The proof of this proposition uses the following property of left and right order (recall that $[1, w]_{\sqsubseteq}$ is a sublattice thereof).

Lemma 3.35 [Le Conte de Poly-Barbut 1994, Lemme 5]. *The maps*

$$(W, \leq_L) \to \mathcal{P}(I), \quad w \mapsto \mathrm{D}_R(w), \quad \textit{and} \quad (W, \leq_R) \to \mathcal{P}(I), \quad w \mapsto \mathrm{D}_L(w)$$

are surjective lattice morphisms.

Proof of Proposition 3.34. By construction, $\overline{\mathrm{D}}_L$ is a section of $K \mapsto w^K$, and these maps form a pair of reciprocal bijections between $\mathfrak{DB}_{\mathfrak{L}}(w)$ and the cutting points of w. Using Lemma 3.35, the map $\overline{\mathrm{D}}_L$ is a lattice antimorphism. Therefore its image set $\mathfrak{DB}_{\mathfrak{R}}(w)$ is a sublattice of the Boolean lattice. The argument on the left is the same. □

3g. *A w-analogue of descent sets.* For each $w \in W$, we now provide a definition of a w-analogue on the interval $[1, w]_R$ of the usual combinatorics of (non)descents on W. From now on, we assume that we have chosen an indexation scheme so that the cutting points of w are given by $(w^K)_{K \in \mathcal{K}(w)}$ or equivalently by $({}^J w)_{J \in \mathcal{J}(w)}$.

Lemma 3.36. *Take a cutting point of w, and write it as $w^K = {}^J w$ for some $J, K \subseteq I$, which are not necessarily blocks. Then*

(i) *for $u \in [1, w]_R$, $u \in [1, {}^J w]_R$ if and only if $\mathrm{D}_L(u) \cap J = \varnothing$;*

(ii) *for $u \in [1, w]_L$, $u \in [1, w^K]_L$ if and only if $\mathrm{D}_R(u) \cap K = \varnothing$.*

Proof. This is a straightforward corollary of Proposition 3.7: Any element u of $[1,w]_R$ can be written uniquely as a product $u'v$ with $u' \in W_J$ and $v \in [1, {}^J w]_R$. So u is in $[1, {}^J w]_R$ if and only if $u' = 1$, which in turn is equivalent to v having no descents in J. This proves (i). The argument for (ii) is analogous. $\square$

Example 3.37. For $w = w_0$, ${}^J w$ is the maximal element of a left descent class, and $[1, {}^J w]_R$ gives all elements of W whose left descent set is a subset of the left descent set of w.

Definition 3.38 (w-nondescent sets). For $u \in [1,w]_R$, define $J^{(w)}(u)$ to be the index $J \in \mathcal{J}^{(w)}$ of the lowest cutting point ${}^J w$ such that $u \in [1, {}^J w]_R$ (or the equivalent condition of Lemma 3.36). Define similarly $K^{(w)}(u)$ as the index in $\mathcal{K}^{(w)}$ of this cutting point.

Example 3.39. When $w = w_0$, $J^{(w_0)}(u)$ and $K^{(w_0)}(u)$ are respectively the sets $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ of left and right nondescents of u.

Problem 3.40. Given J, describe all the elements $w \in W$ such that J is a left block. This essentially only depends on ${}^J w$.

3h. ***Properties of the cutting poset.*** In this section we study the properties of the cutting poset $(W, \sqsubseteq)$ of Theorem 3.19 for the cutting relation $\sqsubseteq$ introduced in Definition 3.5 (see also Figure 2). The following theorem summarizes the results.

Theorem 3.41. $(W, \sqsubseteq)$ *is a meet-distributive meet-semilattice with* 1 *as minimal element, and a subposet of both left and right order.*

Every interval of $(W, \sqsubseteq)$ *is a distributive sublattice and a sublattice of both left and right order.*

Let $w \in W$ *and denote by* $\mathrm{Pred}(w)$ *the set of its* $\sqsubseteq$*-lower covers. Thanks to meet-distributivity, the meet-semilattice* L_w *generated by* $\mathrm{Pred}(w)$ *using* $\wedge_\sqsubseteq$ *(or equivalently* $\wedge_L, \wedge_R$ *if viewed as a sublattice of left or right order) is free, that is, isomorphic to a Boolean lattice.*

In particular, the Möbius function of $(W, \sqsubseteq)$ *is given by* $\mu(u,w) = (-1)^{r(u,w)}$ *if* $u \in L_w$ *and* 0 *otherwise, where* $r(u,w) := |\{v \in \mathrm{Pred}(w) \mid u \sqsubseteq v\}|$.

This Möbius function is used in Section 6d to compute the size of the simple modules of $\mathbb{K}M$.

Since $(W, \sqsubseteq)$ is almost a distributive lattice, Birkhoff's representation theorem suggests that we embed it in the distributive lattice $O(I((W, \sqsubseteq)))$ of the lower sets of its join-irreducible elements (note that a block is join-irreducible if there is only one minimal nontrivial block below it).

Problem 3.42. Describe the set $I(W, \sqsubseteq)$ of join-irreducible elements of $(W, \sqsubseteq)$.

Problem 3.43. Determine the distributive lattice associated with the cutting poset from the join-irreducibles, via Birkhoff's theory.

The join-irreducible elements of $(\mathfrak{S}_n, \sqsubseteq)$, for n small, are counted by the sequence 0, 1, 4, 16, 78, 462, 3224. Figure 2 seems to suggest that they form a tree, but this already fails for $n = 5$. We now briefly comment on the simplest join-irreducible elements, namely the immediate successors w of 1 in the cutting poset. Equivalent statements are that w admits exactly two reduced blocks $\{\}$ and B, possibly with $B = I$, or that the simple module S_w is of dimension $|[1, w]_R| - 1$. For a Coxeter group W, we denote by $S(W)$ the set of elements $w \neq 1$ having no proper reduced blocks, and $T(W)$ those having exactly two reduced blocks. Note that $T(W)$ is the disjoint union of the $S(W_J)$ for $J \subseteq I$.

Example 3.44. In type A, a permutation $w \in S(\mathfrak{S}_n)$ is uniquely obtained by taking a simple permutation, and inflating each 1 of its permutation matrix by an identity matrix. An element of $T(\mathfrak{S}_n)$ has a block diagonal matrix with one block in $S(\mathfrak{S}_m)$ for $m \leq n$, and $n-m$ 1×1 blocks. This gives an easy way to construct the generating series for $S(\mathfrak{S}_n)_{n\in\mathbb{N}}$ and for $T(\mathfrak{S}_n)_{n\in\mathbb{N}}$ from that of the simple permutations given in [Albert and Atkinson 2005].

We now turn to the proof of Theorem 3.41.

Lemma 3.45. $(W, \sqsubseteq)$ *is a partial join-semilattice. That is, when the join exists, it is unique and given by the join in left and in right order*:

$$v \vee_{\sqsubseteq} v' = v \vee_L v' = v \vee_R v'.$$

Proof. Take v and v' with at least one common successor. Applying Corollary 3.27 to the interval $[1, w]_{\sqsubseteq}$ for any such common successor w, one obtains $v, v' \sqsubseteq v \vee_R v' = v \vee_L v' \sqsubseteq w$. Therefore, $v \vee_R v' = v \vee_L v'$ is the join of v and v' in the cutting order. □

Lemma 3.46. $(W, \sqsubseteq)$ *is a meet-semilattice. That is, for* $v, v' \in W$

$$v \wedge_{\sqsubseteq} v' = \bigvee_{u \sqsubseteq v, v'} u,$$

where $\bigvee$ *is the join for the cutting order (or equivalently for left or right order). If further v and v' have a common successor, then*

$$v \wedge_{\sqsubseteq} v' = v \wedge_R v' = v \wedge_L v'.$$

Proof. The first part follows from a general result. Namely, for any poset, the following statements are equivalent (see for example [Pouzet 2013, Proposition 7.3]):

(i) Any bounded nonempty part has an upper bound.

(ii) Any bounded nonempty part has a lower bound.

Here we prove again this fact for the sake of self-containment. Take u and u' two common cutting points for v and v'. Then, using Lemma 3.45, their join exists and $u \vee_{\sqsubseteq} u' = u \vee_R u' = u \vee_L u'$ is also a cutting point for v and v'. The first statement follows by repeated iteration over all common cutting points.

Now assume that v and v' have a common successor w. Then by applying Corollary 3.27 to the interval $[1, w]_{\sqsubseteq}$, we find that $v \wedge_R v' = v \wedge_L v'$ is the meet of v and v' in the cutting order. □

Proof of Theorem 3.41. $(W, \sqsubseteq)$ is a meet-semilattice by Lemma 3.46. Meet-distributivity follows from Corollary 3.29. The argument is in fact general: Any poset with a minimal element 1 such that all intervals $[1, x]$ are distributive lattices and such that any two elements admit either a join or no common successor is a meet-distributive meet-semilattice (see [Edelman 1986] for literature on such). The end of the first statement is Theorem 3.19.

The statement about intervals is Corollary 3.27.

The $\sqsubseteq$-lower covers of an element w correspond to the nontrivial blocks of w that are minimal for inclusion. The top part L_w of an interval $[1, w]_{\sqsubseteq}$ is further described in Corollary 3.29, through the bijection $\phi^{(w)}$ between blocks of w and the interval $[1, w]_{\sqsubseteq}$ of Proposition 3.31. The value of $\mu(u, w)$ depends only on this interval. The remaining statements follow using Rota's crosscut theorem [1964] on Möbius functions for lattices; see also [Blass and Sagan 1997, Theorem 1.3]. □

4. Combinatorics of $M(W)$

In this section we study the combinatorics of the biHecke monoid $M(W)$ of a finite Coxeter group W. In particular, we prove in Sections 4a and 4b that its elements preserve left order and Bruhat order, and derive in Section 4c properties of their image sets and fibers. In Sections 4d and 4e, we prove the key combinatorial ingredients for the enumeration of the simple modules of $\mathbb{K}M(W)$ in Section 7: $M(W)$ is aperiodic and its $\mathcal{J}$-classes of idempotents are indexed by W. Finally, in Section 4f we study Green's relations as introduced in Section 2e and involutions on $M(W)$ in Section 4g.

4a. ***Preservation of left order.*** Recall that $M(W)$ is defined by its right action on elements in W by (1-5) and (1-6). The following key proposition, illustrated in Figure 3, states that it therefore preserves properties on the left.

Proposition 4.1. *Take $f \in M(W)$, $w \in W$, and $j \in I$. Then, $(s_j w).f$ is either $w.f$ or $s_j(w.f)$.*

The proof of Proposition 4.1 is a consequence of the associativity of the 0-Hecke monoid and relies on the following lemma, which is a nice algebraic (partial) formulation of the exchange property [Björner and Brenti 2005, Section 1.5].

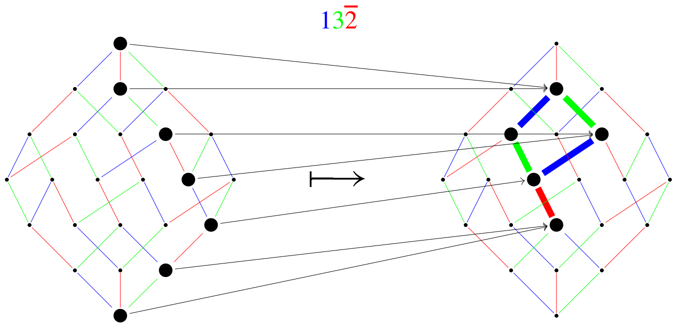


Figure 3. A partial picture of the graph of the element $f := \pi_1\pi_3\overline{\pi}_2$ of the monoid $M(\mathfrak{S}_4)$. On both sides, the underlying poset is left order of $\mathfrak{S}_4$ (with 1 at the bottom, and the same color code as in Figure 1); on the right, the bold dots depict the image set of f. The arrows from the left to the right describe the image of each point along some chain from 1 to w_0.

Lemma 4.2. *Let $w \in W$ and $i, j \in I$ such that $j \notin \mathrm{D}_L(w)$. Then*

$$(s_j w)\,.\,\pi_i = \begin{cases} w\,.\,\pi_i & \text{if } j \in \mathrm{D}_L(w\,.\,\pi_i), \\ s_j(w\,.\,\pi_i) & \text{otherwise.} \end{cases}$$

The same result holds with π_i replaced by $\overline{\pi}_i$.

Proof. Recall that $w\,.\,\pi_v = 1\,.\,(\pi_w\pi_v)$ for any $w, v \in W$. Set $w' = w\,.\,\pi_i$. Then

$$\begin{aligned}(s_j w)\,.\,\pi_i &= 1.(\pi_{s_j w}\pi_i) = 1.((\pi_j\pi_w)\pi_i) = 1\,.\,(\pi_j(\pi_w\pi_i)) = 1\,.\,(\pi_j\pi_{w'}) \\ &= \begin{cases} 1\,.\,\pi_{w'} = w' & \text{if } j \in \mathrm{D}_L(w'), \\ 1\,.\,\pi_{s_j w'} = s_j w' & \text{otherwise.} \end{cases}\end{aligned}$$

The result for $\overline{\pi}_i$ follows from Remark 1.1 and the fact that $w_0 s_j = s_{j'} w_0$ for some $j' \in I$ by Example 3.9 and Lemma 3.3 with $w = w_0$ and $K = \{j\}$. □

Proof of Proposition 4.1. Any element $f \in M(W)$ can be written as a product of π_i and $\overline{\pi}_i$. Lemma 4.2 describes the action of π_i and $\overline{\pi}_i$ on the Hasse diagram of left order. By induction, each π_i and $\overline{\pi}_i$ in the expansion of f satisfies all desired properties, and hence so does f (the statement holds trivially for the identity). □

Proposition 4.3. *For $f \in M(W)$, the following holds.*

(i) *f preserves left order*:

$$w \leq_L w' \quad \textit{implies} \quad w\,.\,f \leq_L w'\,.\,f \quad \textit{for } w, w' \in W.$$

(ii) *Take $w \leq_L w'$ in W, and consider a maximal chain*

$$w\,.\,f = v_1 \xrightarrow{i_1} v_2 \xrightarrow{i_2} \cdots \xrightarrow{i_{k-1}} v_k = w'\,.\,f.$$

Then, there is a maximal chain

$$w = u_{1,1} \to \cdots \to u_{1,\ell_1} \xrightarrow{i_1} u_{2,1} \to \cdots \to u_{2,\ell_2} \xrightarrow{i_2} \cdots$$
$$\cdots \xrightarrow{i_{k-1}} u_{k,1} \to \cdots \to u_{k,\ell_k} = w', \quad \text{(4-1)}$$

such that $u_{j,l} \,.\, f = v_j$ *for all* $1 \le j \le k$ *and* $1 \le l \le \ell_j$.

(iii) f *is length contracting*; *that is*, *for* $w \le_L w'$

$$\ell(w' \,.\, f) - \ell(w \,.\, f) \le \ell(w') - \ell(w).$$

Furthermore, *when equality holds*, $(w' \,.\, f)(w \,.\, f)^{-1} = w'w^{-1}$.

(iv) *Let* $J = [a, b]_L$ *be an interval in left order. Then the image of* J *under* f *denoted by* $J \,.\, f$ *has* $a \,.\, f$ *and* $b \,.\, f$ *as minimal and maximal element*, *respectively. Furthermore*, $J \,.\, f$ *is connected. If* $\ell(b \,.\, f) - \ell(a \,.\, f) = \ell(b) - \ell(a)$, *then* $J \,.\, f$ *is isomorphic to* J, *that is*, $x \,.\, f = (xa^{-1})(a \,.\, f)$ *for* $x \in J$.

Proof. Parts (i) and (ii) are direct consequences of Proposition 4.1, using induction.

Part (iii) follows from (ii).

Part (iv) follows from (i), (ii), and (iii) applied to $a \le_L x$ for all $x \in [a, b]_L$. □

4b. *Preservation of Bruhat order.* Recall the following well-known property of Bruhat order of Coxeter groups.

Proposition 4.4 (lifting property [Björner and Brenti 2005, p. 35]). *Suppose* $u <_B v$ *and* $i \in \mathrm{D}_R(v)$ *but* $i \notin \mathrm{D}_R(u)$. *Then*, $u \le_B vs_i$ *and* $us_i \le_B v$.

The next proposition is a consequence of the lifting property.

Proposition 4.5. *The elements* f *of* $M(W)$ *preserve Bruhat order. That is*, *for* $u, v \in W$

$$u \le_B v \quad \textit{implies} \quad u \,.\, f \le_B v \,.\, f.$$

Proof. It suffices to show the property for π_i and $\bar{\pi}_i$ since they generate $M(W)$. For these, the claim of the proposition is trivial if i is a right descent of u, or i is not a right descent of v. Otherwise, we can apply the lifting property:

$$u \,.\, \pi_i = us_i \le_B v = v \,.\, \pi_i,$$
$$u \,.\, \bar{\pi}_i = u \le_B vs_i = v \,.\, \bar{\pi}_i.$$
□

Remark 4.6. By Lemma 2.3, the preimage of a point is a convex set, but need not be an interval. For example, the preimage of $s_1 s_3 \in \mathfrak{S}_4$ (or 2143 in one-line notation) of $f = \bar{\pi}_1 \pi_2 \pi_1 \pi_3 \bar{\pi}_2 \bar{\pi}_3 \bar{\pi}_1 \bar{\pi}_2$ is

$$\{2413, 2341, 4213, 3412, 3241, 2431, 4312, 4231, 3421, 4321\},$$

which in Bruhat order has two maximal elements 2413 and 2341 and hence is not an interval.

Corollary 4.7 (of Proposition 4.3). *Let $f \in M(W)$.*

(i) *If $1 \,.\, f = 1$, then f is regressive for Bruhat order: $w \,.\, f \leq_B w$ for all $w \in W$.*

(ii) *If $w_0 \,.\, f = w_0$, then f is extensive for Bruhat order: $w \,.\, f \geq_B w$ for all $w \in W$.*

Proof. First suppose that $1 \,.\, f = 1$. Let $w \,.\, f = s_{i_k} \cdots s_{i_1}$ be a reduced decomposition of $w \,.\, f$. This defines a maximal chain

$$1 \,.\, f = 1 = v_0 \xrightarrow{i_1} \cdots \xrightarrow{i_{k-2}} v_{k-2} \xrightarrow{i_{k-1}} v_{k-1} \xrightarrow{i_k} v_k = w \,.\, f$$

in left order. By Proposition 4.3(ii) there is a larger chain from 1 to w so that there is a reduced word for w which contains $s_{i_k} \cdots s_{i_1}$ as a subword. Hence by the subword property of Bruhat order $w \,.\, f \leq_B w$. This proves (i).

Now let $w_0 \,.\, f = w_0$. By arguments similar to the above, constructing a maximal chain from $w \,.\, f$ to $w_0 \,.\, f$ in left order, one finds that $w_0(w \,.\, f)^{-1} \leq_B w_0 w^{-1}$. By [Björner and Brenti 2005, Proposition 2.3.4], the map $v \mapsto w_0 v$ is a Bruhat antiautomorphism and by the subword property $v \mapsto v^{-1}$ is a Bruhat automorphism. This implies $w \leq_B w \,.\, f$ as desired for (ii). □

4c. ***Fibers and image sets.*** Viewing elements of the biHecke monoid $M(W)$ as functions on W, we now study properties of their fibers and image sets.

Proposition 4.8. (i) *The image set* $\mathrm{im}(f)$ *for any $f \in M(W)$ is connected (see Definition 2.1) with a unique minimal and maximal element in left order.*

(ii) *The image set of an idempotent in $M(W)$ is an interval in left order.*

Proof. Part (i) follows immediately from Proposition 4.3(iv) with $J = [1, w_0]_L$.

For part (ii), we let $e \in M(W)$ be an idempotent with image set $\mathrm{im}(e)$. By Proposition 4.3(iv) with $J = [1, w_0]_L$, we have that $1 \,.\, e$ and $w_0 \,.\, e$ are the minimal and maximal, respectively, elements of $\mathrm{im}(e)$. Then by Proposition 4.3(ii), for every maximal chain in left order between $1 \,.\, e$ and $w_0 \,.\, e$, there is a maximal chain in left order of preimage points. Since e is an idempotent, there must be such a chain that contains the original chain. Hence all chains in left order between $1 \,.\, e$ and $w_0 \,.\, e$ are in $\mathrm{im}(e)$, proving that $\mathrm{im}(e)$ is an interval. □

Note that the proof above, in particular Proposition 4.3(ii), heavily uses the fact that the edges in left order are colored.

Definition 4.9. For any $f \in M(W)$, we call the set of fibers of f, denoted by $\mathrm{fibers}(f)$, the (unordered) set-partition of W associated by the equivalence relation $w \equiv w'$ if $w \,.\, f = w' \,.\, f$.

Proposition 4.10. *Take $f \in M(W)$, and consider the Hasse diagram of left order contracted with respect to the fibers of f. Then, this graph is isomorphic to left order restricted on the image set.*

Proof. See Appendix A on colored graphs. □

Proposition 4.11. *Any $f \in M(W)$ is characterized by its set of fibers and $1 \,.\, f$.*

Proof. Fix a choice of fibers. Contract the left order with respect to the fibers. By Proposition 4.10 this graph has to be isomorphic to the left order on the image set.

Once the lowest element in the image set $1 \,.\, f$ is fixed, this isomorphism is forced, since by Proposition 4.8(i) the graphs are (weakly) connected, have a unique minimal element, and there is at most one arrow of a given color leaving each node. □

Proposition 4.11 makes it possible to visualize nontrivial elements of the monoid (see Figure 4).

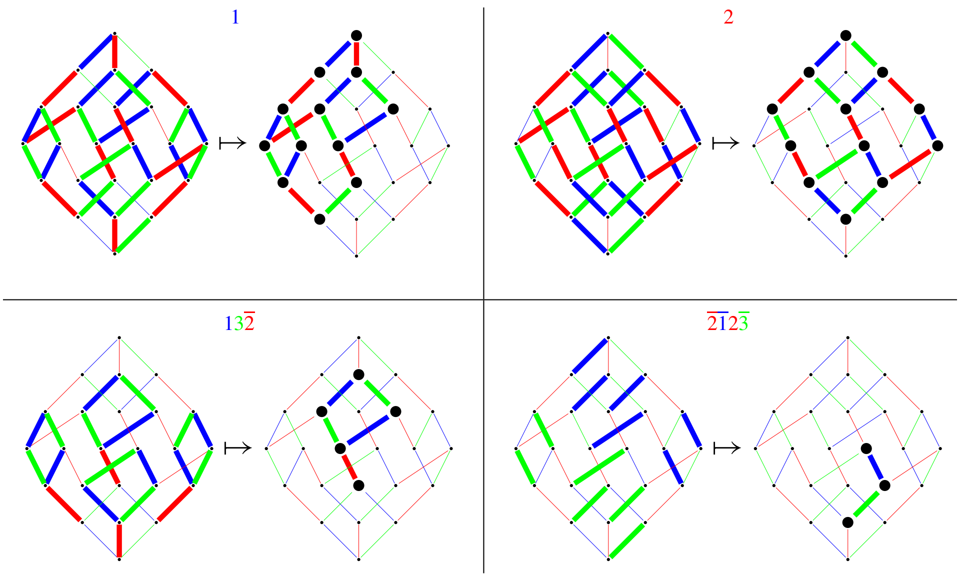


Figure 4. The elements $f = \pi_1$, π_2, $\pi_1\pi_3\overline{\pi}_2$ and $\overline{\pi}_2\overline{\pi}_1\pi_2\overline{\pi}_3$ of $M(\mathfrak{S}_4)$. As in Figure 3, the underlying poset on both sides is left order on $\mathfrak{S}_4$, and the bold dots on the right sides depict the image set of f. On the left side, an edge between two elements of W is thick if they are not in the same fiber. This information completely describes f; indeed $u = 1$ on the left is mapped to the lowest element of the image set on the right; each time one moves u up along a thick edge on the left, its image $u \,.\, f$ is moved up along the edge of the same color on the right.

Recall that a set-partition $\Lambda = \{\Lambda_i\}$ is said to be finer than the set-partition $\Lambda' = \{\Lambda'_i\}$ if for all i there exists a j such that $\Lambda_i \subseteq \Lambda'_j$. This is denoted by $\Lambda \preceq \Lambda'$. The refinement relation is a partial order.

For $f \in M(W)$, define the *type* of f by

$$\operatorname{type}(f) := \operatorname{type}([1\,.\,f, w_0\,.\,f]_L) = (w_0\,.\,f)(1\,.\,f)^{-1}. \tag{4-2}$$

The *rank* of $f \in M(W)$ is the cardinality of the image set $\operatorname{im}(f)$.

Lemma 4.12. *Fix $f \in M(W)$. For $h = fg \in fM(W)$,*

(1) $\operatorname{fibers}(f) \preceq \operatorname{fibers}(h)$,

(2) $\operatorname{type}(h) \leq_B \operatorname{type}(f)$,

(3) $\operatorname{rank}(h) \leq \operatorname{rank}(f)$.

Furthermore, the following are equivalent:

(i) $\operatorname{fibers}(h) = \operatorname{fibers}(f)$,

(ii) $\operatorname{rank}(h) = \operatorname{rank}(f)$,

(iii) $\operatorname{type}(h) = \operatorname{type}(f)$,

(iv) $\ell(w_0\,.h) - \ell(1\,.h) = \ell(w_0\,.\,f) - \ell(1\,.\,f)$.

If any, and therefore all, of the above hold, then h is completely determined (within $fM(W)$) by $1\,.h$.

Proof. For $f, g \in M(W)$, the statement $\operatorname{fibers}(f) \preceq \operatorname{fibers}(fg)$ is obvious.

By Proposition 4.3(iii) and (iv), we know for $f, g \in M(W)$ that either $\operatorname{type}(fg) = \operatorname{type}(f)$ or $\ell(w_0.(fg)) - \ell(1\,.(fg)) < \ell(w_0\,.\,f) - \ell(1\,.\,f)$. In the latter case by Proposition 4.5, $\operatorname{type}(fg) <_B \operatorname{type}(f)$. The second case occurs precisely when $\operatorname{fibers}(f)$ is strictly finer than $\operatorname{fibers}(fg)$, or equivalently $\operatorname{rank}(fg) < \operatorname{rank}(f)$.

The last statement, that if $\operatorname{fibers}(h) = \operatorname{fibers}(f)$ then h is determined by $1\,.h$, follows from Proposition 4.11. □

4d. *Aperiodicity.* Recall from Section 2e that a monoid M is called *aperiodic* if for any $f \in M$, there exists $k > 0$ such that $f^{k+1} = f^k$. Note that, in this case, $f^\omega := f^k = f^{k+1} = \cdots$ is an idempotent.

Proposition 4.13. *The biHecke monoid $M(W)$ is aperiodic.*

Proof. From Proposition 4.3(iv), we know that $\operatorname{im}(f^k)$ has a minimal element $a_k = 1\,.\,f^k$ and a maximal element $b_k = w_0\,.\,f^k$ in left order. Since $\operatorname{im}(f^{k+1}) \subseteq \operatorname{im}(f^k)$, we have $a_{k+1} \geq_L a_k$ and $b_{k+1} \leq_L b_k$. Therefore, both sequences a_k and b_k must ultimately be constant.

This implies that, for N big enough, a_N and b_N are fixed points. Applying Proposition 4.3(iii) yields that all elements in $[a_N, b_N]_L$ are fixed points under f.

It follows successively that $\operatorname{im}(f^N) = [a_N, b_N]_L$, $f^N = f^{N+1} = \cdots$, and $\operatorname{fix}(f) = [a_N, b_N]_L$. □

Corollary 4.14. *The set of fixed points of an element $f \in M(W)$ is an interval in left order.*

Proof. The set of fixed point of f is the image set of f^ω, which is an interval in left order by Proposition 4.8(ii). □

4e. *Idempotents.* We now study the properties of idempotents in $M(W)$.

Proposition 4.15. (i) *For $w \in W$*

$$e_w := \pi_{w^{-1}w_0}\overline{\pi}_{w_0 w}$$

is the unique idempotent such that $1 . e_w = 1$ and $w_0 . e_w = w$. Its image set is $[1, w]_L$, and it satisfies

$$u.e_w = \max_{\leq_B}\big([1, u]_B \cap [1, w]_L\big).$$

(ii) *Similarly, for $w \in W$,*

$$\tilde{e}_w := \overline{\pi}_{w^{-1}}\pi_w$$

is the unique idempotent with image set $[w, w_0]_L$, and it satisfies a dual formula.

(iii) *Furthermore,*

$$e_{a,b} := \overline{\pi}_{a^{-1}} e_{ba^{-1}} \pi_a$$

is an idempotent with image set $[a, b]_L$.

Proof. (i) Clearly, the image of e_w is a subset of $[1, w]_L$. Applying Remark 2.7 shows that $[1, w]_L$ is successively mapped bijectively to $[w^{-1}w_0, w_0]_L$ and back to $[1, w]_L$. So e_w is an idempotent with image set $[1, w]_L$. Reciprocally, let f be an idempotent such that $1 . f = 1$ and $w_0 . f = w$. Then, by Proposition 4.5, f preserves Bruhat order and by Corollary 4.7(i), $u . f \leq_B u$ for all $u \in W$. Furthermore, by Proposition 4.8, the image set of f is the interval $[1, w]_L$. Using Proposition 2.4, uniqueness and the given formula follow.

Statement (ii) is dual to (i) and is proved similarly.

(iii) The image set of $e_{ba^{-1}}$ is $[1, ba^{-1}]_L$; hence the image set of $e_{a,b}$ is a subset of $[a, b]_L$. We conclude by checking that $[a, b]_L$ is mapped bijectively at each step $\overline{\pi}_{a^{-1}}$, $e_{ba^{-1}}$ and π_a (see also Remark 2.7), and therefore consists of fixed points. □

Remark 4.16. For $f \in M(W)$, $f e_v = f e_{u.e_v}$, where $u = w_0 . f$.

Proof. Use the formula of Proposition 4.15(i). □

Corollary 4.17. *For $u, w \in W$, the intersection $[1, u]_B \cap [1, w]_L$ is a $\leq_L$-lower set with a unique maximal element v in Bruhat order. The maximum is given by $v = u \,.\, e_w$.*

4f. *Green's relations.* We have now gathered enough information about the combinatorics of $M(W)$ to give a partial description of its Green's relations, which will be used in the study of the representation theory of $M(W)$.

As an example, Figure 5 completely describes Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ for $M(\mathfrak{S}_3)$. The vertices are the 23 elements of $M(\mathfrak{S}_3)$, each drawn as in Figure 4. The edges give both the left and right Cayley graph of $M(\mathfrak{S}_3)$; for example, there are arrows

$$f \xrightarrow{\times\pi_1} g \quad \text{if } g = f\pi_1 \quad \text{and} \quad f \xrightarrow{\pi_1\times\pi_1} g \quad \text{if } g = f\pi_1 = \pi_1 f.$$

The picture also highlights the $\mathcal{J}$-classes of $M(\mathfrak{S}_3)$, and the corresponding eggbox pictures (that is, the decomposition of the $\mathcal{J}$-classes into $\mathcal{L}$ and $\mathcal{R}$-classes); namely, from top to bottom, there is one $\mathcal{J}$-class of size $1 = 1 \times 1$, two $\mathcal{J}$-classes of size $2 = 1 \times 2$, two $\mathcal{J}$-classes of size $6 = 2 \times 3$, and one $\mathcal{J}$-class of size $6 = 1 \times 6$, where $n \times m$ gives the dimension of the eggbox picture. In other words the $\mathcal{J}$-class splits into n $\mathcal{R}$-classes of size m and also into m $\mathcal{L}$-classes of size n. This example is specific in that all $\mathcal{J}$-classes are regular.

In the sequel, we describe $\mathcal{R}$-classes for general elements, as well as $\mathcal{J}$-order on regular elements. In particular, we obtain that the $\mathcal{J}$-classes of idempotents are indexed by the elements of W, and that $\mathcal{J}$-order on regular classes is given by left-right order $<_{LR}$ on W. Note that the latter is *not* a lattice, unlike for the variety $\mathcal{DA}$ (which consists of all aperiodic monoids all of whose simple modules are dimension 1; see for example [Ganyushkin et al. 2009]).

Proposition 4.18. *Two elements $f, g \in M(W)$ are in the same $\mathcal{R}$-class if and only if they have the same fibers. In particular, the $\mathcal{R}$-class of f is given by*

$$\mathcal{R}(f) = \{h \in f M(W) \mid \operatorname{rank}(h) = \operatorname{rank}(f)\} = \{f_u \mid u \in [1, \operatorname{type}(f)^{-1} w_0]_R\}, \quad (4\text{-}3)$$

where f_u is the unique element of $M(W)$ such that $\operatorname{fibers}(f_u) = \operatorname{fibers}(f)$ *and* $1 \,.\, f_u = u$.

Proof. It is a general easy fact about monoids of functions that elements in the same $\mathcal{R}$-class have the same fibers (see also Lemma 4.12). Reciprocally, if g has the same fibers as f, then one can use Remark 2.7 to define $g' = g\overline{\pi}_{(1\,.\,g)^{-1}}\pi_{1\,.\,f}$ such that $\operatorname{fibers}(g') = \operatorname{fibers}(f)$ and $1 \,.\, g' = 1 \,.\, f$. Also by Proposition 4.11, $f = g' \in gM(W)$, and similarly, $g \in f M(W)$.

Equation (4-3) follows using Lemma 4.12 and Remark 2.7. $\square$

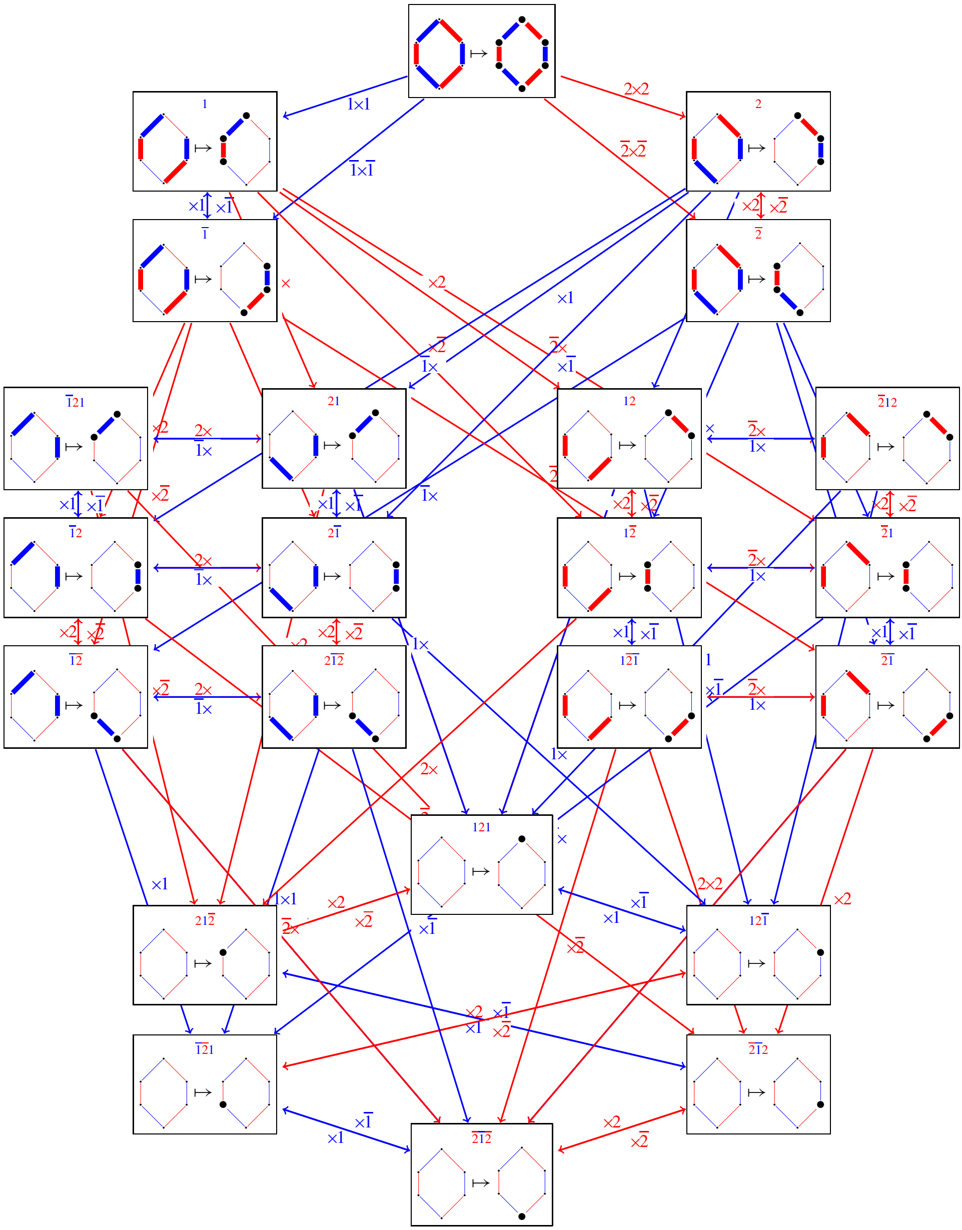


Figure 5. The graph of $\mathscr{J}$-order for $M(\mathfrak{S}_3)$, as described on page 636.

Lemma 4.19. *Let e and f be idempotents of $M(W)$ with respective image sets $[a, b]_L$ and $[c, d]_L$. Then, $f \leq_{\mathscr{J}} e$ if and only if $dc^{-1} \leq_{LR} ba^{-1}$.*

In particular, two idempotents e and f are $\mathscr{J}$-equivalent if and only if the intervals $[a, b]_L$ and $[c, d]_L$ are of the same type: $dc^{-1} = ba^{-1}$.

The properties above extend to any two regular elements (elements whose $\mathcal{J}$-class contains an idempotent).

Proof. First note that an interval $[c, d]_L$ is isomorphic to a subinterval of $[a, b]_L$ if and only $dc^{-1} \leq_{LR} ba^{-1}$. This follows from Proposition 2.5 and the fact that $[c, d]_L$ is a subinterval of $[a, b]_L$ if and only if $[ca^{-1}, da^{-1}]_L$ is a subinterval of $[1, ba^{-1}]_L$. But then dc^{-1} is a subfactor of ba^{-1}.

Assume first that $dc^{-1} \leq_{LR} ba^{-1}$, and let $[c', d']_L$ be a subinterval of $[a, b]_L$ isomorphic to $[c, d]_L$. Using Proposition 2.5, take $u, v \in M(W)$ that induce reciprocal bijections between $[c, d]_L$ and $[c', d']_L$. Then, $f = fuev$, so that f is $\mathcal{J}$-equivalent to e.

Reciprocally, assume that $f = uev$ with $u, v \in M(W)$. Without loss of generality, we may assume that $u = ue$ so that $\operatorname{im}(u) \subseteq [a, b]_L$. Set $c' = c.u$ and $d' = d.u$. Since $f = ff = fuv$, and using Proposition 4.3, the functions u and v must induce reciprocal isomorphisms between $[c, d]_L$ and $[c', d']_L$, the latter being a subinterval of $[a, b]_L$. Therefore, $dc^{-1} \leq_{LR} ba^{-1}$.

To conclude, note that a regular element has the same type as any idempotent in its $\mathcal{J}$-class. □

Corollary 4.20. *The idempotents $(e_w)_{w\in W}$ form a complete set of representatives of regular $\mathcal{J}$-classes in $M(W)$.*

Example 4.21. For $w \in W$, the idempotents e_w and $\tilde{e}_{w^{-1}w_0}$ are in the same $\mathcal{J}$-class. This follows immediately from Lemma 4.19, or by direct computation using the explicit expressions for e_w and $\tilde{e}_{w^{-1}w_0}$ in Proposition 4.15:

$$e_w = e_w^2 = \pi_{w^{-1}w_0}\overline{\pi}_{w_0 w}\pi_{w^{-1}w_0}\overline{\pi}_{w_0 w} = \pi_{w^{-1}w_0}\tilde{e}_{w^{-1}w_0}\overline{\pi}_{w_0 w},$$
$$\tilde{e}_{w^{-1}w_0} = \tilde{e}_{w^{-1}w_0}^2 = \overline{\pi}_{w_0 w}\pi_{w^{-1}w_0}\overline{\pi}_{w_0 w}\pi_{w^{-1}w_0} = \overline{\pi}_{w_0 w} e_w \pi_{w^{-1}w_0}.$$

Corollary 4.22. *The image of a regular element is an interval in left order.*

Proof. A regular element has the same type, and same size of image set as any idempotent in its $\mathcal{J}$-class. □

Remark 4.23. The reciprocal is false: In type B_3, the element $\overline{\pi}_1\overline{\pi}_3\overline{\pi}_2\pi_1\overline{\pi}_3\overline{\pi}_2\overline{\pi}_1$ has the interval $[1, s_2s_3s_2]_L$ as image set, but it is not regular. The same holds in type A_4 with the element $\pi_2\pi_1\overline{\pi}_4\pi_3\overline{\pi}_2\overline{\pi}_1\overline{\pi}_3\pi_4\overline{\pi}_2\overline{\pi}_3\overline{\pi}_4$.

Problem 4.24. Describe $\mathcal{L}$-classes in general, and $\mathcal{L}$-order, $\mathcal{R}$-order, as well as $\mathcal{J}$-order on nonregular elements.

4g. ***Involutions and consequences.*** Define an involution $*$ on W by

$$w \mapsto w^* := w_0 w,$$

where w_0 is the maximal element of W. Moreover, define the *bar* map $M(W) \to M(W)$ as the conjugacy by $*$: For a given $f \in M(W)$

$$w\,.\,\bar{f} := (w^*\,.\,f)^* \quad \text{for all } w \in W.$$

Proposition 4.25. *The bar involution is a monoid endomorphism of $M(W)$ that exchanges π_i and $\overline{\pi}_i$.*

Proof. This is a consequence of the general fact that for any permutation ϕ of W, conjugation by ϕ is an automorphism of the monoid of maps from W to itself. Moreover, it is easy to see that bar exchanges π_i and $\overline{\pi}_i$, so that it fixes $M(W)$. □

The previous proposition has some interesting consequences when applied to idempotents: For any $w \in W$, the bar involution is a bijection from $e_w M(W)$ to $\overline{e}_w M(W)$. But $\overline{e}_w$ fixes w_0 and sends $1 = w_0^*$ to w^*, so that $\overline{e}_w = e_{w^*, w_0} = \tilde{e}_{w_0 w}$. The latter is in turn $\mathcal{J}$-equivalent to $e_{w_0 w^{-1} w_0}$ by Example 4.21. This implies the following result.

Corollary 4.26. *The ideals $e_w M(W)$ and $e_{w_0 w^{-1} w_0} M(W)$ are in bijection.*

5. The Borel submonoid $M_1(W)$ and its representation theory

In the previous section, we outlined the importance of the idempotents $(e_w)_{w \in W}$. A crucial feature is that they live in a "Borel" submonoid $M_1(W) \subseteq M(W)$ of elements of the biHecke monoid $M(W)$ that fix the identity:

$$M_1(W) := \{f \in M(W) \mid 1\,.\,f = 1\}.$$

In this section we study this monoid and its representation theory, as an intermediate step toward the representation theory of $M(W)$ (see Section 6). For the representation theory of $M(W)$, it is actually more convenient to work with the submonoid fixing w_0 instead of 1:

$$M_{w_0}(W) := \{f \in M(W) \mid w_0\,.\,f = w_0\}.$$

However, since both monoids $M_1(W)$ and $M_{w_0}(W)$ are isomorphic under the involution of Section 4g and since the interaction of $M_1(W)$ with Bruhat order is notationally simpler, we focus on $M_1(W)$ in this section.

Note. In the remainder of this paper, unless explicitly stated, we fix a Coxeter group W and use the shorthand notation $M := M(W)$, $M_1 := M_1(W)$ and $M_{w_0} := M_{w_0}(W)$.

From the definition it is clear that M_1 is indeed a submonoid that contains the idempotents $(e_w)_{w \in W}$. Furthermore, by Proposition 4.5 and Corollary 4.7 its elements are both order-preserving and regressive for Bruhat order. In fact, a bit more can be said.

Remark 5.1. For $w \in W$, $w \,.\, M_1$ is the interval $[1, w]_B$ in Bruhat order.

Proof. By Corollary 4.7, for $f \in M_1$, we have $w \,.\, f \leq_B w$. Take reciprocally $v \in [1, w]_B$. Then, using Proposition 4.15, $w \,.\, e_v = v$. □

As a consequence of the preservation and regressiveness on Bruhat order, M_1 is *an ordered monoid with* 1 *on top*. Namely, for $f, g \in M_1$, define the relation $f \leq g$ if $w \,.\, f \leq_B w \,.\, g$ for all $w \in W$. Then, $\leq$ defines a partial order on M_1 such that $f \leq 1$, $fg \leq f$ and $fg \leq g$ for all $f, g \in M_1$. In other words, M_1 is $\mathscr{R}$-trivial (see [Denton et al. 2010/11, Proposition 2.2], as well as Section 2.5 there) and in particular $\mathscr{J}$-trivial.

In the next two subsections, we study the combinatorics of M_1 and then apply the general results on the representation theory of $\mathscr{J}$-trivial monoids of [Denton et al. 2010/11] to M_1.

5a. *$\mathscr{J}$-order on idempotents and minimal generating set.* Recall from Section 2e that $\mathscr{J}$-order is the partial order $\leq_{\mathscr{J}}$ defined by $f \leq_{\mathscr{J}} g$ if there exists $x, y \in M_1$ such that $f = xgy$. The restriction of $\mathscr{J}$-order to idempotents has a very simple description:

Proposition 5.2. *For $u, v \in W$, the following are equivalent*:

$$e_u e_v = e_u, \qquad u \leq_L v,$$
$$e_v e_u = e_u, \qquad e_u \leq_{\mathscr{J}} e_v.$$

Moreover, $(e_u e_v)^\omega = e_{u \wedge_L v}$, *where $u \wedge_L v$ is the meet* (*or greatest lower bound*) *of u and v in left order.*

Proof. This follows from [Denton et al. 2010/11, Theorem 3.4, Lemma 3.6] and Proposition 4.15. □

As a consequence the following definition, which plays a central role in the representation theory of $\mathscr{J}$-trivial monoids [Denton et al. 2010/11], makes sense.

Definition 5.3. For any element $x \in M_1$, define

$$\operatorname{lfix}(x) := \min_{\leq_L}\{u \in W \mid e_u x = x\} \quad \text{and} \quad \operatorname{rfix}(x) := \min_{\leq_L}\{u \in W \mid x e_u = x\} = w_0 \,.\, x,$$

the min being taken for the left order.

Interestingly, M_1 can be defined as the submonoid of M generated by the idempotents $(e_w)_{w \in W}$, and in fact the subset of these idempotents indexed by Grassmannian elements (an element $w \in W$ is *Grassmannian* if it has at most one descent).

Theorem 5.4. *M_1 has a unique minimal generating set that consists of the idempotents e_w where $w^{-1} w_0$ is right Grassmannian.*

In type A_{n-1} this minimal generating set is of size $2^n - n$ (which is the number of Grassmannian elements in this case [Manivel 2001]).

Proof. Define the length $\ell(f)$ of an element $f \in M$ as the length of a minimal expression of f as a product of the generators π_i and $\overline{\pi}_i$. We now prove by induction on the length that M_1 is generated by $\{e_w \mid w \in W\}$.

Take an element $f \in M_1$ of length l. If $l = 0$ we are done. Otherwise, since $1.f = 1$, an expression of f as a product of the π_i and $\overline{\pi}_i$ contains at least one $\overline{\pi}_i$. Write $f = gh$ where $g = \pi_w \overline{\pi}_i$ for some $w \in W$ and $h \in M$ so that $\ell(w)+1+\ell(h) = l$.

Claim. *$f = e_{w_0(ws_i)^{-1}} \pi_w h$ and $\pi_w h \in M_1$.*

It follows from the claim that $\ell(\pi_w h) < l$, and hence since $\pi_w h \in M_1$ we can apply induction to conclude that M_1 is generated by $\{e_w \mid w \in W\}$.

Let us prove the claim. By minimality of l, i is not a descent of w (otherwise, we would obtain a shorter expression for f: $f = \pi_w \overline{\pi}_i h = \pi_{w'} \pi_i \overline{\pi}_i h = \pi_{w'} \overline{\pi}_i h$, where $\ell(w') < \ell(w)$). Therefore, $1.g = 1.(\pi_w \overline{\pi}_i) = w$. Since $f \in M_1$ it follows that $w.h = 1$ and therefore $\pi_w h \in M_1$. It further follows that $\overline{\pi}_{w^{-1}} \pi_w$ acts trivially on the image set $[w, w_0]_L$ of g, and therefore $f = g \overline{\pi}_{w^{-1}} \pi_w h$. Note that $g \overline{\pi}_{w^{-1}} = \pi_w \overline{\pi}_i \overline{\pi}_{w^{-1}} = \pi_w \pi_i \overline{\pi}_i \overline{\pi}_{w^{-1}} = e_{w_0(ws_i)^{-1}}$.

By Proposition 5.2, the idempotents of M_1 are generated by the meet-irreducible idempotents e_w in $\mathcal{J}$ order. Here x is meet-irreducible if and only if $x = a$ or $x = b$ whenever $x = a \wedge b$ for some $a, b \in M_1$. These meet-irreducible elements are indexed by the elements w of W that are meet-irreducible in left order (or equivalently that have at most one left nondescent, that is, $w_0 w^{-1}$ is right Grassmannian).

The uniqueness of the minimal generating set holds for any $\mathcal{J}$-trivial monoid with a minimal generating set [Doyen 1984, Theorem 2; Doyen 1991, Theorem 1]. □

Actually one can be much more precise:

Proposition 5.5. *Any element $f \in M_1$ can be written as a product $e_{w_1} \cdots e_{w_k}$, where*

- *$w_1 >_B \cdots >_B w_k$ is a chain in Bruhat order such that any two consecutive terms w_i and w_{i+1} are incomparable in left order*;
- *$w_i = \mathrm{rfix}(e_{w_1} \cdots e_{w_i}) = \mathrm{lfix}(e_{w_i} \cdots e_{w_k})$.*

Proof. Start from any expression $e_{w_1} \cdots e_{w_k}$ for f. We show that if any of the conditions of the proposition is not satisfied, the expression can be reduced to a strictly smaller (in length, or in Bruhat, term by term) expression, so that induction can be applied.

- If $u \not>_B v$, then by Remark 4.16 $e_u e_v = e_u e_{u.e_v}$ with $u.e_v <_B v$.
- If $u <_L v$, then $e_u e_v = e_u$, and similarly on the right.

- If the left symbol e_u for $e_{w_i} \cdots e_{w_k}$ is not e_{w_i}, then $u <_L w_i$ and

$$e_{w_i} \cdots e_{w_k} = e_u e_{w_i} \cdots e_{w_k} = e_u e_{w_{i+1}} \cdots e_{w_k}.$$

Similarly on the right. □

Corollary 5.6. *For $f \in M_1$, $\operatorname{lfix}(f) \geq_B \operatorname{rfix}(f)$, with equality if and only if f is an idempotent.*

Lemma 5.7. *If $v \leq_B u$ in Bruhat order and $u' = \operatorname{lfix}(e_u e_v)$, then*

$$v \leq_B u' \quad \textit{and} \quad u' \leq_L u.$$

Proof. By Definition 5.3, $u' \leq_L u$ since $e_u(e_u e_v) = e_u e_v$ and for M_1 the minimum is measured in left order. Also by Proposition 4.15

$$v = w_0 . e_u e_v = w_0 . e_{u'} e_u e_v \leq_B u'.$$ □

Lemma 5.8. *If u covers v in Bruhat order and $u' = \operatorname{lfix}(e_u e_v)$, then either $u' = u$, or $u' = v$ and $e_u e_v = e_v e_u$.*

Proof. By Lemma 5.7, we have that either $u' = u$ or $u' = v$, since u covers v in Bruhat order. When $u' = v$, we have again by Lemma 5.7 that $v \leq_L u$. Hence $e_u e_v = e_v = e_v e_u$. □

5b. *Representation theory.* In this subsection, we specialize general results about the representation theory of finite $\mathcal{J}$-trivial monoids to describe some of the representation theory of the Borel submonoid M_1, such as its simple modules, radical, Cartan invariant matrix and quiver. The description also applies to M_{w_0}, mutatis mutandis. We follow the presentation of [Denton et al. 2010/11] (also see this paper for the proofs), though many of the general results have been previously known; see for example [Almeida et al. 2009; Clifford and Preston 1961; Ganyushkin et al. 2009; Lallement and Petrich 1969; Rhodes and Zalcstein 1991] and references therein.

5b1. *Simple modules and radical.* For each $w \in W$ define S_w^1 (written $S_w^{w_0}$ for M_{w_0}) to be the one-dimensional vector space with basis $\{\epsilon_w\}$ together with the right operation of any $f \in M_1$ given by

$$\epsilon_w . f := \begin{cases} \epsilon_w & \text{if } w . f = w, \\ 0 & \text{otherwise.} \end{cases}$$

The basic features of the representation theory of M_1 can be stated as follows:

Theorem 5.9. *The radical of $\mathbb{K}M_1$ is the ideal with basis $f^\omega - f$ for $f \in M_1$ nonidempotent. The quotient of $\mathbb{K}M_1$ by its radical is commutative. Therefore, all simple $\mathbb{K}M_1$-modules are one-dimensional. In fact, the family $\{S_w^1\}_{w \in W}$ forms a complete system of representatives of the simple $\mathbb{K}M_1$-modules.*

5b2. *Cartan matrix and projective modules.* The projective modules and Cartan invariants can be described as follows:

Theorem 5.10. *There exists an explicit basis $(b_x)_{x\in M_1}$ of $\mathbb{K}M_1$ such that, for all $w \in W$,*

- *the family $\{b_x \mid x \in M_1$ with $\operatorname{lfix}(x) = w\}$ is a basis for the right indecomposable projective module P^1_w associated to S^1_w;*
- *the family $\{b_x \mid \operatorname{rfix}(x) = w\}_{x\in M_1}$ is a basis for the left indecomposable projective module associated to S^1_w.*

Moreover, the Cartan invariant of $\mathbb{K}M_1$ defined by $c_{u,v} := \dim(e_u\mathbb{K}M_1e_v)$ for $u, v \in W$ is given by $c_{u,v} = |C_{u,v}|$, where

$$C_{u,v} := \{f \in M_1 \mid \operatorname{lfix}(f) = u \text{ and } \operatorname{rfix}(f) = v\}.$$

In particular, the Cartan matrix of $\mathbb{K}M_1$ is upper-unitriangular with respect to Bruhat order.

Proof. Apply [Denton et al. 2010/11, Section 3.4] and conclude with Corollary 5.6. □

Remark 5.11. In terms of characters, the previous theorem can be restated as

$$[P^1_u] = \sum_{f\in M_1,\operatorname{lfix}(f)=u} [S^1_{w_0\cdot f}], \tag{5-1}$$

which gives the following character for the right regular representation:

$$[\mathbb{K}M_1] = \sum_{f\in M_1} [S^1_{w_0\cdot f}]. \tag{5-2}$$

Problem 5.12. Describe the Cartan matrix and projective modules of $\mathbb{K}M_1$ more explicitly, if at all possible in terms of the combinatorics of the Coxeter group W.

5b3. *Quiver.* We now turn to a description of the quiver of $\mathbb{K}M_1$ in terms of the combinatorics of left and Bruhat order. Recall that M_1 is a submonoid of the monoid of regressive and order preserving functions. As such, it is not only $\mathcal{J}$-trivial but also ordered with 1 on top, that is $\mathcal{B}$-trivial; see [Denton et al. 2010/11, Section 2.5 and Proposition 2.2]. By [Denton et al. 2010/11, Theorem 3.35 and Corollary 3.44] we know that the vertices of the quiver of a $\mathcal{J}$-trivial monoid generated by idempotents are labeled by its idempotents $(e_x)_x$ and there is an edge from vertex e_x to vertex e_z, if $q := e_xe_z$ is not idempotent, has $\operatorname{lfix}(q) = x$ and $\operatorname{rfix}(q) = z$, and does not admit any factorization $q = uv$ that is nontrivial: $eu \neq e$ and $vf \neq f$. By [Denton et al. 2010/11, Proposition 3.31] the condition that q has a nontrivial factorization is equivalent to q having a compatible factorization $q = uv$, meaning that u, v are nonidempotents and $\operatorname{lfix}(q) = \operatorname{lfix}(u)$, $\operatorname{rfix}(u) = \operatorname{lfix}(v)$ and $\operatorname{rfix}(v) = \operatorname{rfix}(q)$.

Let $e_x, e_y, e_z \in M_1$ be idempotents. Call e_y an *intermediate factor* for $q := e_x e_z$ if $e_x e_y e_z = e_x e_z$. Call further e_y a *nontrivial intermediate factor* if $e_x e_y \neq e_x$, and $e_y e_z \neq e_z$.

Lemma 5.13. *The quiver of* $\mathbb{K}M_1$ *is the graph with* W *as vertex set and edges* (x, z) *for all* $x \neq z$ *such that* $q := e_x e_z$ *satisfies* $\operatorname{lfix}(q) = x$ *and* $\operatorname{rfix}(q) = z$ *and admits no nontrivial intermediate factor* e_y *with* $y \in W$.

Proof. Take $q := e_x e_z$ admitting a nontrivial intermediate factor e_y. Then q admits a nontrivial factorization $q = (e_x e_y)(e_y e_z)$ in the sense of [Denton et al. 2010/11, Definition 3.25], and is therefore not in the quiver.

Reciprocally, assume that q admits a compatible factorization, that is $q = uv$ with $\operatorname{lfix}(u) = x$, $\operatorname{rfix}(u) = \operatorname{lfix}(v)$ and $\operatorname{rfix}(v) = z$. By [Denton et al. 2010/11, Lemma 3.29], this factorization is nontrivial: $e_x u \neq e_x$ and $v e_z \neq e_z$. Using Proposition 5.5, write u and v as $u = e_x e_{y_1} \cdots e_{y_k}$ and $v = e_{y_k} \cdots e_{y_\ell} e_z$, with $x >_B y_1 >_B \cdots >_B y_\ell >_B z$. Then, $e_x e_{y_i} e_z = e_x e_z$ for any i; indeed, since M_1 is $\mathcal{B}$-trivial,

$$e_x e_z = e_x e_{y_1} \cdots e_{y_\ell} e_z \leq_{\mathcal{B}} e_x e_{y_i} e_z \leq_{\mathcal{B}} e_x e_z.$$

If any e_{y_i} is a nontrivial intermediate factor for q, we are done by setting $y = y_i$. Otherwise, $e_{y_i} e_z = e_z$ for any i ($e_x e_{y_i} = e_x$ is impossible since $x >_B y_i$). But then, $v = e_{y_k} \cdots e_{y_\ell} e_z = e_z$, a contradiction. □

Problem 5.14. Can Lemma 5.13 be generalized to any $\mathcal{B}$-trivial monoid? Its statement has been tested successfully on the 0-Hecke monoid in type $A_1 - A_6$, $B_3 - B_4$, $D_4 - D_5$, $H_3 - H_4$, G_2, I_{135}, F_4.

Lemma 5.13 admits a combinatorial reformulation in terms of the combinatorics of W. For $x, y, z \in W$ such that $x >_B z$, call $y \in W$ an *intermediate factor* for x, z if $[1, y]_L$ intersects all intervals $[c, a]_B$ with $a \in [1, x]_L$ and $c \in [1, z]_L$ nontrivially. Call further y a *nontrivial intermediate factor* if $x >_B y >_B z$ and $y \not>_L z$.

Theorem 5.15. *The quiver of* $\mathbb{K}M_1$ *is the graph with* W *as vertex set, and edges* (x, z) *for all* $x >_B z$ *and* $x \not>_L z$ *admitting no nontrivial intermediate factor* y. *Each such edge can be associated with the element* $q := e_x e_z$ *of the monoid.*

In particular, the quiver of $\mathbb{K}M_1$ *is acyclic and every cover* $x \succ_B z$ *in Bruhat order that is not a cover in left order contributes one edge to the quiver.*

Proof. Consider a nonidempotent product $e_x e_z$. Using Proposition 5.5, we may assume without loss of generality that $x >_B z$ and $x \not>_L z$, and furthermore that $\operatorname{lfix}(e_x e_z) = x$ and $\operatorname{rfix}(e_x e_z) = z$.

We now show that the combinatorial definition of intermediate factor on an element of $y \in W$ is a reformulation of the monoidal one on the idempotent e_y of M_1.

Assume that e_y is an intermediate factor for $e_x e_z$, that is, $e_x e_y e_z = e_x e_z$. Take $a \in [1, x]_L$ and $c \in [1, z]_L$ with $a \geq_B c$, and write $b = a.e_y \in [1, y]_L$. Using Proposition 4.15, $a \geq_B b$ and $a.e_z \geq_B c$. Furthermore, since a is in the image set of e_x, one has $b.e_z = a.e_y.e_z = a.e_z \geq_B c$. Therefore, $[1, y]_L$ intersects $[c, a]_B$ at least in b. Hence, y is an intermediate factor for x, z.

For the reciprocal, take any $a \in [1, x]_L$. Since M_1 preserves Bruhat order and is regressive, $a.e_y.e_z \leq_B a.e_z$. Set $c = a.e_z$, and take b in $[c, a]_B \cap [1, y]_L$. Using Proposition 4.15,

$$a.e_y.e_z \geq_B b.e_z \geq_B c = a.e_z,$$

and equality holds. Hence, e_y is an intermediate factor for e_x, e_y: $e_x e_y e_z = e_x e_z$.

The combinatorial reformulation of nontriviality for intermediate factors is then straightforward using Proposition 5.5. □

Problem 5.16. Exploit the interrelations between left order and Bruhat order to find a more satisfactory combinatorial description of the quiver of $\mathbb{K}M_1$.

5b4. *Connection with the representation theory of the* 0*-Hecke monoid.* Recall that the 0-Hecke monoid $H_0(W)$ is a submonoid of $M_{w_0}(W)$. As a consequence any $\mathbb{K}M_{w_0}(W)$-module is a $H_0(W)$-module and one can consider the decomposition map $G_0(M_{w_0}(W)) \to G_0(H_0(W))$. It is given by the following formula:

Proposition 5.17. *For $w \in W$, let $S_w^{w_0}$ be the simple $\mathbb{K}M_{w_0}(W)$-module defined by*

$$\epsilon_w . f := \begin{cases} \epsilon_w & \text{if } w.f = w, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $J \subseteq I$, let $S_J^{H_0}$ be the simple $H_0(W)$-module defined by

$$\mu_J . \pi_i := \begin{cases} \mu_I & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the restriction of $S_w^{w_0}$ to $H_0(W)$ is isomorphic to $S_{\mathrm{D}_R(w)}^{H_0}$. The decomposition map is therefore given by

$$G_0(M_{w_0}(W)) \to G_0(H_0(W)), \quad [S_w^{w_0}] \mapsto [S_{\mathrm{D}_R(w)}^{H_0}].$$

Proof. By definition of the action, $w.\pi_i = w$ if and only if $i \in \mathrm{D}_R(W)$. □

5b5. *The tower of $M_1(\mathfrak{S}_n)$ monoids (type A).*

Problem 5.18. The monoids $M_1(\mathfrak{S}_n)$, for $n \in \mathbb{N}$, form a tower of monoids with the natural embeddings $M_1(\mathfrak{S}_n) \times M_1(\mathfrak{S}_m) \hookrightarrow M_1(\mathfrak{S}_{m+n})$. Due to the involution of Section 4g, one has also embeddings $M_{w_0}(\mathfrak{S}_n) \times M_{w_0}(\mathfrak{S}_m) \hookrightarrow M_{w_0}(\mathfrak{S}_{m+n})$. As outlined in the introduction, it would hence be interesting to understand the induction and restriction functors in this setting, and in particular to describe the bialgebra obtained from the associated Grothendieck groups. This would give a representation theoretic interpretation of some bases of FQSym.

In this context, Proposition 5.17 provides an interpretation of the surjective coalgebra morphism FQSym $\twoheadrightarrow$ QSym, through the restriction along the following commutative diagram of monoid inclusions (see [Duchamp et al. 2002] for more details):

$$\begin{tikzcd} H_0(\mathfrak{S}_n) \times H_0(\mathfrak{S}_m) \arrow[r, hook] \arrow[d, hook] & M_{w_0}(\mathfrak{S}_n) \times M_{w_0}(\mathfrak{S}_m) \arrow[d, hook] \\ H_0(\mathfrak{S}_{n+m}) \arrow[r, hook] & M_{w_0}(\mathfrak{S}_{n+m}). \end{tikzcd}$$

6. Translation modules and w-biHecke algebras

The main purpose of this section is to pave the ground for the construction of the simple modules S_w of the biHecke monoid $M := M(W)$ in Section 7a.

As for any aperiodic monoid, each such simple module is associated with some regular $\mathcal{J}$-class D of the monoid, and can be constructed as a quotient of the span $\mathbb{K}\mathcal{R}(f)$ of the $\mathcal{R}$-class of any idempotent f in D, endowed with its natural right $\mathbb{K}M$-module structure (see Section 2g).

In Section 6a, we endow the interval $[1, w]_R$ with a natural structure of a combinatorial $\mathbb{K}M$-module T_w, called *translation module*, and show that, for any $f \in M$, regular or not, the right $\mathbb{K}M$-module $\mathbb{K}\mathcal{R}(f)$ is always isomorphic to some T_w.

The translation modules will play an ubiquitous role for the representation theory of $\mathbb{K}M$ in Section 7: indeed T_w can be obtained by induction from the simple modules S_w of $\mathbb{K}M$, and the right regular representation of $\mathbb{K}M$ admits a filtration in terms of the T_w that mimics the composition series of the right regular representation of $\mathbb{K}M_{w_0}$ in terms of its simple modules S_w. Reciprocally T_w, and therefore the right regular representation of $\mathbb{K}M$, restricts naturally to M_{w_0}. Finally, T_w is closely related to the projective module P_w of $\mathbb{K}M$ (Corollary 7.4).

By taking the quotient of $\mathbb{K}M$ through its representation on T_w, we obtain a w-analogue $\mathcal{H}W^{(w)}$ of the biHecke algebra $\mathcal{H}W$. This algebra turns out to be interesting in its own right, and we proceed by generalizing most of the results of [Hivert and Thiéry 2009] on the representation theory of $\mathcal{H}W$.

As a first step, we introduce in Section 6b a collection of submodules $P_J^{(w)}$ of T_w, which are analogues of the projective modules of $\mathcal{H}W$. Unlike for $\mathcal{H}W$, not any subset J of I yields such a submodule, and this is where the combinatorics of the blocks of w as introduced in Section 3 enters the game. In a second step, we derive in Section 6c a lower bound on the dimension of $\mathscr{H}W^{(w)}$; this requires a (fairly involved) combinatorial construction of a family of functions on $[1, w]_R$ that is triangular with respect to Bruhat order. In Section 6d we combine these results to derive the dimension and representation theory of $\mathscr{H}W^{(w)}$: projective and simple modules, Cartan matrix, quiver, etc. (see Theorem 6.17).

6a. ***Translation modules and w-biHecke algebras.*** In this section we study the combinatorics of the right class modules for the biHecke monoid, in particular a combinatorial model for them. Indeed, we show that the right class modules correspond to uniform translations of image sets, hence the name "translation modules".

Fix $f \in M$. Recall from Definition 2.12 that the right class module associated to f is defined as the quotient

$$\mathbb{K}\mathscr{R}(f) := \mathbb{K} f M / \mathbb{K}\mathscr{R}_{<}(f).$$

The basis of $\mathbb{K}\mathscr{R}(f)$ is the right class $\mathscr{R}(f)$ described in Proposition 4.18. Recall from there that f_u denote the unique element of $M(W)$ such that $\operatorname{fibers}(f_u) = \operatorname{fibers}(f)$ and $1 . f_u = u$.

Proposition 6.1. *Set $w = \operatorname{type}(f)^{-1} w_0$. Then $(f_u)_{u\in[1,w]_R}$ forms a basis of $\mathbb{K}\mathscr{R}(f)$ such that*

$$\begin{aligned} f_u . \pi_i &= \begin{cases} f_u & \text{if } i \in \mathrm{D}_R(u), \\ f_{us_i} & \text{if } i \notin \mathrm{D}_R(u) \text{ and } us_i \in [1, w]_R, \\ 0 & \text{otherwise}; \end{cases} \\ f_u . \overline{\pi}_i &= \begin{cases} f_{us_i} & \text{if } i \in \mathrm{D}_R(u), \\ f_u & \text{if } i \notin \mathrm{D}_R(u) \text{ and } us_i \in [1, w]_R, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6-1}$$

In particular, the action of any $g \in M$ on a basis element f_u of the right class module either annihilates f_u or agrees with the usual action on W: $f_u . g = f_{u.g}$.

Proof. By Definition 2.12 and Proposition 4.18, $(f_u)_{u\in[1,w]_R}$ forms a basis of $\mathbb{K}\mathscr{R}(f)$.

The action of π_i agrees with right multiplication, except when the index v of the new f_v is no longer in $[1, w]_R$, in which case the element is annihilated. The action of $\overline{\pi}_i$ also agrees with right multiplication. However, due to the relations $\pi_i \overline{\pi}_i = \overline{\pi}_i$ and $\overline{\pi}_i \pi_i = \pi_i$, we need that $\overline{\pi}_i$ annihilates f_u if $i \notin \mathrm{D}_R(u)$ and $us_i \notin [1, w]_R$.

The last statement follows by induction writing $f \in M$ in terms of the generators π_i and $\overline{\pi}_i$ and using (6-1). □

Proposition 6.1 gives a combinatorial model for right class modules. It is clear that two functions with the same type yield isomorphic right class modules. The converse also holds:

Proposition 6.2. *For any $f, f' \in M$, the right class modules $\mathbb{K}\mathcal{R}(f)$ and $\mathbb{K}\mathcal{R}(f')$ are isomorphic if and only if* $\operatorname{type}(f) = \operatorname{type}(f')$.

Proof. By Proposition 6.1, it is clear that if $\operatorname{type}(f) = \operatorname{type}(f')$, then $\mathbb{K}\mathcal{R}(f) \cong \mathbb{K}\mathcal{R}(f')$.

Conversely, suppose $\operatorname{type}(f) \neq \operatorname{type}(f')$. Then we also have $w \neq w'$, where $w = \operatorname{type}(f)^{-1} w_0$ and $w' = \operatorname{type}(f')^{-1} w_0$. Without loss of generality, we may assume that $\ell(w) \geq \ell(w')$. Using the combinatorial model of Proposition 6.1, we then have

$$f_1 . \pi_w = f_w \neq 0 \quad \text{and} \quad f'_1 . \pi_w = 0\,,$$

so that $\mathbb{K}\mathcal{R}(f) \not\cong \mathbb{K}\mathcal{R}(f')$. □

It is not obvious from the combinatorial action of π_i and $\overline{\pi}_i$ of Proposition 6.1 that the result indeed gives a module. However, since it agrees with the right action on the quotient space as in Definition 2.12, this is true. By Proposition 6.2, we may choose a canonical representative for right class modules.

Definition 6.3. The module $T_w := \mathbb{K}\mathcal{R}(e_{w,w_0})$ for all $w \in W$ is called the *translation module associated to w*. We identify its basis with $[1, w]_R$ via $u \mapsto f_u$, where $f = e_{w,w_0}$.

For the remainder of this section for $f \in M$ and $u \in [1, w]_R$, unless otherwise specified, $u . f$ means the action of f on u in the translation module T_w.

Definition 6.4. The w-biHecke algebra $\mathcal{H}W^{(w)}$ is the natural quotient of $\mathbb{K}M$ through its representation on T_w. In other words, it is the subalgebra of $\operatorname{End}(T_w)$ generated by the operators π_i and $\overline{\pi}_i$ of Proposition 6.1.

6b. ***Left antisymmetric submodules.*** By analogy with the simple reflections in the Hecke group algebra, we define for each $i \in I$ the operator $s_i := \pi_i + \overline{\pi}_i - 1$. For $u \in [1, w]_R$, the action on the translation module T_w is given by

$$u . s_i = \begin{cases} u s_i & \text{if } u s_i \in [1, w]_R, \\ -u & \text{otherwise.} \end{cases} \tag{6-2}$$

These operators are still involutions, but do not always satisfy the braid relations.

Example 6.5. Take W of type A_2 and $w = s_1$. The translation module T_w has two basis elements $B = (1, s_1)$ and the matrices for s_1 and s_2 on this basis are given by

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is not hard to check that then $s_1 s_2 s_1 \neq s_2 s_1 s_2$.

Similarly, one can define operators $\overleftarrow{s_i}$ acting on the left on the translation module T_w:

$$\overleftarrow{s_i}\,.u = \begin{cases} s_i u & \text{if } s_i u \in [1, w]_R, \\ -u & \text{otherwise.} \end{cases} \tag{6-3}$$

Definition 6.6. For $J \subseteq I$, set $P_J^{(w)} := \{v \in T_w \mid \overleftarrow{s_i}\,.v = -v \text{ for all } i \in J\}$.

For $w = w_0$, these are the projective modules P_J of the biHecke algebra [Hivert and Thiéry 2009].

Proposition 6.7. *Take $w \in W$ and $J \subseteq I$. Then, the following are equivalent*:

(i) *${}^J w$ is a cutting point of w*;

(ii) *$P_J^{(w)}$ is an $\mathbb{K}M$-submodule of T_w.*

Furthermore, when any, and therefore all, of the above hold, $P_J^{(w)}$ is isomorphic to $T_{{}^J w}$, and its basis is indexed by $[1, {}^J w]_R$, that is, assuming $J \in \mathcal{J}^{(w)}$, we have $\{v \in [1, w]_R,\ J \subset J^{(w)}(v)\}$.

Proof. (ii) $\Rightarrow$ (i): Set

$$v_J^w := \sum_{u \in [1, w_J]_R} (-1)^{\ell(u) - \ell(w_J)} u.$$

Up to a scalar factor, this is the unique vector in $P_J^{(w)}$ with support contained in $[1, w_J]_R$. Then,

$$v_J^w\,.\pi_{{}^J w} = \sum_{\substack{u \in [1, w_J]_R \\ \text{s.t. } u\,{}^J w \in [1, w]_R}} (-1)^{\ell(uv) - \ell(w_J v)} u\,{}^J w,$$

$$v_J^w\,.\pi_v \overline{\pi}_{v^{-1}} = \sum_{\substack{u \in [1, w_J]_R \\ \text{s.t. } u\,{}^J w \in [1, w]_R}} (-1)^{\ell(u) - \ell(w_J)} u.$$

Therefore, if ${}^J w \not\leq_R w$, then $v_J^w\,.\pi_{{}^J w} \overline{\pi}_{{}^J w^{-1}}$ is a nonzero vector with support strictly included in $[1, w_J]_R$ and therefore not in $P_J^{(w)}$. By Proposition 3.8 this proves that (ii) implies (i).

(i) $\Rightarrow$ (ii): If (i) holds, then the action of π_i (resp. $\overline{\pi}_i$) on $v_J^w\,.\pi_v$ either leaves it unmodified, kills it (if $vs_i = s_j v$ for some j) or maps it to $v_J^w\,.\pi_{vs_i}$. The vectors $(v_J^w\,.\pi_v)_{v \in [1, {}^J w]_R}$ form a basis of $P_J^{(w)}$ that is stable by M.

The last statement follows straightforwardly. $\square$

It is clear from the definition that $P_{J_1 \cup J_2}^{(w)} = P_{J_1}^{(w)} \cap P_{J_2}^{(w)}$ for $J_1, J_2 \subseteq I$. Since the set $\mathcal{RB}_{\mathcal{L}}(w)$ of left blocks of w is stable under union, the set of $\mathbb{K}M$-modules $(P_J^{(w)})_{J \in \mathcal{RB}_{\mathcal{L}}(w)}$ is stable under intersection. On the other hand, unless J_1 and J_2 are comparable, $P_{J_1 \cup J_2}^{(w)}$ is a strict subspace of $P_{J_1}^{(w)} + P_{J_2}^{(w)}$. This motivates the following definition.

Definition 6.8. For $J \in \mathcal{J}^{(w)}$, we define the module

$$S_J^{(w)} := P_J^{(w)} \Big/ \sum_{J' \supsetneq J,\, J' \in \mathcal{RB}_{\mathcal{L}}(w)} P_{J'}^{(w)}. \tag{6-4}$$

Remark 6.9. By the last statement of Proposition 6.7, and the triangularity of the natural basis of the modules $P_{J'}^{(w)}$, the basis of $S_J^{(w)}$ is given by

$$[1, {}^J w]_R \setminus \bigcup_{v \sqsubset {}^J w} [1, v]_R = \{v \in [1, w]_R,\ J \subset J^{(w)}(v)\}. \tag{6-5}$$

6c. *A (maximal) Bruhat-triangular family of the w-biHecke algebra.* Consider the submonoid F in $\mathcal{H}W^{(w)}$ generated by the operators π_i, $\overline{\pi}_i$, and s_i, for $i \in I$. For $f \in F$ and $u \in [1, w]_R$, we have $u\,.\,f = \pm v$ for some $v \in [1, w]_R$. For our purposes, the signs can be ignored and f be considered as a function from $[1, w]_R$ to $[1, w]_R$.

Definition 6.10. For $u, v \in [1, w]_R$, a function $f \in F$ is called (u, v)-*triangular* (for Bruhat order) if v is the unique minimal element of $\mathrm{im}(f)$ and u is the unique maximal element of $f^{-1}(v)$ (all minimal and maximal elements in this context are with respect to Bruhat order).

Recall the notion of maximal reduced right block $K^{(w)}(u)$ of Definition 3.38.

Proposition 6.11. *Take $u, v \in [1, w]_R$ such $K^{(w)}(u) \subseteq K^{(w)}(v)$. Then, there exists a (u, v)-triangular function $f_{u,v}$ in F.*

For example, for $w = 4312$ in $\mathfrak{S}_4$, the condition on u and v is equivalent to the existence of a path from u to v in the digraph $G^{(4312)}$ (see Figure 1 and Section 6d).

The proof of Proposition 6.11 relies on several remarks and lemmas that are given in the sequel of this section. The construction of $f_{u,v}$ is explicit, and the triangularity derives from $f_{u,v}$ being either in M, or close enough to be bounded below by an element of M. It follows from the upcoming Theorem 6.17 that the condition on u and v is not only sufficient but also necessary.

Remark 6.12. If f is (u, v)-triangular and g is (v, v')-triangular, then fg is (u, v')-triangular.

Remark 6.13. Take $x \in [1, w]_R$ and let $i \in I$. Then, $x\,.\,\overline{\pi}_i \leq_R x\,.\,s_i$.

By repeated application, for $S \subseteq I$, and $i_1, \dots, i_k \in S$, $x\,.\,\overline{\pi}_S \leq_R x\,.\,s_{i_1} \cdots s_{i_k}$, where recall that $\overline{\pi}_S$ is the longest element in the generators $\{\overline{\pi}_j \mid j \in S\}$.

Lemma 6.14. *Take $u \in [1, w]_R$, and define $f_{u,u} := e_{u,w_0} = \overline{\pi}_{u^{-1}} \pi_u$. Then*

(i) *$f_{u,u}$ is (u, u)-triangular*;

(ii) *for $v \in [1, w]_R$, either $v\,.\,f_{u,u} = 0$ or $v\,.\,f_{u,u} \geq_B v$*;

(iii) $\mathrm{im}(f_{u,u}) = [u, w_0]_L \cap [1, w]_R$.

Proof. First consider the case $w = w_0$. Then, (ii) and (iii) hold by Proposition 4.15.

Now take any $w \in W$. By Proposition 6.1 the action of $f \in M$ on the translation module T_w either agrees with the action on W or yields 0. Hence in particular Proposition 4.5 still applies, which yields (ii). This also implies the inclusion $\mathrm{im}(f_{u,u}) \setminus \{0\} \subset [u, w_0]_L \cap [1, w]_R$. The reverse inclusion is straightforward: If $u' = xu$, then $u' \,.\, f_{u,u} = xu \,.\, \overline{\pi}_{u^{-1}} \pi_u = x\pi_u = xu = u'$. Therefore (iii) holds as well.

Finally, (iii) implies that u is the unique minimal element of $\mathrm{im}(f_{u,u})$, and (ii) implies that u is the unique maximal element in $f_{u,u}^{-1}(u)$; therefore (i) holds. $\square$

Lemma 6.15. *If $u >_R v$, then $f_{u,v} := f_{u,u}\overline{\pi}_{u^{-1}v}$ is (u, v)-triangular.*

Proof. By Lemma 6.14(iii), the image set of $f_{u,u}$ is a subset of $[u, w_0]_L$. Therefore, by Remark 2.7, $\overline{\pi}_{u^{-1}v}$ translates it isomorphically to the interval $[v, w_0u^{-1}v]_L$. In particular, the fibers are preserved: $f_{u,v}^{-1}(v) = f_{u,u}^{-1}(u)$, and the triangularity of $f_{u,v}$ follows. $\square$

Lemma 6.16. *Take $u \in [1, w]_R$. Then, either u is a cutting point of w, or there exists a (u, v)-triangular function $f_{u,v}$ in F with $u <_R v \leq_R w$.*

Proof. Let J be the set of short nondescents i of u, and set $V := U_u \cap [1, w]_R$ (recall from Definition 3.15 that $U_u := uW_J$). By Proposition 3.17, V is the set of $w' \in [1, w]_R$ such that $u \sqsubseteq w'$. Furthermore, V is a lattice (it is the intersection of the two lattices $(uW_J, <_R)$ and $[1, w]_R$) with u as unique minimal element; in particular, $V \subset [u, w]_R$.

If $w \in V$ (which includes the case $u = w$ and $J = \{\}$), then u is a cutting point for w and we are done.

Otherwise, consider a shortest sequence $i_1, \dots, i_k$ such that $\{i_1, \dots, i_k\}$ does not intersect $\mathrm{D}_R(u)$, and $v' = us_{i_1} \cdots s_{i_k} \notin V$. Such a sequence must exist since $w \notin V$. Set $S := \{i_1, \dots, i_k\}$. Note that $i_1, \dots, i_{k-1}$ are in J but i_k is not. Furthermore, $u \not\sqsubseteq v'$ while $u = v'^S$ because $v' \in uW_S$ and $S \cap \mathrm{D}_R(u) = \varnothing$.

Case 1: $v' \in \mathrm{im}(f_{u,u})$. Then, $u <_L v'$. Combining this with $u = v'^S$ yields that $u \sqsubseteq v'$, a contradiction.

Case 2: $v' \notin \mathrm{im}(f_{u,u})$. Set $v := us_{i_1}$, and define $f_{u,v} := f_{u,u}\sigma\pi_{i_1}$, where

$$\sigma := s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2}. \tag{6-6}$$

Note that for $k = 1$, we have $\sigma = 1$. We now prove that $f_{u,v}$ is (u, v)-triangular.

First, we consider the fiber $f_{u,v}^{-1}(v)$. By minimality of k, and up to sign, s_{i_k} fixes all the elements of V at distance at most $k - 2$ of u. Hence, $\sigma^{-1}(u) = u$. Simultaneously,

$$v \,.\, \sigma^{-1} = v \,.\, s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2} = v' \,.\, s_{i_{k-1}} \cdots s_{i_2} \in v'W_J. \tag{6-7}$$

Hence, $v\,.\,\sigma^{-1} \notin \mathrm{im}(f_{u,u})$ because $v' \notin \mathrm{im}(f_{u,u})$ and $\mathrm{im}(f_{u,u})$ is stable under right multiplication by s_j for $j \in J$. Putting everything together, we have

$$f_{u,v}^{-1}(v) = f_{u,u}^{-1}(\sigma^{-1}(\pi_{i_1}^{-1}(v))) = f_{u,u}^{-1}(\sigma^{-1}(\{u, v\})) = f_{u,u}^{-1}(\{u\}) = [1, u]_B \cap [1, w]_R.$$

Therefore, u is the unique length maximal element of $f_{u,v}^{-1}(v)$, as desired.

We take now $x \in \mathrm{im}(f_{u,u})$, and apply Proposition 4.5 repeatedly. To start with,

$$u = 1\,.\,f_{u,u} \leq_B x\,.\,f_{u,u}. \tag{6-8}$$

Using Remark 6.13, we have

$$u = u\,.\,\bar{\pi}_S \leq_B (x\,.\,f_{u,u})\,.\,\bar{\pi}_S \leq_B (x\,.\,f_{u,u})\,.\,\sigma = x\,.\,f_{u,u}\,.\,\sigma. \tag{6-9}$$

It follows that

$$v = u\,.\,\pi_{i_1} \leq_B (x\,.\,f_{u,u}\,.\,\sigma)\,.\,\pi_{i_1} = x\,.\,f_{u,v}. \tag{6-10}$$

In particular, v is the unique Bruhat minimal element of $\mathrm{im}(f_{u,v})$, as desired. □

Proof of Proposition 6.11. Since W is finite, repeated application of Lemma 6.16 yields a finite sequence of triangular functions

$$f_{u,u_1}, \ldots, f_{u_{k-1},u_k}, \quad \text{where} \quad u <_R u_1 <_R \cdots <_R u_k$$

and u_k is a cutting point w^J of w. Since $u <_R w^J$, one has $J \subset K^{(w)}(u) \subset K^{(w)}(v)$, and therefore $u_k = w^J >_R v$. Then, applying Lemma 6.15 one can construct a (u_k, v)-triangular function $f_{u_k,v}$. Finally, by Remark 6.12, composing all these triangular functions gives a (u, v)-triangular function $f_{u,u_1} \cdots f_{u_{k-1},u_k} f_{u_k,v}$. □

6d. *Representation theory of the w-biHecke algebra.* Consider the digraph $G^{(w)}$ on $[1, w]_R$ with an edge $u \mapsto v$ if $u = vs_i$ for some i and $J^{(w)}(u) \subseteq J^{(w)}(v)$. Up to orientation, this is the Hasse diagram of right order (see for example Figure 1). The following theorem is a generalization of [Hivert and Thiéry 2009, Section 3.3].

Theorem 6.17. *$\mathcal{H}W^{(w)}$ is the maximal algebra stabilizing all modules $P_J^{(w)}$ for $J \in \mathcal{RB}_{\mathcal{L}}(w)$*

$$\mathcal{H}W^{(w)} = \{f \in \mathrm{End}(T_w) \mid f(P_J^{(w)}) \subseteq P_J^{(w)}\}$$

The elements $f_{u,v}$ of Proposition 6.11 form a basis $\mathcal{H}W^{(w)}$; in particular,

$$\dim \mathcal{H}W^{(w)} = \left|\{(u, v) \mid J^{(w)}(u) \subseteq J^{(w)}(v)\}\right|. \tag{6-11}$$

$\mathcal{H}W^{(w)}$ is the digraph algebra of the graph $G^{(w)}$.

The family $(P_J^{(w)})_{J\in\mathcal{RB}_{\mathcal{L}}(w)}$ forms a complete system of representatives of the indecomposable projective modules of $\mathcal{H}W^{(w)}$.

The family $(S_J^{(w)})_{J\in\mathcal{RB}_{\mathcal{L}}(w)}$ forms a complete system of representatives of the simple modules of $\mathcal{H}W^{(w)}$. The dimension of $S_J^{(w)}$ is the size of the corresponding w-nondescent class.

$\mathscr{H}W^{(w)}$ is Morita equivalent to the poset algebra of the lattice $[1, w]_{\sqsubseteq}$. In particular, its Cartan matrix is the incidence matrix and its quiver the Hasse diagram of this lattice.

Proof. From Proposition 6.11, one derives by triangularity that $\dim \mathscr{H}W^{(w)} \geq |\{(u, v) \mid K^{(w)}(u) \subseteq K^{(w)}(v)\}|$. The stability of all the subspaces $P_J^{(w)}$ imposes the converse equality. Hence, $\mathscr{H}W^{(w)}$ is exactly the subalgebra of $\mathrm{End}(T_w)$ stabilizing each $P_J^{(w)}$. The remaining statements follow straightforwardly, as in [Hivert and Thiéry 2009, Section 3.3]. See also for example [Denton et al. 2010/11, Section 3.7.4] for the Cartan matrix and quiver of a poset algebra. □

7. Representation theory of $M(W)$

In this section, we gather all results of the preceding sections in order to describe the representation theory of $M := M(W)$. The main result is Theorem 7.1, which gives the simple modules of $\mathbb{K}M$. We further relate the representation theory of $\mathbb{K}M$ to the representation theory of $\mathbb{K}M_{w_0}$. In particular, we prove that the translation modules are exactly the modules induced by the simple modules of $\mathbb{K}M_{w_0}$. We then conclude by computing some characters and the decomposition map from $\mathbb{K}M$ to $\mathbb{K}M_{w_0}$.

7a. *Simple modules.* We now study the simple modules of the biHecke monoid $\mathbb{K}M$ and also show that the translation modules are indecomposable.

Theorem 7.1. (i) *The biHecke monoid M admits $|W|$ nonisomorphic simple modules $(S_w)_{w\in W}$ (resp. projective indecomposable modules $(P_w)_{w\in W}$).*

(ii) *The simple module S_w is isomorphic to the top simple module*

$$S_{\{\}}^{(w)} = T_w \Big/ \sum_{v \sqsubset w} T_v$$

of the translation module T_w. Its dimension is given by

$$\dim S_w = \Big|[1, w]_R \setminus \bigcup_{v \sqsubset w} [1, v]_R\Big|.$$

In general, the simple quotient module $S_J^{(w)}$ of T_w is isomorphic to S_{J_w} of M.

Proof. Since M is aperiodic (Proposition 4.13), we may apply the special form of Clifford, Munn, and Ponizovskiĭ's construction of the simple modules (see Theorem 2.14). Namely, the simple modules are indexed by the regular $\mathscr{J}$- classes of M; by Corollary 4.20, there are $|W|$ of them. This yields (i), since for any finite-dimensional algebra, the simple and indecomposable projective modules share the same indexing set (see [Curtis and Reiner 1962, Corollary 54.14]).

Clifford, Munn, and Ponizovskiĭ further construct S_w as the top of the right class modules, that is, in our case, of the translation module T_w. Our explicit description of the radical of T_w as $\sum_{v \sqsubset w} T_v$ in (ii) is a straightforward application of Theorem 6.17. The dimension formula follows using Remark 6.9. □

For a direct proof that $\operatorname{rad} T_w = \sum_{v \sqsubset w} T_v$, without using Theorem 6.17, one would want to show that $\sum_{v \sqsubset w} T_v$ is exactly the annihilator of $\mathscr{J}(e_{w,w_0})$. One inclusion is easy, thanks to the following remark.

Remark 7.2. The submodule T_v is annihilated by $\mathscr{J}(e_{w,w_0}) = \mathscr{J}(\overline{\pi}_w)$.

Proof. Fix w and take v such that $v \sqsubset w$. Then $\overline{\pi}_w$ annihilates $T_v \subset T_w$. Indeed, combining $\overline{\pi}_w(w) = 1$ with Propositions 6.1 and 4.5, one obtains that $\overline{\pi}_w$ either annihilates f_u or maps it to f_1. Take now $x \in T_v$, and write $x\,.\overline{\pi}_w = \lambda f_1$. Since T_v is a submodule, λf_1 lies in T_v; however the basis elements of T_v have disjoint support and since $v \sqsubset w$ none of them are collinear to f_1. Therefore $x\,.\overline{\pi}_w = 0$. □

Type	$\lvert W\rvert$	$\lvert M_{w_0}\rvert$	$\lvert M\rvert$	$(\dim S_w)_w$	$\sum \dim S_w$
A_0	1	1	1	1	1
A_1	2	2	3	1^2	2
$A_2 = I_2(3)$	6	8	23	$1^4 2^2$	8
A_3	24	71	477	$1^8 2^4 3^4 4^6 5^2$	62
A_4	120	1646	31103	$1^{16} 2^{10} 3^8 4^{16} 5^{16} 6^6 \cdots 20^6$	770
A_5	720	118929	7505009	$1^{32} 2^{24} 3^{20} 4^{42} 5^{38} 6^{40}$	
				$\cdots 120^2$	13080
$B_2 = I_2(4)$	8	14	49	$1^4 2^2 3^2$	14
B_3	48	498	5455	$1^8 2^4 3^4 4^6 5^7 6^4 7^4 8^4 9^1$	
				$\cdot 10^2 11^2 12^2$	246
B_4	384	149622	6664553	$1^{16} 2^{10} 3^{10} 4^{14} 5^{17} 6^{16} \cdots 80^2$	6984
$G_2 = I_2(6)$	12	32	153	$1^4 2^2 3^2 4^2 5^2$	32
H_3	120	87	1039	$1^8 2^4 3^4 4^8 5^6 6^7 \cdots 36^2$	1404
$A_1 \times A_1$	4	4	9	$1^2 1^2$	4
$I_2(p)$	$2p$	p^2-p+2	$\frac{2}{3}p^3+\frac{4}{3}p+1$	$1^4 2^2 \cdots (p-1)^2$	p^2-p+2

Table 1. Statistics on the biHecke monoids $M := M(W)$ for the small Coxeter groups. In column four, $1^8 2^4 \cdots 5^2$ means that there are 8 simple modules of dimension 1, 4 of dimension 2, and so on. The sum in the last column is over w. The sequence p^2-p+2 is #A014206 in [OEIS Foundation 2012].

Example 7.3. The simple module S_{4312} is of dimension 3, with basis indexed by $\{4312, 4132, 1432\}$ (see Figure 1). The other simple modules S_{3412}, S_{4123}, and S_{1234} are of dimension 5, 3, and 1, respectively. See also Table 1.

In general, the two extreme cases are, on the one hand, when w is the maximal element of a parabolic subgroup, in which case the simple module is of dimension 1 and, on the other hand, when w is an immediate successor of 1 in the cutting poset (see Example 3.44), in which case the simple module is of dimension $|T_w| - 1$. In the other cases, one can use Theorem 3.41 to calculate the dimension of S_w by inclusion-exclusion from the sizes of the intervals $[1, {}^J w]_R$, where ${}^J w$ runs through the free sublattice at the top of the interval $[1, w]_\sqsubseteq$ of the cutting poset. Note that the sizes of the intervals in W can also be computed by a similar inclusion-exclusion (the Möbius function for right order is given by $\mu(u, w) = (-1)^k$ if the interval $[u, w]_R$ is isomorphic to some W_J with $|J| = k$, and 0 otherwise). This may open the door for some generating series manipulations to derive statistics like the sum of the dimension of the simple modules.

Corollary 7.4. *The translation module T_w is an indecomposable $\mathbb{K}M$-module, quotient of the projective module P_w of $\mathbb{K}M$.*

Proof. Direct application of Corollary 2.15 □

7b. ***From $M_{w_0}(W)$ to $M(W)$.*** In this section, we use our knowledge of M_{w_0} to learn more about M.

Proposition 7.5. *The translation module T_w is isomorphic to the induction to $\mathbb{K}M$ of the simple module $S_w^{w_0}$ of $\mathbb{K}M_{w_0}$.*

The proof of this proposition follows from the upcoming lemmas giving some simple conditions on a general inclusion of monoids $B \subseteq A$ under which the (regular) right class modules of $\mathbb{K}A$ are induced from those of $\mathbb{K}B$.

Lemma 7.6. *Let $B \subseteq A$ be two finite monoids and $f \in B$. If*

$$\mathbb{K}\mathcal{R}^A_<(f) = \mathbb{K}\mathcal{R}^B_<(f)A,$$

then the right class module $\mathbb{K}\mathcal{R}^A(f)$ is isomorphic to the induction from $\mathbb{K}B$ to $\mathbb{K}A$ of the right class module $\mathbb{K}\mathcal{R}^B(f)$:

$$\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f)\uparrow_{\mathbb{K}B}^{\mathbb{K}A}.$$

Proof. Recall that for a $\mathbb{K}B$-module Y, the module $Y\uparrow_{\mathbb{K}B}^{\mathbb{K}A}$ induced by Y from $\mathbb{K}B$ to $\mathbb{K}A$ is given by $Y\uparrow_{\mathbb{K}B}^{\mathbb{K}A} := Y \otimes_{\mathbb{K}B} \mathbb{K}A$.

By construction of the right class modules (see Definition 2.12), we have the following exact sequences:

$$0 \to \mathbb{K}\mathcal{R}^B_<(f) \to \mathbb{K}fB \to \mathbb{K}\mathcal{R}^B(f) \to 0, \tag{7-1}$$

$$0 \to \mathbb{K}\mathcal{R}^A_<(f) \to \mathbb{K}fA \to \mathbb{K}\mathcal{R}^A(f) \to 0. \tag{7-2}$$

Consider now the sequence obtained by tensoring (7-1) by $\mathbb{K}A$:

$$0 \to \mathbb{K}\mathcal{R}^B_<(f) \otimes_{\mathbb{K}B} \mathbb{K}A \to \mathbb{K}fB \otimes_{\mathbb{K}B} \mathbb{K}A \to \mathbb{K}\mathcal{R}^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A \to 0\,. \tag{7-3}$$

We want to prove that it is exact and isomorphic to (7-2).

First note that, since $\mathbb{K}B$ is a subalgebra of $\mathbb{K}A$, we have $b \otimes a = 1 \otimes ba$ for $b \in B$ and $a \in A$. Therefore the product map

$$\mu : \mathbb{K}fB \otimes_{\mathbb{K}B} \mathbb{K}A \to \mathbb{K}fA, \qquad fb \otimes a \to fba$$

is an isomorphism of $\mathbb{K}A$-modules.

Consider the restriction of μ to $\mathbb{K}\mathcal{R}^B_<(f) \otimes_{\mathbb{K}B} \mathbb{K}A$. Its image set is $\mathbb{K}\mathcal{R}^B_<(f)A$, which is equal to $\mathbb{K}\mathcal{R}^A_<(f)$ by hypothesis. Therefore, μ restricts to an A-module isomorphism from $\mathbb{K}\mathcal{R}^B_<(f) \otimes_{\mathbb{K}B} \mathbb{K}A$ to $\mathbb{K}\mathcal{R}^A_<(f)$. As a consequence, the following diagram is commutative, all vertical arrows being isomorphisms (for short we write here $\otimes$ for $\otimes_{\mathbb{K}B}$):

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K}\mathcal{R}^B_<(f) \otimes \mathbb{K}A \arrow[r] \arrow[d, "\mu"'] & \mathbb{K}fB \otimes \mathbb{K}A \arrow[r] \arrow[d, "\mu"'] & \mathbb{K}\mathcal{R}^B(f) \otimes \mathbb{K}A \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & \mathbb{K}\mathcal{R}^A_<(f) \arrow[r] & \mathbb{K}fA \arrow[r] & \mathbb{K}\mathcal{R}^B(f) \otimes \mathbb{K}A \arrow[r] & 0
\end{tikzcd}$$

It is a well-known fact that the functor $\cdot \otimes_{\mathbb{K}B} \mathbb{K}A$ is right exact, so that the middle and right part of the top sequence is exact. The left part of the bottom sequence is clearly exact. Therefore they are both exact sequences.

Comparing with (7-2), we obtain that

$$\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A,$$

where the latter is isomorphic to $\mathbb{K}\mathcal{R}^B(f)\uparrow^{\mathbb{K}A}_{\mathbb{K}B}$ by definition. □

In the next lemma we denote by $<_{\mathcal{R}^A}$ the strict right preorder on a monoid A; that is, $x <_{\mathcal{R}^A} y$ if $x \leq_{\mathcal{R}^A} y$ but $x \notin \mathcal{R}^A(y)$.

Lemma 7.7. *Let $B \subseteq A$ be two finite monoids and assume that*:

(i) *$\mathcal{R}$-order on B is induced by $\mathcal{R}$-order on A*; *that is, for all $x, y \in B$*,

$$x <_{\mathcal{R}^A} y \quad \Longleftrightarrow \quad x <_{\mathcal{R}^B} y.$$

(ii) *Any $\mathcal{R}$-class of A intersects B.*

Then, for any $f \in B$, the equality $\mathbb{K}\mathcal{R}^B_<(f)A = \mathbb{K}\mathcal{R}^A_<(f)$ holds. In particular,

$$\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f)\uparrow^{\mathbb{K}A}_{\mathbb{K}B}.$$

Moreover, condition (i) may be replaced by the stronger condition

(i′) $$x \leq_{\mathcal{R}^A} y \quad \Longleftrightarrow \quad x \leq_{\mathcal{R}^B} y.$$

Proof. Inclusion $\subseteq$: Take $b \in B$ with $b <_{\mathcal{R}^B} f$ and $a \in A$. Then, using (i), we have $ba \in \mathbb{K}\mathcal{R}^A_<(f)$, since

$$ba \leq_{\mathcal{R}^A} b <_{\mathcal{R}^A} f.$$

Inclusion $\supseteq$: Take $a \in A$ with $a <_{\mathcal{R}^A} f$. Using (ii) choose an element $b \in B$ such that $b \,\mathcal{R}^A\, a$. Then $b \leq_{\mathcal{R}^A} a <_{\mathcal{R}^A} f$ and therefore, by (i), $b \in \mathbb{K}\mathcal{R}^B_<(f)$. It follows that $a \in \mathbb{K}\mathcal{R}^B_<(f)A$.

The statement $\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f)\uparrow^{\mathbb{K}A}_{\mathbb{K}B}$ follows from Lemma 7.6. □

Here is an example of what can go wrong when Condition (i) fails.

Example 7.8. Let A be the (multiplicative) submonoid of $M_2(\mathbb{Z})$ with elements given by the matrices

$$1 := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\; b_{11} := \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\; b_{12} := \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\; a_{21} := \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\; b_{22} := \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right),\; 0 := \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right).$$

Alternatively, A is the aperiodic Rees matrix monoids (see Definition 2.16) whose nontrivial $\mathcal{J}$-class is described by

$$\begin{pmatrix} b^*_{11} & b_{12} \\ a_{21} & b^*_{22} \end{pmatrix},$$

where the * marks the elements that are idempotent. In other words, $A = M(P)$, where $P := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and for convenience the matrix above specifies names for the elements of the nontrivial $\mathcal{J}$-class. Recall that the nontrivial left and right classes of A are given respectively by the columns and rows of this matrix.

Let B be the submonoid $\{1, b_{11}, b_{12}, b_{22}, 0\}$. Then B satisfies condition (ii) but not condition (i): indeed $b_{11} \,\mathcal{R}^A\, b_{12}$ whereas $b_{11} <_{\mathcal{R}^B} b_{12}$. Then, taking $f = b_{11}$, one obtains $\mathcal{R}^B_<(b_{11}) = \{0, b_{12}\}$ so that $\mathcal{R}^B_<(b_{11})A = \{0, b_{11}, b_{12}\}$, and therefore

$$\mathbb{K}\{0\} = \mathbb{K}\mathcal{R}^A_<(b_{11}) \subset \mathbb{K}\mathcal{R}^B_<(b_{11})A = \mathbb{K}\{0, b_{11}, b_{12}\}.$$

Now $\mathbb{K}\mathcal{R}^B(b_{11}) = \mathbb{K}\{0, b_{11}, b_{12}\}/\mathbb{K}\{0, b_{12}\}$, so that $\mathbb{K}\mathcal{R}^B(b_{11})$ is one-dimensional, spanned by $x := b_{11} \mod (\mathbb{K}\{0, b_{12}\})$. The action of B is given by $x\,.\,1 = x\,.\,b_{11} = x$ and $x\,.\,m = 0$ for any $m \in B \setminus \{1, b_{11}\}$.

We claim that

$$\mathbb{K}\mathcal{R}^B(b_{11})\uparrow^{\mathbb{K}A}_{\mathbb{K}B} = \mathbb{K}\mathcal{R}^B(b_{11}) \otimes_{\mathbb{K}B} \mathbb{K}A = 0.$$

Indeed, $x \otimes 1 = x\,.b_{11} \otimes 1 = x \otimes b_{11} = x \otimes b_{12}a_{21} = x\,.b_{12} \otimes a_{21} = 0$. Thus

$$\mathbb{K}\mathcal{R}^A(b_{11}) \ncong \mathbb{K}\mathcal{R}^B(b_{11})\uparrow_{\mathbb{K}B}^{\mathbb{K}A}.$$

As shown in the following example, Condition (i′) may be strictly stronger than Condition (i) because $<_{\mathcal{R}}$ is only a preorder.

Example 7.9. Let A be the aperiodic Rees matrix monoid with nontrivial $\mathcal{J}$-class given by

$$\begin{pmatrix} a_{11}^* & b_{12} & b_{13} \\ a_{21}^* & b_{22}^* & a_{23} \\ a_{31}^* & a_{32} & b_{33}^* \end{pmatrix},$$

Let B be the submonoid $\{1, b_{12}, b_{13}, b_{22}, b_{33}, 0\}$. Then B satisfies conditions (i) and (ii), but not condition (i′): b_{12} and b_{13} are incomparable for $\leq_{\mathcal{R}^B}$ whereas they are in the same right class for A.

We now turn to the proof of Proposition 7.5 by showing that $M_{w_0}(W) \subseteq M(W)$ satisfy the conditions of Lemma 7.7. We use the stronger condition (i′).

Lemma 7.10. *The biHecke monoid and its Borel submonoid $M_{w_0}(W) \subseteq M(W)$ satisfy conditions* (i′) and (ii) *of Lemma 7.7.*

Proof. By Proposition 4.18, for any $f \in M$ there exists a unique $f_1 \in \mathcal{R}(f) \cap M_1$. Using the bar involution of Section 4g, one finds similarly a unique $\overline{f_1} \in \mathcal{R}(f) \cap M_{w_0}$. This proves condition (ii).

We now prove the nontrivial implication in condition (i′). Take $f, g \in M_{w_0}$ with $f \leq_{\mathcal{R}^M} g$. Then, $f = gx$ for some $x \in M$. Note that $w_0 . f = w_0 . g = w_0$, which implies that $w_0 . x = w_0$ as well. Hence x is in fact in M_{w_0} and $f \leq_{\mathcal{R}^{M_{w_0}}} g$. □

Proof of Proposition 7.5. Let $g_w := e_{w,w_0}$. By definition, the translation module is the quotient $T_w = \mathbb{K}g_w M/\mathbb{K}\mathcal{R}_<(g_w)$, whereas $S_w^{w_0} = \mathbb{K}g_w M_{w_0}/\mathbb{K}\mathcal{R}_<^{w_0}(g_w)$. By Lemma 7.10, $M_{w_0} \subseteq M$ satisfy the two conditions of Lemma 7.7; Proposition 7.5 follows. □

Theorem 7.11. *The right regular representation of $\mathbb{K}M$ admits a filtration with factors all isomorphic to translation modules, and its character is given by*

$$[\mathbb{K}M] = \sum_{f \in M_{w_0}} [T_{1.f}]. \tag{7-4}$$

Proof. As any monoid algebra, $\mathbb{K}M$ admits a filtration where each composition factor is given by (the linear span of) an $\mathcal{R}$-class of M. By Proposition 6.2, each such composition factor is isomorphic to the translation module $T_{1.f}$, where f is the unique element of the $\mathcal{R}$-class that lies in M_{w_0}. The character formula follows.

Alternatively, it can be obtained using Proposition 7.5 and the character formula for the right regular representation of M_{w_0} (see Remark 5.11):

$$[\mathbb{K}M_{w_0}]_{M_{w_0}} = \sum_{f \in M_{w_0}} [S^{w_0}_{1.f}]_{M_{w_0}}, \tag{7-5}$$

which completes the proof. □

Proposition 7.12. *For any $w \in W$, the translation module T_w is multiplicity-free as an $\mathbb{K}M_{w_0}$-module and its character is given by*

$$[T_w]_{M_{w_0}} = \sum_{u \in [1,w]_R} [S^{w_0}_u]_{M_{w_0}}. \tag{7-6}$$

Proof. Let f be an element in M that yields the translation module T_w, and define f_u as in Proposition 4.18.

Take some sequence $u_1, \dots, u_m$ (for $m = |[1, w]_R|$) of the elements of $[1, w]_R$ that is length increasing, and define the corresponding sequence of subspaces by $X_i := \mathbb{K}\{u_1, \dots, u_i\}$. Using Lemma 6.14, each such subspace is stable by M_{w_0}, and $X_0 \subset \cdots \subset X_m$ forms an M_{w_0}-composition series of T_w since X_i/X_{i-1} is of dimension 1.

Consider now a composition factor X_i/X_{i-1}. Again, by Lemma 6.14, e_{v,w_0} fixes u_i if and only if $v \le_L u_i$ (that is, if the image set $[u_i, w^{-1}w_0u_i]_L$ of f_{u_i} is contained in the image set $[v, w_0]_L$ of e_{v,w_0}), and kills it otherwise. Hence, X_i/X_{i-1} is isomorphic to $S^{w_0}_{u_i}$. □

Theorem 7.13. *The decomposition map of $\mathbb{K}M$ over $\mathbb{K}M_{w_0}$ is lower uni-triangular for right order, with* 0, 1 *entries. More explicitly*,

$$[S_w]_{M_{w_0}} = \sum_{u \in [1,w]_R \setminus \bigcup_{v \sqsubset w} [1,v]_R} [S^{w_0}_u]_{M_{w_0}}. \tag{7-7}$$

Proof. Since S_w is a quotient of T_w, its composition factors form a subset of the composition factors for T_w. Hence, using Proposition 7.12, the decomposition matrix of M over M_{w_0} is lower triangular for right order, with 0, 1 entries. Furthermore, by construction (see Remark 6.9 and Theorem 7.1(ii)), $S_w = T_w / \sum_{v \sqsubset w} T_v$; using Proposition 7.12 the sum on the right hand side contains at least one composition factor isomorphic to $S^{w_0}_u$ for each u in $[1, v]_R$ with $v \sqsubset w$; therefore S_w has no such composition factor. We conclude using the dimension formula of Theorem 7.1(ii). □

Example 7.14. Following up on Example 7.3, the decomposition of the $\mathbb{K}M$-simple module S_{4312} over $\mathbb{K}M_{w_0}$ is given by $[S_{4312}]_{M_{w_0}} = [S^{w_0}_{4312}] + [S^{w_0}_{4132}] + [S^{w_0}_{1432}]$. See also Figure 1 and the decomposition matrices given in Appendix A.

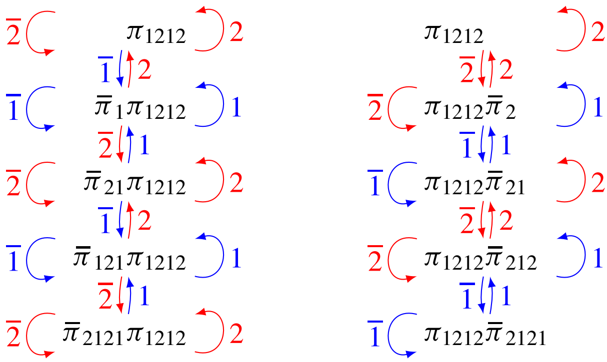


Figure 6. The left and right class modules indexed by $w := s_1s_2s_1s_2$ for the biHecke monoid $M(I_p)$ with $p \geq 5$. The left picture also describes the left simple module S_w of $M(I_p)$, and the projective module $P_w^{w_0}$ of the Borel submonoid $M_{w_0}(I_p)$.

7c. ***Example: the rank 2 Coxeter groups.*** We now give a complete description of the representation theory of the biHecke monoid for each rank 2 Coxeter group I_p. The proofs are left as exercises for the reader.

Example 7.15. Let M be the biHecke monoid for the dihedral group $W := I_p$ of order $2p$. Then, M is a regular monoid.

The right class module $\mathbb{K}\mathcal{R}_w := \mathbb{K}\mathcal{R}(e_{w,w_0})$ is the translation module spanned by $[1, w]_R$. It is of dimension $2p$ for $w = w_0$, and $\ell(w)$ otherwise. The left class modules $\mathbb{K}\mathcal{L}_1$ and $\mathbb{K}\mathcal{L}_{w_0}$ are respectively the trivial module spanned by 1 and the zero module spanned by w_0. For $w \neq 1, w_0$, the left class module $\mathbb{K}\mathcal{L}_w$ is of dimension $\ell(w) - 1$, and its structure is as in Figure 6. In particular,

$$|M| = 2p + 1 + 2\sum_{k=1}^{p-1} k(k+1) = \tfrac{2}{3}p^3 + \tfrac{4}{3}p + 1.$$

The simple right module S_w can be constructed from the cutting poset. Namely, S_1 is the trivial module spanned by 1, while S_{w_0} is the zero module spanned by w_0 and, for $w \neq 1, w_0$, S_w is the quotient of the right class module by the line spanned by alternating sum of $[1, w]_R$. The simple left module S_w is directly given by the left class module L_w.

The quiver of M is given by the cutting poset (see Figure 7). The q-Cartan matrix is given by the path algebra of this quiver; namely, there is an extra arrow from 1 to w_0 with weight q^2. In particular, it is upper unitriangular and of determinant 1.

Example 7.16. Let M_{w_0} be the Borel submonoid of the biHecke monoid for the dihedral group $W := I_p$ of order $2p$.

$$
\begin{tikzcd}[column sep=small]
 & & & & w_0 & & & & \\
s_1s_2s_1 & s_1s_2 & s_1 & w_0s_1 \arrow[ur] & & w_0s_2 \arrow[ul] & s_2 & s_2s_1 & s_2s_1s_2 \\
 & & & & 1 \arrow[ullll] \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] \arrow[urrr] \arrow[urrrr] & & & &
\end{tikzcd}
$$

Figure 7. The Hasse diagram of the cutting poset for the dihedral group $W := I_5$. This is also the quiver of the biHecke monoid for that group.

$$
\begin{tikzcd}
 & w_0 & \\
s_1s_2s_1s_2 \arrow[d] & & s_2s_1s_2s_1 \arrow[d] \\
s_1s_2s_1 \arrow[d] & & s_2s_1s_2 \arrow[d] \\
s_1s_2 \arrow[d] & & s_2s_1 \arrow[d] \\
s_1 & & s_2 \\
 & 1 &
\end{tikzcd}
$$

Figure 8. The quiver of the Borel submonoid $M_{w_0}(I_5)$ of the bi-Hecke monoid for the dihedral group I_5.

The projective module P_w of M_{w_0} is given by the left simple modules S_w, or equivalently the left-class-module L_w of M. In particular,

$$|M_{w_0}| = 1 + 1 + 2\sum_{k=1}^{p-1} k = p^2 - p + 2.$$

The quiver of M_{w_0} is given by the cover relations in Bruhat order (or equivalently right order) that are not covers in left order (see Figure 8); this gives two chains of length $p-1$. The monoid algebra is isomorphic to the path algebra of this quiver, which gives right away its radical filtration. Combinatorially speaking, every non-idempotent element f of the monoid admits a unique minimal factorization $e_w e_u$, with $\ell(u) < \ell(w)$ and $u \not\leq_L w$; namely, $u := f(1)$ and w is the smallest element such that $f(w) = 1$.

8. Research in progress

Our guiding problem is the search for a formula for the cardinality of the biHecke monoid. Using a standard result of the representation theory of finite-dimensional algebras together with the results of this paper, we can now write

$$|M(W)| = \sum_{w\in W} \dim S_w \dim P_w,$$

where $\dim S_w$ is given by an inclusion-exclusion formula. It remains to determine the dimensions of the projective modules P_w.

While studying the representation theory of the Borel submonoid M_1 as an intermediate step, the authors realized that many of the combinatorial ingredients that arose were well-known in the semigroup community (for example the Green's relations and related classes, automorphism groups, etc.), and hence the representation theory of M_1 is naturally expressed in the context of $\mathscr{J}$-trivial monoids; see [Denton et al. 2010/11]. This sparked their interest in the representation theory of more general classes of monoids, in particular aperiodic monoids.

At the current stage, it appears that the Cartan matrix of an aperiodic monoid (and therefore the composition series of its projective modules, and by consequence their dimensions) is completely determined by the knowledge of the composition series for both left and right class modules. In other words, the study in this paper of right class modules (that is, translation modules), whose original purpose was to construct the simple modules using [Ganyushkin et al. 2009, Theorem 7], turns out to complete half of this program. The remaining half, in progress, is the decomposition of left class modules.

At the combinatorial level, this requires one to control $\mathscr{L}$-order. Loosely speaking, $\mathscr{L}$-order is essentially given by left and right order in W; however, within $\mathscr{L}$-classes the structure seems more elusive, in particular because fibers are more difficult to describe than image sets. Another difficulty is that, unlike for $\mathscr{R}$-class modules, $\mathscr{L}$-class modules are not all isomorphic to regular ones (that is, classes containing idempotents).

Yet, the general theory gives that the decomposition matrix should be upper triangular for left-right order for regular classes, and upper triangular for Bruhat order for nonregular ones, with no left-right "arrow" for left-right order. Pushing this further gives that the Cartan matrix has determinant 1.

We conclude by illustrating the above for $W = \mathfrak{S}_4$ in Figure 9. The blue arrows are the covering relations of the cutting poset, which encode the composition series of the translation modules (that is, right class modules). Namely, the character of T_w is given by the sum of $q^k[S_u]$ for u below w in the cutting poset, with k the

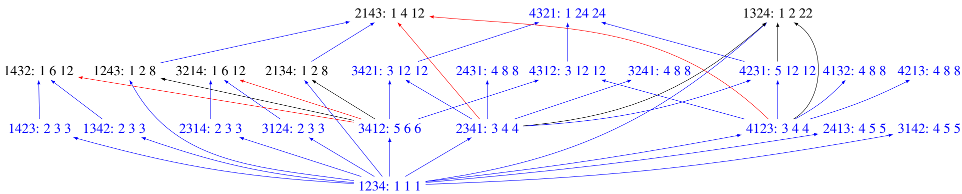


Figure 9. Graph encoding the characters of left and right class modules, and therefore the Cartan invariant matrix for $M(\mathfrak{S}_4)$. See the text for details.

distance from u to w in that poset. For example,

$$[T_{2143}] = [S_{2143}] + q[S_{1243}] + q[S_{2134}] + q^2[S_{1234}],$$
$$[T_{2341}] = [S_{2341}] + q[S_{1234}],$$
$$[T_{4123}] = [S_{4123}] + q[S_{4123}].$$

Similarly the black and red arrows encode the composition series of regular and, respectively, nonregular left classes. In this simple example, the q-character of a right projective module P_w is then given by

$$[P_w] = [T_w] + \sum_u q[T_u],$$

where (u, w) is a black or red arrow in the graph. For example,

$$\begin{aligned}[P_{2143}] &= [T_{2143}] + q[T_{2341}] + q[T_{4123}] \\ &= [S_{2143}] + q[S_{1243}] + q[S_{2134}] + q[S_{2341}] + q[S_{4123}] + 3q^2[S_{1234}].\end{aligned}$$

Acknowledgments

A short version with the announcements of (some of) the results of this paper was published in [Hivert et al. 2010]. The discovery and analysis of the cutting poset is due to Schilling and Thiéry. The study of the biHecke monoid and of its representation theory is joint work of all three authors, under an initial suggestion of Alain Lascoux.

We would like to thank Jason Bandlow, Mahir Can, Brant Jones, Jean-Christophe Novelli, Jean-Éric Pin, Nathan Reading, Vic Reiner, Franco Saliola, Benjamin Steinberg, and Jean-Yves Thibon for enlightening discussions. In particular, Saliola pointed us to the large body of literature on semigroups, and the recent progress in their representation theory. Special thanks to the anonymous referee for carefully reading the paper, suggesting many improvements, and many insightful comments. We are grateful to Aladin Virmaux for pointing out several misprints and helping to work out the details of Examples 7.15 and 7.16.

This research was driven by computer exploration, using the open-source mathematical software Sage [Stein et al. 2009] and its algebraic combinatorics features developed by the Sage-Combinat community [Sage-Combinat 2008].

Schilling would like to thank the Université Paris Sud, Orsay and the Kastler foundation for support during her visit in November 2008. Thiéry would like to thank the Department of Mathematics at UC Davis for hospitality during his visit in 2007–2008 and spring 2009.

Appendix A. Monoid of edge surjective morphism of a colored graph

Let C be a set whose elements are called colors. We consider colored simple digraphs without loops. More precisely, a C-colored graph is a triple $G = (V, E, c)$, where V is the set of vertices of G, $E \subset V \times V / \{(x, x) \mid x \in V\}$ is the set of (oriented) edges of G, and $c : E \to C$ is the coloring map.

Definition A.1. Let $G = (V, E, c)$ and $G' = (V', E', c')$ be two colored graphs. An *edge surjective* morphism (or ES-morphism) from G to G' is a map $f : V \to V'$ such that

- For any edge $(a, b) \in E$, either $f(a) = f(b)$, or $(f(a), f(b)) \in E'$ and $c(a, b) = c'(f(a), f(b))$.
- For any edge $(a', b') \in E'$ with a' and b' in the image set of f there exists an edge $(a, b) \in E$ such that $f(a) = a'$ and $f(b) = b'$.

Note that by analogy to categories, instead of ES-morphism, we can speak about full morphisms.

The following proposition shows that colored graphs together with edge surjective morphisms form a category.

Proposition A.2. *For any colored graphs G, G_1, G_2, G_3,*

- *the identity* id $: G \to G$ *is an ES-morphism*;
- *for any ES-morphism $f : G_1 \to G_2$ and $g : G_2 \to G_3$ the composed function $g \circ f : G_1 \to G_3$ is an ES-morphism.*

Corollary A.3. *For any colored graph G, the set of ES-morphisms from G to G is a submonoid of the monoid of the functions from G to G.*

Here are some general properties of ES-morphisms:

Proposition A.4. *Let G_1 and G_2 be two colored graphs and f an ES-morphism from G_1 to G_2. Then the image of any path in G_1 is a path in G_2.*

In our particular case, we have some more properties:

(i) The graph is acyclic, with unique source and sink. In particular, it is (weakly) connected.

	12	21	Proj.
12	1	.	1
21	.	1	1
Proj.	1	1	

Table 2. q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_2)$ (type A_1).

	123	132	213	231	312	321	Proj.
123	1	.	.	.	.	.	1
132	.	1	.	.	q	.	2
213	.	.	1	q	.	.	2
231	.	.	.	1	.	.	1
312	.	.	.	.	1	.	1
321	.	.	.	.	.	1	1
Proj.	1	1	1	2	2	1	

Table 3. q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_3)$ (type A_2).

(ii) The graph is ranked by the integers, and edges occur only between two consecutive ranks.

(iii) The graph is C-regular, which means that for any vertex v

Remarks A.5. Proposition 4.1 gives that our monoid is a submonoid of the $M(G)$ monoid for left order.

Propositions 4.3 and 4.11 are generic, and would apply to any $M(G)$. For the latter, we just need that G is C-regular.

A natural source of colored graphs are crystal graphs. A question that arises is what the G-monoid of a crystal looks like.

Appendix B. Tables

B1. ***q-Cartan invariant matrices.*** In Tables 2–7, we give the Cartan invariant matrix for $\mathbb{K}M_{w_0}$ and $\mathbb{K}M$ in types A_1, A_2 and A_3. The q-parameter records the layer in the radical filtration. The extra rows and columns entitled "Simp." and "Proj." give the dimension of the simple and projective modules, on the right for right modules and below for left modules. When all simple modules are one-dimensional, the column is omitted.

Using [Thiéry 2012], it is possible to go further, and compute for example the Cartan invariant matrix for M in type A_4 in about one hour (though at $q = 1$ only).

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Proj.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1243	.	1	.	.	q	.	.	.	.	.	q	q^2	.	.	.	.	q	.	q^2	.	.	.	.	.	6
1324	.	.	1	q	.	.	.	.	.	q	.	.	q	q^2	.	q^2	q^3	.	q	q^2	.	q	.	.	10
1342	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	q^2	.	.	q	.	.	.	.	4
1423	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	2
1432	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	q	.	.	q	.	.	q^2	.	4
2134	.	.	.	.	.	.	1	.	q	q^2	q	.	.	.	.	.	q	.	.	.	q^2	.	.	.	6
2143	.	.	.	.	.	.	.	1	.	q	q	q^2	.	.	.	.	.	.	q	.	q^2	q^3	.	.	7
2314	.	.	.	.	.	.	.	.	1	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
2341	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2413	.	.	.	.	.	.	.	.	.	.	1	q	.	.	.	.	.	.	.	.	q	q^2	.	.	4
2431	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	2
3124	.	.	.	.	.	.	.	.	.	.	.	.	1	q	.	q	q^2	.	.	.	.	.	.	.	4
3142	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	q	.	.	.	.	.	.	.	2
3214	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	q	q	q^2	.	.	.	.	.	.	4
3241	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	q	.	.	.	.	.	.	2
3412	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1
3421	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1
4123	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
4132	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	q	.	2
4213	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	q	.	.	2
4231	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
4312	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1
4321	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
Proj.	1	1	1	2	2	1	1	1	2	5	4	4	2	4	1	4	9	3	5	4	4	6	3	1	

Table 4. q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_4)$ (type A_3).

	12	21	Simp.	Proj.
12	1	.	1	1
21	q	1	1	2
Simp.	1	1		
Proj.	2	1		

Table 5. q-Cartan invariant matrix of $M(\mathfrak{S}_2)$ (type A_1).

B2. *Decomposition matrices.* Since M_{w_0} is a submonoid of M, any simple M-module is also a simple M_{w_0}-module. The matrices of Tables 8–10 give the (generalized) M_{w_0} character of the simple M-module. The table reads as follows: for any two permutations σ, τ, the coefficient $m_{\sigma,\tau}$ gives the Jordan–Hölder multiplicity of the M_{w_0}-module $S_\tau^{w_0}$ in the M-module S_σ. In particular, since the simple M_{w_0}-modules are of dimension 1, summing each line one recovers the dimension of the simple M-modules.

	123	132	213	231	312	321	Simp.	Proj.
123	1	.	.	.	.		1	1
132	q	1	.	.	.		1	2
213	q	.	1	.	.		1	2
231	q	.	.	1	.		2	3
312	q	.	.	.	1		2	3
321	q^2	.	.	q	q	1	1	6
Simp.	1	1	1	2	2	1		
Proj.	8	1	1	3	3	1		

Table 6. q-Cartan invariant matrix of $M(\mathfrak{S}_3)$ (type A_2).

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Simp.	Proj.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
1243	q^2+q	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	8
1324	q^3+2q^2+q	.	1	.	.	.	.	.	.	q^2+q	.	.	.	.	.	.	.	.	q^2+q	.	.	q	.	.	1	22
1342	q	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
1423	q	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
1432	$2q^2$	.	.	q	q	1	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	12
2134	q^2+q	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	8
2143	$3q^2$	q	.	.	.	.	q	1	.	q	.	.	.	.	.	.	.	.	q	.	.	.	.	.	1	12
2314	q	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
2341	q	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	4
2413	q	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	4	5
2431	q^2	.	.	.	.	.	.	.	.	q	.	1	.	.	.	.	.	.	.	.	.	.	.	.	4	8
3124	q	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2	3
3142	q	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	4	5
3214	$2q^2$	.	.	.	.	.	.	.	q	.	.	.	q	.	1	.	q	.	.	.	.	.	.	.	1	12
3241	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	1	.	.	.	.	.	.	.	.	4	8
3412	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	5	6
3421	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	q	1	.	.	.	.	.	.	3	12
4123	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	3	4
4132	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	1	.	.	.	.	4	8
4213	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	1	.	.	.	4	8
4231	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	.	q	.	.	1	.	.	5	12
4312	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	q	.	.	.	1	.	3	12
4321	q^3	.	.	.	.	.	.	.	.	q^2	.	.	.	.	.	.	q^2	q	q^2	.	.	q	q	1	1	24
Simp.	1	1	1	2	2	1	1	1	2	3	4	4	2	4	1	4	5	3	3	4	4	5	3	1		
Proj.	71	2	1	3	3	1	2	1	3	23	4	4	3	4	1	4	16	4	23	4	4	7	4	1		

Table 7. q-Cartan invariant matrix of $M(\mathfrak{S}_4)$ (type A_3).

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	12	21	Simp.
12	1	.	1
21	.	1	1

Table 8. Decomposition matrix of $M(\mathfrak{S}_2)$ on $M_{w_0}(\mathfrak{S}_2)$ (type A_1).

	123	132	213	231	312	321	Simp.
123	1	.	.	.	.	.	1
132	.	1	.	.	.	.	1
213	.	.	1	.	.	.	1
231	.	.	1	1	.	.	2
312	.	1	.	.	1	.	2
321	.	.	.	.	.	1	1

Table 9. Decomposition matrix of $M(\mathfrak{S}_3)$ on $M_{w_0}(\mathfrak{S}_3)$ (type A_2).

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Simp.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1243	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1324	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1342	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
1423	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
1432	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2134	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2143	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2314	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
2341	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3
2413	.	1	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	4
2431	.	1	.	.	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	4
3124	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2
3142	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	4
3214	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1
3241	.	.	1	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	4
3412	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	5
3421	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.	3
4123	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	3
4132	.	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	4
4213	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	4
4231	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	5
4312	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	3
4321	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1

Table 10. Decomposition matrix of $M(\mathfrak{S}_4)$ on $M_{w_0}(\mathfrak{S}_4)$ (type A_3).

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Communicated by Victor Reiner
Received 2011-06-08 Revised 2012-02-20 Accepted 2012-04-04

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

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Algebra & Number Theory

Volume 7 No. 3 2013