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On differential modules associated to de Rham representations in the imperfect residue field case

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Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with possibly imperfect residue fields, and let G_K the absolute Galois group of K. In the first part of this paper, we prove that Scholl's generalization of fields of norms over K is compatible with Abbes–Saito's ramification theory. In the second part, we construct a functor $\mathbb{N}_{\mathrm{dR}}$ that associates a de Rham representation V to a (φ, ∇)-module in the sense of Kedlaya. Finally, we prove a compatibility between Kedlaya's differential Swan conductor of $\mathbb{N}_{\mathrm{dR}}(V)$ and the Swan conductor of V, which generalizes Marmora's formula.

Introduction

Hodge theory relates the singular cohomology of complex projective manifolds X to the spaces of harmonic forms on X. Its p-adic analogue, p-adic Hodge theory, enables us to compare the p-adic étale cohomology $H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ of proper smooth varieties X over the p-adic field $\mathbb{Q}_p$ with the de Rham cohomology of X. Precisely speaking, the natural action of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ on the p-adic étale cohomology can be recovered after tensoring both cohomologies with $\mathbb{B}_{\mathrm{dR}}$, which is the ring of p-adic periods introduced by Jean-Marc Fontaine. If X has

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semistable reduction, then one can obtain a more precise comparison theorem between the p-adic étale cohomology of X and the log-cristalline cohomology of the special fiber of X. Thus, we have a satisfactory p-adic étale cohomology theory on proper smooth varieties over $\mathbb{Q}_p$.

A p-adic representation V of $G_{\mathbb{Q}_p}$ is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous linear $G_{\mathbb{Q}_p}$-action. Fontaine [1994] defined the notions of de Rham, crystalline, and semistable representations, which form important subcategories of the category of p-adic representations of $G_{\mathbb{Q}_p}$. Then, he associated linear algebraic objects such as filtered vector spaces with extra structures to objects in each category. Fontaine's classification is compatible with geometry in the following sense: for a proper smooth variety X over $\mathbb{Q}_p$, the p-adic representation $H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ of $G_{\mathbb{Q}_p}$ is only de Rham in general. However, if X has a semistable reduction (resp. good reduction), then $H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is semistable (resp. crystalline).

There also exists a more analytic description of general p-adic representations. Let $\mathbb{B}_{\mathbb{Q}_p}$ be the fraction field of the p-adic completion of $\mathbb{Z}_p[\![t]\!][1/t]$. We define the action of $\Gamma_{\mathbb{Q}_p} := G_{\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p}$ on $\mathbb{B}_{\mathbb{Q}_p}$ by $\gamma(t) = (1+t)^{\chi(g)} - 1$, where $\chi : \Gamma_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character. We also define a Frobenius lift φ on $\mathbb{B}_{\mathbb{Q}_p}$ by $\varphi(t) = (1+t)^p - 1$. An étale $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathbb{B}_{\mathbb{Q}_p}$ is a finite dimensional $\mathbb{B}_{\mathbb{Q}_p}$-vector space M endowed with compatible actions of φ and $\Gamma_{\mathbb{Q}_p}$such that the Frobenius slopes of M are all zero. Using Fontaine–Wintenberger's isomorphism

$$G_{\mathbb{Q}_p(\mu_{p^\infty})} \cong G_{\mathbb{F}_p((t))},$$

of Galois groups, Fontaine [1990] proved an equivalence between the category of p-adic representations and the category of étale $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $\mathbb{B}_{\mathbb{Q}_p}$. We consider the overconvergent subring

$$\mathbb{B}^\dagger_{\mathbb{Q}_p} := \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \in \mathbb{B}_{\mathbb{Q}_p};\ a_n \in \mathbb{Q}_p,\, |a_n|\rho^n \to 0 \text{ for some } \rho \in (0,1] \text{ and } n \to -\infty \right\}$$

of $\mathbb{B}_{\mathbb{Q}_p}$. Frédéric Cherbonnier and Pierre Colmez [1998] proved that the category of étale $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $\mathbb{B}_{\mathbb{Q}_p}$ is equivalent to the category of étale $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $\mathbb{B}^\dagger_{\mathbb{Q}_p}$. As a consequence of Cherbonnier–Colmez' theorem, p-adic analysis over the Robba ring

$$\mathcal{R}_{\mathbb{Q}_p} := \bigcup_{\rho' \in (0,1)} \left\{ \sum_{n\in\mathbb{Z}} a_n t^n;\ a_n \in \mathbb{Q}_p,\, |a_n|\rho^n \to 0 \text{ for all } \rho \in (\rho', 1] \text{ and } n \to \pm\infty \right\}$$

comes into play. Actually, via the above equivalences, Laurent Berger [2002] associated a p-adic differential equation $\mathbb{N}_{\mathrm{dR}}(V)$ over $\mathcal{R}_{\mathbb{Q}_p}$ to a de Rham representation V. By using this functor $\mathbb{N}_{\mathrm{dR}}$ and the quasi-unipotence of p-adic differential equations due to Yves André, Zoghman Mebkhout and Kiran Kedlaya, Berger proved Fontaine's p-adic local monodromy conjecture, which is a p-adic analogue of Grothendieck's l-adic monodromy theorem. We note that in the above theory,

$G_{\mathbb{Q}_p}$ is usually replaced by G_K, where K is a complete valuation field of mixed characteristic $(0, p)$ with a perfect residue field.

Recently, based on earlier work of Gerd Faltings and Osamu Hyodo, Fabrizio Andreatta and Olivier Brinon [2008] started to generalize Fontaine's theory in the relative situation: instead of complete discrete valuation rings with perfect residue fields, they work over higher dimensional ground rings R such as the generic fiber of the Tate algebra $\mathbb{Z}_p\{T_1, T_1^{-1}, \dots, T_d, T_d^{-1}\}$. In this paper, we work in the most basic case of Andreatta–Brinon's setup. That is, our ground ring K is still a complete valuation field, but it has a nonperfect residue field k_K such that $p^d = [k_K : k_K^p] < \infty$. Such a complete discrete valuation field arises as the completion of a ground ring along the special fiber in Andreatta–Brinon's setup.

Even in our situation, a generalization of Fontaine's theory could be useful as in the proof of Kato's divisibility result [2004] in the Iwasawa main conjecture for GL_2. Using compatible systems of K_2 of affine modular curves $Y(p^nN)$ for varying n, Kato defines (p-adic) Euler systems in Galois cohomology groups over $\mathbb{Q}_p$ whose coefficients are related to cusp forms. A key ingredient in this paper is that Kato's Euler systems are related with some products of Eisenstein series via Bloch–Kato's dual exponential map $\exp^*$. In the proof of this fact, p-adic Hodge theory for "the field of q-expansions" $\mathcal{K}$ plays an important role, where $\mathcal{K}$ is the fraction field of the p-adic completion of $\mathbb{Z}_p[\zeta_{p^N}][\![q^{1/N}]\!][q^{-1}]$. Roughly speaking, Tate's universal elliptic curve together with its torsion points induces a morphism $\mathrm{Spec}(\mathcal{K}(\zeta_{p^n}, q^{p^{-n}})) \to Y(p^nN)$. Using a generalization of Fontaine's ring $\mathbb{B}_{\mathrm{dR}}$ over $\mathcal{K}$, Kato defines a dual exponential map for Galois cohomology groups over $\mathcal{K}(\zeta_{p^n}, q^{p^{-n}})$, and proves its compatibility with $\exp^*$. Then, the image of Kato's Euler system under $\exp^*$ is calculated by using Kato's generalized explicit reciprocity law for p-divisible groups over $\mathcal{K}(\zeta_{p^n}, q^{p^{-n}})$.

To explain our results, we recall Anthony Scholl's theory [2006] of field of norms, which is a generalization of Fontaine–Wintenberger's theorem when k_K is nonperfect. In the rest of the introduction we restrict ourselves for simplicity to the "Kummer tower case": we choose a lift $\{t_j\}_{1\le j\le d}$ of a p-basis of k_K and define a tower $\mathfrak{K} := \{K_n\}_{n>0}$ of fields by $K_n := K(\mu_{p^n}, t_1^{p^{-n}}, \dots, t_d^{p^{-n}})$ for $n > 0$, and set $K_\infty := \bigcup_n K_n$. Then, the Frobenius on $\mathcal{O}_{K_{n+1}}/p\mathcal{O}_{K_{n+1}}$ factors through $\mathcal{O}_{K_n}/p\mathcal{O}_{K_n} \hookrightarrow \mathcal{O}_{K_{n+1}}/p\mathcal{O}_{K_{n+1}}$, and the limit $X_{\mathfrak{K}}^+ := \varprojlim_n \mathcal{O}_{K_n}/p\mathcal{O}_{K_n}$ is a complete valuation ring of characteristic p. Here, we denote the integer ring of a valuation field F by $\mathcal{O}_F$. Let $X_{\mathfrak{K}}$ be the fraction field of $X_{\mathfrak{K}}^+$. Then, Scholl proved that a similar limit procedure gives an equivalence of categories $\mathbf{F\acute{E}t}_{K_\infty} \cong \mathbf{F\acute{E}t}_{X_{\mathfrak{K}}}$, where $\mathbf{F\acute{E}t}_A$ denotes the category of finite étale algebras over A. In particular, we obtain an isomorphism of Galois groups

$$\tau : G_{K_\infty} \xrightarrow{\sim} G_{X_{\mathfrak{K}}}.$$

The Galois group of a complete valuation field F is canonically endowed with nonlog and log ramification filtrations in the sense of [Abbes and Saito 2002]. By using the ramification filtrations, one can define Artin and Swan conductors of Galois representations, which are important arithmetic invariants. It is natural to ask that Scholl's isomorphism τ is compatible with ramification theory. The first goal of this paper is to answer this question in the following form:

Theorem 0.0.1 (Theorem 3.5.3). *Let V be a p-adic representation of G_K, where the G_K-action of V factors through a finite quotient. Then, the Artin and Swan conductors of $V|_{K_n}$ are stationary and their limits coincide with the Artin and Swan conductors of $\tau^*(V|_{K_\infty})$.*

We briefly mention the idea of the proof in the Artin case. Note that in the perfect residue field case, the result follows from the fact that the upper numbering ramification filtration is a renumbering of the lower numbering one, and this latter filtration is compatible with the field of norms construction; see [Marmora 2004, Lemme 5.4]. However, in the imperfect residue field case, since Abbes–Saito's ramification filtration is not a renumbering of the lower numbering one, we proceed as follows. Let L/K be a finite Galois extension. Let $X_{\mathfrak{L}}$ be an extension of $X_{\mathfrak{K}}$ corresponding to the tower $\mathfrak{L}=\{L_n := LK_n\}_{n>0}$ under Scholl's equivalence. Then, it suffices to prove that the nonlog ramification filtrations of G_{L_n/K_n} and $G_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$ coincide with each other. Abbes–Saito's nonlog ramification filtration of a finite extension E/F of complete discrete valuation fields is described by a certain family of rigid analytic spaces $as^a_{E/F}$ for $a \in \mathbb{Q}_{\geq 0}$ attached to E/F. In terms of Abbes–Saito's setup, we only have to prove that the number of connected components of $as^a_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$ and $as^a_{L_n/K_n}$ are the same for sufficiently large n. An optimized proof of this assertion is as follows: we construct a characteristic 0 lift R of $X^+_{\mathfrak{K}}$, which is realized as the ring of functions on the open unit ball over a complete valuation ring. We can find a prime ideal $\mathfrak{p}_n$ of R such that $R/\mathfrak{p}_n$ is isomorphic to $\mathcal{O}_{K_n}$. Then, we construct a lift $AS^a_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$ over $\mathrm{Spec}(R)$ of $as^a_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$, whose generic fiber at $\mathfrak{p}_n$ is isomorphic to $as^a_{L_n/K_n}$. We may also regard $AS^a_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$ as a family of rigid spaces parametrized by $\mathrm{Spec}(R)$. What we actually prove is that in such a family of rigid spaces over $\mathrm{Spec}(R)$, the number of the connected components of the fiber varies "continuously". This is done by Gröbner basis arguments over complete regular local rings, extending the method of Liang Xiao [2010]. The continuity result implies our assertion since the point $\mathfrak{p}_n \in \mathrm{Spec}(R)$ "converges" to the point $(p) \in \mathrm{Spec}(R)$.

Note that Shin Hattori reproved [2014] the above ramification compatibility of Scholl's isomorphism τ by using Peter Scholze's perfectoid spaces [2012], which form a geometric interpretation of the Fontaine–Wintenberger theorem. We briefly explain Hattori's proof. Let $\mathbb{C}_p$ (resp. $\mathbb{C}^\flat_p$) be the completion of the algebraic closure of K_∞ (resp. $X_{\mathfrak{K}}$). Scholze proved the tilting equivalence between certain adic spaces (resp. perfectoid spaces) over $\mathbb{C}_p$ and $\mathbb{C}_p^\flat$. Let C be a perfectoid field and Y a subvariety of $\mathbb{A}^n_C$. A perfection of Y is the perfectoid space defined as the pullback of Y under the canonical projection $\varprojlim_{T_i \mapsto T_i^p} \mathbb{A}^n_C \to \mathbb{A}^n_C$, where $T_1, \dots, T_n$ denotes a coordinate of $\mathbb{A}^n_C$. Hattori proved that the tilting of the perfections of $(as^a_{L_n/K_n})_{\mathbb{C}_p}$ and $(as^a_{X_{\mathfrak{L}}/X_{\mathfrak{K}}})_{\mathbb{C}_p^\flat}$ are isomorphic under the tilting equivalence. Since the underlying topological spaces are homeomorphic under taking perfections and the tilting equivalence, he obtained the ramification compatibility.

The second goal of this paper is to generalize Berger's functor $\mathbb{N}_{\mathrm{dR}}$ and prove a ramification compatibility of $\mathbb{N}_{\mathrm{dR}}$ which extends Theorem 0.0.1. Precisely, we construct a functor from the category of de Rham representations to the category of (φ, ∇)-modules over the Robba ring. Our target objects (φ, ∇)-modules are defined by Kedlaya [2007] as generalizations of p-adic differential equations. Kedlaya also defined the differential Swan conductor $\mathrm{Swan}^\nabla(M)$ for a (φ, ∇)-module M, which is a generalization of the irregularity of p-adic differential equations. Then, we prove the following de Rham version of Theorem 0.0.1:

Theorem 0.0.2 (Theorem 4.7.1). *Let V be a de Rham representation of G_K. Then we have*

$$\mathrm{Swan}^\nabla(\mathbb{N}_{\mathrm{dR}}(V)) = \lim_{n\to\infty} \mathrm{Swan}(V|_{K_n}),$$

where Swan *on the right-hand side means Abbes–Saito's Swan conductor. Moreover, the sequence* $\{\mathrm{Swan}(V|_{K_n})\}_{n>0}$ *is eventually stationary.*

Both Theorems 0.0.1 and 0.0.2 are due to Adriano Marmora [2004] when the residue field is perfect. Even when the residue field is perfect, our proof of Theorem 0.0.2 is slightly different from Marmora's proof since we use a dévissage argument to reduce to the pure slope case. As is addressed in [Kedlaya 2007, §3.7], it seems to be possible to define a ramification invariant of $\mathbb{N}_{\mathrm{dR}}(V)$ in terms of (φ, Γ_K)-modules so that one can compute $\mathrm{Swan}(V)$ instead of $\mathrm{Swan}(V|_{K_n})$. It is also important to extend the construction of $\mathbb{N}_{\mathrm{dR}}$ to the general relative case: one may expect that a relative version of slope theory, described in [Kedlaya 2013] for example, will be an important tool.

Structure of the paper

In Section 1, we gather various basic results used in this paper. These contain some p-adic Hodge theory, Abbes–Saito's ramification theory, Kedlaya's theory of overconvergent rings, and Scholl's fields of norms. In Section 2, we prove some ring theoretic properties of overconvergent rings by using Kedlaya's slope theory. In Section 3, we develop a Gröbner basis argument over complete regular local rings and overconvergent rings. We apply the Gröbner basis argument to study families of rigid spaces, and use it to prove Theorem 0.0.1. In Section 4, we generalize Berger's gluing argument to construct a differential module $\mathbb{N}_{\mathrm{dR}}(V)$ for de Rham representations V. We also study the graded pieces of $\mathbb{N}_{\mathrm{dR}}(V)$ with respect to Kedlaya's slope filtration to reduce Theorem 0.0.2 to Theorem 0.0.1 by dévissage.

Convention

Throughout this paper, let p be a prime number. All rings are assumed to be commutative unless otherwise stated. For a ring R, denote by $\pi_0^{\mathrm{Zar}}(R)$ the set of connected components of $\mathrm{Spec}(R)$ with respect to the Zariski topology. For a field E, fix an algebraic closure, denoted by E^{alg} or $\overline{E}$, and a separable closure E^{sep}. Let $G_{F/E}$ be the Galois group of a finite extension F/E, and let G_E be the absolute Galois group of E. For a field k of characteristic p, let $k^{\mathrm{pf}} := k^{p^{-\infty}}$ be the perfect closure in a fixed algebraic closure of k.

For a complete valuation field K, we let $\mathcal{O}_K$ be its integer ring, π_K a uniformizer, and k_K the residue field. Let $v_K : K \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation satisfying $v_K(\pi_K) = 1$. We let K^{ur} be the p-adic completion of the maximal unramified extension of K and denote by I_K the inertia subgroup of G_K. We assume that K is of mixed characteristic $(0, p)$ and that $[k_K : k_K^p] = p^d < \infty$ in the rest of this paragraph. Let e_K be the absolute ramification index. Let $\mathbb{C}_p$ be the p-adic completion of K^{alg} and let v_p be the p-adic valuation of $\mathbb{C}_p$, normalized by $v_p(p) = 1$. We fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n>0}$ in K^{alg}, i.e., ζ_p is a primitive p-th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \in \mathbb{N}_{>0}$. Let $\chi : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character defined by $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for all $n > 0$. We denote the fraction field of a Cohen ring of k_K by K_0. Denote a lift of a p-basis of k_K in $\mathcal{O}_K$ by $\{t_j\}_{1\le j\le d}$. For a given $\{t_j\}_{1\le j\le d}$, we can choose an embedding $K_0 \hookrightarrow K$ such that $\{t_j\}_{1\le j\le d} \subset \mathcal{O}_{K_0}$, see [Ohkubo 2013, §1.1]. Unless otherwise stated, we always choose $\{t_j\}_{1\le j\le d}$ and an embedding $K_0 \hookrightarrow K$ in this way, and we fix sequences of p-power roots $\{t_j^{p^{-n}}\}_{n\ge 0, 1\le j\le d}$ of $\{t_j\}_{1\le j\le d}$ in K^{alg}, i.e., we have $\bigl(t_j^{p^{-n-1}}\bigr)^p = t_j^{p^{-n}}$ for all $n > 0$. For such a sequence, we define K^{pf} as the p-adic completion of $\bigcup_n K\bigl(\{t_j^{p^{-n}}\}_{1\le j\le d}\bigr)$. This is a complete discrete valuation field with perfect residue field k_K^{pf}, and we regard $\mathbb{C}_p$ as the p-adic completion of the algebraic closure of K^{pf}.

For a ring R, let $W(R)$ be the Witt ring with coefficients in R. If R is of characteristic p, then we denote the absolute Frobenius on R by φ and also denote the ring homomorphism $W(\varphi) : W(R) \to W(R)$ by φ. Let $[x] \in W(R)$ be the Teichmüller lift of $x \in R$.

For an integer $h > 0$, define $\mathbb{Q}_{p^h} := W(\mathbb{F}_{p^h})[1/p]$. Let K be a complete discrete valuation field, and $F/\mathbb{Q}_p$ a finite extension. A finite dimensional F-vector space V with continuous semilinear G_K-action is called an F-representation of G_K. If moreover $F = \mathbb{Q}_p$, then we call V a p-adic representation of G_K. We denote the category of F-representations of G_K by $\mathrm{Rep}_F(G_K)$. We say that V is finite (resp.

of finite geometric monodromy) if G_K (resp. I_K) acts on V via a finite quotient. We denote the category of finite (resp. finite geometric monodromy) F-representations of G_K by $\mathrm{Rep}_F^{f}(G_K)$ (resp. $\mathrm{Rep}_F^{f.g.}(G_K)$).

For homomorphisms $f, g : M \to N$ of abelian groups, we denote by $M^{f=g}$ the kernel of the map $f-g$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor := \inf\{n \in \mathbb{Z}; n \geq x\}$ be the least integer greater than or equal to x. We let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ be the set of all natural numbers.

1. Preliminaries

In this section, we fix notation and recall basic results needed in this paper.

1.1. ***Fréchet spaces.*** We will define some basic terminology of topological vector spaces. Although we will use both valuations and norms to consider topologies, we will define our terminology in terms of valuations for simplicity. See [Kedlaya 2010] or [Schneider 2002] for details.

Notation 1.1.1. Let M be an abelian group. A valuation v of M is a map $v : M \to \mathbb{R} \cup \{\infty\}$ such that $v(x-y) \geq \inf\{v(x), v(y)\}$ for all $x, y \in R$ and $v(x) = \infty$ if and only if $x = 0$. Moreover, when $M = R$ is a ring, v is multiplicative if $v(xy) = v(x) + v(y)$ for all $x, y \in R$. A ring equipped with a multiplicative valuation is called a valuation ring. If (R, v) is a valuation ring and (M, v_M) is an R-module with valuation v_M, then we say that v_M is an R-valuation if $v_M(\lambda x) = v(\lambda) + v_M(x)$ holds for all $\lambda \in R$ and $x \in M$.

Let (R, v) be a valuation ring and M a finite free R-module. For an R-basis $e_1, \dots, e_n$ of M, we define the R-valuation v_M on M (compatible with v) associated to $e_1, \dots, e_n$ by $v_M\left(\sum_{1 \leq i \leq n} a_i e_i\right) = \inf_i v(a_i)$ for $a_i \in R$ ([Kedlaya 2010, Definition 1.3.2]). The topology defined by v_M is independent of the choice of a basis of M ([Kedlaya 2010, Definition 1.3.3]). Hence, we do not refer to a basis to consider v_M and we just denote v_M by v unless otherwise stated.

For any valuation v on M, we define the associated nonarchimedean norm $|\cdot| : M \to \mathbb{R}$ by $|x| := a^{-v(x)}$ for a fixed $a \in \mathbb{R}_{>1}$ (nonarchimedean means that it satisfies the strong triangle inequality). Conversely, for any nonarchimedean norm $|\cdot|$, $v(\cdot) = -\log_a |\cdot|$ is a valuation. We will apply various definitions made for norms to valuations, and vice versa in this manner.

Notation 1.1.2. Let (K, v) be a complete valuation field. Let $\{w_r\}_{r \in I}$ be a family of K-valuations of a K-vector space V. Consider the topology $\mathcal{T}$ of V whose neighborhoods at 0 are generated by $\{x \in V; w_r(x) \geq n\}$ for all $r \in I$ and $n \in \mathbb{N}$. We call $\mathcal{T}$ the topology of V defined by $\{w_r\}_{r \in I}$ and denote the topological space V with this topology by $(V, \{w_r\}_{r \in I})$, or simply by V. If $\mathcal{T}$ is equivalent to the topology defined by $\{w_r\}_{r \in I_0}$ for some countable subset $I_0 \subset I$, we call $\mathcal{T}$ the K-Fréchet topology defined by $\{w_r\}_{r \in I}$. For a K-vector space, it is well-known that a K-Fréchet topology is metrizable (and vice versa). Moreover, when V is complete, we call V a K-Fréchet space. Note that V is just a K-Banach space when $\#I_0 = 1$. Also, note that a topological K-vector space V is a K-Fréchet space if and only if V is isomorphic to an inverse limit of K-Banach spaces whose transition maps consist of bounded K-linear maps. More precisely, let V be a K-Fréchet space with valuations $w_0 \geq w_1 \geq \dots$, and V_n the completion of V with respect to w_n. Then the canonical map $V \to \varprojlim_n V_n$ is an isomorphism of K-Fréchet spaces. Also, note that if V and W are K-Fréchet spaces, then $\mathrm{Hom}_K(V, W)$ is again a K-Fréchet space with respect to the operator norm.

Let $(R, \{w_r\}_{r\in I})$ be a K-Fréchet space for a ring R. If $\{w_r\}_{r\in I}$ are multiplicative, then we call R a K-Fréchet algebra. For a finite free R-module M, we choose a basis of M and let $\{w_{r,M}\}_{r\in I}$ be the R-valuations compatible with $\{w_r\}_{r\in I}$. Obviously, $(M, \{w_{r,M}\}_{r\in I})$ is a K-Fréchet space. Unless otherwise stated, we always endow a finite free R-module with such a family of valuations.

In the rest of the paper, we omit the prefix "K" unless otherwise stated.

Recall that the category of Fréchet spaces is closed under quotients, completed tensor products and direct sums and that the open mapping theorem holds.

1.2. *Continuous derivations over K.* In this subsection, we recall the continuous Kähler differentials ([Hyodo 1986, §4]). Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ such that $[k_K : k_K^p] = p^d < \infty$.

Definition 1.2.1. Let $\widehat{\Omega}^1_{\mathcal{O}_K}$ be the p-adic Hausdorff completion of $\Omega^1_{\mathcal{O}_K/\mathbb{Z}}$ and put $\widehat{\Omega}^1_K := \widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$. Let $d : K \to \widehat{\Omega}^1_K$ be the canonical derivation.

Recall that $\widehat{\Omega}^1_K$ is a finite K-vector space with basis $\{dt_j\}_{1\leq j\leq d}$. Moreover, if K is absolutely unramified, then $\widehat{\Omega}^1_{\mathcal{O}_K}$ is a finite free $\mathcal{O}_K$-module with basis $\{dt_j\}_{1\leq j\leq d}$. Also, $\widehat{\Omega}^1_{\bullet}$ is compatible with base change, i.e., $L \otimes_K \widehat{\Omega}^1_K \cong \widehat{\Omega}^1_L$ for any finite extension L/K.

Notation 1.2.2. Let R be a topological ring and M a topological R-module. We let $\mathrm{Der}_{\mathrm{cont}}(R, M)$ be the R-module of continuous derivations $d : R \to M$.

One can prove the next lemma by dévissage and the universality of the usual Kähler differentials.

Lemma 1.2.3. *For an inductive limit M of K-Fréchet spaces, we have the canonical isomorphism*

$$d^* : \mathrm{Hom}_K(\widehat{\Omega}^1_K, M) \xrightarrow{\sim} \mathrm{Der}_{\mathrm{cont}}(K, M).$$

Definition 1.2.4. Let $\{\partial_j\}_{1\leq j\leq d} \subset \mathrm{Der}_{\mathrm{cont}}(K_0, K_0) \cong \mathrm{Hom}_{K_0}(\Omega^1_{K_0}, K_0)$ be the dual basis of $\{dt_j\}_{1\leq j\leq d}$. We call $\{\partial_j\}$ the derivations associated to $\{t_j\}$. We also denote by ∂_j the canonical extension of ∂_j to $\partial_j : K^{\mathrm{alg}} \to K^{\mathrm{alg}}$. Since $\partial_j(t_i) = \delta_{ij}$, we may denote ∂_j by $\partial/\partial t_j$.

1.3. ***Some Galois extensions.*** In this subsection, we will fix some notation of a certain Kummer extension which will be studied later. See [Hyodo 1986, §1] for details. In this subsection, let $\widetilde{K}$ be an absolutely unramified complete discrete valuation field of mixed characteristic $(0, p)$ with $[k_{\widetilde{K}} : k_{\widetilde{K}}^p] = p^d < \infty$. We put

$$\widetilde{K}_n := \widetilde{K}(\zeta_{p^n}, t_1^{p^{-n}}, \dots, t_1^{p^{-n}}) \text{ for } n > 0,\ \ \widetilde{K}_\infty := \bigcup_{n>0} \widetilde{K}_n,\ \ \widetilde{K}_{\mathrm{arith}} := \bigcup_{n>0} \widetilde{K}(\zeta_{p^n}),$$
$$\Gamma_{\widetilde{K}}^{\mathrm{geom}} := G_{\widetilde{K}_\infty/\widetilde{K}_{\mathrm{arith}}}, \quad \Gamma_{\widetilde{K}}^{\mathrm{arith}} := G_{\widetilde{K}_{\mathrm{arith}}/\widetilde{K}},$$
$$\Gamma_{\widetilde{K}} := G_{\widetilde{K}_\infty/\widetilde{K}}, \qquad H_{\widetilde{K}} := G_{\widetilde{K}^{\mathrm{alg}}/\widetilde{K}_\infty}.$$

Then, we have isomorphisms

$$\Gamma_{\widetilde{K}}^{\mathrm{arith}} \cong \mathbb{Z}_p^\times, \quad \Gamma_{\widetilde{K}}^{\mathrm{geom}} \cong \mathbb{Z}_p^d,$$

which are compatible with the action of $\Gamma_{\widetilde{K}}^{\mathrm{arith}}$ on $\Gamma_{\widetilde{K}}^{\mathrm{geom}}$. The isomorphisms are defined as follows: an element $a \in \mathbb{Z}_p^\times$ corresponds to $\gamma_a \in \Gamma_{\widetilde{K}}^{\mathrm{arith}}$ such that $\gamma_a(\zeta_{p^n}) = \zeta_{p^n}^a$ for all n. An element $b = (b_j) \in \mathbb{Z}_p^d$ corresponds to $\gamma_b \in \Gamma^{\mathrm{geom}}$ for $1 \le j \le d$ such that $\gamma_b(\zeta_{p^n}) = \zeta_{p^n}$ for all $n \in \mathbb{N}$ and $\gamma_b(t_j^{p^{-n}}) = \zeta_{p^n}^{b_j} t_j^{p^{-n}}$. By regarding $\Gamma_{\widetilde{K}}^{\mathrm{arith}}$ as a subgroup $G_{\widetilde{K}_\infty/\bigcup_n \widetilde{K}\left(t_1^{p^{-n}},\dots,t_1^{p^{-n}}\right)}$ of $\Gamma_{\widetilde{K}}$, we obtain isomorphisms

$$\eta = (\eta_0, \dots, \eta_d) : \Gamma_{\widetilde{K}} \cong \Gamma_{\widetilde{K}}^{\mathrm{arith}} \ltimes \Gamma_{\widetilde{K}}^{\mathrm{geom}} \cong \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^d.$$

Since we have a canonical isomorphism

$$\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^d \cong \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \dots & \mathbb{Z}_p \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \le \mathrm{GL}_{d+1}(\mathbb{Z}_p),$$

the group $\Gamma_{\widetilde{K}}$ can be regarded as a classical p-adic Lie group with Lie algebra

$$\mathfrak{g} := \mathrm{Lie}(\Gamma_{\widetilde{K}}) \cong \mathbb{Q}_p \ltimes \mathbb{Q}_p^d = \begin{pmatrix} \mathbb{Q}_p & \dots & \mathbb{Q}_p \\ & 0 & \end{pmatrix} \subset \mathfrak{gl}_{d+1}(\mathbb{Q}_p).$$

For an integer $n > 0$ and a finite extension $L/\widetilde{K}$, we put

$$L_n := \widetilde{K}_n L, \qquad L_\infty := \widetilde{K}_\infty L,$$
$$\Gamma_L := G_{L_\infty/L}, \quad H_L := G_{\widetilde{K}^{\mathrm{alg}}/L_\infty}.$$

Then, Γ_L is an open subgroup of $\Gamma_{\widetilde{K}}$. Hence, there exists an open normal subgroup of Γ_L which is isomorphic to an open subgroup of $(1 + 2p\mathbb{Z}_p) \ltimes \mathbb{Z}_p^d$ by the map η. Also, we may identify the p-adic Lie algebra of Γ_L with $\mathfrak{g}$. Finally, we define closed subgroups of Γ_L

$$\Gamma_{L,0} := \{\gamma \in \Gamma_L;\ \eta_j(\gamma) = 0 \text{ for all } 1 \le j \le d\},$$
$$\Gamma_{L,j} := \{\gamma \in \Gamma_L;\ \eta_0(\gamma) = 1, \eta_i(\gamma) = 0 \text{ for all } 1 \le i \le d, i \ne j\} \text{ for } 1 \le j \le d.$$

1.4. ***Basic construction of Fontaine's rings.*** In this subsection, we recall the definition of rings of p-adic periods due to Fontaine, see [Ohkubo 2013, §3] for details.

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with $[k_K : k_K^p] = p^d < \infty$. Let $\widetilde{\mathbb{E}}^+ := \varprojlim_n \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, where the transition maps are given by the Frobenius. This is a complete valuation ring of characteristic p whose (algebraically closed) fractional field is denoted by $\widetilde{\mathbb{E}}$. We have a canonical identification

$$\widetilde{\mathbb{E}} \cong \left\{(x^{(n)})_{n\in\mathbb{N}} \in \mathbb{C}_p^{\mathbb{N}};\ (x^{(n+1)})^p = x^{(n)} \text{ for all } n \in \mathbb{N}\right\}.$$

For $x \in \mathbb{C}_p$, we denote by $\tilde{x} \in \widetilde{\mathbb{E}}$ an element $\tilde{x} = (x^{(n)})$ such that $x^{(0)} = x$. In particular, we put $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$, $\tilde{t}_j := (t_j, t_j^{1/p}, \dots) \in \widetilde{\mathbb{E}}^+$. We define the valuation $v_{\widetilde{\mathbb{E}}}$ of $\widetilde{\mathbb{E}}$ by $v_{\widetilde{\mathbb{E}}}((x^{(n)})) = v_p(x^{(0)})$. We put

$$\widetilde{\mathbb{A}}^+ := W(\widetilde{\mathbb{E}}^+) \subset \widetilde{\mathbb{A}} := W(\widetilde{\mathbb{E}}),$$
$$\widetilde{\mathbb{B}}^+ := \widetilde{\mathbb{A}}^+[1/p] \subset \widetilde{\mathbb{B}} := \widetilde{\mathbb{A}}[1/p],$$
$$\pi := [\varepsilon] - 1, \quad q := \pi/\varphi^{-1}(\pi) = \sum_{0\le i<p} \left[\varepsilon^{1/p}\right]^i \in \widetilde{\mathbb{A}}^+$$

and we define a surjective ring homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$$
$$\sum_{n \gg -\infty} p^n[x_n] \mapsto p^n x_n^{(0)},$$

which maps $\widetilde{\mathbb{A}}^+$ to $\mathcal{O}_{\mathbb{C}_p}$. Note that q is a generator of the kernel of $\theta|_{\widetilde{\mathbb{A}}^+}$.

Let $\mathcal{K}$ be a closed subfield of $\mathbb{C}_p$ whose value group $v_p(\mathcal{K}^\times)$ is discrete. We will define rings

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}.$$

Let $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ be the universal p-adically formal pro-infinitesimal $\mathcal{O}_\mathcal{K}$-thickening of $\mathcal{O}_{\mathbb{C}_p}$. More precisely, if $\theta_{\mathbb{C}_p/\mathcal{K}} : \mathcal{O}_\mathcal{K} \otimes_\mathbb{Z} \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ denotes the linear extension of θ, then $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is the $(p, \ker\theta_{\mathbb{C}_p/\mathcal{K}})$-adic Hausdorff completion of $\mathcal{O}_\mathcal{K} \otimes_\mathbb{Z} \widetilde{\mathbb{A}}^+$. The map $\theta_{\mathbb{C}_p/\mathcal{K}}$ extends to $\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathcal{O}_{\mathbb{C}_p}$. Note that $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ is canonically identified with $\widetilde{\mathbb{A}}^+$. Let $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ be the $\ker\theta_{\mathbb{C}_p/\mathcal{K}}$-adic Hausdorff completion of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[1/p]$ and $\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \mathbb{C}_p$ the canonical map induced by $\theta_{\mathbb{C}_p/\mathcal{K}}$.

Let

$$u_j := t_j - [\tilde{t}_j] \in \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/K_0},$$
$$t := \log([\varepsilon]) := \sum_{n\geq 1}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \subset \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}.$$

Finally, we define $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}[1/t]$. These constructions are functorial with respect to $\mathbb{C}_p$ and $\mathcal{K}$. In particular:

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p} \subset \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \subset \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \subset \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}.$$

Therefore, any continuous $\mathcal{K}$-algebra automorphism of $\mathbb{C}_p$ acts on these rings. We also have the following explicit descriptions:

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/K_0} \cong \widetilde{\mathbb{A}}^+[\![u_1,\dots,u_d]\!], \quad \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \cong \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}[\![u_1,\dots,u_d]\!]$$

and $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is a complete discrete valuation field with uniformizer t and residue field $\mathbb{C}_p$. Also, $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ and $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ are invariant after replacing K by a finite extension. In particular, these rings are endowed with canonical K^{alg}-algebra structures.

For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we define $\mathbb{D}_{\mathrm{dR}}(V) := (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V)^{G_K}$, which is a finite dimensional K-vector space with $\dim_K \mathbb{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$. When the dimensions are equal, we call V de Rham and denote the category of de Rham representations of G_K by $\mathrm{Rep}_{\mathrm{dR}}(G_K)$.

We endow $\varprojlim_k \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[1/p]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^k$ with the inverse limit topology, equipping $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[1/p]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^k$ with the $\mathcal{K}$-Banach space structure whose unit disc is the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. The identification of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\varprojlim_k \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[1/p]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^k$ gives $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ its canonical topology and it is a $\mathcal{K}$-Fréchet algebra.

The ring $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is endowed with a continuous $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear connection

$$\nabla^{\mathrm{geom}} : \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \otimes_{\mathcal{K}} \widehat{\Omega}^1_{\mathcal{K}},$$

which is induced by the canonical derivation $d : \mathcal{K} \to \widehat{\Omega}^1_{\mathcal{K}}$. More precisely, if we denote by $\{\partial_j\}_{1\leq j\leq d}$ the derivations of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ given by $\nabla^{\mathrm{geom}}(x) = \sum_j \partial_j(x) \otimes dt'_j$, then ∂_j is the unique $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear extension of $\partial/\partial t_j : K \to K$. Thus, we can regard the above connection as a connection associated to a "coordinate" $t_1,\dots,t_d$ of K, hence we put the superscript "geom". We denote the kernel of ∇^{geom} by $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, which coincides with the image of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$. Therefore, we may identify $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$.

We also define a subring $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/\mathbb{Q}_p}$ of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ as follows: let $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ be the universal p-adically formal $\mathbb{Z}_p$-thickening of $\mathcal{O}_{\mathbb{C}_p}$, i.e., the p-adic Hausdorff completion of the PD-envelope of $\widetilde{\mathbb{A}}^+$ with respect to the ideal $\ker\theta_{\mathbb{C}_p/\mathbb{Q}_p}$, compatible with the canonical PD-structure on the ideal (p). Since the construction is functorial, the Frobenius $\varphi:\widetilde{\mathbb{A}}^+\to\widetilde{\mathbb{A}}^+$ acts on both $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}:=\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}[1/p]$. We define $\mathbb{B}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/\mathbb{Q}_p}:=\bigcap_{n\in\mathbb{N}}\varphi^n(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p})$, which is the maximal subring of $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ that is stable under φ. By construction, $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/\mathbb{Q}_p}$ is a subring of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\cong\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$.

Finally, for simplicity, we denote

$$\mathbb{B}^{\nabla+}_{\mathrm{dR}}:=\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p},\quad \mathbb{B}^{\nabla}_{\mathrm{dR}}:=\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p},\quad \mathbb{B}^{+}_{\mathrm{dR}}:=\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K},$$
$$\mathbb{B}_{\mathrm{dR}}:=\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K},\quad \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}:=\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/\mathbb{Q}_p}$$

when no confusion arises.

1.5. ***Ramification theory of Abbes–Saito.*** In this subsection, we review Abbes–Saito's ramification theory, see [Abbes and Saito 2002, 2003] for details.

Let K be a complete discrete valuation field with residue field of characteristic p. Let L/K be a finite separable extension. Let $Z=\{z_0,\dots,z_n\}$ be a set of generators of $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra. View $\mathcal{O}_K\langle Z_0,\dots,Z_n\rangle$ as a Tate algebra, and let $Z_i\mapsto z_i$ be the corresponding surjective $\mathcal{O}_K$-algebra homomorphism from $\mathcal{O}_K\langle Z_0,\dots,Z_n\rangle$ to $\mathcal{O}_L$ with kernel I_Z. For $a\in\mathbb{Q}_{>0}$, we define the nonlog Abbes–Saito space by

$$as^a_{L/K,Z}:=D^{n+1}(|\pi_K|^{-a}f;\ f\in I_Z)=\{x\in D^{n+1};\ |f(x)|\le|\pi_K|^a\ \ \forall f\in I_Z\},$$

which is an affinoid subdomain of the $(n+1)$-dimensional polydisc D^{n+1}. Let $\pi_0^{\mathrm{geom}}(as^a_{L/K,Z})$ be the geometric connected components of $as^a_{L/K,Z}$, i.e., the connected components of $as^a_{L/K,Z}\times_K K^{\mathrm{alg}}$ with respect to the Zariski topology. We define a G_K-set $\mathcal{F}^a(L):=\pi_0^{\mathrm{geom}}(as^a_{L/K,Z})$ and let

$$b(L/K):=\inf\{a\in\mathbb{R};\ \#\mathcal{F}^a(L)=[L:K]\}\in\mathbb{Q}.$$

be the nonlog ramification break. If L/K is Galois, then $\mathcal{F}^a(L)$ can be identified with a quotient of $G_{L/K}$. Moreover, the system $\{\mathcal{F}^a(L)\}_a$ of G_K-sets defines a filtration $\{G^a_{L/K}\}_{a\in\mathbb{Q}_{\ge0}}$ of $G_{L/K}$ such that $\mathcal{F}^a(L)\cong G_{L/K}/G^a_{L/K}$ as G_K-sets.

There exists a log variation of this construction by considering the following log structure. Let $P\subset Z$ be a subset containing a uniformizer, and take a lift $g_j\in\mathcal{O}_K\langle Z_0,\dots,Z_n\rangle$ of $z_j^{e_K}/\pi_K^{v_L(z_j)}$ for each $z_j\in P$. For each pair $(z_i,z_j)\in P\times P$, we take a lift $h_{i,j}\in\mathcal{O}_K\langle Z_0,\dots,Z_n\rangle$ of $z_j^{v_L(z_i)}/z_i^{v_L(z_j)}$. For $a\in\mathbb{Q}_{>0}$, we define the log Abbes–Saito space by

$$as^a_{L/K,Z,P}:=D^{n+1}\left(\begin{matrix}|\pi_K|^{-a}f\\ |\pi_K|^{-a-v_L(z_i)}(X_i^{e_{L/K}}-\pi_K g_i)\\ |\pi_K|^{-a-v_L(z_i)v_L(z_j)/e_{L/K}}(X_j^{v_L(z_i)}-X_i^{v_L(z_j)}h_{i,j})\end{matrix}\right)$$

as an affinoid subdomain of D^{n+1}. Here, f ranges over I_Z and the indices z_i and (z_i, z_j) range over P and $P \times P$ respectively. As before, we define the G_K-set $\mathcal{F}^a_{\log}(L) := \pi_0^{\mathrm{geom}}(as^a_{L/K,Z,P})$ and define the log ramification break by

$$b_{\log}(L/K) := \inf\{a \in \mathbb{R};\ \#\mathcal{F}^a_{\log}(L) = [L : K]\} \in \mathbb{Q}.$$

A similar procedure as before defines the log ramification filtration $\{G^a_{L/K,\log}\}_{a\in\mathbb{Q}_{\geq 0}}$ of $G_{L/K}$.

In this paper, we consider only the following simple Abbes–Saito spaces. With the notation as above, let $p_0, \dots, p_m$ be a system of generators of the kernel of the surjection $\mathcal{O}_K\langle X_0, \dots, X_n\rangle \to \mathcal{O}_L$. Assume that z_0 is a uniformizer of L and $p_0 = X_0^{e_{L/K}} - \pi_K g_0$ for some $g_0 \in \mathcal{O}_K\langle X_0, \dots, X_n\rangle$. In this case, we have a simple log structure: we put $P := \{z_0\}$ and we choose g_0 as a lift of $z_0^{e_{L/K}}/\pi_K$. We also choose 1 as $h_{1,1}$. Hence, Abbes–Saito spaces are given by

$$as^a_{L/K,Z} = D^{n+1}(|\pi_K|^{-a} p_j \text{ for } 0 \leq j \leq m),$$

$$as^a_{L/K,Z,P} = D^{n+1}(|\pi_K|^{-a-1} p_0, |\pi_K|^{-a} p_j \text{ for } 1 \leq j \leq m).$$

Let $F/\mathbb{Q}_p$ be a finite extension and V an F-representation of G_K with finite local monodromy. We define Abbes–Saito's Artin and Swan conductors by

$$\mathrm{Art}^{\mathrm{AS}}(V) := \sum_{a\in\mathbb{Q}_{\geq 0}} a \cdot \dim_F(V^{\bigcap_{b>a} G^b_K}/V^{G^a_K}),$$
$$\mathrm{Swan}^{\mathrm{AS}}(V) := \sum_{a\in\mathbb{Q}_{\geq 0}} a \cdot \dim_F(V^{\bigcap_{b>a} G^b_{K,\log}}/V^{G^a_{K,\log}}).$$

Note that the above construction does not depend on other choices, like Z and P. Also, note that both the Artin and Swan conductors are additive and compatible with unramified base change. When k_K is perfect, the log (resp. nonlog) ramification filtration is compatible with the usual upper numbering filtration (resp. shift by one). Moreover, our Artin and Swan conductors coincide with the classical Artin and Swan conductors when k_K is perfect.

Theorem 1.5.1 (Hasse–Arf theorem, [Xiao 2012, Theorem 4.5.14]). *Assume that K is of mixed characteristic. Let $F/\mathbb{Q}_p$ be a finite extension and $V \in \mathrm{Rep}^{f.g.}_F(G_K)$. Then, we have* $\mathrm{Art}(V) \in \mathbb{Z}$ *if K is not absolutely unramified*; *we have* $\mathrm{Swan}^{\mathrm{AS}}(V) \in \mathbb{Z}$ *if $p \neq 2$, and* $\mathrm{Swan}^{\mathrm{AS}}(V) \in 2^{-1}\mathbb{Z}$ *if $p = 2$.*

Xiao gives more precise results in the equal characteristic case, as we will see in Theorem 1.7.10.

1.6. *Overconvergent rings.* In this subsection, we will recall basic definitions of overconvergent rings associated to complete valuation fields of characteristic p, following [Kedlaya 2004, §2–3] and [Kedlaya 2005, §2].

Construction 1.6.1 ([Kedlaya 2005, §2.1–2.2]). Let (E, v) be a complete valuation field of characteristic p. Assume that either E is perfect or that v is a discrete valuation. We will construct an overconvergent ring associated to E. We first consider the case where E is perfect. Note that any element of $W(E)[1/p]$ is uniquely expressed as $\sum_{k\gg-\infty} p^k[x_k]$ with $x_k \in E$. For $n \in \mathbb{Z}$, we define a "partial valuation" on $W(E)[1/p]$ by

$$v^{\leq n}\biggl(\sum_{k\gg-\infty} p^k[x_k]\biggr) := \inf_{k\leq n} v(x_k).$$

For $r \in \mathbb{R}_{>0}$, we define

$$w_r(x) := \inf_n \{r v^{\leq n}(x) + n\},$$

$$W(E)_r := \{x \in W(E);\, w_r(x) < \infty\}.$$

Then, $W(E)_r[1/p]$ is a subring of $W(E)[1/p]$ and w_r is a multiplicative valuation of $W(E)_r[1/p]$. Moreover, we have $W(E)_r \subset W(E)_{r'}$ for $r' \leq r$. We put $W_{\mathrm{con}}(E) := \varinjlim_{r\to 0} W(E)_r$.

Next, we consider the general case, i.e., we do not need to assume that E is perfect in the following. Let Γ be a Cohen ring of E with a Frobenius lift φ. Then, we can obtain a Frobenius-compatible embedding $\Gamma \hookrightarrow W(E^{\mathrm{pf}}) \hookrightarrow W(\widehat{E}^{\mathrm{alg}})$, where $\widehat{E}^{\mathrm{alg}}$ is the completion of E^{alg}. By using this embedding, we can define $v^{\leq n}$ and w_r on Γ. Moreover, we define $\Gamma_r := \Gamma \cap W(\widehat{E}^{\mathrm{alg}})_r$ and $\Gamma_{\mathrm{con}} := \varinjlim_{r\to 0} \Gamma_r = \Gamma \cap W_{\mathrm{con}}(\widehat{E}^{\mathrm{alg}})$. We say that Γ has enough r-units if the canonical map $\Gamma_r \to E$ is surjective. We say that Γ has enough units if Γ has enough r-units for some $r > 0$. Note that if E is perfect, then Γ has enough r-units for any r. In general, by [Kedlaya 2004, Proposition 3.11], Γ has enough r-units for all sufficiently small r. In the following, we fix r_0 such that Γ has enough r-units for all $r \leq r_0$. Note that Γ_r is a PID for $r < r_0$, and Γ_{con} is a Henselian local ring with maximal ideal (p), residue field E and fraction field $\Gamma_{\mathrm{con}}[1/p]$ [Kedlaya 2005, Lemma 2.1.12]. We endow $\Gamma_r[1/p]$ with the Fréchet topology defined by the family of valuations $\{w_s\}_{0<s\leq r}$. Let $\Gamma_{\mathrm{an},r}$ be the completion of $\Gamma_r[1/p]$ with respect to the Fréchet topology and $\Gamma_{\mathrm{an,con}} := \varinjlim_{r\to 0} \Gamma_{\mathrm{an},r}$. We extend $v^{\leq n}$ and w_r to $v^{\leq n}, w_r : \Gamma_{\mathrm{an},r} \to \mathbb{R}$ and we endow $\Gamma_{\mathrm{an},r}$ (resp. $\Gamma_{\mathrm{an,con}}$) with the Fréchet topology defined by $\{w_s\}_{0<s\leq r}$ (resp. the inductive limit topology of Fréchet topologies). Note that $\varphi(\Gamma_r) \subset \Gamma_{r/p}$; hence, φ extends to a map $\varphi : \Gamma_{\mathrm{an},r} \to \Gamma_{\mathrm{an},r/p}$. In particular, Γ_{con} and $\Gamma_{\mathrm{an,con}}$ are canonically endowed with endomorphisms φ. Also, note that $\Gamma_{\mathrm{an},r}$ for all $r < r_0$ and hence, $\Gamma_{\mathrm{an,con}}$ are Bézout integral domains [Kedlaya 2005, Theorem 2.9.6].

In the rest of this subsection, we will see explicit descriptions of Γ_{con}, together with its finite étale extensions, by using rings of overconvergent power series ring.

Notation 1.6.2. Let $\mathcal{O}$ be a complete discrete valuation ring of mixed characteristic $(0, p)$. Let $\mathcal{O}\{\{S\}\}$ be the p-adic Hausdorff completion of $\mathcal{O}((S)) := \mathcal{O}[\![S]\!][S^{-1}]$. For $r \in \mathbb{Q}_{>0}$, we define the ring of overconvergent power series over $\mathcal{O}$ as

$$\mathcal{O}((S))^{\dagger,r} := \{f \in \mathcal{O}\{\{S\}\};\ f \text{ converges on } 0 < v_p(S) \leq r\},\ \mathcal{O}((S))^{\dagger} := \bigcup_{r>0} \mathcal{O}((S))^{\dagger,r}.$$

Recall that $\big(\mathcal{O}((S))^{\dagger}, (\pi_{\mathcal{O}})\big)$ is a Henselian discrete valuation ring [Matsuda 1995, Proposition 2.2]. We also define the Robba ring $\mathcal{R}$ associated to $\mathcal{O}((S))^{\dagger}$ by

$$\mathcal{R} := \Big\{ f = \sum_{n\in\mathbb{Z}} a_n S^n;\ a_n \in \mathrm{Frac}(\mathcal{O}),\ f \text{ converges on } 0 < v_p(S) \leq r \text{ for some } r > 0 \Big\}.$$

Construction 1.6.3. We construct a realization of a finite étale extension of $\mathcal{O}((S))^{\dagger}$ as an overconvergent power series ring. Let Γ be a Cohen ring of a complete discrete valuation field E of characteristic p. By fixing an isomorphism $f : \Gamma \cong \mathcal{O}\{\{S\}\}$, where $\mathcal{O}$ is a Cohen ring of k_E, we identify Γ and E with $\mathcal{O}\{\{S\}\}$ and $k_E((S))$. Let Γ'/Γ be a finite étale extension, with Γ' connected and F/E the corresponding residue field extension. Then, Γ' is again a Cohen ring of F. We identify F with $k_F((T))$ and fix a Cohen ring $\mathcal{O}'$ of k_F. We claim that there exists an isomorphism $f' : \Gamma' \cong \mathcal{O}'\{\{T\}\}$ such that f' modulo p is the identity, $f'(\mathcal{O}[\![S]\!]) \subset \mathcal{O}'[\![T]\!]$ and $f' : \mathcal{O}[\![S]\!] \to \mathcal{O}'[\![T]\!]$ is finite flat. We can write $S = T^{e_{F/E}}\bar{u}$ in $\mathcal{O}_F$ with some $\bar{u} \in \mathcal{O}_F^{\times}$. We fix a lift $u \in \mathcal{O}'[\![T]\!]^{\times}$ of $\bar{u}$ with respect to the projection $\mathcal{O}'[\![T]\!] \to \mathcal{O}_F$ and let $s' : \mathbb{Z}[S_0] \to \mathcal{O}'[\![T]\!];\ S_0 \mapsto T^{e_{F/E}}u$ be a ring homomorphism. Let $s : \mathbb{Z}[S_0] \to \mathcal{O}[\![S]\!]$ be the ring homomorphism sending S_0 to S. By the formal smoothness of s (see [Ohkubo 2013, §1A]), there exists a local ring homomorphism $\beta : \mathcal{O}[\![S]\!] \to \mathcal{O}'[\![T]\!]$:

$$\begin{tikzcd} \mathcal{O}[\![S]\!] \arrow[r] \arrow[drr, dashed, "\beta"] & \mathcal{O}_E \arrow[r] & \mathcal{O}_F \\ \mathbb{Z}[S_0] \arrow[u, "s"] \arrow[rr, "s'"] & & \mathcal{O}'[\![T]\!] \arrow[u] \end{tikzcd}$$

By the local criteria of flatness and Nakayama's lemma, β is finite flat. By the definition of s and s', β induces a map $\beta : \mathcal{O}((S)) \to \mathcal{O}'((T))$, and hence a map $\hat{\beta} : \mathcal{O}\{\{S\}\} \to \mathcal{O}'\{\{T\}\}$. Since $\hat{\beta}$ is finite étale with residue field extension F/E, there exists a canonical isomorphism $f' : \Gamma' \cong \mathcal{O}'\{\{T\}\}$, which satisfies the desired properties by the construction of β.

The relation $S = T^{e_{F/E}}u$ for $u \in \mathcal{O}'[\![T]\!]^{\times}$ gives $f'(\mathcal{O}((S))^{\dagger,r}) \subset \mathcal{O}'((T))^{\dagger,r/e_{F/E}}$. In the limit $r \to \infty$, we obtain a flat morphism $f' : \mathcal{O}((S))^{\dagger} \to \mathcal{O}'((T))^{\dagger}$. Finally, we prove the finiteness of $f' : \mathcal{O}((S))^{\dagger} \to \mathcal{O}'((T))^{\dagger}$. We fix a basis $\omega_1, \ldots, \omega_g$ of $\mathcal{O}'[\![T]\!]$ as an $\mathcal{O}[\![S]\!]$-module. Then, we only have to prove that $x \in \mathcal{O}'((T^{e_{F/E}}))^{\dagger,r}$

can be written as $\sum_i \omega_i \sum_n a_{i,n} S^n$ with $\sum_n a_{i,n} S^n \in \mathcal{O}((S))^{\dagger, r e_{F/E}}$. By the relation $Su^{-1} = T^{e_{F/E}}$ again, any element $x \in \mathcal{O}'((T^{e_{F/E}}))^{\dagger, r}$ can be written as $\sum_{n \in \mathbb{Z}} a_n S^n$ with $a_n \in \mathcal{O}'[\![T]\!]$ such that $|a_n||p|^{e_{F/E} nr} \to 0$ for $n \to -\infty$, where $|\cdot|$ is a norm of $\mathcal{O}'[\![T]\!]$ associated to the p-adic valuation. We write $a_n = \sum_i a_{n,i} \omega_i$. Then, we have $|a_n| = \sup_i |a_{n,i}|$, where $|\cdot|$ on the RHS is a norm of $\mathcal{O}[\![S]\!]$ associated to the p-adic valuation. Hence, $\sum_n a_{n,i} S^n$ belongs to $\mathcal{O}((S))^{\dagger, r e_{F/E}}$, which implies the assertion.

Lemma 1.6.4 [Kedlaya 2005, Lemma 2.3.5, Corollary 2.3.7]. *Let Γ be a Cohen ring of a complete discrete valuation field E of characteristic p and $\varphi : \Gamma \to \Gamma$ a Frobenius lift. By fixing an isomorphism $f : \Gamma \cong \mathcal{O}\{\{S\}\}$, we identify Γ and E with $\mathcal{O}\{\{S\}\}$ and $k_E((S))$. Assume that $\varphi(S) \in \mathcal{O}((S))^{\dagger}$. Then, we have*

$$\Gamma_r = \mathcal{O}((S))^{\dagger, r}, \quad \Gamma_{\mathrm{con}} = \mathcal{O}((S))^{\dagger}$$

for all sufficiently small $r > 0$.

Moreover, let F/E be a finite separable extension, Γ'/Γ the corresponding finite étale extension and $\varphi : \Gamma' \to \Gamma'$ the corresponding Frobenius lift extending φ. We fix an isomorphism $f' : \Gamma' \cong \mathcal{O}'\{\{T\}\}$ as in Construction 1.6.3. Then, f' induces isomorphisms

$$\Gamma'_r \cong \mathcal{O}'((T))^{\dagger, r/e_{F/E}}, \quad \Gamma'_{\mathrm{con}} \cong \mathcal{O}'((T))^{\dagger}$$

for all sufficiently small $r > 0$.

Proof. Let φ be the Frobenius lift of $\mathcal{O}'\{\{T\}\}$ obtained by identifying $\mathcal{O}'\{\{T\}\}$ with Γ'. We only have to check that the assumption $\varphi(T) \in \mathcal{O}'((T))^{\dagger}$ in [Kedlaya 2005, Convention 2.3.1] is satisfied. This follows from the fact that $\mathcal{O}'((T))^{\dagger}$ is integrally closed in $\mathcal{O}'\{\{T\}\}$, which in turn is a consequence of Raynaud's criteria of integral closedness for Henselian pairs [Raynaud 1970, Théorème 3(b), Chapitre XI]. □

Finally, we define (pure) φ-modules over overconvergent rings.

Definition 1.6.5 [Kedlaya 2005, Definition 4.6.1]. Let R be $\Gamma[1/p]$, $\Gamma_{\mathrm{con}}[1/p]$, or $\Gamma_{\mathrm{an,con}}$ (Construction 1.6.1) and let $\sigma := \varphi^h$ for some $h \in \mathbb{N}_{>0}$. A σ-module over R is a finite free R-module M endowed with a semilinear σ-action such that $1 \otimes \sigma : M \otimes_{R,\sigma} R \to M$ is an isomorphism. Assume that E is algebraically closed. Then, any σ-module over $\Gamma[1/p]$ or $\Gamma_{\mathrm{an,con}}$ admits a Dieudonné–Manin decomposition [Kedlaya 2005, Theorem 4.5.7] and we define the slope multiset of M as the multiset of the p-adic valuations of the "eigenvalues". For a σ-module M over $\Gamma_{\mathrm{con}}[1/p]$, we define the slope multiset of M as the slope multiset of $\Gamma \otimes_{\Gamma_{\mathrm{con}}[1/p]} M$, which coincides with that of $\Gamma_{\mathrm{an,con}} \otimes_{\Gamma_{\mathrm{con}}[1/p]} M$. For a general E, we define the slope multiset after the base change $\Gamma \to W(\widehat{E}^{\mathrm{alg}})$. A σ-module over R is pure of slope s if the slope multiset consists of only s. If M is a σ-module that is pure of slope 0, then we call M étale.

Let φ be a Frobenius lift of $\Gamma := \mathcal{O}\{\{S\}\}$ with $\varphi(S) \subset \mathcal{O}((S))^\dagger$. By Lemma 1.6.4, we can view $\mathcal{O}((S))^\dagger[1/p]$ and $\mathcal{R}$ in Notation 1.6.2 as $\Gamma_{\mathrm{con}}[1/p]$ and $\Gamma_{\mathrm{an,con}}$, and we can give similar definitions for $R = \mathcal{O}((S))^\dagger[1/p]$ and $\mathcal{R}$.

When R is one of the above rings, we denote the category of σ-modules (resp. étale σ-modules, σ-modules of pure slope s) over R by $\mathrm{Mod}_R(\sigma)$ (resp. $\mathrm{Mod}_R^{\mathrm{et}}(\sigma)$, $\mathrm{Mod}_R^s(\sigma)$).

1.7. ***Differential Swan conductor.*** The aim of this subsection is to recall the definition of the differential Swan conductor. The following coordinate free definition of the continuous Kähler differentials for overconvergent rings will be useful.

Definition 1.7.1. Let Γ be an absolutely unramified complete discrete valuation ring of mixed characteristic $(0, p)$. For a subring R of Γ, we define Ω^1_R as the R-submodule of $\widehat{\Omega}^1_\Gamma$ generated by the image of R under $d : \Gamma \to \widehat{\Omega}^1_\Gamma$.

Lemma 1.7.2. *Let $\Gamma := \mathcal{O}\{\{S\}\}$ and $\Gamma^\dagger := \mathcal{O}((S))^\dagger$, where $\mathcal{O}$ is a Cohen ring of a field k of characteristic p. Assume that $[k : k^p] = p^d < \infty$. Then, $\Omega^1_{\Gamma^\dagger}$ is the unique $\Gamma^\dagger$-submodule $\mathcal{M}$ of $\widehat{\Omega}^1_\Gamma$ such that*

(i) *$\mathcal{M}$ is of finite type over $\Gamma^\dagger$.*

(ii) *The image of $\Gamma^\dagger$ under $d : \Gamma \to \widehat{\Omega}^1_\Gamma$ is contained in $\mathcal{M}$.*

(iii) *The canonical map $\Gamma \otimes_{\Gamma^\dagger} \mathcal{M} \to \widehat{\Omega}^1_\Gamma$ is an isomorphism.*

Also, if $\varphi : \Gamma \to \Gamma$ is a Frobenius lift $\varphi(\Gamma^\dagger) \subset \Gamma^\dagger$, $\Omega^1_{\Gamma^\dagger}$ is stable under $\varphi : \widehat{\Omega}^1_\Gamma \to \widehat{\Omega}^1_\Gamma$.

We omit the proof since it is elementary. Note that if $\{t_j\} \subset \mathcal{O}$ is a lift of a p-basis of k, then $\Omega^1_{\Gamma^\dagger}$ is a free of rank $d+1$ with basis $dS, dt_1, \ldots, dt_d$.

Corollary 1.7.3. *With the notation as in Lemma 1.6.4, the canonical isomorphism $\Gamma' \otimes_\Gamma \widehat{\Omega}^1_\Gamma \cong \widehat{\Omega}^1_{\Gamma'}$ descends to a canonical isomorphism $\Gamma'_{\mathrm{con}} \otimes_{\Gamma_{\mathrm{con}}} \Omega^1_{\Gamma_{\mathrm{con}}} \cong \Omega^1_{\Gamma'_{\mathrm{con}}}$.*

Notation 1.7.4. In the rest of this section, let the notation be as in Lemma 1.7.2. We fix a Frobenius lift $\varphi : \Gamma \to \Gamma$ satisfying $\varphi(\Gamma^\dagger) \subset \Gamma^\dagger$. Let $\mathcal{R}$ be the Robba ring associated to $\Gamma^\dagger$ and assume that $\varphi(\mathcal{R}) \subset \mathcal{R}$. We put $\Omega^1_{\mathcal{R}} := \mathcal{R} \otimes_{\Gamma^\dagger} \Omega^1_{\Gamma^\dagger}$. Note that the canonical derivation $d : \Gamma^\dagger \to \Omega^1_{\Gamma^\dagger}$ extends to $d : \mathcal{R} \to \Omega^1_{\mathcal{R}}$.

Definition 1.7.5. A ∇-module M over $\mathcal{R}$ is a finite free module over $\mathcal{R}$ together with a connection $\nabla = \nabla_M : M \to M \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}$ such that the composition of ∇_M with the map $M \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}} \to M \otimes_{\mathcal{R}} \bigwedge^2_{\mathcal{R}} \Omega^1_{\mathcal{R}}$ induced by ∇ is the zero map. For $h \in \mathbb{N}_{>0}$, a (φ^h, ∇)-module M over $\mathcal{R}$ is a φ^h-module over $\mathcal{R}$ endowed with a ∇-module structure commuting with the action of φ^h. We call a (φ^h, ∇)-module pure (resp. étale) if the underlying φ^h-module is pure (resp. étale). Similarly, we define étale or pure (φ^h, ∇)-modules over $\Gamma^\dagger$ and Γ. Denote by $\mathrm{Mod}^s_R(\varphi^h, \nabla)$ the category of pure (φ^h, ∇)-modules over R, where R is either Γ, $\Gamma^\dagger[1/p]$ or $\mathcal{R}$.

Theorem 1.7.6 [Kedlaya 2007, Theorem 3.4.6]. *For a* (φ, ∇)*-module* M *over* $\mathcal{R}$, *there exists a canonical slope filtration*

$$0 = \mathrm{Fil}^0(M) \subset \cdots \subset \mathrm{Fil}^l(M) = M,$$

whose graded pieces are (φ, ∇)*-modules of pure slope* $s_1 < \cdots < s_l$.

Construction 1.7.7 [Kedlaya 2007, Definition 3.3.4]. Let $F/\mathbb{Q}_p$ be a finite unramified extension and $V \in \mathrm{Rep}_F^{f.g.}(G_E)$. Let $\Gamma^{\dagger,\mathrm{ur}}$ be the maximal unramified extension of $\Gamma^\dagger$. We put $\Omega^1_{\Gamma^{\dagger,\mathrm{ur}}} := \varinjlim \Omega^1_{\Gamma_1^\dagger}$, where the limit runs all the finite étale extensions $\Gamma_1^\dagger/\Gamma^\dagger$ with $\Gamma_1^\dagger$ connected. We consider the connection

$$\begin{aligned} \nabla : \Gamma^{\dagger,\mathrm{ur}} \otimes_{\mathcal{O}_F} V &\to \Omega^1_{\Gamma^{\dagger,\mathrm{ur}}} \otimes_{\mathcal{O}_F} V \\ \lambda \otimes y &\mapsto d\lambda \otimes y. \end{aligned} \tag{*}$$

Since $\Omega^1_{\Gamma^{\dagger,\mathrm{ur}}} \cong \Gamma^{\dagger,\mathrm{ur}} \otimes_{\Gamma^\dagger} \Omega^1_{\Gamma^\dagger}$ as G_E-modules by Corollary 1.7.3, we obtain a connection

$$\nabla : D^\dagger(V) \to \Omega^1_{\Gamma^\dagger} \otimes_{\Gamma^\dagger} D^\dagger(V),$$

where $D^\dagger(V) := (\Gamma^{\dagger,\mathrm{ur}} \otimes_{\mathcal{O}_F} V)^{G_E}$ is a finite dimensional $\Gamma^\dagger[1/p]$-module of rank $\dim_F V$, by taking G_E-invariants of (*). Thus, we obtain a rank preserving functor

$$D^\dagger : \mathrm{Rep}_F^{f.g.}(G_E) \to \mathrm{Mod}_{\Gamma^\dagger[1/p]}(\nabla).$$

By extending scalars, we also obtain a rank preserving functor

$$D^\dagger_{\mathrm{rig}} : \mathrm{Rep}_F^{f.g.}(G_E) \to \mathrm{Mod}_{\mathcal{R}}(\nabla).$$

Note that if V is endowed with a semilinear action of φ^h for $h \in \mathbb{N}$, then $D^\dagger(V)$ and $D^\dagger_{\mathrm{rig}}(V)$ are also endowed with semilinear φ^h-actions.

Definition 1.7.8. For a ∇-module M over $\mathcal{R}$, let $\mathrm{Swan}^\nabla(M)$ be the differential Swan conductor of M as in [Kedlaya 2007, Definition 2.8.1].

Recall that the differential Swan conductor is defined in terms of the behavior of the logarithmic radius of convergence [Xiao 2010, Definition 2.3.20], which depends only on the Jordan–Hölder factors of a given ∇-module by definition. In particular, we have:

Lemma 1.7.9 (The additivity of the differential Swan conductor). *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* ∇*-modules over* $\mathcal{R}$. *Then, we have* $\mathrm{Swan}^\nabla(M) = \mathrm{Swan}^\nabla(M') + \mathrm{Swan}^\nabla(M'')$.

The following is Xiao's Hasse–Arf Theorem in the characteristic p case.

Theorem 1.7.10 [Xiao 2010, Theorem 4.4.1, Corollary 4.4.3]. *Let* V *be an* F*-representation of* G_E *of finite local monodromy. Then, we have*

$$\mathrm{Swan}^{\mathrm{AS}}(V) = \mathrm{Swan}^\nabla(D^\dagger_{\mathrm{rig}}(V)).$$

Moreover, these invariants are nonnegative integers.

1.8. *Scholl's fields of norms.* In this subsection, we recall some results of [Scholl 2006, §1.3], which are a generalization of Fontaine–Wintenberger's fields of norms. Throughout this subsection, let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with $[k_K : k_K^p] = p^d < \infty$.

Definition 1.8.1. Let $K_1 \subset K_2 \subset \dots$ be finite extensions of K and put $K_\infty = \bigcup K_n$. We say that a tower $\mathfrak{K} := \{K_n\}_{n>0}$ is strictly deeply ramified if there exists $n_0 > 0$ and an element $\xi \in \mathcal{O}_{K_{n_0}}$ such that $0 < v_p(\xi) \leq 1$, and such that the following condition holds: for every $n \geq n_0$, the extension K_{n+1}/K_n has degree p^{d+1}, and there exists a surjection $\Omega^1_{\mathcal{O}_{K_{n+1}}/\mathcal{O}_{K_n}} \to (\mathcal{O}_{K_{n+1}}/\xi\mathcal{O}_{K_{n+1}})^{d+1}$ of $\mathcal{O}_{K_{n+1}}$-modules.

Let $\mathfrak{K} = \{K_n\}_{n>0}$ be a strictly deeply ramified tower. For $n \geq n_0$, we have $e_{K_{n+1}/K_n} = p$ and $k_{K_{n+1}} = k_K^{1/p}$, and the Frobenius $\mathcal{O}_{K_{n+1}}/\xi\mathcal{O}_{K_{n+1}} \to \mathcal{O}_{K_{n+1}}/\xi\mathcal{O}_{K_{n+1}}$ induces a surjection $f_n : \mathcal{O}_{K_{n+1}}/\xi\mathcal{O}_{K_{n+1}} \to \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n}$. We also choose a uniformizer π_{K_n} of K_n with $\pi_{K_{n+1}}^p \equiv \pi_{K_n} \mod \xi\mathcal{O}_{K_n}$. Then, we define $X^+ := X^+(\mathfrak{K}, \xi, n_0) := \varprojlim_{n\geq n_0} \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n}$, with transition maps $\{f_n\}$. Let $\mathrm{pr}_n : X^+ \to \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n}$ be the n-th projection for $n \geq n_0$. We put $\Pi := (\pi_{K_n} \mod \xi\mathcal{O}_{K_n}) \in X^+$. Let $k_{\mathfrak{K}} := \varprojlim_{n\geq n_0} k_{K_n}$ where the transition maps are induced by f_n's. Since $k_{K_{n+1}} = k_{K_n}^{1/p}$, the projections $\mathrm{pr}_n : k_{\mathfrak{K}} \to k_{K_n}$ are isomorphisms for all $n \geq n_0$. Moreover, X^+ is a complete discrete valuation ring of characteristic p, with uniformizer Π and residue field $k_{\mathfrak{K}}$. The construction does not depend on ξ or n_0, and X^+ is invariant after changing $\{K_n\}_n$ by $\{K_{n+m}\}_n$ for some m. Hence, we may denote $X^+(\mathfrak{K}, \xi, n_0)$ by $X^+_{\mathfrak{K}}$ and denote the fractional field of $X^+_{\mathfrak{K}}$ by $X_{\mathfrak{K}}$. Note that if K_n/K is Galois for all n, then $X^+_{\mathfrak{K}}$ and $X_{\mathfrak{K}}$ are canonically endowed with $G_{K_\infty/K}$-actions by construction.

Example 1.8.2 (Kummer tower case). Let $K = \widetilde{K}$ and $\{L_n\}$ be as in Section 1.3. Then, $\{L_n\}$ is strictly deeply ramified [Ohkubo 2010, Example 6.2].

Let L_∞/K_∞ be a finite extension. We choose a finite extension L/K such that $L_\infty = LK_\infty$. Then, the tower $\mathfrak{L} := \{L_n := LK_n\}$ depends only on L_∞ up to shifting, and is also strictly deeply ramified with respect to any $\xi' \in K_{n_0}$ with $0 < v_p(\xi') < v_p(\xi)$ ([Scholl 2006, Theorem 1.3.3]). Note that if L_n/K is Galois for all n, then $X^+_{\mathfrak{L}}$ and $X_{\mathfrak{L}}$ are canonically endowed with $G_{L_\infty/K}$-actions by construction.

Theorem 1.8.3 [Scholl 2006, Theorem 1.3.4]. *Let the notation be as above. Denote the category of finite étale algebras over K_∞ (resp. $X_{\mathfrak{K}}$) by $\boldsymbol{F\acute{E}t}_{K_\infty}$ (resp. $\boldsymbol{F\acute{E}t}_{X_{\mathfrak{K}}}$). Then, the functor*

$$X_\bullet : \boldsymbol{F\acute{E}t}_{K_\infty} \to \boldsymbol{F\acute{E}t}_{X_{\mathfrak{K}}}$$
$$L_\infty \mapsto X_{\mathfrak{L}}$$

is an equivalence of Galois categories. In particular, the corresponding fundamental groups are isomorphic, i.e., $G_{K_\infty} \cong G_{X_{\mathfrak{K}}}$. Moreover, the sequences $\{[L_n : K_n]\}_n$, $\{e_{L_n/K_n}\}_n$ and $\{[k_{L_n} : k_{K_n}]\}_n$ are stationary for sufficiently large n. Their limits are equal to $[X_{\mathfrak{L}} : X_{\mathfrak{K}}]$, $e_{X_{\mathfrak{L}}/X_{\mathfrak{K}}}$ and $[k_{X_{\mathfrak{L}}} : k_{X_{\mathfrak{K}}}]$.

1.9. (φ, Γ_K)*-modules.* Throughout this subsection, let K be a complete discrete valuation field of mixed characteristic $(0, p)$. In this subsection, we will recall the theory of (φ, Γ_K)-modules in the Kummer tower case [Andreatta 2006]. To avoid complications, especially when verifying the assumption [Scholl 2006, (2.1.2)], we will assume the following to work under the settings of [Andreatta 2006; Andreatta and Brinon 2008; 2010].

Assumption 1.9.1 [Andreatta 2006, §1]. Let $\mathcal{V}$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field. Let R_0 be the p-adic Hausdorff completion of $\mathcal{V}[T_1, \dots, T_d][1/T_1 \dots T_d]$ and $\widetilde{R}$ a ring obtained from R_0 by iterating finitely many times the following operations:

(ét) The p-adic Hausdorff completion of an étale extension.

(loc) The p-adic Hausdorff completion of the localization with respect to a multiplicative system.

(comp) The Hausdorff completion with respect to an ideal containing p.

We assume that there exists a finite flat morphism $\widetilde{R} \to \mathcal{O}_K$, which sends T_j to t_j.

Note that $\widetilde{R}$ is an absolutely unramified complete discrete valuation ring. Denote $\widetilde{R}$ by $\mathcal{O}_{\widetilde{K}}$ and $\mathrm{Frac}(\widetilde{R})$ by $\widetilde{K}$. Let $L/\widetilde{K}$ be a finite extension. In the rest of this subsection, we will use the notation from Sections 1.3 and 1.4. We also apply the results of Section 1.8 to the Kummer tower $\{L_n\}_{n>0}$.

Notation 1.9.2 [Andreatta and Brinon 2008, §4.1]. We will denote

$$\mathbb{E}_L^+ := X_{\mathfrak{L}}^+, \quad \mathbb{E}_L := X_{\mathfrak{L}}.$$

For any nonzero $\xi \in p\mathcal{O}_{L_\infty}$, we put

$$\widetilde{\mathbb{E}}_L^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{L_\infty}/\xi\mathcal{O}_{L_\infty}, \quad \widetilde{\mathbb{E}}_L := \mathrm{Frac}(\widetilde{\mathbb{E}}_L^+),$$

where both rings are independent of the choice of ξ. We also put

$$\widetilde{\mathbb{A}}_L^+ := W(\widetilde{\mathbb{E}}_L^+), \quad \widetilde{\mathbb{A}}_L := W(\widetilde{\mathbb{E}}_L), \quad \widetilde{\mathbb{B}}_L := \widetilde{\mathbb{A}}_L[1/p].$$

By definition, we have $\mathbb{E}_L^+ \subset \widetilde{\mathbb{E}}_L^+$ and $\mathbb{E}_L \subset \widetilde{\mathbb{E}}_L$, and $\widetilde{\mathbb{E}}_L$ can be viewed as a closed subring of $\widetilde{\mathbb{E}}$. In particular, the rings $\widetilde{\mathbb{A}}_L^+$, $\widetilde{\mathbb{A}}_L$ and $\widetilde{\mathbb{B}}_L$ can be viewed as subrings of $\widetilde{\mathbb{A}}^+$, $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$. Note that the completion of an algebraic closure of $\mathbb{E}_L$ coincides with $\widetilde{\mathbb{E}}$. Moreover, $\widetilde{\mathbb{E}}$ is perfect and $(\widetilde{\mathbb{E}}_L, v_{\widetilde{\mathbb{E}}})$ is a perfect complete valuation field, whose integer ring is $\widetilde{\mathbb{E}}_L^+$. By using the $G_{\widetilde{K}}$-actions on $\widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{A}}$, we can write

$$\widetilde{\mathbb{E}}_L^+ = (\widetilde{\mathbb{E}}^+)^{H_L}, \quad \widetilde{\mathbb{E}}_L = \widetilde{\mathbb{E}}^{H_L}, \quad \widetilde{\mathbb{A}}_L = \widetilde{\mathbb{A}}^{H_L}, \quad \widetilde{\mathbb{B}}_L = \widetilde{\mathbb{B}}^{H_L},$$

see [Andreatta and Brinon 2008, Lemma 4.1].

Lemma 1.9.3 (a special case of [Andreatta and Brinon 2008, Proposition 4.42]). *We put $\mathbb{A}^+_{W(k_{\mathcal{V}})} := W(k_{\mathcal{V}})[\![\pi]\!] \subset \widetilde{\mathbb{A}}^+$, where $\pi = [\varepsilon]-1 \in \widetilde{\mathbb{A}}^+$. Let $L/\widetilde{K}$ be a finite extension. The weak topology of $\widetilde{\mathbb{A}}_L \cong \widetilde{\mathbb{E}}_L^{\mathbb{N}}$ is the product topology $\widetilde{\mathbb{E}}_L^{\mathbb{N}}$, where $\widetilde{\mathbb{E}}_L$ is endowed with the valuation topology. Then, there exists a unique subring $\mathbb{A}_L$ of $\widetilde{\mathbb{A}}_L$ such that*:

(i) *$\mathbb{A}_L$ is complete for the weak topology.*

(ii) *$p\widetilde{\mathbb{A}}_L \cap \mathbb{A}_L = p\mathbb{A}_L$.*

(iii) *One has an commutative diagram*

$$\begin{array}{ccc} \mathbb{A}_L & \twoheadrightarrow & \mathbb{E}_L \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{A}}_L & \twoheadrightarrow & \widetilde{\mathbb{E}}_L \end{array}$$

(iv) *$[\varepsilon], [\tilde{t}_j] \in \mathbb{A}_L$ for all j.*

(v) *There exists an $\mathbb{A}^+_{W(k)}$-subalgebra $\mathbb{A}^+_L$ of $\mathbb{A}_L$ and $r_L \in \mathbb{Q}_{>0}$ such that*:
   (a) *There exists $a \in \mathbb{N}$ such that $p/\pi^a \in \mathbb{A}^+_L$ and $\mathbb{A}^+_L/(p/\pi^a) \cong \mathbb{E}^+_L$.*
   (b) *If $\alpha, \beta \in \mathbb{N}_{>0}$ are such that $\alpha/\beta < pr_L/(p-1)$, then $\mathbb{A}^+_L \subset \widetilde{\mathbb{A}}^+_L\{p^\alpha/\pi^\beta\}$; here, $\widetilde{\mathbb{A}}^+_L\{p^\alpha/\pi^\beta\}$ denotes the p-adic Hausdorff completion of $\widetilde{\mathbb{A}}^+_L[p^\alpha/\pi^\beta]$.*
   (c) *$\mathbb{A}^+_L$ is complete for the weak topology.*

Moreover, by the uniqueness, $\mathbb{A}_L$ is stable under the actions of φ and $G_{L_\infty/K}$ if $L/\widetilde{K}$ is Galois.

Definition 1.9.4. Let $\mathbb{A}$ be the p-adic Hausdorff completion of $\bigcup_{L/\widetilde{K}} \mathbb{A}_L$, which is a subring of $\widetilde{\mathbb{A}}$ that is stable under the actions of both G_K and φ. We also put $\mathbb{B}_L := \mathbb{A}_L[1/p]$ and $\mathbb{B} := \mathbb{A}[1/p]$.

Remark 1.9.5. (i) As remarked in [Andreatta and Brinon 2008, §4.3], $\mathbb{A}_L$ is the unique finite étale $\mathbb{A}_{\widetilde{K}}$-algebra corresponding to $\mathbb{E}_L/\mathbb{E}_{\widetilde{K}}$; in particular, $\mathbb{A}_L$ is a Cohen ring of $\mathbb{E}_L$.

(ii) The action of $\Gamma_{\widetilde{K}}$ on $\mathbb{A}_{\widetilde{K}}$ is determined by the action of $\Gamma_{\widetilde{K}}$ on $\pi, [\tilde{t}_1], \dots, [\tilde{t}_d]$, since $\varepsilon - 1, \tilde{t}_1, \dots, \tilde{t}_d$ form a p-basis of $\mathbb{E}_{\widetilde{K}}$. Explicit descriptions are given by:

$$\gamma_a(\pi) = (1+\pi)^a - 1, \quad \gamma_a([\tilde{t}_j]) = [\tilde{t}_j] \text{ for } a \in \mathbb{Z}_p^\times,$$

$$\gamma_b(\pi) = \pi, \quad \gamma_b([\tilde{t}_j]) = (1+\pi)^{b_j}[\tilde{t}_j] \text{ for } b = (b_j) \in \mathbb{Z}_p^d.$$

Definition 1.9.6. For $h \in \mathbb{N}_{>0}$, an étale (φ^h, Γ_L)-module M over $\mathbb{B}_L$ is an étale φ^h-module over $\mathbb{B}_L$ endowed with a semilinear continuous G_K-action that commutes with the action of φ^h. Denote by $\mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}_L}(\varphi^h, \Gamma_L)$ the category of étale (φ^h, Γ_L)-modules over $\mathbb{B}_L$.

For $V \in \mathrm{Rep}_{\mathbb{Q}_{p^h}}(G_L)$, let $\mathbb{D}(V) := (\mathbb{B} \otimes_{\mathbb{Q}_{p^h}} V)^{H_L}$. For $M \in \mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}_L}(\varphi^h, \Gamma_L)$, let $\mathbb{V}(M) := (\mathbb{B} \otimes_{\mathbb{B}_K} M)^{\varphi^h=1}$.

Theorem 1.9.7 ([Andreatta 2006, Theorem 7.11] or [Andreatta and Brinon 2008, Théorème 4.34]). *Let $h \in \mathbb{N}_{>0}$. Then, the functor $\mathbb{D}$ gives a rank preserving equivalence of categories*

$$\mathbb{D} : \mathrm{Rep}_{\mathbb{Q}_{p^h}}(G_L) \to \mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}_L}(\varphi^h, \Gamma_L)$$

with quasi-inverse $\mathbb{V}$.

1.10. *Overconvergence of p-adic representations.* In this subsection, we will recall the overconvergence of p-adic representations in [Andreatta and Brinon 2008]. We still keep the notations of Section 1.9 and Assumption 1.9.1.

Definition 1.10.1. We apply Construction 1.6.1 to $(\widetilde{\mathbb{E}}, v_{\widetilde{\mathbb{E}}})$ with $\Gamma = \widetilde{\mathbb{A}}$ and write

$$\widetilde{\mathbb{A}}^{\dagger,r} := \Gamma_r,\ \widetilde{\mathbb{A}}^{\dagger} := \Gamma_{\mathrm{con}},\quad \widetilde{\mathbb{B}}^{\dagger,r} := \Gamma_r[1/p],\quad \widetilde{\mathbb{B}}^{\dagger} := \Gamma_{\mathrm{con}}[1/p],$$

$$\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} := \Gamma_{\mathrm{an},r},\quad \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} := \Gamma_{\mathrm{an,con}}.$$

We define $v^{\leq n}_{\widetilde{\mathbb{E}}}$ and w_r the same way. For a finite extension $L/\widetilde{K}$, we apply a similar construction to the following $(E, v_{\widetilde{\mathbb{E}}})$ with Γ and we denote:

Γ	E	Γ_r	Γ_{con}	$\Gamma_r[1/p]$	$\Gamma_{\mathrm{con}}[1/p]$	$\Gamma_{\mathrm{an},r}$	$\Gamma_{\mathrm{an,con}}$
$\mathbb{A}$	$\mathbb{E}$	$\mathbb{A}^{\dagger,r}$	$\mathbb{A}^{\dagger}$	$\mathbb{B}^{\dagger,r}$	$\mathbb{B}^{\dagger}$	$\mathbb{B}^{\dagger,r}_{\mathrm{rig}}$	$\mathbb{B}^{\dagger}_{\mathrm{rig}}$
$\widetilde{\mathbb{A}}_L$	$\widetilde{\mathbb{E}}_L$	$\widetilde{\mathbb{A}}^{\dagger,r}_L$	$\widetilde{\mathbb{A}}^{\dagger}_L$	$\widetilde{\mathbb{B}}^{\dagger,r}_L$	$\widetilde{\mathbb{B}}^{\dagger}_L$	$\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},L}$	$\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},L}$
$\mathbb{A}_L$	$\mathbb{E}_L$	$\mathbb{A}^{\dagger,r}_L$	$\mathbb{A}^{\dagger}_L$	$\mathbb{B}^{\dagger,r}_L$	$\mathbb{B}^{\dagger}_L$	$\mathbb{B}^{\dagger,r}_{\mathrm{rig},L}$	$\mathbb{B}^{\dagger}_{\mathrm{rig},L}$

By construction, we have $\widetilde{\mathbb{B}}^{\dagger} = \bigcup_r \widetilde{\mathbb{B}}^{\dagger,r}$, $\mathbb{B}^{\dagger} = \bigcup_r \mathbb{B}^{\dagger,r}$, $\widetilde{\mathbb{B}}^{\dagger,r}_K = \widetilde{\mathbb{B}}_K \cap \widetilde{\mathbb{B}}^{\dagger,r}$, $\widetilde{\mathbb{B}}^{\dagger}_K = \bigcup_r \widetilde{\mathbb{B}}^{\dagger,r}_K$, $\mathbb{B}^{\dagger,r}_K = \mathbb{B}_K \cap \mathbb{B}^{\dagger,r}$ and $\mathbb{B}^{\dagger}_K = \bigcup_r \mathbb{B}^{\dagger,r}_K$. We endow $\widetilde{\mathbb{B}}^{\dagger,r}$, $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$, ..., etc. with the Fréchet topologies defined by $\{w_s\}_{0<s\leq r}$.

We can describe $\mathbb{A}^{\dagger}_L$ by the ring of overconvergent power series.

Lemma 1.10.2 (cf. [Berger 2002, Proposition 1.4]). *Let $\mathcal{O}$ be a Cohen ring of $k_{\mathbb{E}_{\widetilde{K}}}$. Then, there exists an isomorphism $\mathbb{A}_{\widetilde{K}} \cong \mathcal{O}\{\{\pi\}\}$, which induces an isomorphism $\mathbb{A}^{\dagger,r}_{\widetilde{K}} \cong \mathcal{O}((\pi))^{\dagger}$ for all sufficiently small $r > 0$. Similarly, there exists an isomorphism $\mathbb{A}_L \cong \mathcal{O}'\{\{\pi'\}\}$, which induces an isomorphism $\mathbb{A}^{\dagger,r}_L \cong \mathcal{O}'((\pi'))^{\dagger,r/e_{\mathbb{E}_L/\mathbb{E}_{\widetilde{K}}}}$, where $\mathcal{O}'$ is a Cohen ring of $k_{\mathbb{E}_L}$.*

Proof. Fix any isomorphism $\mathbb{A}_{\widetilde{K}} \cong \mathcal{O}\{\{\pi\}\}$ (Remark 1.9.5(i)). Since $\varphi(\pi) = [\varepsilon]^p - 1 = (1+\pi)^p - 1 \in \mathcal{O}\{\{\pi\}\}^{\dagger}$, the assertion follows from Lemma 1.6.4. $\square$

Notation 1.10.3. The isomorphism in Lemma 1.10.2 enables us to apply the results of Section 1.7. In particular, for any finite extension $L/\widetilde{K}$, we have a canonical continuous derivation

$$d : \mathbb{B}^{\dagger}_{\mathrm{rig},L} \to \Omega^1_{\mathbb{B}^{\dagger}_{\mathrm{rig},L}},$$

with $\Omega^1_{\mathbb{B}^{\dagger}_{\mathrm{rig},L}} := \mathbb{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbb{A}^{\dagger}_L} \Omega^1_{\mathbb{A}^{\dagger}_L}$ a free $\mathbb{B}^{\dagger}_{\mathrm{rig},L}$-module with basis $d\pi, d[\tilde{t}_1], \dots, d[\tilde{t}_d]$. Hence, we have a notion of (φ, ∇)-modules over $\mathbb{B}^{\dagger}_{\mathrm{rig},L}$ and the associated differential Swan conductors.

Definition 1.10.4. Let $h \in \mathbb{N}_{>0}$. An étale (φ^h, Γ_L)-module M over $\mathbb{B}^{\dagger}_L$ is an étale φ^h-module over $\mathbb{B}^{\dagger}_L$ endowed with a continuous semilinear G_K-action that commutes with the φ^h-action. Denote by $\mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}^{\dagger}_L}(\varphi^h, \Gamma_L)$ the category of étale (φ^h, Γ_L)-modules over $\mathbb{B}^{\dagger}_L$.

For $V \in \mathrm{Rep}_{\mathbb{Q}_{p^h}}(G_L)$, let

$$\mathbb{D}^{\dagger,r}(V) := (\mathbb{B}^{\dagger,r} \otimes_{\mathbb{Q}_{p^h}} V)^{H_L}, \qquad \mathbb{D}^{\dagger}(V) = \bigcup_r \mathbb{D}^{\dagger,r}(V),$$

$$\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V) := \mathbb{B}^{\dagger,r}_{\mathrm{rig},L} \otimes_{\mathbb{B}^{\dagger,r}_L} \mathbb{D}^{\dagger,r}(V), \quad \mathbb{D}^{\dagger}_{\mathrm{rig}}(V) = \bigcup_r \mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V).$$

For $M \in \mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}^{\dagger}_L}(\varphi^h, \Gamma_L)$, let $\mathbb{V}(M) := (\mathbb{B}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_L} M)^{\varphi^h=1}$.

Theorem 1.10.5 [Andreatta and Brinon 2008, Theorem 4.35]. *Let $h \in \mathbb{N}_{>0}$. The functor $\mathbb{D}^{\dagger}$ gives a rank preserving equivalence of categories*

$$\mathbb{D}^{\dagger} : \mathrm{Rep}_{\mathbb{Q}_{p^h}}(G_L) \to \mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}^{\dagger}_L}(\varphi^h, \Gamma_L)$$

with quasi-inverse $\mathbb{V}$. Moreover, $\mathbb{D}^{\dagger}$ and $\mathbb{V}$ are compatible with $\mathbb{D}$ and $\mathbb{V}$ in Theorem 1.9.7. Furthermore, $\mathbb{D}^{\dagger,r}(V)$ is free of rank $\dim_{\mathbb{Q}_{p^h}} V$ over $\mathbb{B}^{\dagger,r}_K$ for all sufficiently small r, and we have a canonical isomorphism $\mathbb{B}^{\dagger}_K \otimes_{\mathbb{B}^{\dagger,r}_K} \mathbb{D}^{\dagger,r}(V) \xrightarrow{\sim} \mathbb{D}^{\dagger}(V)$.

The functor $\mathbb{D}^{\dagger}_{\mathrm{rig}}$ will be studied in Section 4.5.

2. Adequateness of overconvergent rings

In this section, we will prove the "adequateness", which ensures that the elementary divisor theorem holds, for overconvergent rings defined in Section 1.6. The adequateness of overconvergent rings seems to be well-known to the experts: at least when the overconvergent ring is isomorphic the Robba ring, the adequateness follows from Lazard's results [1962] as in [Berger 2002, Proposition 4.12(5)]). Since the author could not find an appropriate reference, we give a proof.

Definition 2.0.1 [Helmer 1943, §2]. An integral domain R is adequate if the following hold:

(i) R is a Bézout ring, that is, any finitely generated ideal of R is principal.

(ii) For any $a, b \in R$ with $a \neq 0$, there exists a decomposition $a = a_1 a_2$ such that $(a_1, b) = R$ and $(a_3, b) \neq R$ for any nonunit factor a_3 of a_2.

Recall that if R is an adequate integral domain, then the elementary divisor theorem holds for free R-modules, see [Helmer 1943, Theorem 3]. Precisely speaking, let $N \subset M$ be finite free R-modules of ranks n and m respectively. Then, there exists a basis of $e_1, \dots, e_m$ (resp. $f_1, \dots, f_n$) of M (resp. N) and nonzero elements $\lambda_1 \mid \cdots \mid \lambda_n \in R$ such that $f_i = \lambda_i e_i$ for $1 \leq i \leq n$.

In the rest of this section, let the notation be as in Construction 1.6.1. We fix $r_0 > 0$ such that Γ has enough r_0-units and let $r \in (0, r_0)$ unless otherwise stated. Recall that $\Gamma_{\mathrm{an},r}$ is a Bézout integral domain.

Definition 2.0.2. We recall basic terminologies, see also [Kedlaya 2004, §3.5]. For $x \in \Gamma_{\mathrm{an},r}$ nonzero, we define the Newton polygon of x as the lower convex hull of the set of points $(v^{\leq n}(x), n)$, minus any segments of slope less than $-r$ on the left end and/or any segments of nonnegative slope on the right end of the polygon. We define the slopes of x as the negatives of the slopes of the Newton polygon of x. We also define the multiplicity of a slope $s \in (0, r]$ of x as the positive difference in y-coordinates between the endpoints of the segment of the Newton polygon of slope $-s$, or 0 if there is no such segment. If x has only one slope s, we say that x is pure of slope s.

A slope factorization of a nonzero element x of $\Gamma_{\mathrm{an},r}$ is a Fréchet-convergent product $x = \prod_{1 \leq i \leq n} x_i$ for n either a positive integer or ∞, where each x_i is pure of slope s_i with $s_1 > s_2 > \dots$ (cf. an explanation before [Kedlaya 2004, Lemma 3.26]).

Recall that the multiplicity is compatible with multiplication, i.e., the multiplicity of a slope s of xy is the sum of its multiplicities as a slope of x and of y [Kedlaya 2004, Corollary 3.22]. Also, recall that $x \in \Gamma_{\mathrm{an},r}$ is a unit if and only if x has no slopes [Kedlaya 2005, Corollary 2.5.12].

Lemma 2.0.3 [Kedlaya 2004, Lemma 3.26]. *Every nonzero element of $\Gamma_{\mathrm{an},r}$ has a slope factorization.*

For simplicity, we denote $\Gamma_{\mathrm{an},r}$ by R in the rest of this subsection. The lemma below is an immediate consequence of R being Bézout and the additivity of the multiplicity of a slope.

Lemma 2.0.4. (i) *Let $x, y \in R$ such that x is pure of slope s and let z be a generator of (x, y). Then, z is also pure of slope s, with multiplicity less than or equal to the multiplicity of slope s of x. In particular, if the multiplicity of the slope s of y is equal to zero, then z is a unit and we have $(x, y) = R$.*

(ii) *Let $x, y \in R$ such that x is pure of slope s. Then, the decreasing sequence of the ideals $\{(x, y^n)\}_{n \in \mathbb{N}}$ is eventually stationary.*

Lemma 2.0.5 (the uniqueness of slope factorizations). *Let $x \in R$ be a nonzero element. Let $x = \prod_i x_i = \prod_i x'_i$ be slope factorizations whose slopes are $s_1 > s_2 > \dots$ and $s'_1 > s'_2 > \dots$. Let m_i and m'_i be the multiplicities of s_i and s'_i for x_i and x'_i. Then, we have $s_i = s'_i$ and $x_i = x'_i u_i$ for some $u_i \in R^\times$. In particular, we have $m_i = m'_i$.*

Proof. We can easily reduce to the case $i = 1$. Since the multiplicity of the slope s_1 of $\prod_{i>1} x'_i$ is equal to zero, we have $(x_1, \prod_{i>1} x'_i) = R$ by Lemma 2.0.4(i). Hence, we have $(x_1, x) = (x_1, x_1 \prod_{i>1} x_i) = (x_1)$. By assumption, we have $s_1 \neq s'_j$ except for at most one j. Just as before, we have

$$(x_1, x) = \Big(x_1, x'_j \prod_{i\neq j} x'_i\Big) = (x_1, x'_j) = \Big(x_1 \prod_{i>1} x_i, x'_j\Big) = (x, x'_j) = \Big(x'_j \prod_{i\neq j} x'_i, x'_j\Big) = (x'_j),$$

i.e., $(x_1) = (x'_j)$. Hence, there exists $u \in R^\times$ such that $x_1 = x'_j u$. By the same argument, $x'_1 = x_l u'$ for some l and $u' \in R^\times$. Since $\{s_i\}$ and $\{s'_i\}$ are strictly decreasing, we must have $j = l = 1$, which implies the assertion. □

Lemma 2.0.6. *The integral domain $\Gamma_{\mathrm{an},r}$ is adequate. In particular, the elementary divisor theorem holds over $\Gamma_{\mathrm{an},r}$.*

Proof. We only have to prove that condition (ii) in Definition 2.0.1 is satisfied. Let $a, b \in R$ with $a \neq 0$. If $b = 0$, then it suffices to put $a_1 = 1$, $a_2 = a$. If b is a unit, then it suffices to put $a_1 = a$, $a_2 = 1$. Therefore, we may assume that b is neither a unit nor zero. Let $b = \prod_{i>0} b_i$ be a slope factorization with slopes $s_1 > s_2 > \dots$. By Lemma 2.0.4(ii), there exists $z_i \in R$ such that $(a, b_i^n) = (z_i)$ for all sufficiently large n. By [Kedlaya 2004, Proposition 3.13], we may assume that z_i admits a semi-unit decomposition, meaning that z_i is equal to a convergent sum of the form $1 + \sum_{j<0} u_{i,j} p^j$, where $u_{i,j} \in R^\times \cup \{0\}$. As in the proof of [Kedlaya 2004, Lemma 3.26], we can prove that $\{z_1 \dots z_i\}_{i>0}$ converges. Next, we claim that there exists $u_i \in R$ such that $a = z_1 \dots z_i u_i$. We proceed by induction on i. By definition, we have $a = z_1 u_1$ for some u_1. Assume that we have defined u_i. Since the multiplicity of the slope s_{i+1} of z_j is equal to zero for $1 \leq j \leq i$, we have $(z_j, z_{i+1}) = R$ for $1 \leq j \leq i$. Hence, we have $(z_{i+1}) = (a, z_{i+1}) = (z_1 \dots z_i u_i, z_{i+1}) = (u_i, z_{i+1})$, which implies $z_{i+1} \mid u_i$. Therefore, $u_{i+1} := u_i / z_{i+1}$ satisfies the condition. By this proof, we can choose $u_i = u_1/(z_1 \dots z_i)$. We put $a_1 := \lim_{i\to\infty} u_i = u_1 / \prod_{i>1} z_i$ and $a_2 := \prod_{i>0} z_i$, which is a slope factorization of a_2. We prove that the factorization $a = a_1 a_2$ satisfies the condition. We first prove $(a_1, b) = R$. By the uniqueness of slope factorizations, we only have to prove $(a_1, b_i) = R$ for all i. Fix $i \in \mathbb{N}_{>0}$. Then, for all sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned}(z_i) = (a, b_i^n) = (a, b_i^{n+1}) = (a_1 a_2, b_i^{n+1}) &\subset (a_1, b_i)(a_2, b_i^n)\\ &\subset (a_1, b_i)(z_i, b_i^n) = (a_1, b_i)(z_i).\end{aligned}$$

Since $z_i \neq 0$, we have $R \subset (a_1, b_i)$, which implies the assertion. Finally, we prove $(a_3, b) \neq R$ for any nonunit $a_3 \in R$ dividing a_2. By replacing a_3 by any factor of a slope factorization of a_3, we may assume that a_3 is pure. By the uniqueness of slope factorizations, a_3 divides z_i for some i. Since $z_i \mid b_i^n$ for sufficiently large n, we also have $a_3 \mid b_i^n$. Hence, we have $(a_3, b_i) \neq R$, and in particular, $(a_3, b) \neq R$. $\square$

3. Variations of Gröbner basis argument

In this section, we will systematically develop a basic theory of Gröbner bases over various rings. Our theory generalizes the basic theory of Gröbner bases over fields ([Cox et al. 1997], particularly, §2). As a first application, we will prove the continuity of connected components of flat families of rigid analytic spaces over annuli (Proposition 3.4.5(iii)). As a second application, we prove the ramification compatibility of Scholl's fields of norms (Theorem 3.5.3).

The idea to use a Gröbner basis argument to study Abbes–Saito's rigid spaces of positive characteristic is in [Xiao 2010, §1]. Some results of this section, particularly Sections 3.2 and 3.3, are already proved there, however we do not use Xiao's results. We will work under a slightly stronger assumption and deduce stronger results, with much clearer and simpler proofs, than Xiao's. Note that this section is independent from the other parts of this paper, except Sections 1.5 and 1.8.

Notation 3.0.1. Throughout this section, we will use multi-index notation. We write $\underline{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$, $|\underline{n}| := n_1 + \dots + n_l$ and $\underline{X}^{\underline{n}} = X_1^{n_1} \dots X_l^{n_l}$ for variables $\underline{X} = (X_1, \dots, X_l)$. We also denote by $\underline{X}^{\mathbb{N}}$ the set of monic monomials $\{\underline{X}^{\underline{n}} \mid \underline{n} \in \mathbb{N}^l\}$.

In this section, when we consider a topology on a ring, we will use a norm $|\cdot|$ rather than a valuation.

3.1. *Convergent power series.*

In this subsection, we consider rings of strictly convergent power series over the ring of rigid analytic functions over annuli, which play an analogous role to Tate algebra in the classical situation. We also gather basic definitions and facts on these rings for the rest of this section.

Definition 3.1.1. Let R be a ring. For $f = \sum_{\underline{n}} a_{\underline{n}} \underline{X}^{\underline{n}} \in R[\![\underline{X}]\!]$ with $a_{\underline{n}} \in R$, we call each $a_{\underline{n}} \underline{X}^{\underline{n}}$ a term of f. If $f = a_{\underline{n}} \underline{X}^{\underline{n}}$ with $a_{\underline{n}} \in R$, then we call f a monomial. If $a_{\underline{n}} = 1$, then f is called monic.

Definition 3.1.2 [Bosch et al. 1984, Section 1.4.1, Definition 1]. Let $(R, |\cdot|)$ be a normed ring. We define a Gauss norm on $R[\underline{X}]$ by $|\sum_{\underline{n}} a_{\underline{n}} X^{\underline{n}}| := \sup_{\underline{n}} |a_{\underline{n}}|$. A formal power series $\sum_{\underline{n}} a_{\underline{n}} X^{\underline{n}} \in R[\![\underline{X}]\!]$ is strictly convergent if $|a_{\underline{n}}| \to 0$ as $|\underline{n}| \to \infty$. We denote the ring of strictly convergent power series over R by $R\langle \underline{X} \rangle$. The above norm $|\cdot|$ can be uniquely extended to $|\cdot| : R\langle \underline{X} \rangle \to \mathbb{R}_{\geq 0}$. Note that if R is complete with respect to $|\cdot|$, then $R\langle \underline{X} \rangle$ is also complete with respect to $|\cdot|$, see [Bosch et al. 1984, Section 1.4.1, Proposition 3].

We recall basic facts on rings of strictly convergent power series. Let R be a complete normed ring, whose topology is equivalent to the $\mathfrak{a}$-adic topology for an ideal $\mathfrak{a}$. Then, $R\langle \underline{X}\rangle$ is canonically identified with the $\mathfrak{a}$-adic Hausdorff completion of $R[\underline{X}]$. We further assume that R is Noetherian. Then, $R\langle \underline{X}\rangle$ is R-flat. Moreover, for any ideal $\mathfrak{b}$ of R, we have a canonical isomorphism

$$R\langle \underline{X}\rangle \otimes_R (R/\mathfrak{b}) \cong (R/\mathfrak{b})\langle \underline{X}\rangle,$$

where the RHS means the $\mathfrak{a}$-adic Hausdorff completion of $(R/\mathfrak{b})[\underline{X}]$.

For a complete discrete valuation ring $\mathcal{O}$ with $F = \mathrm{Frac}(\mathcal{O})$, we denote by $\mathcal{O}\langle \underline{X}\rangle$ (resp. $F\langle \underline{X}\rangle$) the rings of convergent power series over $\mathcal{O}$ (resp. F).

Lemma 3.1.3. *Assume that R is a complete normed Noetherian ring, whose topology is equivalent to the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a}$ of R. Let $I \subset R\langle \underline{X}\rangle$ be an ideal such that $R\langle \underline{X}\rangle/I$ is R-flat. Then, I is also R-flat. Moreover, for any ideal $J \subset R$, we have $I \cap J \cdot R\langle \underline{X}\rangle = JI$. In particular, if $f \in I$ is divisible by $s \in R$ in $R\langle \underline{X}\rangle$, then $f/s \in I$.*

We omit the proof since it is an easy exercise in flatness.

Notation 3.1.4. In the rest of this subsection, we fix the notation as follows. Let $\mathcal{O}$ be a Cohen ring of a field k of characteristic p and fix a norm $|\cdot|$ on $\mathcal{O}$ corresponding to the p-adic valuation. We put

$$R^+ := \mathcal{O}[\![S]\!] \subset R := \mathcal{O}(\!(S)\!)$$

and for $r \in \mathbb{Q}_{>0}$, we define a norm

$$\begin{aligned} |\cdot|_r : R &\to \mathbb{R}_{\geq 0} \\ \sum_{n \gg -\infty} a_n S^n &\mapsto \sup_n |a_n||p|^{rn}, \end{aligned}$$

which is multiplicative by [Kedlaya 2010, Proposition 2.1.2]. Recall the definition

$$R^{\dagger,r} = \left\{ \sum_{n\in\mathbb{Z}} a_n S^n \in \mathcal{O}\{\{S\}\};\ |a_n S^n|_r \to 0 \text{ as } n \to -\infty \right\}$$

from Notation 1.6.2. Note that we may canonically identify $R^{\dagger,r}/pR^{\dagger,r}$ with $k(\!(S)\!)$. We can extend $|\cdot|_r$ to $|\cdot|_r : R^{\dagger,r} \to \mathbb{R}_{\geq 0}$ by $|\sum_n a_n S^n|_r := \sup_n |a_n S^n|_r$. We define subrings of $R^{\dagger,r}$ by

$$R_0^{\dagger,r} := \{f \in R^{\dagger,r};\ |f|_r \leq 1\},$$

$$\mathcal{R}_0^{\dagger,r} := R_0^{\dagger,r} \cap R = \{f \in R;\ |f|_r \leq 1\}.$$

Note that for $a, b \in \mathbb{N}$ with $b > 0$, $|p^a/S^b|_r \leq 1$ if and only if $a/b \geq r$. Also, note that $R^{\dagger,r} = R_0^{\dagger,r}[S^{-1}]$ since $|S|_r < 1$. We may regard $R^{\dagger,r}$ as the ring of rigid analytic functions on the annulus $[p^r, 1)$ whose values at the boundary $|S| = 1$ are bounded by 1.

Lemma 3.1.5. (i) *The R^+-algebra $\mathcal{R}_0^{\dagger,r}$ is finitely generated.*

(ii) *The topologies of $R_0^{\dagger,r}$ defined by $|\cdot|_r$ and by the ideal (p, S) are equivalent.*

(iii) *The rings $R_0^{\dagger,r}$ and $R^{\dagger,r}$ are complete with respect to $|\cdot|_r$, and $\mathcal{R}_0^{\dagger,r}$ is dense in $R_0^{\dagger,r}$.*

(iv) *The rings $\mathcal{R}_0^{\dagger,r}$, $R_0^{\dagger,r}$, and $R^{\dagger,r}$ are Noetherian integral domains.*

Proof. Let $a, b \in \mathbb{N}_{>0}$ denote relatively prime integers such that $r = a/b$.

(i) It is straightforward to check that $\mathcal{R}_0^{\dagger,r}$ is generated as an R^+-algebra by $p^{\lfloor rb' \rfloor}/S^{b'}$ for $b' \in \{0, \dots, b\}$.

(ii) For $n \in \mathbb{N}$, we have

$$\sup\left\{|x|_r;\ x \in (p, S)^n R_0^{\dagger,r}\right\} \le \left\{\inf(|p|, |S|_r)\right\}^n$$

and the RHS converges to 0 as $n \to \infty$. Hence, the (p, S)-adic topology of $R_0^{\dagger,r}$ is finer than the topology defined by $|\cdot|_r$. To prove that the topology of $R_0^{\dagger,r}$ defined by $|\cdot|_r$ is finer than the (p, S)-adic topology, it suffices to prove that

$$\left\{x \in R_0^{\dagger,r};\ |x|_r \le |(pS)^n|_r\right\} \subset (p, S)^n R_0^{\dagger,r}$$

for all $n \in \mathbb{N}$. Let $x = \sum_{m\in\mathbb{Z}} a_m S^m \in \text{LHS}$ with $a_m \in \mathcal{O}$. Then, we have $|a_m S^{m-n}|_r \le |p^n| \le 1$. Hence, $x = S^n \sum_{m\in\mathbb{Z}} a_m S^{m-n} \in S^n \cdot R_0^{\dagger,r}$, which implies the assertion.

(iii) If $f = \sum_{n\in\mathbb{Z}} a_n S^n \in R_0^{\dagger,r}$ with $a_n \in \mathcal{O}$, then $\left\{\sum_{n\ge -m} a_n S^n\right\}_{m\in\mathbb{N}} \subset \mathcal{R}_0^{\dagger,r}$ converges to f, which implies the last assertion. Since $R_0^{\dagger,r}$ is an open subring of $R^{\dagger,r}$, we only have to prove completeness of $R_0^{\dagger,r}$. Let $\{f_m\}_{m\in\mathbb{N}} \subset R_0^{\dagger,r}$ be a sequence such that $|f_m|_r \to 0$ as $m \to \infty$. We only have to prove that the limit $\sum_m f_m$ exists in $R_0^{\dagger,r}$ with respect to $|\cdot|_r$. Write $f_m = \sum_{n\in\mathbb{Z}} a_n^{(m)} S^n$ with $a_n^{(m)} \in \mathcal{O}$. For $n \in \mathbb{Z}$, we have

$$|a_n^{(m)}| \le \frac{|f_m|_r}{|S^n|_r} = |p|^{-nr} |f_m|_r,$$

hence, $|a_n^{(m)}| \to 0$ as $m \to \infty$. Moreover, $a_n := \sum_{m\in\mathbb{N}} a_n^{(m)} \in \mathcal{O}$ converges to 0 as $n \to -\infty$. Hence, the formal Laurent series $f := \sum_{n\in\mathbb{Z}} a_n S^n$ belongs to $\mathcal{O}\{\{S\}\}$. Since

$$|a_n S^n|_r \le \sup_{m\in\mathbb{N}} |a_n^{(m)} S^n|_r \le \sup_{m\in\mathbb{N}} |f_m|_r \le 1,$$

we have $f \in R_0^{\dagger,r}$. For $m \in \mathbb{N}$, we have

$$|f-(f_0+\cdots+f_m)|_r \le \sup_n |a_nS^n-(a_n^{(0)}+\cdots+a_n^{(m)})S^n|_r$$
$$\le \sup_n \sup_{l>m} |a_n^{(l)}S^n|_r = \sup_{l>m}\sup_n |a_n^{(l)}S^n|_r \le \sup_{l>m}|f_l|_r$$

and the last term converges to 0 as $m \to \infty$, which implies $f=\sum_m f_m$.

(iv) This follows from (i), (ii) and (iii). □

Definition 3.1.6. Let $R^+\langle \underline{X}\rangle$ be the (p,S)-adic Hausdorff completion of $R^+[\underline{X}]$. We also define $R_0^{\dagger,r}\langle \underline{X}\rangle$ and $R^{\dagger,r}\langle \underline{X}\rangle$ as the rings of strictly convergent power series over $R_0^{\dagger,r}$ and $R^{\dagger,r}$ with respect to $|\cdot|_r$. We endow $R_0^{\dagger,r}\langle \underline{X}\rangle$ and $R^{\dagger,r}\langle \underline{X}\rangle$ with the topology defined by the norm $|\cdot|_r$. By Lemma 3.1.5(iii), $R_0^{\dagger,r}\langle \underline{X}\rangle$ and $R^{\dagger,r}\langle \underline{X}\rangle$ are complete. By Lemma 3.1.5(ii), $R_0^{\dagger,r}\langle \underline{X}\rangle$ can be regarded as the (p,S)-adic Hausdorff completion of $R_0^{\dagger,r}[\underline{X}]$, hence, $R_0^{\dagger,r}\langle \underline{X}\rangle$ and $R^{\dagger,r}\langle \underline{X}\rangle = R_0^{\dagger,r}\langle \underline{X}\rangle[S^{-1}]$ are Noetherian integral domains by Lemma 3.1.5(iv). Also, we may view $R^+\langle \underline{X}\rangle$ as a subring of $R_0^{\dagger,r}\langle \underline{X}\rangle$.

The following lemma seems to be used implicitly in [Xiao 2010, §1].

Lemma 3.1.7. *The canonical map $R^+\langle \underline{X}\rangle \to R^{\dagger,r}\langle \underline{X}\rangle$ is flat.*

Proof (due to Liang Xiao). We may regard $R_0^{\dagger,r}\langle \underline{X}\rangle$ as the (p,S)-adic Hausdorff completion of $R^+\langle \underline{X}\rangle \otimes_{R^+} R_0^{\dagger,r}$. Since $\mathcal{R}_0^{\dagger,r}$ is dense in $R_0^{\dagger,r}$ by Lemma 3.1.5(iii), $R_0^{\dagger,r}\langle \underline{X}\rangle$ can be viewed as the (p,S)-adic Hausdorff completion of $R^+\langle \underline{X}\rangle \otimes_{R^+} \mathcal{R}_0^{\dagger,r}$, which is Noetherian by Lemma 3.1.5(i). Hence, the canonical map

$$\alpha : R^+\langle \underline{X}\rangle \otimes_{R^+} \mathcal{R}_0^{\dagger,r} \to R_0^{\dagger,r}\langle \underline{X}\rangle$$

is flat. Since $\mathcal{R}_0^{\dagger,r}[S^{-1}] = \mathcal{R}$ and $R_0^{\dagger,r}\langle \underline{X}\rangle[S^{-1}] = R^{\dagger,r}\langle \underline{X}\rangle$, the canonical map $\alpha[S^{-1}]$ is also flat, which implies the assertion. □

Next, we consider prime ideals corresponding to good "points" of the open unit disc $R^+ = \mathcal{O}[\![S]\!]$.

Definition 3.1.8. An Eisenstein polynomial in R^+ is a polynomial in $\mathcal{O}[S]$ of the form $P(S) = S^e + a_{e-1}S^{e-1} + \cdots + a_0$ with $a_i \in \mathcal{O}$ such that $p \mid a_i$ for all i and $p^2 \nmid a_0$. We call $\mathfrak{p} \in \mathrm{Spec}(R^+)$ an Eisenstein prime ideal if $\mathfrak{p}$ is generated by an Eisenstein polynomial $P(S)$. Then, we put $\deg(\mathfrak{p}) := e$ if $e \neq 0$ and $\deg(\mathfrak{p}) := \infty$ if $e = 0$. Note that we may regard $\kappa(\mathfrak{p}) := R/\mathfrak{p}R$ as a complete discrete valuation field with integer ring $R^+/\mathfrak{p}R^+$. We denote by $\pi_{\mathfrak{p}} \in \mathcal{O}_{\kappa(\mathfrak{p})}$ the image of S, which is a uniformizer of $\mathcal{O}_{\kappa(\mathfrak{p})}$. Note that $\deg(\mathfrak{p}) < \infty$ if and only if the characteristic of $R/\mathfrak{p}$ is zero. For simplicity, we write $\kappa(p)$ and S instead of $\kappa((p))$ and $\pi_{\kappa((p))}$.

Lemma 3.1.9. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be Eisenstein prime ideals of R^+. If*

$$\inf(v_{\kappa(\mathfrak{p})}(x \bmod \mathfrak{p}), v_{\kappa(\mathfrak{q})}(x \bmod \mathfrak{q})) < \inf(\deg \mathfrak{p}, \deg \mathfrak{q}),$$

for $x \in R^+$, then we have $v_{\kappa(\mathfrak{p})}(x \bmod \mathfrak{p}) = v_{\kappa(\mathfrak{q})}(x \bmod \mathfrak{q})$.

Proof. Let $x \in R^+$ and $i \in \mathbb{N}$ such that $0 \leq i < \deg \mathfrak{p}$. Then, we have the following equivalences:

$$\begin{aligned} v_{\kappa(p)}(x \bmod p) = i &\iff x \in (p, S^i) \setminus (p, S^{i+1}) \\ &\iff x \in (\mathfrak{p}, S^i) \setminus (\mathfrak{p}, S^{i+1}) \iff v_{\kappa(\mathfrak{p})}(x \bmod \mathfrak{p}) = i, \end{aligned}$$

where the second equivalence follows from the fact $(\mathfrak{p}, S^i) = (p, S^i)$, and the other equivalences follow from the definitions. By replacing $\mathfrak{q}$ by $\mathfrak{p}$, we obtain similar equivalences. As a result, $v_{\kappa(\mathfrak{p})}(x \bmod \mathfrak{p}) = i \Leftrightarrow v_{\kappa(\mathfrak{q})}(x \bmod \mathfrak{q}) = i$ for $x \in R^+$ and $i < \inf(\deg(\mathfrak{p}), \deg(\mathfrak{q}))$, which implies the assertion. $\square$

The ring $R^{\dagger,r}\langle \underline{X} \rangle$ can be considered as a family of Tate algebras:

Lemma 3.1.10. *Let $\mathfrak{p}$ be an Eisenstein prime ideal of R^+ with $e = \deg(\mathfrak{p})$. Let $r \in \mathbb{Q}_{>0}$ satisfy $1/e \leq r$. Then, there exists a canonical isomorphism*

$$R^{\dagger,r}\langle \underline{X} \rangle / \mathfrak{p} R^{\dagger,r}\langle \underline{X} \rangle \xrightarrow{\sim} \kappa(\mathfrak{p})\langle \underline{X} \rangle.$$

In particular, $\mathfrak{p} R^{\dagger,r} \neq R^{\dagger,r}$.

Proof. We will briefly recall a result in [Lazard 1962]. Let F be a complete discrete valuation field of mixed characteristic $(0, p)$. Recall that $L_F[0, r]$ is the ring of Laurent series with variable S and coefficients in F, which converge in the annulus $|p|^r \leq |S| < 1$, see [Lazard 1962, §1.3]. For $r' \in \mathbb{Q}_{>0}$, a polynomial $P \in F[S]$ is said to be r'-extremal if all zeroes x of P in F^{alg} satisfy $v(x) = r'$, see [Lazard 1962, §2.7]. Let $r' \leq r$ be a positive rational number and $P \in F[S]$ an r'-extremal polynomial. Then, for $f \in L_F[0, r]$, there exist a unique $g \in L_F[0, r]$ and a unique polynomial $Q \in F[S]$ of degree less than $\deg P$ such that $f = Pg + Q$, which is a special case of [Lazard 1962, Lemme 2]. Note that if $f \in F[S]$ with $\deg(f) < \deg(P)$, then we have $g = 0$ and $Q = f$ by the uniqueness. In particular, the canonical map $\delta : F[S]/P \cdot F[S] \to L_F[0, r]/P \cdot L_F[0, r]$ is an isomorphism.

We prove the assertion. We can easily reduce to the case $\underline{X} = \phi$. That is, we only have to prove that the canonical map

$$R^{\dagger,r}/\mathfrak{p} R^{\dagger,r} \to \kappa(\mathfrak{p})$$

is an isomorphism. The assertion is trivial when $\mathfrak{p} = (p)$. Hence, we may assume $\mathfrak{p} \neq (p)$. Since p is invertible in $\kappa(\mathfrak{p})$, p is also invertible in $R^{\dagger,r}/\mathfrak{p} R^{\dagger,r}$. Hence, we have $R^{\dagger,r}/\mathfrak{p} R^{\dagger,r} = R^{\dagger,r}[1/p]/\mathfrak{p} R^{\dagger,r}[1/p]$. Note that $R^{\dagger,r}[1/p]$ coincides, by definition, with $L_F[0, r]$ with $F := \mathrm{Frac}(\mathcal{O})$. Let $P \in \mathcal{O}[S]$ be an Eisenstein polynomial which generates $\mathfrak{p}$. Then, P is $1/e$-extremal by a property of Eisenstein polynomials. Hence, the assertion follows from the isomorphisms

$$L_F[0,r]/\mathfrak{p}L_F[0,r] \cong F[S]/P\cdot F[S] \\ \cong (\mathcal{O}[S]/P\cdot\mathcal{O}[S])[1/p] \cong (R^+/\mathfrak{p})[1/p] = \kappa(\mathfrak{p}).$$

Here, the first isomorphism is Lazard's δ, with $r' = 1/e$. □

3.2. ***Gröbner basis argument over complete regular local rings.*** In this subsection, we will develop a basic theory of Gröbner bases over complete regular local rings R, which generalizing that over fields. This is done in [Xiao 2010, §1.1], when R is a 1-dimensional complete regular local ring of characteristic p. We assume knowledge of the classical theory of Gröbner bases over fields; our basic reference is [Cox et al. 1997].

Recall that the classical theory of Gröbner bases on $F[\underline{X}]$ for a field F can be regarded as a multi-variable version of the Euclidean division algorithm of the 1-variable polynomial ring $F[X]$. To obtain an appropriate division algorithm in $F[\underline{X}]$, we need to fix a "monomial order" on $F[\underline{X}]$ in order to define a leading term, which is the analogue of the naïve degree function in the 1-variable case. Hence, we should first define a notion of leading terms over the ring of convergent power series.

Definition 3.2.1. A monomial order $\succeq$ on a commutative monoid $(M,+)$ is an well-order such that if $\alpha \succeq \beta$, then $\alpha+\gamma \succeq \beta+\gamma$. When $\alpha \succeq \beta$ and $\alpha \neq \beta$, we write $\alpha \succ \beta$.

In the following, we restrict to the case where M is isomorphic to $\mathbb{N}^l$. Moreover, the reader may assume that $\succ$ is a lexicographic order; the lexicographic order $\succeq_{\mathrm{lex}}$ on $\mathbb{N}^l$ is defined by $(a_1,\ldots,a_l) \succ_{\mathrm{lex}} (a'_1,\ldots,a'_l)$ if $a_1 = a'_1, \ldots, a_i = a'_i, a_{i+1} > a'_{i+1}$. A lexicographic order is a monomial order, see [Cox et al. 1997, §2.2, Proposition 4].

For convenience, we define a monoid $M \cup \{\infty\}$ by $\alpha + \infty = \infty$ for any $\alpha \in M \cup \{\infty\}$. We extend any monomial order $\succeq$ on M to $M \cup \{\infty\}$ by $\infty \succ \alpha$ for any $\alpha \in M$.

Construction 3.2.2. Let R be a complete regular local ring of Krull dimension d with fixed regular system of parameters $\{s_1,\ldots,s_d\}$. We put $R_i := R/(s_1,\ldots,s_i)R$, which is also a regular local ring. We denote the image of $s_{i+1},\ldots,s_d$ in R_i by $s_{i+1},\ldots,s_d$ again and we regard these as a fixed regular system of parameters. Let $v_{s_i} : R_i \twoheadrightarrow \mathbb{N} \cup \{\infty\}$ be the multiplicative valuation associated to the divisor $s_i = 0$. For a nonzero $f \in R$ and $0 \le i \le d$, we define a nonzero $f^{(i)} \in R_i$ inductively as follows. We put $f^{(0)} := f$, and define $f^{(i+1)}$ as the image of $f^{(i)}/s_{i+1}^{v_{s_{i+1}}(f^{(i)})}$ in R_{i+1}, which is nonzero by definition. We put $\underline{v}_R(f) := (v_{s_1}(f^{(0)}), v_{s_2}(f^{(1)}), \ldots, v_{s_d}(f^{(d-1)})) \in \mathbb{N}^d$ and $\underline{v}_R(0) := \infty$. Thus, we obtain a map $\underline{v}_R : R \to \mathbb{N}^d \cup \{\infty\}$. We also apply this construction to each R_i. Note that we have a formula

$$\underline{v}_R(f) = \big(v_{s_1}(f), \underline{v}_{R_1}(f^{(1)})\big). \tag{1}$$

Also, note that $\underline{v}_R$ is multiplicative, i.e., $\underline{v}_R(fg) = \underline{v}_R(f) + \underline{v}_R(g)$, which follows by induction on d and by using the formula.

Let $R\langle \underline{X}\rangle$ be the $\mathfrak{m}_R$-adic Hausdorff completion of $R[\underline{X}]$. We fix a monomial order $\succeq$ on $\underline{X}^{\mathbb{N}} \cong \mathbb{N}^l$. For any nonzero $f = \sum_{\underline{n}} a_{\underline{n}} \underline{X}^{\underline{n}} \in R\langle \underline{X}\rangle$ with $a_{\underline{n}} \in R$, we define $\underline{v}_R(f) := \inf_{\succeq_{\text{lex}}} \underline{v}_R(a_{\underline{n}})$, where $\succeq_{\text{lex}}$ is the lexicographic order on $\mathbb{N}^d$, and $\underline{\deg}_R(f) := \inf_{\succeq}\{\underline{n} \in \mathbb{N}^l;\ \underline{v}_R(a_{\underline{n}}) = \underline{v}_R(f)\}$. We put $\underline{\deg}_R(0) := \infty$. Note that when $f \neq 0$, we have a formula

$$\underline{\deg}_R(f) = \underline{\deg}_R(f^{(0)}) = \underline{\deg}_R(f^{(1)}) = \cdots = \underline{\deg}_R(f^{(d)}), \tag{2}$$

which follows from (1). Also, note that $\underline{\deg}_R$ is multiplicative. Indeed, formula (2) allows us to reduce to the case where R is a field; here $\underline{\deg}_R$ is multiplicative by [Cox et al. 1997, Chapter 2, Lemma 8]. Thus, we obtain a multiplicative map

$$\underline{v}_R \times \underline{\deg}_R : R\langle \underline{X}\rangle \to (\mathbb{N}^d \times \mathbb{N}^l) \cup \{\infty\},$$

where ∞ in the RHS denotes (∞, ∞). We endow $\mathbb{N}^d \times \mathbb{N}^l$ with a total order $\succeq$ by

$$(\underline{a}, \underline{n}) \succeq (\underline{a}', \underline{n}') \text{ if } \underline{a} \preceq_{\text{lex}} \underline{a}' \text{ or } \underline{a} = \underline{a}' \text{ and } \underline{n} \succeq \underline{n}'$$

and extend it to $(\mathbb{N}^d \times \mathbb{N}^l) \cup \{\infty\}$ as in Definition 3.2.1. Note that this order is an extension of the fixed order on $\mathbb{N}^l = \{0\} \times \dots \{0\} \times \mathbb{N}^l$. As in the classical notation, we also define

$$\mathrm{LT}_R(f) := \underline{s}^{\underline{v}_R(f)} \underline{X}^{\underline{\deg}_R(f)} \quad \text{for } f \neq 0, \mathrm{LT}_R(0) := 0,$$

where $\underline{s} = (s_1, \dots, s_d)$. Note that LT_R is also multiplicative by the multiplicativities of $\underline{v}_R$ and $\underline{\deg}_R$. We also have the formula

$$\mathrm{LT}_R(f) \equiv \mathrm{LT}_{R_i}(f \bmod (s_1, \dots, s_i)) \bmod (s_1, \dots, s_i),\ \forall f \in R\langle \underline{X}\rangle. \tag{3}$$

Indeed, if $s_i \mid f^{(i-1)}$ for some i, then both sides are zero. If $s_i \nmid f^{(i-1)}$ for all i, then the formula follows from (1) and (2). The map LT_R takes values in the subset $\underline{s}^{\mathbb{N}}\underline{X}^{\mathbb{N}} \cup \{0\}$ of $R\langle \underline{X}\rangle$. We identify $\underline{s}^{\mathbb{N}}\underline{X}^{\mathbb{N}} \cup \{0\}$ with $(\mathbb{N}^d \times \mathbb{N}^l) \cup \{\infty\}$ as a monoid and consider the total order $\succeq$ on $\underline{s}^{\mathbb{N}}\underline{X}^{\mathbb{N}} \cup \{0\}$.

When R is a field, the above definition coincides with the classical definition as in [Cox et al. 1997, §2].

Remark 3.2.3. LT stands for "leading term" with respect to a given monomial order in the classical case $d = 0$. To define an appropriate LT in the case $d > 0$, we should consider a suitable order on the coefficient ring R, which is defined by using an ordered regular system of parameters as above. Our definition is compatible with dévissage, namely, compatible with parameter-reducing maps $R \to R_1 \to \cdots \to R_d$. This property enables us to reduce everything about Gröbner bases to the classical case by assuming a certain "flatness" as we will see below.

In the rest of this subsection, let the notation be as in Construction 3.2.2. In particular, we fix a monomial order $\succeq$ on $\underline{X}^{\mathbb{N}}$.

Definition 3.2.4. For I an ideal of $R\langle \underline{X} \rangle$, we denote by $\mathrm{LT}_R(I)$ the ideal of $R\langle \underline{X} \rangle$ generated by $\{\mathrm{LT}_R(f);\, f \in I\}$. Assume that $R\langle \underline{X} \rangle / I$ is R-flat. We say that $f_1, \dots, f_s \in I$ form a Gröbner basis if $(\mathrm{LT}_R(f_1), \dots, \mathrm{LT}_R(f_s)) = \mathrm{LT}_R(I)$. Note that a Gröbner basis always exists since $R\langle \underline{X} \rangle$ is Noetherian.

Note that for monomials $f, f_1, \dots, f_s \in R\langle \underline{X} \rangle$, we have $f \in (f_1, \dots, f_s)$ if and only if f is divisible by some f_i. Indeed, any term of $g \in (f_1, \dots, f_s)$ is divisible by some f_i, which implies the necessity.

Notation 3.2.5. Let I be an ideal of $R\langle \underline{X} \rangle$ such that $R\langle \underline{X} \rangle / I$ is R-flat. We write $I_i := I/(s_1, \dots, s_i)I$. We may identify $R\langle \underline{X} \rangle \otimes_R R_i$ and $I \otimes_R R_i$ with $R_i\langle \underline{X} \rangle$ and I_i, respectively. Note that $R_i\langle \underline{X} \rangle / I_i$ is R_i-flat.

Lemma 3.2.6. *Let I be an ideal of $R\langle \underline{X} \rangle$ such that $R\langle \underline{X} \rangle / I$ is R-flat. The following are equivalent for $f_1, \dots, f_s \in I$:*

(i) *$f_1, \dots, f_s$ form a Gröbner basis of I.*

(ii) *The images of $f_1, \dots, f_s$ form a Gröbner basis of $I_i \subset R_i\langle \underline{X} \rangle$ for some i.*

Moreover, when $f_1, \dots, f_s$ is a Gröbner basis of I, $f_1, \dots, f_s$ generate I.

Proof. We prove the first assertion by induction on $d = \dim R$. When $d = 0$, there is nothing to prove. Assume the assertion is true for dimension $< d$. By the induction hypothesis, we only have to prove the equivalence between (i) and (ii) with $i = 1$.

We first prove (i) $\Rightarrow$ (ii). Let $\bar{f} \in I_1$ be a nonzero element and $f \in I$ a lift of $\bar{f}$. By assumption, we have $\mathrm{LT}_R(f_j) \mid \mathrm{LT}_R(f)$ for some j. Then, $\mathrm{LT}_{R_1}(f_j \bmod s_1)$ divides $\mathrm{LT}_{R_1}(\bar{f})$ by formula (3).

We prove (ii) $\Rightarrow$ (i). Let $f \in I$ be a nonzero element. By Lemma 3.1.3, we have $f^{(1)} = f/s_1^{v_{s_1}(f)} \in I$. By assumption, we have $\mathrm{LT}_{R_1}(f_j \bmod s_1) \mid \mathrm{LT}_{R_1}(f^{(1)} \bmod s_1)$ for some j. Since $\mathrm{LT}_{R_1}(f^{(1)} \bmod s_1) \neq 0$, s_1 does not divides f_j, i.e., $v_{s_1}(f_j) = 0$. By formulas (1) and (2), $\mathrm{LT}_R(f_j)$ divides $\mathrm{LT}_R(f^{(1)})$, and hence divides $\mathrm{LT}_R(f)$, which implies the assertion.

We prove the last assertion. By Nakayama's lemma and (ii) with $i = d$, the assertion is reduced to the case where R is a field. In this case, the assertion follows from [Cox et al. 1997, §2.5, Corollary 6]. $\square$

Remark 3.2.7. By Lemma 3.2.6, $f_1, \dots, f_s$ is a Gröbner basis of I if and only if $f_1 \bmod \mathfrak{m}_R, \dots, f_s \bmod \mathfrak{m}_R$ is a Gröbner basis of $I/\mathfrak{m}_R I$. In particular, the definition of Gröbner basis does not depend on the choice of a regular system of parameters $\{s_1, \dots, s_d\}$.

We can generalize the classical division algorithm, which is a basic tool in many Gröbner basis arguments.

Proposition 3.2.8 (division algorithm). *Let I be an ideal of $R\langle\underline{X}\rangle$ such that $R\langle\underline{X}\rangle/I$ is R-flat. Let $f_1, \dots, f_s \in I$ be a Gröbner basis of I. Then, for any nonzero $f \in R\langle\underline{X}\rangle$, there exist $a_i, r \in R\langle\underline{X}\rangle$ for all i such that*

$$f = \sum_{1 \le i \le s} a_i f_i + r,$$

with $\mathrm{LT}_R(f) \succeq \mathrm{LT}_R(a_i f_i)$ if $a_i f_i \neq 0$, and any nonzero term of r is not divisible by any $\underline{X}^{\deg_R(f_i)}$. Moreover, such r is uniquely determined (but the a_i's are not), and $f \in I$ if and only if $r = 0$.

Proof. When $d = 0$, i.e, R is a field, the assertion is well known (see [Cox et al. 1997, §2.6, Proposition 1] for example). We prove the first assertion by induction on $d = \dim R$. Assume that the assertion is true for dimension $< d$. We may assume $s_1 \nmid f_i$ for all i. Indeed, by Lemma 3.2.6, the set $\{f_i;\ s_1 \nmid f_i\}$ forms a Gröbner basis of I. Moreover, any $\mathrm{LT}_R(f_j)$ is divisible by some $\mathrm{LT}_R(f_i)$ with $s_1 \nmid f_i$. Therefore, if we can write $f = \sum_{i: s_1 \nmid f_i} a_i f_i + r$ with respect to $\{f_i;\ s_1 \nmid f_i\}$, then we can write f in the same way with respect to $f_1, \dots, f_s$. First, we construct $g_n \in R\langle\underline{X}\rangle$ by induction on $n \in \mathbb{N}$. For $h \in R\langle\underline{X}\rangle$, let $\bar{h}$ be its image in $R_1\langle\underline{X}\rangle$. Put $g_0 := f$. Assume that g_n has been defined. Put $g_n' := g_n / s_1^{v_{s_1}(g_n)}$. By applying the induction hypothesis to $I_1 = (\bar{f}_1, \dots, \bar{f}_s)$, we have $\bar{a}_{i,n}, \bar{r}_n \in R_1\langle\underline{X}\rangle$ with

$$\bar{g}_n' = \sum_i \bar{a}_{i,n} \bar{f}_i + \bar{r}_n,$$

such that no nonzero terms of $\bar{r}_n$ are divisible by any $\underline{X}^{\deg_{R_1}(\bar{f}_i)}$, and such that $\mathrm{LT}_{R_1}(\bar{g}_n') \succeq \mathrm{LT}_{R_1}(\bar{a}_{i,n} \bar{f}_i)$ if $\bar{a}_{i,n} \bar{f}_i \neq 0$. We choose lifts $a_{i,n}$ and r_n in $R\langle\underline{X}\rangle$ of $\bar{a}_{i,n}$ and $\bar{r}_n$, respectively, such that no nonzero terms of $a_{i,n}$ and r_n are divisible by s_1. Then, we put $g_{n+1} := g_n - s_1^{v_{s_1}(g_n)} \bigl(\sum_i a_{i,n} f_i + r_n\bigr)$. By construction, we have $v_{s_1}(g_{n+1}) > v_{s_1}(g_n)$, hence, $\{g_n\}$ converges s_1-adically to zero. Moreover, $a_i := \sum_n s_1^{v_{s_1}(g_n)} a_{i,n}$ and $r := \sum_n s_1^{v_{s_1}(g_n)} r_n$ converge s_1-adically and we have $f = \sum_i a_i f_i + r$. We will check that a_i and r satisfy the condition. Since $s_1 \nmid f_i$ and since no nonzero term of r_n is divisible by s_1, no nonzero term of r is divisible by $\underline{X}^{\deg_R(f_i)}$ for all i. We have $v_{s_1}(f_i) = 0$ by assumption and $v_{s_1}(a_i) \geq v_{s_1}(f)$ by definition. If $v_{s_1}(a_i) > v_{s_1}(f)$, then we have $\underline{v}_R(f) \preceq_{\mathrm{lex}} \underline{v}_R(a_i f_i)$, hence, $\mathrm{LT}_R(f) \succeq \mathrm{LT}_R(a_i f_i)$. If $v_{s_1}(a_i) = v_{s_1}(f)$, then we have $a_i^{(0)} \equiv a_{i,0} \bmod s_1$, hence, $\underline{v}_R(f) \preceq \underline{v}_R(a_i f_i)$ by formulas (1), (2) and the choice of $\bar{a}_{i,0}$. In particular, $\mathrm{LT}_R(f) \succeq \mathrm{LT}_R(a_i f_i)$. Thus, we obtain the first assertion.

We prove the rest of the assertion. We first prove the uniqueness of r. Let $f = \sum a_i f_i + r = \sum a_i' f_i + r'$ be expressions satisfying the conditions. Then, we have $r-r' \in I$, hence, $\mathrm{LT}_R(r-r) \in \mathrm{LT}_R(I)$. Therefore, $r-r'$ is divisible by $\mathrm{LT}_R(f_i)$ for some i. Since no nonzero term of $r-r'$ is divisible by any $\mathrm{LT}_R(f_i)$, we must have $r=r'$. We prove the equivalence $r=0 \Leftrightarrow f \in I$. We only have to prove the necessity. Since $r \in I$, we have $\mathrm{LT}_R(r) \in \mathrm{LT}_R(I)$. Hence, $\mathrm{LT}_R(r)$ is divisible by $\mathrm{LT}_R(f_i)$ for some i. Since all nonzero terms of r are divisible by $\underline{X}^{\deg_R(f_i)}$, we must have $r=0$. $\square$

Definition 3.2.9. We call the above expression $f=\sum a_i f_i+r$ a standard expression (of f) and call r the remainder of f (with respect to $f_1,\dots,f_s$). Note that standard expressions are additive and compatible with scalar multiplications, that is, if $f=\sum_i a_i f_i+r$ and $g=\sum_i a'_i f_i+r'$ are standard expressions, then $f+g=\sum_i (a_i+a'_i) f_i+r+r'$ is also a standard expression of $f+g$, and $\lambda f=\sum_i \lambda a_i f_i+\lambda r$ is a standard expression of λf for $\lambda \in R$ by formulas (1) and (2). The remainder of f depends only on the class $f \bmod I$ by Proposition 3.2.8 and the above additive property. Therefore, we may call r the remainder of $f \bmod I$.

As in the classical case, we have the following.

Lemma 3.2.10. *Let I be an ideal of $R\langle\underline{X}\rangle$ such that $R\langle\underline{X}\rangle/I$ is R-flat. Let $f_1,\dots,f_s \in I$ be a Gröbner basis of I. Let $f \in R\langle\underline{X}\rangle$ be a nonzero element. For $r \in R\langle\underline{X}\rangle$, the following are equivalent*:

(i) *r is the remainder of f.*

(ii) *$f-r \in I$ and no nonzero term of r is divisible by $\underline{X}^{\deg(f_i)}$ for all i.*

Proof. Since the assertion (i) $\Rightarrow$ (ii) is trivial, we prove the converse. By applying the division algorithm to $f-r$, we have $f-r=\sum a_i f_i$ such that $\mathrm{LT}_R(f) \succeq \mathrm{LT}_R(a_i f_i)$ if $a_i f_i \neq 0$. This means exactly that r is the remainder of f. $\square$

Corollary 3.2.11. *Let the notation be as in Lemma 3.2.10. We regard $f_1 \bmod s_1,\dots,f_s \bmod s_1$ as a Gröbner basis of I_1. For $f \in R\langle\underline{X}\rangle$ with $s_1 \nmid f$, denote by r and r' the remainders of f and $f \bmod s_1$, respectively. Then, we have $r \bmod s_1 \equiv r'$.*

Finally, we give a concrete example of a Gröbner basis, which will appear in Section 3.5.

Proposition 3.2.12. *Let $I=(f_1,\dots,f_s) \subset R\langle\underline{X}\rangle$ be an ideal. Assume that there exists relatively prime monic monomials $T_1,\dots,T_s$ and units $u_1,\dots,u_s \in R^{\times}$ such that $\mathrm{LT}_R(f_i)=u_i T_i$ for $1 \le i \le s$. Then, we have the following*:

(i) *$R\langle\underline{X}\rangle/I$ is R-flat.*

(ii) *$f_1,\dots,f_s$ is a Gröbner basis of I.*

(iii) *$f_1,\dots,f_s$ is a regular sequence in $R\langle\underline{X}\rangle$.*

Proof. We may assume that $\mathrm{LT}_R(f_1), \dots, \mathrm{LT}_R(f_s)$ are relatively prime monic monomials by replacing f_i by f_i/u_i. We first note that in the case of $d = 0$, the assertion is basic, since condition (i) is automatically satisfied. Condition (ii) directly follows from [Cox et al. 1997, §2.9, Theorem 3 and Proposition 4]. Condition (iii) follows from [Eisenbud 1995, Proposition 15.15] with $F = S = R[\underline{X}]$ and $M = 0$, $h_j = f_j$, where F, S and M, h_j's are as in the reference. We prove the assertion by induction on s. In the case of $s = 1$, we have only to prove condition (i). We proceed by induction on d. By the local criteria of flatness and the induction hypothesis, we only have to prove that the multiplication by s_1 on $R\langle\underline{X}\rangle/I$ is injective. Let $f \in R\langle\underline{X}\rangle$ such that $s_1 f \in I$. Write $s_1 f = f_1 h$ for some $h \in R\langle\underline{X}\rangle$. By taking v_{s_1}, we have $s_1 \mid h$ since $s_1 \nmid f_1$. This implies $f_1 \mid f$, i.e., $f \in I$. This finishes the case $s = 1$. We assume that the assertion is true when the cardinality of f_i's is $< s$. We proceed by induction on d. The case $d = 0$ can be done as above. Assume that the assertion is true for dimension $< d$. For $h \in R\langle X\rangle$, denote by $\bar{h}$ its image in $R_1\langle\underline{X}\rangle$. By assumption, $s_1 \nmid f_i$ for all i, hence, we can apply the induction hypothesis to $\bar{f}_1, \dots, \bar{f}_s \in I_1 := (\bar{f}_1, \dots, \bar{f}_s) \subset R\langle\underline{X}\rangle$ by formula (3). Hence, $R_1\langle\underline{X}\rangle/I_1$ is R_1-flat, $\bar{f}_1, \dots, \bar{f}_s$ are a Gröbner basis of I_1, and $\bar{f}_1, \dots, \bar{f}_s$ is a regular sequence in $R_1\langle\underline{X}\rangle$. Condition (ii) follows from Lemma 3.2.6. Next, we check condition (i). By the local criteria of flatness, we only have to prove that multiplication by s_1 on $R\langle\underline{X}\rangle/I$ is injective. It suffices to prove $I \cap s_1 \cdot R\langle\underline{X}\rangle \subset s_1 I$. Denote by $C_\bullet$ and $\bar{C}_\bullet$ Koszul complexes for $\{f_1, \dots, f_s\}$ and $\{\bar{f}_1, \dots, \bar{f}_s\}$ [Matsumura 1980, 18.D]. Then, we have $\bar{C}_i = C_i/s_1 C_i$ for $i \geq 1$ by definition, and $\bar{C}_\bullet$ is exact since $\bar{f}_1, \dots, \bar{f}_s$ is a regular sequence. We also have a morphism of complexes $C_\bullet \to \bar{C}_\bullet$, whose first few terms are

$$\begin{tikzcd}
\dots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, two heads, "d_1"] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r] & \bar{C}_2 \arrow[r, "\bar{d}_2"] & \bar{C}_1 \arrow[r, "\bar{d}_1"] & I_1 \arrow[r] & 0
\end{tikzcd}$$

Let $f \in I \cap s_1 \cdot R\langle\underline{X}\rangle$. Then, there exists $a \in C_1$ such that $d_1(a) = f$. Since $\bar{d}_1(\bar{a}) \equiv 0 \bmod s_1$, there exists $\bar{b} \in C_2$ with $\bar{d}_2(\bar{b}) = \bar{a}$. Let $b \in C_2$ be a lift of $\bar{b}$. Then, there exists $a' \in C_1$ such that $a - d_2(b) = s_1 a'$. Therefore, we have $f = d_1(a - d_2(b)) = s_1 d_1(a') \in s_1 I$. Thus, condition (i) is proved. Finally, we check condition (iii). We only have to prove that if $f_i f \in (f_1, \dots, f_{i-1})$ for some $f \in R\langle\underline{X}\rangle$ and $1 \leq i \leq s$, then we have $f \in (f_1, \dots, f_{i-1})$. Note that $f_1, \dots, f_{i-1}$ is a Gröbner basis of $(f_1, \dots, f_{i-1})$ by the induction hypothesis. Let $f = \sum_{1 \leq j < i} a_j f_j + r$ be a standard expression of f with respect to $f_1, \dots, f_{i-1}$. It suffices to prove that $r = 0$. We suppose the contrary and deduce a contradiction. No nonzero term of r is divisible by $\mathrm{LT}_R(f_j)$ for any $1 \leq j < i$; in particular, we have $\mathrm{LT}_R(f_j) \nmid \mathrm{LT}_R(r)$. By assumption, $f_i f = f_i\left(\sum_{1 \leq j < i} a_j f_j\right) + f_i r \in (f_1, \dots, f_{i-1})$. We therefore have

$f_i r \in (f_1, \dots, f_{i-1})$. In particular, there exists $1 \le j < i$ with $\mathrm{LT}_R(f_j) \mid \mathrm{LT}_R(f_i r)$. Since $\mathrm{LT}_R(f_i)$ and $\mathrm{LT}_R(f_j)$ are relatively prime, we have $\mathrm{LT}_R(f_j) \mid \mathrm{LT}_R(r)$, which is a contradiction. Thus, we obtain assertion (iii). $\square$

A remarkable feature of the remainder is the compatibility with quotient norms:

Lemma 3.2.13. *Let I be an ideal of $R\langle \underline{X}\rangle$ such that $R\langle \underline{X}\rangle/I$ is R-flat. Let $f_1, \dots, f_s \in I$ be a Gröbner basis of I. Let $|\cdot| : R \to \mathbb{R}_{\ge 0}$ be any nonarchimedean norm satisfying $|R| \le 1$ and $|\mathfrak{m}_R| < 1$. We extend $|\cdot|$ to a norm on $R\langle \underline{X}\rangle$ by $|\sum_{\underline{n}} a_{\underline{n}} \underline{X}^{\underline{n}}| := \sup_{\underline{n}} |a_{\underline{n}}| < \infty$. If we denote by $|\cdot|_{\mathrm{qt}} : R\langle \underline{X}\rangle/I \to \mathbb{R}_{\ge 0}$ the quotient norm of $|\cdot|$, then the remainder r of $f \in R\langle \underline{X}\rangle$ achieves the quotient norm of $f \bmod I$, i.e.,*

$$|r| = |f \bmod I|_{\mathrm{qt}}.$$

Proof. Let $f = \sum \lambda_{\underline{n}} \underline{X}^{\underline{n}}$ with $\lambda_{\underline{n}} \in R$. Let $\underline{X}^{\underline{n}} = \sum a_{\underline{n},i} f_i + r_{\underline{n}}$ be a standard expression of $\underline{X}^{\underline{n}}$. Let $a_i := \sum_{\underline{n}} \lambda_{\underline{n}} a_{\underline{n},i}$ and $r := \sum_{\underline{n}} \lambda_{\underline{n}} r_{\underline{n}}$, which converge since $\lambda_{\underline{n}} \to 0$ as $|\underline{n}| \to \infty$. Then, $f = \sum a_i f_i + r$ is a standard expression of f by Lemma 3.2.10. We have $|a_i f_i| \le |a_i| \le \sup_{\underline{n}} |\lambda_{\underline{n}} a_{\underline{n},i}| \le \sup_{\underline{n}} |\lambda_{\underline{n}}| = |f|$. Hence, we have $|r| \le |f|$. Since the remainder depends only on the class $f \bmod I$, we have

$$|f \bmod I|_{\mathrm{qt}} = \inf_{g \in I} |f + g| \ge |r| \ge |f \bmod I|_{\mathrm{qt}},$$

which implies the assertion. $\square$

3.3. ***Gröbner basis argument over annuli.*** In this subsection, we will give an analogue of a Gröbner basis argument over rings of overconvergent power series. We use the notations of Section 3.1 and 3.2 and further use the following notation.

Notation 3.3.1. Let $\mathcal{O}$, R^+, and R be as in Notation 3.1.4. Fix $\{p, S\}$ as a regular system of parameters of R^+. Let $I \subset R^+\langle \underline{X}\rangle$ be an ideal such that $R^+\langle \underline{X}\rangle/I$ is R^+-flat. For $r \in \mathbb{Q}_{>0}$, we give $R^{\dagger,r}$ the topology defined by the norm $|\cdot|_r$, and write

$$A := R^+\langle \underline{X}\rangle/I, \quad I^{\dagger,r} := I \otimes_{R^+\langle \underline{X}\rangle} R^{\dagger,r}\langle \underline{X}\rangle, \quad A^{\dagger,r} := A \otimes_{R\langle \underline{X}\rangle} R^{\dagger,r}\langle \underline{X}\rangle.$$

(When $I = 0$, $R^{\dagger,r}\langle \underline{X}\rangle$ is denoted by $R\langle \underline{X}\rangle^{\dagger,r}$ in this notation. However, we use this notation for simplicity.) Since $R^+\langle \underline{X}\rangle \to R^{\dagger,r}\langle \underline{X}\rangle$ is flat (Lemma 3.1.7), we may identify $I^{\dagger,r}$ and $A^{\dagger,r}$ with $I \cdot R^{\dagger,r}\langle \underline{X}\rangle$ and $R^{\dagger,r}\langle \underline{X}\rangle/I^{\dagger,r}$. Since R^+ is an integral domain, A and hence, $A^{\dagger,r}$ are R^+-torsion free by flatness.

Let $|\cdot|_{r,\mathrm{qt}} : A^{\dagger,r} \to \mathbb{R}_{\ge 0}$ be the quotient norm of $|\cdot|_r$. Note that $A^{\dagger,r}$ is complete with respect to $|\cdot|_{r,\mathrm{qt}}$ by [Bosch et al. 1984, Section 1.1.7, Proposition 3].

Lemma 3.3.2 (cf. [Xiao 2010, Lemma 1.1.22]). *Let $f_1, \dots, f_s \in I$ be a Gröbner basis of I. For $f \in R^{\dagger,r}\langle \underline{X}\rangle$, there exists a unique $\mathfrak{r} \in R^{\dagger,r}\langle \underline{X}\rangle$ such that $f - \mathfrak{r} \in I^{\dagger,r}$ and no nonzero term of $\mathfrak{r}$ is divisible by $\underline{X}^{\underline{\deg}_R(f_i)}$. Moreover, we have $|\mathfrak{r}|_{r'} = |f|_{r',\mathrm{qt}}$ for $r' \in \mathbb{Q} \cap (0, r]$, and $\mathfrak{r} = 0$ if and only if $f \in I^{\dagger,r}$. We call $\mathfrak{r}$ the remainder of f (with respect to $f_1, \dots, f_s$).*

Proof. We first construct $\mathfrak{r}$. Let $f=\sum_{\underline{n}}\lambda_{\underline{n}}\underline{X}^{\underline{n}}\in R^{\dagger,r}\langle\underline{X}\rangle$ with $\lambda_{\underline{n}}\in R^{\dagger,r}$. Let

$$\underline{X}^{\underline{n}}=\sum_i a_{\underline{n},i}f_i+r_{\underline{n}}$$

be the standard expression of $\underline{X}^{\underline{n}}$ in $R^+\langle\underline{X}\rangle$ with respect to $f_1,\dots,f_s$. Since $\lambda_{\underline{n}}\to 0$ as $|\underline{n}|\to\infty$, the series

$$a_i:=\sum_{\underline{n}}\lambda_{\underline{n}}a_{\underline{n},i},\quad \mathfrak{r}:=\sum_{\underline{n}}\lambda_{\underline{n}}r_{\underline{n}}$$

converge in $R^{\dagger,r}\langle\underline{X}\rangle$ with respect to the topology defined by $|\cdot|_r$. Then, we have

$$|\mathfrak{r}|_{r'}\le\sup_{\underline{n}}|\lambda_{\underline{n}}r_{\underline{n}}|_{r'}\le\sup_{\underline{n}}|\lambda_{\underline{n}}|_{r'}=|f|_{r'}. \tag{4}$$

Obviously, no nonzero term of $\mathfrak{r}$ is divisible by any $\underline{X}^{\underline{\deg}_R(f_i)}$ and we have $f-\mathfrak{r}=\sum_i a_if_i\in I^{\dagger,r}$.

We prove the uniqueness of $\mathfrak{r}$. We suppose the contrary and deduce a contradiction. Let $\mathfrak{r}'\in R^{\dagger,r}\langle\underline{X}\rangle$ be an element such that $f-\mathfrak{r}'\in I^{\dagger,r}$ and such that no nonzero term of $\mathfrak{r}'$ is divisible by any $\underline{X}^{\underline{\deg}_R(f_i)}$. We choose $m\in\mathbb{N}$ such that $\delta:=S^m(\mathfrak{r}-\mathfrak{r}')$ belongs to $I_0^{\dagger,r}:=I\otimes_{R^+\langle\underline{X}\rangle}R_0^{\dagger,r}\langle\underline{X}\rangle$. If we write $\delta=p^n\delta'$ such that $\delta'\in R_0^{\dagger,r}\langle\underline{X}\rangle$ is not divisible by p in $R_0^{\dagger,r}\langle\underline{X}\rangle$, then we have $\delta'\in I_0^{\dagger,r}$ by Lemma 3.1.3. We may identify $I_0^{\dagger,r}/pI_0^{\dagger,r}$ with I/pI by Lemma 3.1.10. We write $\bar{\delta}':=\delta'\bmod pI_0^{\dagger,r}\in I/pI$. We also write $R_1^+:=R^+/pR^+$, which is a complete discrete valuation ring with uniformizer S. Then, no nonzero term of $\bar{\delta}'$ is divisible by $\underline{X}^{\underline{\deg}_{R_1^+}(f_i\bmod p)}$. Hence, $\bar{\delta}'$ is the remainder of 0 with respect to $f_1\bmod p,\dots,f_s\bmod p$ in $R_1\langle\underline{X}\rangle$. By Lemma 3.2.10, $\bar{\delta}'=0$, i.e., $\delta'\in\bmod pI_0^{\dagger,r}$, contradicting $p\nmid\delta'$.

We prove $f=I^{\dagger,r}\Leftrightarrow\mathfrak{r}=0$. If $f\in I^{\dagger,r}$, then 0 satisfies the required property for the remainder, and hence $\mathfrak{r}=0$ by uniqueness. If $\mathfrak{r}=0$, then $f\in I^{\dagger,r}$ by definition.

We prove $|\mathfrak{r}|_{r'}=|f\bmod I^{\dagger,r}|_{r',\mathrm{qt}}$. Let $\alpha\in I^{\dagger,r}$. Since $\mathfrak{r}$ satisfies the required condition for the remainder of $f+\alpha$, the remainder of $f+\alpha$ is equal to $\mathfrak{r}$ by uniqueness. In particular, the remainder depends only on the of class $f\bmod I^{\dagger,r}$. Hence, the assertion follows from

$$|f\bmod I^{\dagger,r}|_{r',\mathrm{qt}}=\inf_{\alpha\in I^{\dagger,r}}|\mathfrak{r}+\alpha|_{r'}\ge|\mathfrak{r}|_{r'}\ge|f\bmod I^{\dagger,r}|_{r',\mathrm{qt}},$$

where the first equality follows from (4) and the second inequality follows by definition. □

The following is an immediate consequence of the above lemma.

Lemma 3.3.3. *Let $f_1,\dots,f_s$ be a Gröbner basis of I. Let $f,g\in R^{\dagger,r}\langle\underline{X}\rangle$ and let $\mathfrak{r},\mathfrak{r}'$ be their remainders with respect to $f_1,\dots,f_s$. Then, we have the following*:

(i) *The remainder of $f+g$ is equal to $\mathfrak{r}+\mathfrak{r}'$.*

(ii) *The remainder* $\mathfrak{r}$ *depends only on* $f \bmod I^{\dagger,r}$. *One may call the remainder of* f *the remainder of* $f \bmod I^{\dagger,r}$.

(iii) *For* $\lambda \in R^{\dagger,r}$, *the remainder of* λf *is equal to* $\lambda\mathfrak{r}$. *Moreover, if* $f \bmod I^{\dagger,r}$ *is divisible by* $\lambda \in R^{\dagger,r}$, *then* $\mathfrak{r}$ *is also divisible by* λ.

Corollary 3.3.4. *Let* $\mathfrak{a} \subsetneq R^{\dagger,r}$ *be a principal ideal. Then, we have* $\bigcap_{n\in\mathbb{N}} \mathfrak{a}^n \cdot A^{\dagger,r} = 0$.

Proof. Fix a Gröbner basis $f_1, \dots, f_s$ of I. Let $f \in \bigcap_{n\in\mathbb{N}} \mathfrak{a}^n \cdot A^{\dagger,r}$ and let $\mathfrak{r}$ be the remainder of f with respect to $f_1, \dots, f_s$. By Lemma 3.3.3(iii) and the assumption, we have $\mathfrak{r} \in \bigcap_{n\in\mathbb{N}} \mathfrak{a}^n = 0$. □

Remark 3.3.5. Using [Kedlaya 2005, Proposition 2.6.5], one can prove that $R^{\dagger,r}$ is a principal ideal domain. We do not use this fact in this paper.

3.4. ***Continuity of connected components for families of affinoids.*** In this subsection, we will apply the previous results to prove a continuity of connected components of fibers of families of affinoids.

Lemma 3.4.1. *Let* $f : R \to S$ *be a morphism of Noetherian rings and let* $\mathrm{Idem}(T)$ *denote the set of idempotents of a ring* T. *If the canonical map* $f_* : \mathrm{Idem}(R) \to \mathrm{Idem}(S)$ *is surjective and* $f_*^{-1}(\{0\}) = \{0\}$, *then* $f^* : \pi_0^{\mathrm{Zar}}(S) \to \pi_0^{\mathrm{Zar}}(R)$ *is bijective.*

Proof. We first recall a basic fact on commutative algebras. For a ring A, finite partitions of $\mathrm{Spec}(A)$ into nonempty open subspaces as a topological space correspond to finite sets of nonzero idempotents $e_1, \dots, e_n$ of A such that $\sum_i e_i = 1$ and $e_i e_j = 0$ for all $i \neq j$. Precisely, $e_1, \dots, e_n$ correspond to $\mathrm{Spec}(Ae_1) \sqcup \cdots \sqcup \mathrm{Spec}(Ae_n)$ (for details, see [Bourbaki 1998, Proposition 15, II, §4, no 3]).

Decompose $\mathrm{Spec}(R)$ into connected components and choose the corresponding idempotents $e_1, \dots, e_n$ as above. Since the nonzero idempotents $f(e_1), \dots, f(e_n)$ satisfy $\sum_{1\le i\le n} f(e_i) = 1$ and $f(e_i)f(e_j) = 0$ for $i \neq j$, we obtain a finite partition $\mathrm{Spec}(S) = \mathrm{Spec}(Sf(e_1)) \sqcup \cdots \sqcup \mathrm{Spec}(Sf(e_n))$. Hence, we only have to prove that $\mathrm{Spec}(Sf(e_i))$ is connected for all $1 \le i \le n$. Let $e' \in \mathrm{Idem}(Sf(e_i))$. By regarding e' as an element of $\mathrm{Idem}(S)$, we obtain an $x \in \mathrm{Idem}(R)$ such that $e' = f(x)$. Since $xe_i \in \mathrm{Idem}(Re_i)$ and $\mathrm{Spec}(Re_i)$ is connected by definition, we either have $xe_i = 0$ or $xe_i = e_i$. Since we have $e' = e'f(e_i) = f(x)f(e_i) = f(xe_i)$, we either have $e' = 0$ or $e' = f(e_i)$. Hence, $Sf(e_i)$ has only trivial idempotents, which implies the assertion. □

Notation 3.4.2. In the remainder of this subsection, we let the notation be as in Notation 3.3.1 and Definition 3.1.8, unless otherwise stated. For an Eisenstein prime ideal $\mathfrak{p}$ of R^+, we fix a norm $|\cdot|_{\mathfrak{p}}$ of the complete discrete valuation field $\kappa(\mathfrak{p})$ and write

$$A_{\kappa(\mathfrak{p})} := (A/\mathfrak{p}A)[S^{-1}].$$

We identify $R^+\langle\underline{X}\rangle/\mathfrak{p}R^+\langle\underline{X}\rangle$ with $\mathcal{O}_{\kappa(\mathfrak{p})}\langle\underline{X}\rangle$, and denote the Gauss norm on $\kappa(\mathfrak{p})\langle\underline{X}\rangle$ by $|\cdot|_{\mathfrak{p}}$. We also denote the quotient (resp. spectral) norm of $|\cdot|_{\mathfrak{p}}$ on $A/\mathfrak{p}A$ and $A_{\kappa(\mathfrak{p})}$ by $|\cdot|_{\mathfrak{p},\mathrm{qt}}$ (resp. $|\cdot|_{\mathfrak{p},\mathrm{sp}}$). For simplicity, we also write $|f \bmod I/\mathfrak{p}I|_{\mathfrak{p},\mathrm{qt}}$ (resp. $|f \bmod I/\mathfrak{p}I|_{\mathfrak{p},\mathrm{qt}}$) by $|f|_{\mathfrak{p},\mathrm{qt}}$ (resp. $|f|_{\mathfrak{p},\mathrm{qt}}$) for $f\in\kappa(\mathfrak{p})\langle\underline{X}\rangle$.

For $f=\sum_{\underline{n}}a_{\underline{n}}\underline{X}^{\underline{n}}\in\mathcal{O}_{\kappa(\mathfrak{p})}\langle\underline{X}\rangle$ with nonzero $a_{\underline{n}}\in\mathcal{O}_{\kappa(\mathfrak{p})}$, let $\tilde{a}_{\underline{n}}\in R^+$ be a lift of $a_{\underline{n}}$. Then, $\tilde{f}:=\sum_{\underline{n}}\tilde{a}_{\underline{n}}\underline{X}^{\underline{n}}\in R^+\langle\underline{X}\rangle$ is called a minimal lift of f.

We may apply Construction 3.2.2 to $R=\mathcal{O}_{\kappa(\mathfrak{p})}$ and $s_1=\pi_{\mathfrak{p}}$ with the same monomial order $\succeq$ for $\mathcal{O}[\![S]\!]$. Let $f_1,\dots,f_s$ be a Gröbner basis of I. Then, the images of f_i's in $R^+/\mathfrak{m}_{R^+}[\underline{X}]$ form a Gröbner basis by Lemma 3.2.6. Hence, the images of f_i's in $\mathcal{O}_{\kappa(\mathfrak{p})}\langle\underline{X}\rangle$ form a Gröbner basis of $I/\mathfrak{p}I$ by Lemma 3.2.6 again. In particular, if $\mathfrak{r}$ is the remainder of $f\in R^+\langle\underline{X}\rangle$ with respect to $f_1,\dots,f_s$, then the image of $\mathfrak{r}$ in $\mathcal{O}_{\kappa(\mathfrak{p})}\langle\underline{X}\rangle$ is the remainder of $f \bmod \mathfrak{p}$ with respect to $f_1 \bmod \mathfrak{p},\dots,f_s \bmod \mathfrak{p}$.

By using our Gröbner basis argument, Lemma 3.1.9 can be converted into the following form:

Lemma 3.4.3. *Let $c\in\mathbb{N}$ and let $\mathfrak{p}$, $\mathfrak{q}$ be Eisenstein prime ideals of R^+ such that $c<\inf(\deg\mathfrak{p},\deg\mathfrak{q})$. Assume that for $n\in\mathbb{N}$, we have*

$$|f^n|_{\mathfrak{p},\mathrm{qt}}\geq|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^c|f|_{\mathfrak{p},\mathrm{qt}}^n,\quad\forall f\in A_{\kappa(\mathfrak{p})}.$$

Then, we have

$$|f^n|_{\mathfrak{q},\mathrm{qt}}\geq|\pi_{\mathfrak{q}}|_{\mathfrak{q}}^c|f|_{\mathfrak{q},\mathrm{qt}}^n,\quad\forall f\in A_{\kappa(\mathfrak{q})}.$$

Proof. We fix a Gröbner basis $f_1,\dots,f_s$ of I. We may regard the $f_i \bmod \mathfrak{p}$'s (resp. $f_i \bmod \mathfrak{q}$'s) as a Gröbner basis of $I/\mathfrak{p}I$ (resp. $I/\mathfrak{q}I$). To prove the assertion, we may assume that $f\in A/\mathfrak{q}A$. Let $\mathfrak{r}\in\mathcal{O}_{\kappa(\mathfrak{q})}\langle\underline{X}\rangle$ be the remainder of f. We have $|f|_{\mathfrak{q},\mathrm{qt}}=|\mathfrak{r}|_{\mathfrak{q}}=|\pi_{\mathfrak{q}}|_{\mathfrak{q}}^m$ for some $m\in\mathbb{N}$. To prove the assertion, we may assume $|f|_{\mathfrak{q},\mathrm{qt}}=|\mathfrak{r}|_{\mathfrak{q}}=1$ by replacing f, $\mathfrak{r}$ by $f/\pi_{\mathfrak{q}}^m$, $\mathfrak{r}/\pi_{\mathfrak{q}}^m$.

Let $\tilde{\mathfrak{r}}\in R^+\langle\underline{X}\rangle$ be a minimal lift of $\mathfrak{r}$ and let $\tilde{f}\in A$ denote the image of $\tilde{\mathfrak{r}}$. Denote by $\mathfrak{r}_n\in R^+\langle\underline{X}\rangle$ the remainder of $\tilde{f}^n$. Then, we have

$$|\mathfrak{r}_n \bmod \mathfrak{p}|_{\mathfrak{p}}=|\tilde{f}^n \bmod \mathfrak{p}|_{\mathfrak{p},\mathrm{qt}}\geq|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^c|\tilde{f} \bmod \mathfrak{p}|_{\mathfrak{p},\mathrm{qt}}^n$$

by Lemma 3.2.13 and by assumption. Since $|\mathfrak{r}|_{\mathfrak{q}}=1$, the coefficient of some $\underline{X}^{\underline{n}}$ in $\mathfrak{r}$ belongs to $\mathcal{O}_{\kappa(\mathfrak{p})}^{\times}$. Therefore, the coefficient of $\underline{X}^{\underline{n}}$ in $\tilde{\mathfrak{r}}$, hence, in $\tilde{\mathfrak{r}} \bmod \mathfrak{p}$ is a unit. Therefore, we have

$$|\tilde{f} \bmod \mathfrak{p}|_{\mathfrak{p},\mathrm{qt}}=|\tilde{\mathfrak{r}} \bmod \mathfrak{p}|_{\mathfrak{p}}=1,$$

hence, $|\mathfrak{r}_n \bmod \mathfrak{p}|_{\mathfrak{p}}\geq|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^c$. By applying Lemma 3.1.9 to the coefficient λ of $\mathfrak{r}_n$ that satisfies $|\lambda \bmod \mathfrak{p}|_{\mathfrak{p}}\geq|\pi_{\mathfrak{p}}|^c$, we obtain $|\mathfrak{r}_n \bmod \mathfrak{q}|_{\mathfrak{q}}\geq|\pi_{\mathfrak{q}}|_{\mathfrak{q}}^c$. Since $\mathfrak{r}_n \bmod \mathfrak{q}$ is the remainder of f^n, we have $|f^n|_{\mathfrak{q},\mathrm{qt}}=|\mathfrak{r}_n \bmod \mathfrak{q}|_{\mathfrak{q}}\geq|\pi_{\mathfrak{q}}|_{\mathfrak{q}}^c$ by Lemma 3.2.13, which implies the assertion. □

The following lemma can be considered as an analogue of Hensel's lemma.

Lemma 3.4.4 (cf. [Xiao 2010, Theorem 1.2.11]). *Assume that there exists $c \in \mathbb{R}_{\geq 0}$ such that*

$$|\cdot|_{\mathfrak{p},\mathrm{sp}} \geq |\pi_{\mathfrak{p}}|^{c}|\cdot|_{\mathfrak{p},\mathrm{qt}} \;\; on\; A_{\kappa(\mathfrak{p})}.$$

Then, for all $\mathfrak{r} \in \mathbb{Q}_{>0} \cap [1/\deg \mathfrak{p}, 1/2c)$, there exists a canonical bijection

$$\pi_0^{\mathrm{Zar}}(A_{\kappa(\mathfrak{p})}) \to \pi_0^{\mathrm{Zar}}(A^{\dagger,r}).$$

Proof. Replacing c by $\lfloor c \rfloor$, we may assume $c \in \mathbb{N}$. Denote by α the canonical map $\mathrm{Idem}(A^{\dagger,r}) \to \mathrm{Idem}(A_{\kappa(\mathfrak{p})})$. By Lemma 3.4.1, we only have to prove that we have $\alpha^{-1}(\{0\}) = \{0\}$ and that α is surjective. Let $e \in \mathrm{Idem}(A^{\dagger,r})$ satisfy $\alpha(e) = 0$. Then, we have $e \in \mathfrak{p} \cdot A^{\dagger,r}$. Since $e = e^n$, we have $e \in \bigcap_{n \in \mathbb{N}} \mathfrak{p}^n \cdot A^{\dagger,r} = 0$ by Corollary 3.3.4, which implies the first assertion. We will prove the surjectivity of α. Let $e \in \mathrm{Idem}(A_{\kappa(\mathfrak{p})})$. Since $|e|_{\mathfrak{p},\mathrm{sp}} = 1 \geq |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{c}|e|_{\mathfrak{p},\mathrm{qt}}$ by assumption, we have $e \in \pi_{\mathfrak{p}}^{-c}A/\mathfrak{p}A$. Hence, we can choose $e' \in A$ such that $e \equiv S^{-c}e' \bmod \mathfrak{p}$. Put $h_0 := S^{-2c}(e'^2 - S^c e') \in A[S^{-1}]$. Since

$$e'^2 - S^c e' \equiv (S^c e)^2 - S^c \cdot S^c e \equiv S^{2c}(e^2 - e) \equiv 0 \bmod \mathfrak{p},$$

we have $h_0 \in \mathfrak{p}S^{-2c} \cdot A$. Since $\mathfrak{p} \subset (p, S^e)R^+$, we obtain

$$|h_0|_{r,\mathrm{qt}} \leq \sup(|S|^e, |p|)|S|^{-2c} = |p^{1-2cr}| < 1.$$

We define sequences $\{f_n\}$ and $\{h_n\}$ in $A[S^{-1}]$ inductively as follows. Put $f_0 := S^{-c}e'$ and let h_0 be as above. For $n \geq 0$, we put

$$f_{n+1} := f_n + h_n - 2h_n f_n, \quad h_{n+1} := f_{n+1}^2 - f_{n+1} \in A[S^{-1}].$$

Note that for $n \in \mathbb{N}$, we have

$$f_{n+1} = -f_n^2(2f_n - 3), \quad f_{n+1} - 1 = -(f_n - 1)^2(2f_n + 1),$$

hence, $h_{n+1} = f_n^2(f_n - 1)^2(4f_n^2 - 4h_n - 3) = h_n^2(4h_n - 3)$. Then, we have

$$|h_{n+1}|_{r,\mathrm{qt}} \leq |h_n|_{r,\mathrm{qt}}^2 \sup(|h_n|_{r,\mathrm{qt}}, 1).$$

Therefore, by induction on n, we have $|h_n|_r < 1$, hence, $|h_{n+1}|_r \leq |h_n|_r^3$. In particular, we have $|h_n|_r \to 0$ for $n \to \infty$. We also have

$$\sup(|f_{n+1}|_{r,\mathrm{qt}}, 1) \leq \sup(|f_n|_{r,\mathrm{qt}}, |h_n|_{r,\mathrm{qt}}, |h_n|_{r,\mathrm{qt}}|f_n|_{r,\mathrm{qt}}, 1) = \sup(|f_n|_{r,\mathrm{qt}}, 1),$$

hence, $\sup(|f_n|_{r,\mathrm{qt}}, 1) \leq \sup(|f_0|_{r,\mathrm{qt}}, 1)$. Therefore, we have

$$|f_{n+1} - f_n|_{r,\mathrm{qt}} = |h_n(1 - 2f_n)|_{r,\mathrm{qt}} \leq |h_n|_{r,\mathrm{qt}} \sup(|f_n|_{r,\mathrm{qt}}, 1) \leq |h_n|_{r,\mathrm{qt}} \sup(|f_0|_{r,\mathrm{qt}}, 1).$$

In particular, $\{f_n\}_n$ is a Cauchy sequence in $A^{\dagger,r}$ with respect to $|\cdot|_{r,\mathrm{qt}}$. The element $f := \lim_{n\to\infty} f_n$ satisfies $f^2 - f = \lim_{n\to\infty} h_n = 0$ and is an idempotent of $A^{\dagger,r}$. Since we have $h_n \in \mathfrak{p} \cdot A^{\dagger,r}$ by induction on n, $f \equiv f_0 \equiv e \bmod \mathfrak{p}$, i.e., $\alpha(f) = e$. $\square$

Proposition 3.4.5 (Continuity of connected components). *Let $A_{\kappa(p)}$ be reduced.*

(i) *There exists $c \in \mathbb{R}_{\geq 0}$ such that*

$$|\cdot|_{(p),\mathrm{sp}} \geq |S|_{(p)}^{c}\,|\cdot|_{(p),\mathrm{qt}}$$

on $A_{\kappa(p)}$. We fix such c in the following.

(ii) *Let $n \in \mathbb{N}_{\geq 2}$ and $\mathfrak{p}$ an Eisenstein prime ideal of R^+ with* $\deg \mathfrak{p} > nc$. *Then*:

$$|\cdot|_{\mathfrak{p},\mathrm{sp}} \geq |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{nc}{n-1}}\,|\cdot|_{\mathfrak{p},\mathrm{qt}} \quad \text{on } A_{\kappa(\mathfrak{p})}.$$

(iii) *Let $\mathfrak{p}$ be an Eisenstein prime ideal of R^+ such that* $\deg \mathfrak{p} > 3c$. *Then*, *for $r \in \mathbb{Q}_{>0} \cap [1/\deg \mathfrak{p}, \frac{1}{2}c)$, there exists a canonical bijection*

$$\pi_0^{\mathrm{Zar}}(A_{\kappa(\mathfrak{p})}) \to \pi_0^{\mathrm{Zar}}(A^{\dagger,r}).$$

In particular, *we have*

$$\#\pi_0(A_{\kappa(\mathfrak{p})}) = \#\pi_0(A_{\kappa(p)}) = \#\pi_0^{\mathrm{Zar}}(A^{\dagger,r}).$$

Proof.

(i) By assumption, $|\cdot|_{(p),\mathrm{sp}}$ is equivalent to $|\cdot|_{(p),\mathrm{qt}}$ on $A_{\kappa(p)}$. Hence, there exists $\lambda \in \mathbb{R}_{>0}$ such that $|\cdot|_{\mathrm{sp}} \geq \lambda|\cdot|_{\mathrm{qt}}$. From $|1|_{\mathrm{sp}} = |1|_{\mathrm{qt}} = 1$, we deduce $\lambda \leq 1$. Hence, $c = \log_{|S|}\lambda \geq 0$ satisfies the condition.

(ii) By (i), we have

$$|f^n|_{(p),\mathrm{qt}} \geq |f^n|_{(p),\mathrm{sp}} = |f|_{(p),\mathrm{sp}}^{n} \geq |S|_{(p)}^{nc}\,|f|_{(p),\mathrm{qt}}^{n}, \quad \forall f \in A_{\kappa(p)}.$$

From Lemma 3.4.3, we obtain

$$|f^n|_{\mathfrak{p},\mathrm{qt}} \geq |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{nc}\,|f|_{\mathfrak{p},\mathrm{qt}}^{n}, \quad \forall f \in A_{\kappa(\mathfrak{p})}.$$

By using this inequality iteratively, we obtain

$$|f^{n^i}|_{\mathfrak{p},\mathrm{qt}} \geq |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{nc+n^2c+\cdots+n^ic}\,|f|_{\mathfrak{p},\mathrm{qt}}^{n^i} = |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{nc(n^i-1)}{n-1}}\,|f|_{\mathfrak{p},\mathrm{qt}}^{n^i}, \quad \forall f \in A_{\kappa(\mathfrak{p})}.$$

Hence, for all $f \in A_{\kappa(\mathfrak{p})}$, we have $|f|_{\mathfrak{p},\mathrm{sp}} = \inf_{i\in\mathbb{N}} |f^{n^i}|_{\mathfrak{p},\mathrm{qt}}^{1/n^i} \geq |\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{nc/(n-1)}\,|f|_{\mathfrak{p},\mathrm{qt}}$.

(iii) When $\mathfrak{p} = (p)$, the assertion follows from (i) and Lemma 3.4.4. We consider the case $\mathfrak{p} \neq (p)$. By applying Lemma 3.4.4 to the inequality in (ii) with $n = 3$, we obtain the assertion for $r \in \mathbb{Q}\cap[1/\deg \mathfrak{p}, \frac{1}{3}c)$. For general $r \in \mathbb{Q}\cap[1/\deg \mathfrak{p}, \frac{1}{2}c)$, the assertion is reduced to the previous case by taking π_0^{Zar} of the commutative diagram

$$\begin{tikzcd} A_{\kappa(p)} \arrow[d, equal, "\mathrm{id}"] & A^{\dagger,r} \arrow[l, "\text{can.}"'] \arrow[r, "\text{can.}"] \arrow[d, "\text{can.}"] & A_{\kappa(\mathfrak{p})} \arrow[d, equal, "\mathrm{id}"] \\ A_{\kappa(p)} & A^{\dagger,\frac{1}{\deg \mathfrak{p}}} \arrow[l, "\text{can.}"'] \arrow[r, "\text{can.}"] & A_{\kappa(\mathfrak{p})} \end{tikzcd}$$

□

Remark 3.4.6. In Theorem 1.2.11 of [Xiao 2010], Xiao proves $\#\pi_0(A_{\kappa(p)}) = \#\pi_0^{\mathrm{Zar}}(A^{\dagger,r})$ under the slightly mild Hypothesis 1.1.10 on A by a similar idea. To generalize Xiao's result for Eisenstein prime ideals, it seems needed to assume that A is flat over R.

To obtain a geometric version of this proposition, we need the following lifting lemma.

Lemma 3.4.7. *Let $\mathfrak{p}$ be an Eisenstein prime ideal of R^+ and $L/\kappa(\mathfrak{p})$ a finite extension. Let $\mathcal{O}'$ be a Cohen ring of k_L and put $R' := \mathcal{O}'[\![T]\!]$. Then, there exists a finite flat morphism $\alpha : R^+ \to R'$ and an isomorphism $R'/\mathfrak{p}R' \cong \mathcal{O}_L$ of $R^+/\mathfrak{p}$-algebras. Moreover, for any Eisenstein prime $\mathfrak{q}$ of R^+, $\mathfrak{q}R'$ is again an Eisenstein prime ideal with degree $e_{L/\kappa(\mathfrak{p})}\deg(\mathfrak{q})$.*

Proof. We can define α similar to the definition of the homomorphism β in Construction 1.6.3: we fix an $\mathcal{O}'$-algebra structure on $\mathcal{O}_L$, and let $f : R' \to \mathcal{O}_L$ be the local $\mathcal{O}'$-algebra homomorphism, which maps T to a uniformizer π_L of L. Write $\pi_{\mathfrak{p}} = \pi_L^{e_{L/\kappa(\mathfrak{p})}}\bar{u}$ with $u \in \mathcal{O}_L^{\times}$. Since f is surjective by Nakayama's lemma, we can choose a lift $u \in (R')^{\times}$ of $\bar{u}$. Since R^+ is p-adically formally smooth over $\mathbb{Z}[S]$, we can define a morphism $\alpha : R^+ \to R'$, which maps S to $T^{e_{L/\kappa(\mathfrak{p})}}u$, by the lifting property.

We claim that $\mathfrak{p}R'$ is an Eisenstein prime. Let P be an Eisenstein polynomial of $\mathcal{O}[S]$ that generates $\mathfrak{p}$. We have $P \equiv T^{\deg(\mathfrak{p})e_{L/\kappa(\mathfrak{p})}}u \bmod pR'$ for some unit $u \in R'$. By the Weierstrass preparation theorem, there exists a distinguished polynomial $Q(T)$ of degree $\deg(\mathfrak{p})e_{L/\kappa(\mathfrak{p})}$ and a unit $U(T) \in R'$ such that $P = Q(T)U(T)$. By evaluating at $T = 0$, we see that $Q(0)$ is equal to p times a unit of $\mathcal{O}'$, which implies the claim. In particular, $R'/\mathfrak{p}R'$ is a discrete valuation ring. Hence, the canonical surjection $R'/\mathfrak{p}R' \to \mathcal{O}_L$ induced by f is an isomorphism. By Nakayama's lemma and the local criteria of flatness, α is finite flat. The second assertion also follows from the Weierstrass preparation theorem. □

The following is our main result of this subsection:

Proposition 3.4.8 (continuity of geometric connected components). *Assume that $A_{\kappa(p)}$ is geometrically reduced.*

(i) *If all connected components of $A_{\kappa(p)}$ are geometrically connected, then all connected components of $A_{\kappa(\mathfrak{p})}$ are also geometrically connected for all Eisenstein prime ideals $\mathfrak{p}$ of R^+ with* $\deg\mathfrak{p} \gg 0$.

(ii) *For all Eisenstein prime ideals $\mathfrak{p}$ of R^+ with* $\deg\mathfrak{p} \gg 0$, *we have*

$$\#\pi_0^{\mathrm{geom}}(A_{\kappa(\mathfrak{p})}) = \#\pi_0^{\mathrm{geom}}(A_{\kappa(p)}).$$

Proof.

(i) By assumption, there exists $c \in \mathbb{R}_{\geq 0}$ such that $|\cdot|_{(p),\mathrm{sp}} \geq |S|_{(p)}^{c}|\cdot|_{(p),\mathrm{qt}}$ on $A_{\kappa(p)} \otimes_{\kappa(p)} \kappa(p)^{\mathrm{alg}}$. We prove that all Eisenstein prime ideals $\mathfrak{p}$ of R^+ with $\deg(\mathfrak{p}) > 3c$ satisfy the condition. Let $L/\kappa(\mathfrak{p})$ be a finite extension. Let R' be as in Lemma 3.4.7. Since R' is finite flat over R^+, we have $R^+\langle \underline{X}\rangle \otimes_{R^+} R' \cong R'\langle \underline{X}\rangle$ and $I' := I \otimes_{R^+\langle \underline{X}\rangle} R'\langle \underline{X}\rangle \cong I \cdot R'\langle \underline{X}\rangle$. Hence, we can apply Proposition 3.4.5 to $R^+ = R'$, $I = I'$ and $A = A' := A \otimes_{R^+} R' \cong R'\langle \underline{X}\rangle / I'$. Note that $ce_{L/\kappa(\mathfrak{p})}$ can be taken as c in Proposition 3.4.5(i). Therefore, Proposition 3.4.5(iii) yields

$$\begin{aligned}\#\pi_0^{\mathrm{Zar}}(A_{\kappa(\mathfrak{p})} \otimes_{\kappa(\mathfrak{p})} L) = \#\pi_0^{\mathrm{Zar}}(A'_{\kappa(\mathfrak{p}R')}) = \pi_0^{\mathrm{Zar}}(A'_{\kappa(p)}) \\ = \#\pi_0^{\mathrm{Zar}}(A_{\kappa(p)}) = \#\pi_0^{\mathrm{Zar}}(A_{\kappa(\mathfrak{p})}),\end{aligned}$$

where the third equality follows from the assumption. Therefore, we have $\#\pi_0^{\mathrm{geom}}(A_{\kappa(\mathfrak{p})}) = \#\pi_0(A_{\kappa(\mathfrak{p})})$, which implies the assertion.

(ii) Let $L/\kappa(p)$ be a finite extension such that all connected components of $A_{\kappa(p)} \otimes_{\kappa(p)} L$ are geometrically connected. Let R' be a lifting of $\mathcal{O}_L$ as in Lemma 3.4.7 and A' as in the proof of (i). Part (i) and Proposition 3.4.5(iii) give the assertion. □

3.5. ***Application: Ramification compatibility of fields of norms.*** In this subsection, we prove Theorem 3.5.3, which is the ramification compatibility of Scholl's equivalence in Theorem 1.8.3, as an application of our Gröbner basis argument.

We first construct a characteristic zero lift of the Abbes–Saito space in characteristic p.

Lemma 3.5.1. *Let F/E be a finite extension of complete discrete valuation fields of characteristic p. Assume that the residue field extension k_F/k_E is either trivial or purely inseparable. For $m \in \mathbb{N}$, we put $\underline{X} := (X_0, \dots, X_m)$ and $\underline{Y} := (Y_0, \dots, Y_m)$.*

(i) [*Xiao 2010, Notation* 3.3.8] *For some $m \in \mathbb{N}$, there exist a set of generators $\{z_0, \dots, z_m\}$ of $\mathcal{O}_F$ as an $\mathcal{O}_E$-algebra, with z_0 a uniformizer of F, and a set of generators $\{p_0, \dots, p_m\}$ of the kernel of the $\mathcal{O}_E$-algebra homomorphism $\mathcal{O}_E\langle \underline{X}\rangle \twoheadrightarrow \mathcal{O}_F$ defined by $X_j \mapsto z_j$ such that*

$$\begin{aligned}p_0 &= X_0^{e_{F/E}} + \pi_E \eta_0,\\ p_j &= X_j^{f_j} - \varepsilon_j + X_0 \delta_j + \pi_E \eta_j \quad \textit{for } 1 \leq j \leq m,\end{aligned}$$

where $\delta_j, \eta_j \in \mathcal{O}_E\langle \underline{X}\rangle$, $\varepsilon_j \in \mathcal{O}_E\langle X_0, \dots, X_{j-1}\rangle$ and $f_j \in \mathbb{N}$.

(ii) *Let $\succeq$ be the lexicographic order on $\mathcal{O}_E\langle \underline{X}\rangle$ defined by $X_m \succ \cdots \succ X_0$. We view π_E as a regular system of parameters of $\mathcal{O}_E$ and apply Construction 3.2.2. Then, we have $\mathrm{LT}_{\mathcal{O}_E}(p_0^n) = X_0^{ne_{F/E}}$ for all $n \in \mathbb{N}$. Let $l, n \in \mathbb{N}_{>0}$ satisfy*

$p^l n \geq e_{F/E}$. Then, for $1 \leq j \leq m$, there exists $\theta_{j,l,n} \in \mathcal{O}_E\langle \underline{X} \rangle$ such that $\mathrm{LT}_{\mathcal{O}_E}(p_j^{p^l n} - p_0^{\lfloor p^l n/e_{F/E} \rfloor}\theta_{j,l,n}) = uX_j^{f_j p^l n}$ for some unit $u \in 1 + \pi_E\mathcal{O}_E$.

(iii) (*cf.* [Xiao 2010, *Example* 1.3.4.]). *Fix an isomorphism $E \cong k_E((S))$. Let $\mathcal{O}$ be a Cohen ring of k_E and let $R := \mathcal{O}[\![S]\!]$ with canonical projection $R \to \mathcal{O}_E$. Fix a lift $P_j \in R\langle \underline{X} \rangle$ of p_j for all j. Let $\underline{\alpha} \in \mathbb{N}^{m+1}$, $\underline{\beta} \in \mathbb{N}_{>0}^{m+1}$. Assume that $\lfloor \beta_j/e_{F/E} \rfloor \geq \beta_0$ for all $1 \leq j \leq m$, and assume that there exists $l \in \mathbb{N}_{>0}$ such that $p^l \mid \beta_j$ for all $1 \leq j \leq m$. Then, the R-algebra*

$$A_{\underline{\alpha},\underline{\beta}} := R\langle \underline{X}, \underline{Y} \rangle/(S^{\alpha_j}Y_j - P_j^{\beta_j},\ 0 \leq j \leq m).$$

is R-flat. Moreover, the fiber of $A_{\underline{\alpha},\underline{\beta}}$ at any Eisenstein prime $\mathfrak{p}$ of R is an affinoid variety, which gives rise to the following affinoid subdomain of $D^{m+1}_{\kappa(\mathfrak{p})}$:

$$D^{m+1}(|\pi_{\mathfrak{p}}|^{-\alpha_j/\beta_j}(P_j \bmod \mathfrak{p}),\ 0 \leq j \leq m).$$

Proof.

(i) See [Xiao 2010, Construction 3.3.5] for details.

(ii) Since the coefficient of $X_0^{ne_{F/E}}$ in p_0^n is equal to 1, the first assertion follows from $p_0^n \equiv X_0^{ne_{F/E}} \bmod \pi_E$. For the second, we put $\theta_{j,l,n} := X_0^{p^l n - e_{F/E}\lfloor p^l n/e_{F/E}\rfloor}\delta_j^{p^l n}$. Since

$$p_j^{p^l n} \equiv X_j^{p^l n f_j} - \varepsilon_j^{p^l n} + X_0^{p^l n}\delta_j^{p^l n} \equiv X_j^{p^l n f_j} - \varepsilon_j^{p^l n} + p_0^{\lfloor p^l n/e_{F/E}\rfloor}\theta_{j,l,n} \bmod \pi_E,$$

we have $\mathrm{LT}_{k_E}(p_j^{p^l n} - p_0^{\lfloor p^l n/e_{F/E}\rfloor}\theta_{j,l,n} \bmod \pi_E) = \mathrm{LT}_{k_E}(X_j^{p^l n f_j} - \varepsilon_j^{p^l n} \bmod \pi_E) = X_j^{f_j p^l n}$, which implies the assertion.

(iii) The last assertion is trivial. We prove the first assertion. Let $\succeq$ be the lexicographic order on $\mathcal{O}_E\langle \underline{X}, \underline{Y} \rangle$ defined by $X_m \succ \cdots \succ X_0 \succ Y_m \succ \cdots \succ Y_0$. We view $\{p, S\}$ as a regular system of parameters of R and apply Construction 3.2.2. For $1 \leq j \leq m$, we choose a lift of $\theta_{j,l,\beta_j/p^l}$ and denote it by Θ_j for simplicity. Then, the ideal $(S^{\alpha_j}Y_j - P_j^{\beta_j},\ 0 \leq j \leq m)$ is generated by $Q_0 := S^{\alpha_0}Y_0 - P_0^{\beta_0}$ and

$$Q_j := S^{\alpha_j}Y_j - P_j^{\beta_j} - (S^{\alpha_0}Y_0 - P_0^{\beta_0})P_0^{\lfloor \beta_j/e_{F/E}\rfloor - \beta_0}\Theta_j$$

for $1 \leq j \leq m$. It follows from Proposition 3.2.12 that we only have to prove that $\mathrm{LT}_{R/\mathfrak{m}_R}(-Q_j \bmod \mathfrak{m}_R)$ are relatively prime monic monomials. We have $\mathrm{LT}_{R/\mathfrak{m}_R}(Q_0 \bmod \mathfrak{m}_R) = -\mathrm{LT}_{R/\mathfrak{m}_R}(p_0^{\beta_0}) = -X_0^{e_{F/E}\beta_0}$. Since

$$Q_j \equiv -p_j^{\beta_j} + p_0^{\lfloor \beta_j/e_{F/E}\rfloor}\theta_{j,l,\beta_j/p^l} \bmod \mathfrak{m}_R,$$

we have $\mathrm{LT}_{R/\mathfrak{m}_R}(Q_j \bmod \mathfrak{m}_R) = -X_j^{f_j\beta_j}$ by (ii), which yields the assertion. $\square$

In the rest of this subsection, let the notation be as in Definition 1.8.1.

Lemma 3.5.2. *Fix an isomorphism $X_{\mathfrak{K}} \cong k_{\mathfrak{K}}((\Pi))$, let $\mathcal{O}$ be a Cohen ring of $k_{\mathfrak{K}}$ and put $R := \mathcal{O}[\![\Pi]\!]$.*

(i) *There exists a surjective local ring homomorphism $\phi_n : R \to \mathcal{O}_{K_n}$ for all sufficiently large n such that diagram*

$$\begin{tikzcd} R \arrow[r, two heads, "\text{can.}"] \arrow[d, "\phi_n"] & X_{\mathfrak{K}}^{+} \arrow[d, "\mathrm{pr}_n"] \\ \mathcal{O}_{K_n} \arrow[r, two heads, "\text{can.}"] & \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n} \end{tikzcd}$$

commutes, and $\ker(\phi_n)$ *is an Eisenstein prime ideal of R. We fix ϕ_n in the following and put $\mathfrak{p}_n := \ker(\phi_n)$.*

(ii) *Let $r \in \mathbb{Q}_{>0}$ and let L_∞/K_∞ be a finite extension and $\mathfrak{L} = \{L_n\}_{n>0}$ a corresponding strictly deeply ramified tower. Assume that the residue field extension of $X_{\mathfrak{L}}/X_{\mathfrak{K}}$ is either trivial or purely inseparable. Then, there exists a flat R-algebra AS^r (resp. $AS^r_{\log}$) of the form $R\langle \underline{X}\rangle/I$ for an ideal $I \subset R\langle\underline{X}\rangle$, whose fibers at (p) and $\mathfrak{p}_n$ are isomorphic to the Abbes–Saito spaces $as^r_{X_{\mathfrak{L}}/X_{\mathfrak{K}},\bullet}$ and $as^r_{L_n/K_n,\bullet}$ (resp. $as^r_{X_{\mathfrak{L}}/X_{\mathfrak{K}},\bullet,\bullet}$ and $as^r_{L_n/K_n,\bullet,\bullet}$) for all sufficiently large n.*

(iii) *With the notation and assumption of (ii), we have for all sufficiently large n:*

$$\#\mathcal{F}^r(X_{\mathfrak{L}}) = \#\mathcal{F}^r(L_n), \quad \#\mathcal{F}^r_{\log}(X_{\mathfrak{L}}) = \#\mathcal{F}^r_{\log}(L_n).$$

Proof. Put $E := X_{\mathfrak{K}}$ and $F := X_{\mathfrak{L}}$.

(i) For all sufficiently large n, the projection $\mathrm{pr}_n : \mathcal{O}_E \to \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n}$ induces an isomorphism $\Phi_n : k_{\mathfrak{K}} \to k_{K_n}$ of the residue fields. Hence, we can choose an embedding $\mathcal{O} \to \mathcal{O}_{K_n}$ that lifts Φ_n. Let π_{K_n} be a uniformizer of $\mathcal{O}_{K_n}$, which is a lift of $\mathrm{pr}_n(\Pi) \in \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n}$. Since the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\Pi] \to R$; $\Pi \mapsto \Pi$ is formally étale, we have a map ϕ_n sending Π to π_{K_n}. Since $\mathcal{O}_{K_n}/\mathcal{O}$ is totally ramified, the kernel of ϕ_n is generated by an Eisenstein polynomial.

(ii) Fix $\xi' \in \mathcal{O}_{K_\infty}$ such that $0 < v_p(\xi') < v_p(\xi)$ and such that $\{L_n\}_{n>0}$ is strictly deeply ramified with respect to ξ'. We denote the composite $\mathrm{can} \circ \mathrm{pr}_n : \mathcal{O}_E \to \mathcal{O}_{K_n}/\xi\mathcal{O}_{K_n} \to \mathcal{O}_{K_n}/\xi'\mathcal{O}_{K_n}$ by pr_n again, and fix an expression $r = a/b$ with $a, b \in \mathbb{N}$ and $b > 0$. Also, fix $l \in \mathbb{N}$ with $p^l \geq e_{F/E}$. Define $\underline{\alpha}, \underline{\alpha}_{\log}, \underline{\beta}, \underline{\beta}_{\log} \in \mathbb{N}^l$ via $\alpha_0 := a$, $\alpha_{\log,0} := a + b$, $\beta_0 := \beta_{\log,0} := b$, and $\alpha_j = \alpha_{\log,j} = ap^l$, $\beta_j = \beta_{\log,j} := bp^l$ for $1 \leq j \leq m$. Then, we can apply Lemma 3.5.1 to the finite extension F/E. In the following, we use the notation as of that lemma. We will prove that $A_{\underline{\alpha},\underline{\beta}}$ (resp. $A_{\underline{\alpha}_{\log},\underline{\beta}_{\log}}$) satisfies the desired condition. We first consider the nonlog case. By Lemma 3.5.1(iii), the fiber of $A_{\underline{\alpha},\underline{\beta}}$ at (p) is isomorphic to

$as^r_{F/E,Z}$, where $Z=\{z_0,\dots,z_m\}$. Recall that we have a canonical surjection $\mathrm{pr}_n:\mathcal{O}_F\to\mathcal{O}_{L_n}/\xi'\mathcal{O}_{L_n}$ for all sufficiently large n. We choose a lift $z_j^{(n)}\in\mathcal{O}_{L_n}$ of $\mathrm{pr}_n(z_j)\in\mathcal{O}_{L_n}/\xi'\mathcal{O}_{L_n}$. Then, the $z_j^{(n)}$'s are generators of $\mathcal{O}_{L_n}$ as an $\mathcal{O}_{K_n}$-algebra by Nakayama's lemma and, by lemma Lemma 3.5.1(i), $z_j^{(0)}$ is a uniformizer of $\mathcal{O}_{L_n}$. We consider the surjection $\varphi_n:\mathcal{O}_{K_n}\langle\underline{X}\rangle\to\mathcal{O}_{L_n}$; $X_j\mapsto z_j^{(n)}$ and choose a lift $p_j^{(n)}\in\ker(\varphi_n)$ of $\mathrm{pr}_n(p_j)\in\mathcal{O}_{K_n}/\xi'\mathcal{O}_{K_n}[\underline{X}]$:

$$\begin{tikzcd}
\mathcal{O}_E\langle\underline{X}\rangle \arrow[rr, two heads, "X_j\mapsto z_j"] \arrow[d, "\mathrm{pr}_n"'] & & \mathcal{O}_F \arrow[d, "\mathrm{pr}_n"] \\
\mathcal{O}_{K_n}/\xi'\mathcal{O}_{K_n}[\underline{X}] \arrow[rr, two heads, "X_j\mapsto \mathrm{pr}_n(z_j)"] & & \mathcal{O}_{L_n}/\xi'\mathcal{O}_{L_n} \\
\mathcal{O}_{K_n}\langle\underline{X}\rangle \arrow[rr, two heads, "\varphi_n;X_j\mapsto z_j^{(n)}"] \arrow[u, two heads, "\text{can.}"] & & \mathcal{O}_{L_n}. \arrow[u, two heads, "\text{can.}"']
\end{tikzcd}$$

By Nakayama's lemma, the $p_j^{(n)}$'s are generators of $\ker(\varphi_n)$. We may assume $v_{K_n}(\xi')\ge r$ by choosing n sufficiently large. Since $\phi_n(P_j)\equiv p_j^{(n)} \bmod (\xi')$, we have $|\phi_n(P_j)(x)|\le|\pi_{K_n}|^r$ if and only if $|p_j^{(n)}(x)|\le|\pi_{K_n}|^r$ for any $x\in\mathcal{O}_{\overline{K}}^{m+1}$. This implies that the fiber of AS^r at $\mathfrak{p}_n$ is isomorphic to $as^r_{L_n/K_n,Z^{(n)}}$, where $Z^{(n)}=\{z_0^{(n)},\dots,z_m^{(n)}\}$, which implies the assertion. In the log case, a similar proof works if we choose n sufficiently large such that $v_{K_n}(\xi')\ge r+1$.

(iii) This follows from applying Proposition 3.4.8 to AS^r and $AS^r_{\log}$. □

The following is the main theorem in this subsection. See [Hattori 2014, §6] for an alternative proof.

Theorem 3.5.3. *Let L_∞/K_∞ be a finite separable extension and $\mathfrak{L}=\{L_n\}_{n>0}$ a corresponding strictly deeply ramified tower. Then, the sequence $\{b(L_n/K_n)\}_{n>0}$ (resp. $\{b_{\log}(L_n/K_n)\}_{n>0}$) converges to $b(X_{\mathfrak{L}}/X_{\mathfrak{K}})$ (resp. $b_{\log}(X_{\mathfrak{L}}/X_{\mathfrak{K}})$).*

Proof. Since the nonlog and log ramification filtrations are invariant under base change, so are the nonlog and log ramification breaks. Hence, we may assume that the residue field extension of $X_{\mathfrak{L}}/X_{\mathfrak{K}}$ is either trivial or purely inseparable by replacing K_∞ and L_∞ by their maximal unramified extensions. We first prove the nonlog case. Recall that we have $[X_{\mathfrak{L}}:X_{\mathfrak{K}}]=[L_n:K_n]$ for all sufficiently large n by Theorem 1.8.3. For $r\in\mathbb{Q}_{>0}$ with $b(X_{\mathfrak{L}}/X_{\mathfrak{K}})<r$, we have $\#\mathcal{F}^r(L_n)=\#\mathcal{F}^r(X_{\mathfrak{L}})=[L_n:K_n]$ for all sufficiently large n by Lemma 3.5.2. Hence, we have $\limsup_n b(L_n/K_n)\le b(X_{\mathfrak{L}}/X_{\mathfrak{K}})$. For $r\in\mathbb{Q}_{>0}$ with $b(X_{\mathfrak{L}}/X_{\mathfrak{K}})>r$, we have $\#\mathcal{F}^r(L_n)=\#\mathcal{F}^r(X_{\mathfrak{L}})<[L_n:K_n]$ for all sufficiently large n by Lemma 3.5.2 and the definition of $\mathcal{F}^r$. Hence, we have $\liminf_n b(L_n/K_n)\ge b(X_{\mathfrak{L}}/X_{\mathfrak{K}})$. Therefore, we have $b(X_{\mathfrak{L}}/X_{\mathfrak{K}}) \le \liminf_n b(L_n/K_n) \le \limsup_n b(L_n/K_n) \le b(X_{\mathfrak{L}}/X_{\mathfrak{K}})$, which implies the assertion. In the log case, the same argument with b and $\mathcal{F}^r$ replaced by $b_{\log}$ and $\mathcal{F}^r_{\log}$ works. $\square$

The following representation version of Theorem 3.5.3 will be used in the proof of Theorem 4.7.1.

Lemma 3.5.4. *Let $F/\mathbb{Q}_p$ be a finite extension and let $V \in \mathrm{Rep}^f_F(G_{K_n})$ a finite F-representation for some n. We identify $G_{X_{\mathfrak{K}}}$ with G_{K_∞} via the equivalence in Theorem 1.8.3.*

(i) *For $m \ge n$, let L_m (resp. L_∞, X') be the finite Galois extension corresponding to the kernel of the action of G_{K_m} (resp. G_{K_∞}, $G_{X_{\mathfrak{K}}}$) on V. Then, L_∞ corresponds to X' under the equivalence in Theorem 1.8.3 and $\{L_m\}_{m\ge n}$ is a strictly deeply ramified tower corresponding to L_∞.*

(ii) *The sequences $\left\{\mathrm{Art}^{\mathrm{AS}}(V|_{K_m})\right\}_{m\ge n}$ and $\left\{\mathrm{Swan}^{\mathrm{AS}}(V|_{K_m})\right\}_{m\ge n}$ are eventually stationary and their limits are equal to $\mathrm{Art}^{\mathrm{AS}}(V|_{X_{\mathfrak{K}}})$ and $\mathrm{Swan}^{\mathrm{AS}}(V|_{X_{\mathfrak{K}}})$.*

Proof.

(i) The first assertion is trivial. We prove the second assertion. Since $G_{L_n} \cap G_{K_m} = G_{L_m}$ for all $m \ge n$, we have $L_m = L_n K_m$. Therefore, $\{L_m\}$ is a strictly deeply ramified tower corresponding to $L'_\infty := \bigcup_m L_m$. Hence, we only have to prove that $L_\infty = L'_\infty$. Let $\rho : G_{K_n} \to \mathrm{GL}(V)$ be a matrix presentation of V. By the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G_{L_\infty} \arrow[r, "\text{inc.}"] & G_{K_\infty} \arrow[r, "\rho"] \arrow[d, hook, "\text{can.}"] & \mathrm{GL}(V) \arrow[d, "\text{id}"] \\ 1 \arrow[r] & G_{L_m} \arrow[r, "\text{inc.}"] & G_{K_m} \arrow[r, "\rho|_{G_{K_m}}"] & \mathrm{GL}(V), \end{tikzcd}$$

where the horizontal sequences are exact, we obtain a canonical injection $G_{L_\infty} \hookrightarrow G_{L_m}$. Therefore, we have $L_m \subset L_\infty$, hence, $L'_\infty \subset L_\infty$. To prove the converse, we only have to prove $[L_\infty : K_\infty] \le [L'_\infty : K_\infty]$. Since $(K_\infty \cap L_n)/K_n$ is finite, we have $K_\infty \cap L_n = K_m \cap L_n$ for sufficiently large m. In particular,

$$\begin{aligned}[L'_\infty : K_\infty] &= [L_n K_\infty : K_\infty] = [L_n : K_\infty \cap L_n] \\ &= [L_n : K_m \cap L_n] = [L_n K_m : K_m] = [L_m : K_m].\end{aligned}$$

Then, the assertion follows from

$$[L_\infty : K_\infty] = \#\rho(G_{K_\infty}) \le \#\rho(G_{K_m}) = [L_m : K_m].$$

(ii) By Maschke's theorem, there exists an irreducible decomposition $V|_{X_{\mathfrak{K}}} = \bigoplus_\lambda V^\lambda$ with $V^\lambda \in \mathrm{Rep}^f_F(G_{X_{\mathfrak{K}}})$. We choose $m_0 \in \mathbb{N}$ such that the canonical map $G_{L_\infty/K_\infty} \to G_{L_m/K_m}$ is an isomorphism for all $m \geq m_0$. Then, V^λ is G_{K_m}-stable for all $m \geq m_0$. Moreover, $V^\lambda|_{K_m} \in \mathrm{Rep}^f_F(G_{K_m})$ is irreducible. For $m \geq m_0$, let L^λ_m/K_m be the finite Galois extension corresponding to the kernel of the action of G_{K_m} on V^λ. By (i), $\mathfrak{L}^\lambda = \{L^\lambda_m\}_{m\geq m_0}$ is a strictly deeply ramified tower and $X_{\mathfrak{L}^\lambda}$ corresponds to the kernel of the action of $G_{X_\mathfrak{K}}$ on V^λ. By the irreducibility of the action of G_{K_m} (resp. $G_{X_\mathfrak{K}}$) on V^λ, we have

$$\mathrm{Art}^{\mathrm{AS}}(V^\lambda|_{K_m}) = b(L^\lambda_m/K_m)\dim_F(V),$$
$$\mathrm{Art}^{\mathrm{AS}}(V^\lambda|_{X_\mathfrak{K}}) = b(X_{\mathfrak{L}^\lambda}/X_\mathfrak{K})\dim_F(V)$$

for $m \geq m_0$. We apply Theorem 3.5.3 to each $\mathfrak{L}^\lambda$, to get $\lim_{m\to\infty}\mathrm{Art}(V|_{K_m}) = \mathrm{Art}(V|_{X_\mathfrak{K}})$. Note that K_m is not absolutely unramified for sufficiently large m. Indeed, the definition of strictly deeply ramified implies that K_{m+1}/K_m is not unramified. By Theorem 1.5.1, the convergence of $\{\mathrm{Art}(V|_{K_m})\}$ implies that $\{\mathrm{Art}(V|_{K_m})\}$ is eventually stationary, which implies the assertion for the Artin conductor. The assertion for the Swan conductor follows from the same argument by replacing Art and b by Swan and $b_{\log}$. □

Remark 3.5.5 (a Hasse–Arf property). Let the notation be as in Lemma 3.5.4 and let $p = 2$. By Theorem 1.7.10 and Lemma 3.5.4(ii), $\mathrm{Swan}(V|_{K_m})$ is an integer for all sufficiently large m (cf. Theorem 1.5.1).

4. Differential modules associated to de Rham representations

In this section, we first construct $\mathbb{N}_{\mathrm{dR}}(V)$ as a (φ, Γ_K)-module for de Rham representations $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, see Section 4.2. Then, we prove that $\mathbb{N}_{\mathrm{dR}}(V)$ can be endowed with a (φ, ∇)-module structure (Section 4.4). Then, we define Swan conductors of de Rham representations (Section 4.6) and we prove that the differential Swan conductor of $\mathbb{N}_{\mathrm{dR}}(V)$ and Swan conductor of V are compatible (Section 4.7).

Throughout this section, let K be a complete discrete valuation field of mixed characteristic $(0, p)$. Except for Section 4.6, we assume that K satisfies Assumption 1.9.1, and we use the notation of Section 1.3.

4.1. ***Calculation of horizontal sections.*** For perfect k_K, $\mathbb{N}_{\mathrm{dR}}(V)$ is constructed by gluing a certain family of vector bundles over $K_n[\![t]\!]$ for $n \gg 0$, see [Berger 2008b, Section II.1]. When k_K is not perfect, $K_n[\![t]\!]$ should be replaced by the ring of horizontal sections of $K_n[\![u, t_1, \dots, t_d]\!]$ with respect to the connection ∇^{geom}, which will be studied in this subsection.

Definition 4.1.1. (i) We have a canonical K_n-algebra injection

$$K_n[\![t, u_1, \dots, u_d]\!] \to \mathbb{B}^+_{\mathrm{dR}}$$

since $\mathbb{B}_{\mathrm{dR}}^{+}$ is a complete local K^{alg}-algebra. The topology of $K_n[\![t, u_1, \dots, u_d]\!]$ as a subring of $\mathbb{B}_{\mathrm{dR}}^{+}$ (endowed with the canonical topology) is called the canonical topology. Note that $K_n[\![t, u_1, \dots, u_d]\!]$ is stable under the G_K-action, and that the G_K-action factors through Γ_K.

(ii) Let F be a complete valuation field. The Fréchet topology on

$$F[\![X_1, \dots, X_n]\!] \cong \varprojlim_m F[X_1, \dots, X_n]/(X_1, \dots, X_n)^m$$

is the inverse limit topology, where $F[X_1, \dots, X_n]/(X_1, \dots, X_n)^m$ is endowed with a (unique) topological F-vector space structure. Note that $F[\![X_1, \dots, X_n]\!]$ is a Fréchet space, and that the $(X_1, \dots, X_n)$-adic topology of $F[\![X_1, \dots, X_n]\!]$ is finer than the Fréchet topology.

Lemma 4.1.2. *The canonical topology of $K_n[\![t, u_1, \dots, u_d]\!]$ and the Fréchet topology are equivalent. In particular, $K_n[\![t, u_1, \dots, u_d]\!]$ is a closed subring of $\mathbb{B}_{\mathrm{dR}}^{+}$.*

Proof. Put $V_m := K_n[t, u_1, \dots, u_d]/(t, u_1, \dots, u_d)^m$ and identify $K_n[\![t, u_1, \dots, u_d]\!]$ with $\varprojlim_m V_m$. If we endow V_m with a (unique) topological K_n-vector space structure, then the resulting inverse limit topology is the Fréchet topology. We have a canonical injection $V_m \to \mathbb{B}_{\mathrm{dR}}^{+}/(t, u_1, \dots, u_d)^m$. If we endow V_m with the subspace topology as a subset of $\mathbb{B}_{\mathrm{dR}}^{+}/(t, u_1, \dots, u_d)^m$, which is endowed with the canonical topology, then the resulting inverse limit topology is the canonical topology. Since $\mathbb{B}_{\mathrm{dR}}^{+}/(t, u_1, \dots, u_d)^m$ is K_n-Banach space by definition, V_m endowed with this topology is a topological K_n-vector space. This implies the assertion. □

Notation 4.1.3. The subring $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla^{\mathrm{geom}}=0} = \mathbb{B}_{\mathrm{dR}}^{\nabla+} \cap K_n[\![t, u_1, \dots, u_d]\!]$ of $\mathbb{B}_{\mathrm{dR}}^{\nabla+}$ is denoted by $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ for $n \in \mathbb{N}$. We call the subspace topology of $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ as a subring of $\mathbb{B}_{\mathrm{dR}}^{+}$ (endowed with the canonical topology) the canonical topology. Note that $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ is a closed subring of $\mathbb{B}_{\mathrm{dR}}^{\nabla+}$ since the connection $\nabla^{\mathrm{geom}} : \mathbb{B}_{\mathrm{dR}}^{+} \to \mathbb{B}_{\mathrm{dR}}^{+} \otimes_K \widehat{\Omega}_K^1$ is continuous and $\mathbb{B}_{\mathrm{dR}}^{\nabla+}$ is closed in $\mathbb{B}_{\mathrm{dR}}^{+}$.

Lemma 4.1.4. *The ring $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ is a complete discrete valuation ring with residue field K_n and uniformizer t.*

Proof. We define a map

$$f : K_n[t, u_1, \dots, u_d] \to K_n[\![t, u_1, \dots, u_d]\!]$$
$$x \mapsto \sum_{(n_1,\dots,n_d)\in\mathbb{N}^d} \frac{(-1)^{n_1+\dots+n_d}}{n_1! \dots n_d!} u_1^{n_1} \dots u_d^{n_d}\, \partial_1^{n_1} \circ \dots \circ \partial_d^{n_d}(x).$$

It is easy to check that this is an abstract ring homomorphism such that $\mathrm{Im}(f) \subset K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$, $f(tx) = tf(x)$ for all $x \in K_n[t, u_1, \dots, u_d]$ and $f(u_j) = 0$ for all j. In particular, f is $(t, u_1, \dots, u_d)$-adically continuous. Passing to the completion, we obtain a ring homomorphism $f : K_n[\![t, u_1, \dots, u_d]\!] \to K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$. Since f is identity on $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$, f is surjective and f induces a surjection

$$\bar{f} : K_n[\![t]\!] \cong K_n[\![t, u_1, \dots, u_d]\!]/(u_1, \dots, u_d) \to K_n[\![t, u_1, \dots, u_d]\!]^{\nabla},$$

where the first isomorphism is induced by the inclusion $K_n[\![t]\!] \subset K_n[\![t, u_1, \dots, u_d]\!]$. Since $\bar{f}(t) = t$ is nonzero, $\bar{f}$ is an isomorphism, which implies the assertion. □

Lemma 4.1.5. *The t-adic topology on $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ is finer than the canonical topology.*

Proof. Denote $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ by R and identify R with $\varprojlim_m R/t^m R$. If we endow $R/t^m R$ with the discrete topology, then the resulting inverse limit topology is the t-adic topology. By Lemma 4.1.4 and dévissage, the canonical map $R/t^m R \to K_n[t, u_1, \dots, u_d]/(t, u_1, \dots, u_d)^m$ is injective. If we endow $R/t^m R$ with the subspace topology as a subset of $K_n[t, u_1, \dots, u_d]/(t, u_1, \dots, u_d)^m$, endowed with a (unique) topological K_n-vector space structure, then the resulting inverse limit topology is the canonical topology. Since the discrete topology is the finest topology, we obtain the assertion. □

The map f defined in the proof of Lemma 4.1.4 is continuous when $K = \widetilde{K}$:

Lemma 4.1.6. *Let $\varphi : \mathcal{O}_{\widetilde{K}} \to \mathcal{O}_{\widetilde{K}}$ be the unique Frobenius lift, characterized by $\varphi(t_j) = t_j^p$ for all $1 \le j \le d$. Then, the map $f : \widetilde{K}_n[\![t, u_1, \dots, u_d]\!] \to \widetilde{K}_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ defined in the proof of Lemma 4.1.4 is continuous with respect to the Fréchet topologies.*

Proof. By the definition of f, we only have to prove the following claim: for all $m \in \mathbb{N}$ and $1 \le j \le d$, we have

$$\partial_j^m(\mathcal{O}_{\widetilde{K}}) \subset m!\,\mathcal{O}_{\widetilde{K}}.$$

We first note since $d : \mathcal{O}_{\widetilde{K}} \to \widehat{\Omega}^1_{\mathcal{O}_{\widetilde{K}}}$ and $\varphi_* : \widehat{\Omega}^1_{\mathcal{O}_{\widetilde{K}}} \to \widehat{\Omega}^1_{\mathcal{O}_{\widetilde{K}}}$ commute, we have

$$\partial_j \circ \varphi^i = p^i t_j^{p^i - 1} \varphi^i \circ \partial_j \tag{5}$$

for all $i \in \mathbb{N}$ and $1 \le j \le d$. We prove the claim. Fix m and choose $i \in \mathbb{N}$ such that $v_p(m!) \le i$. Since $k_{\widetilde{K}} = k_{\widetilde{K}}^{p^i}[\bar{t}_1, \dots, \bar{t}_d]$, we have $\mathcal{O}_{\widetilde{K}} = \varphi^i(\mathcal{O}_{\widetilde{K}})[t_1, \dots, t_d]$ by Nakayama's lemma. By Leibniz's rule, we have

$$\partial_j^m(\varphi^i(\lambda) t_1^{a_1} \dots t_d^{a_d}) = \sum_{0 \le m_0 \le m} \binom{m}{m_0} \partial_j^{m_0}(\varphi^i(\lambda)) t_1^{a_1} \dots \partial_j^{m-m_0}(t_j^{a_j}) \dots t_d^{a_d} \tag{6}$$

for $\lambda \in \mathcal{O}_{\widetilde{K}}$ and $a_1, \dots, a_d \in \mathbb{N}$. We have $\partial_j^{m_0}(\varphi^i(\lambda)) \in p^i \mathcal{O}_{\widetilde{K}} \subset m!\mathcal{O}_{\widetilde{K}}$, unless $m_0 = 0$, by (5), and $\partial_j^m(t_j^{a_j}) \in m!\,\mathcal{O}_{\widetilde{K}}$. Hence, the RHS of (6) belongs to $m!\,\mathcal{O}_{\widetilde{K}}$, which implies the claim. □

4.2. *Construction of* $\mathbb{N}_{\mathrm{dR}}$. In this subsection, we construct $\mathbb{N}_{\mathrm{dR}}(V)$ as a (φ, Γ_K)-module for de Rham representations V. The idea is similar to [Berger 2008b, §II], i.e., gluing a compatible family of vector bundles over $K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ to obtain vector bundles over $\mathbb{B}^{\dagger,r}_{\mathrm{rig}}$.

Notation 4.2.1. For $n \in \mathbb{N}$, put $r(n) := 1/p^{n-1}(p-1)$. For $r \in \mathbb{Q}_{>0}$, let $n(r) \in \mathbb{N}$ be the smallest integer n with $r \geq r(n)$.

For each K, we fix r_0 such that $\mathbb{A}_K$ has enough r_0-units (Construction 1.6.1) and $\mathbb{A}^{\dagger,r}_K \cong \mathcal{O}'((\pi'))^{\dagger,r/e_{K/\tilde{K}}}$ for all $r \in \mathbb{Q}_{>0} \cap (0, r_0)$ (Lemma 1.10.2), where $\mathcal{O}'$ is a Cohen ring of $k_{\mathbb{E}_K}$. In the rest of this section, let $r \in \mathbb{Q}_{>0}$, and when we consider $\mathbb{A}^{\dagger,r}_K$, $\mathbb{B}^{\dagger,r}_K$ and $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$, we tacitly assume $r \in \mathbb{Q}_{>0} \cap (0, r_0)$ unless otherwise stated. Moreover, for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we further choose r_0 sufficiently small (dependent on V though) such that $\mathbb{D}^{\dagger,r}(V)$ admits a $\mathbb{B}^{\dagger,r}_K$-basis for all $r \in (0, r_0)$. Note that $\mathbb{A}^{\dagger,r}_K$, $\mathbb{B}^{\dagger,r}_K$ are PID's and that $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ is a Bézout integral domain.

Definition 4.2.2. Let $r > 0$ and $n \in \mathbb{N}$ with $n \geq n(r)$. For $x = \sum_{k \gg -\infty} p^k [x_k] \in \widetilde{\mathbb{B}}^{\dagger,r}$, the sequence $\{\sum_{k \leq N} p^k [x_k^{p^{-n}}]\}_{N \in \mathbb{Z}}$ converges in $\mathbb{B}^{\nabla+}_{\mathrm{dR}}$. Moreover, if we put

$$\begin{aligned} \iota_n : \widetilde{\mathbb{B}}^{\dagger,r} &\to \mathbb{B}^{\nabla+}_{\mathrm{dR}} \\ x &\mapsto \sum_{k \gg -\infty} p^k [x_k^{p^{-n}}], \end{aligned}$$

then ι_n is a continuous ring homomorphism (see the proof of [Andreatta and Brinon 2010, Lemme 7.2] for details). Since $\mathbb{B}^{\nabla+}_{\mathrm{dR}}$ is Fréchet complete, ι_n extends to a continuous ring homomorphism

$$\iota_n : \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \to \mathbb{B}^{\nabla+}_{\mathrm{dR}}.$$

We also denote by ι_n the restriction of ι_n to $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$ or $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Unless otherwise stated, we also denote by ι_n the composite of ι_n and the inclusion $\mathbb{B}^{\nabla+}_{\mathrm{dR}} \subset \mathbb{B}^{+}_{\mathrm{dR}}$.

Lemma 4.2.3. *For $x \in \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$, we have*

$$x \in (\mathbb{B}^{\dagger,r}_K)^{\times} \Leftrightarrow x \in (\mathbb{B}^{\dagger,r}_{\mathrm{rig},K})^{\times} \Leftrightarrow x \text{ has no slopes} \Leftrightarrow x \in (\widetilde{\mathbb{B}}^{\dagger,r}_K)^{\times} \Leftrightarrow x \in (\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K})^{\times}.$$

Proof. Note that the slopes of x as an element of $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ or $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$ are the same by definition (see Section 2). Therefore, the assertion follows from [Kedlaya 2005, Corollary 2.5.12]. □

Lemma 4.2.4. *For $B = \mathbb{B}^{\dagger,r}_K, \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}, \widetilde{\mathbb{B}}^{\dagger,r}_K, \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$, we have*

$$\ker(\theta \circ \iota_n : B \to \mathbb{C}_p) = \varphi^{n-1}(q)B$$

for $n \geq n(r)$.

Proof. Note that since $\widetilde{\mathbb{E}}_K$ and $\widetilde{\mathbb{E}}_{K\widetilde{K}^{\mathrm{pf}}}$ are isomorphic, the associated analytic rings $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$ and $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K\widetilde{K}^{\mathrm{pf}}}$ are isomorphic. Hence, in the case of $B=\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$, the claim follows from [Berger 2008b, Proposition 4.8]. By regarding $\mathbb{C}_p$ as the completion of an algebraic closure of $\widetilde{K}^{\mathrm{pf}}$ and applying [Berger 2008b, Remarque 2.14], we have $\ker(\theta\circ\iota_n:\widetilde{\mathbb{B}}^{\dagger,r}\to\mathbb{C}_p)=\varphi^{n-1}(q)\widetilde{\mathbb{B}}^{\dagger,r}$. Since $(\widetilde{\mathbb{B}}^{\dagger,r})^{H_K}=\widetilde{\mathbb{B}}^{\dagger,r}_K$ and $\varphi^{n-1}(q)\in\widetilde{\mathbb{B}}^{\dagger,r}_K$, we obtain the assertion for $B=\widetilde{\mathbb{B}}^{\dagger,r}_K$. We will prove the assertion for $B=\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Let $x\in\ker(\theta\circ\iota_n:\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}\to\mathbb{C}_p)$. Since $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ is a Bézout integral domain, we have $(x,\varphi^{n-1}(q))=(y)$ for some $y\in\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Let $y'\in\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ such that $\varphi^{n-1}(q)=yy'$. Since $y\in\ker(\theta\circ\iota_n:\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}\to\mathbb{C}_p)=\varphi^{n-1}(q)\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$, we have $y=\varphi^{n-1}(q)y''$ for some $y''\in\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$, hence, $y'y''=1$. By Lemma 4.2.3, y' is a unit in $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Hence, we have $x\in\varphi^{n-1}(q)\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ for any $x\in\ker(\theta\circ\iota_n:\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}\to\mathbb{C}_p)$, which implies the assertion. For $B=\mathbb{B}^{\dagger,r}_K$, a similar proof works since $\mathbb{B}^{\dagger,r}_K$ is a PID, hence, a Bézout integral domain. □

Lemma 4.2.5. *The image of $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ under ι_n is contained in $K_n[\![t,u_1,\dots,u_d]\!]$ for $n\geq n(r)$. In particular, ι_n induces a morphism $\iota_n:\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}\to K_n[\![t,u_1,\dots,u_d]\!]^{\nabla}$ for $n\geq n(r)$.*

Proof. Since $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}\subset\mathbb{B}^{\dagger,r(n)}_{\mathrm{rig},K}$, we may assume $r=r(n)$. By [Andreatta and Brinon 2010, Lemme 8.5], there exists a subring $\mathcal{A}_{R,(1,(p-1)p^{n-1})}$ of $\widetilde{\mathbb{A}}$ such that $\mathbb{A}^{\dagger,r(n)}_K=\mathcal{A}_{R,(1,(p-1)p^{n-1})}[[\bar{\pi}]^{-1}]$. The inclusion $\iota_n(\mathbb{B}^{\dagger,r}_K)\subset K_n[\![t,u_1,\dots,u_d]\!]$ is proved in Proposition 8.6 of the same paper. Since $K_n[\![r,u_1,\dots,u_d]\!]$ is closed in $\mathbb{B}^+_{\mathrm{dR}}$, we obtain the assertion. □

Lemma 4.2.6. *For $h\in\mathbb{N}$ and $n\geq n(r)$, the morphism*

$$\mathrm{pr}_h\circ\iota_n:\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}\to K_n[\![t,u_1,\dots,u_d]\!]^{\nabla}/t^hK_n[\![t,u_1,\dots,u_d]\!]^{\nabla}$$

is surjective.

Proof. Since $t\in\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ we may assume $h=1$ by Lemma 4.1.4. Put $\theta_n:=\theta\circ\iota_n$. Let $\mathbb{A}^+_K\subset\mathbb{A}^{\dagger,r}_K$ be as in [Andreatta and Brinon 2008, Proposition 4.42]. By the proof of [Andreatta and Brinon 2010, Lemme 8.2], $\theta_n:\mathbb{A}^+_K\to\mathcal{O}_{K_n}$ is surjective after taking the reduction modulo some power of p. Since $\mathbb{A}^+_K$ is Noetherian and $(p/\pi^a,p)$-adically Hausdorff complete, $\mathbb{A}^+_K$ is p-adically Hausdorff complete, which implies the surjectivity of $\theta_n:\mathbb{A}^+_K\to\mathcal{O}_{K_n}$ by Nakayama's lemma. □

Lemma 4.2.7. *The image of $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ under ι_n is dense in $K_n[\![t,u_1,\dots,u_d]\!]^{\nabla}$ with respect to the canonical topology for $n\geq n(r)$.*

Proof. By Lemma 4.1.5, the assertion follows from Lemma 4.2.6. □

Lemma 4.2.8 ([Kedlaya 2005, Corollary 2.8.5, Definition 2.9.5], see also [Berger 2008a, Proposition 1.1.1]). *For $B = \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$, $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$, $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$, $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ and a B-submodule M of a finite free B-module, the following are equivalent*:

(i) *M is finite free.*

(ii) *M is closed.*

(iii) *M is finitely generated.*

Lemma 4.2.9. *Let B be either $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ or $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. If I is a principal ideal of B which divides (t^h) for some $h \in \mathbb{N}$, then I is generated by an element of the form $\prod_{n \geq n(r)} (\varphi^{n-1}(q)/p)^{j_n}$ with $j_n \leq h$.*

Proof. Note that we have a slope factorization $t = \pi \prod_{n\geq 1} (\varphi^{n-1}(q)/p)$ in $\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ (see the proof of [Berger 2008b, Proposition I. 2.2]). For $n < n(r)$, $\varphi^{n-1}(q)/p$ is a unit in $\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ and for $n \geq n(r)$, $\varphi^{n-1}(q)/p$ generates a prime ideal of B by Lemma 4.2.4. Hence, the assertion follows from the uniqueness of slope factorizations, see Lemma 2.0.5. □

Lemma 4.2.10 (The existence of a partition of unity). *Let $n \in \mathbb{N}$ and $r > 0$ satisfy $n \geq n(r)$. For $w \in \mathbb{N}_{>0}$, there exists $t_{n,w} \in \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ such that $\iota_n(t_{n,w}) = 1 \bmod t^w K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ and $\iota_m(t_{n,w}) \in t^w K_m[\![t, u_1, \dots, u_d]\!]^{\nabla}$ if $m \neq n$ and $m \geq n(r)$.*

Proof. Since $\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p} \subset \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ and $\mathbb{Q}_p(\zeta_{p^m})[\![t]\!] \subset K_m[\![t, u_1, \dots, u_d]\!]^{\nabla}$, we may assume $K = \mathbb{Q}_p$. The assertion then follows from [Berger 2008b, Lemma I.2.1]. □

Lemma 4.2.11. *Let B be either $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ or $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. For $n \geq n(r)$, write $\iota_n : B := \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \to B_n := \mathbb{B}^{\nabla+}_{\mathrm{dR}}$ in the first case and $\iota_n : B := \mathbb{B}^{\dagger,r}_{\mathrm{rig},K} \to B_n := K_n[\![t, u_1, \dots, u_d]\!]^{\nabla}$ in the second case. Let D be a φ-module over B of rank d' and $D^{(1)}$ and $D^{(2)}$ two B-submodules of rank d' stable by φ on $D[1/t] = B[1/t] \otimes_B D$ such that*

(i) *$D^{(1)}[1/t] = D^{(2)}[1/t] = D[1/t]$;*

(ii) *$B_n \otimes_{\iota_n, B} D^{(1)} = B_n \otimes_{\iota_n, B} D^{(2)}$ for all $n \geq n(r)$.*

Then, we have $D^{(1)} = D^{(2)}$.

Proof. Since $D^{(1)} + D^{(2)}$ is finite free by Lemma 4.2.8 and satisfies the same condition as $D^{(2)}$, we may assume that $D^{(1)} \subset D^{(2)}$ by replacing $D^{(2)}$ by $D^{(1)} + D^{(2)}$. Then, the proof of [Berger 2008b, Proposition I.3.4] works by using the ingredients Lemma 2.0.6 and Lemma 4.2.9 instead of [Berger 2008b, Proposition I.2.2]. □

Proposition 4.2.12 (cf. [Berger 2008b, Théorème II.1.2]). *Let $V \in \mathrm{Rep}_{\mathrm{dR}}(G_K)$ be a de Rham representation with negative Hodge–Tate weights. Let B be either $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ or $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Let B_n and $\iota_n : B \to B_n$ be as in Lemma 4.2.11. In the first case, let $D_n := (\mathbb{B}^{+}_{\mathrm{dR}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V))^{\nabla^{\mathrm{geom}}=0}$, and let $D_n := (K_n[\![t, u_1, \dots, u_d]\!] \otimes_K \mathbb{D}_{\mathrm{dR}}(V))^{\nabla^{\mathrm{geom}}=0}$ in* *the second case. Put* $D := \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$ *in the first case and* $D := \mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)$ *in the second case. Then, the following holds.*

(i) *There exists* $h \in \mathbb{N}$ *such that*

$$t^h B_n \otimes_{\iota_n, B} D \subset D_n \subset B_n \otimes_{\iota_n, B} D$$

for all $n \geq n(r)$.

(ii) *Let* $\iota_n : D \to B_n \otimes_{\iota_n, B} D$ *be given by* $x \mapsto 1 \otimes x$ *and put*

$$\mathcal{N} := \{x \in D;\ \iota_n(x) \in D_n \ \textit{for all}\ n \geq n(r)\}.$$

Then, $\mathcal{N}$ *is a finite free* B*-submodule of* D*, whose rank is equal to* $\dim_{\mathbb{Q}_p} V$*. Moreover, there exists a canonical isomorphism*

$$B_n \otimes_{\iota_n, B} \mathcal{N} \to D_n$$

for all $n \geq n(r)$.

Proof.

(i) Since the inclusion $B_n \subset \mathbb{B}^+_{\mathrm{dR}}$ is faithfully flat by Lemma 4.1.4, we only have to prove the assertion after tensoring $\mathbb{B}^+_{\mathrm{dR}}$ over B_n. We have the following isomorphisms:

$$\begin{aligned}\mathbb{B}^+_{\mathrm{dR}} \otimes_{B_n} B_n \otimes_{\iota_n, B} D &\cong \mathbb{B}^+_{\mathrm{dR}} \otimes_{\iota_n, \mathbb{B}^{\dagger,r}} \mathbb{B}^{\dagger,r} \otimes_{\mathbb{B}^{\dagger,r}} D^{\dagger,r} \\ &\cong \mathbb{B}^+_{\mathrm{dR}} \otimes_{\iota_n, \mathbb{B}^{\dagger,r}} \mathbb{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V = \mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V,\end{aligned}$$

where $D^{\dagger,r} := \widetilde{\mathbb{B}}^{\dagger,r} \otimes_{\mathbb{Q}_p} V$ in the first case and $D^{\dagger,r} := \mathbb{D}^{\dagger,r}(V)$ in the second case. Since $\mathbb{B}^+_{\mathrm{dR}} \otimes_{B_n} D_n \subset \mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ by assumption and $\mathbb{B}^+_{\mathrm{dR}} \otimes_{B_n} D_n[1/t] \cong \mathbb{B}^+_{\mathrm{dR}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V)[1/t] = \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$, there exists $h \in \mathbb{N}$ such that

$$t^h \mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \subset \mathbb{B}^+_{\mathrm{dR}} \otimes_{B_n} D_n \subset \mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V,$$

which implies the assertion.

(ii) Since $\mathcal{N}$ is a closed B-submodule of D containing $t^h D$, $\mathcal{N}$ is free of rank $\dim_{\mathbb{Q}_p} V$ by Lemma 4.2.8. To prove the second assertion, we only have to prove that the canonical map $B_n \otimes_{\iota_n, B} \mathcal{N} \to D_n / t D_n$ is surjective for all $n \geq n(r)$ since B_n is a t-adically complete discrete valuation ring. Fix n and let $x \in D_n$. Note that $\mathrm{pr}_{h+1} \circ \iota_n : B \to B_n / t^{h+1} B_n$ is surjective. Indeed, when $B = \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$, this follows from Lemma 4.2.6. When $B = \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$, it is reduced to the case $h = 0$, and $\mathrm{pr}_1 \circ \iota_n = \theta \circ \iota_n : \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \to \mathbb{C}_p$ is surjective since $\widetilde{\mathbb{B}}^+ \subset \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$. Hence, there exists $y \in D$ such that $\iota_n(y) - x \in t^{h+1} B_n \otimes_{\iota_n, B} D \in t D_n$. We put

$z := t_{n,h+1}y \in D$, where $t_{n,h+1}$ is as in Lemma 4.2.10. By the property of $t_{\bullet,\bullet}$, we have

$$\iota_n(z) - x = \bigl(\iota_n(t_{n,h+1}) - 1\bigr)\iota_n(y) + \iota_n(y) - x \in tD_n$$

and for $m \neq n$,

$$\iota_m(z) \in t^{h+1} B_n \otimes_{\iota_n, B} D \subset tD_n.$$

These imply $z \in \mathcal{N}$; hence, we obtain the assertion. □

Definition 4.2.13. In the context of Proposition 4.2.12, we denote $\mathcal{N}$ by $\widetilde{\mathbb{N}}^{\dagger,r}_{\mathrm{rig}}(V)$ in the first case and by $\mathbb{N}_{\mathrm{dR},r}(V)$ in the second case. For a de Rham representation V with arbitrary Hodge–Tate weights, we put $\widetilde{\mathbb{N}}^{\dagger,r}_{\mathrm{rig}}(V) := \widetilde{\mathbb{N}}^{\dagger,r}_{\mathrm{rig}}(V(-n))(n)$ and $\mathbb{N}_{\mathrm{dR},r}(V) := \mathbb{N}_{\mathrm{dR},r}(V(-n))(n)$ for sufficiently large $n \in \mathbb{N}$. These definitions are independent of the choice of n. We also put $\widetilde{\mathbb{N}}^{\dagger}_{\mathrm{rig}}(V) := \bigcup_r \widetilde{\mathbb{N}}^{\dagger,r}_{\mathrm{rig}}(V)$ and $\mathbb{N}_{\mathrm{dR},r}(V) := \bigcup_r \mathbb{N}_{\mathrm{dR},r}(V)$. We note that for $0 < s \leq r$, the canonical map $\mathbb{B}^{\dagger,s}_{\mathrm{rig},K} \otimes_{\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbb{N}_{\mathrm{dR},r}(V) \to \mathbb{N}_{\mathrm{dR},s}(V)$ is an isomorphism by Lemma 4.2.11 and Proposition 4.2.12. So, the canonical morphism $\mathbb{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{B}^{\dagger,r}_{\mathrm{rig}}} \mathbb{N}_{\mathrm{dR},r}(V) \to \mathbb{N}_{\mathrm{dR}}(V)$ is an isomorphism, and in particular, $\mathbb{N}_{\mathrm{dR}}(V)$ is a finite free $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$-module of rank $\dim_{\mathbb{Q}_p} V$. Since the map $\varphi : \mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V) \to \mathbb{D}^{\dagger,r/p}_{\mathrm{rig}}(V)$ induces a map $\varphi : \mathbb{N}_{\mathrm{dR},r}(V) \to \mathbb{N}_{\mathrm{dR},r/p}(V)$ by the formula $\iota_{n+1} \circ \varphi = \iota_n$, $\mathbb{N}_{\mathrm{dR}}(V)$ is stable under the (φ, Γ_K)-action of $\mathbb{D}^{\dagger}_{\mathrm{rig}}(V)$. Similarly, $\widetilde{\mathbb{N}}^{\dagger}_{\mathrm{rig}}(V)$ is free of rank $\dim_{\mathbb{Q}_p} V$ and is stable under the (φ, G_K)-action of $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$. Thus, we obtain a (φ, G_K)-module $\widetilde{\mathbb{N}}^{\dagger}_{\mathrm{rig}}(V)$ over $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ and a (φ, Γ_K)-module $\mathbb{N}_{\mathrm{dR}}(V)$ over $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$.

4.3. ***Differential action of a p-adic Lie group.*** In this subsection, we recall basic facts on the differential action of a certain p-adic Lie group. Throughout this subsection, let $\mathcal{G}$ be a p-adic Lie group, which is isomorphic to an open subgroup of $(1 + 2p\mathbb{Z}_p) \ltimes \mathbb{Z}_p^d$ via a continuous group homomorphism $\eta : \mathcal{G} \hookrightarrow \mathbb{Z}_p^{\times} \ltimes \mathbb{Z}_p^d$. Denote $\eta(\gamma) = (\eta_0(\gamma), \dots, \eta_d(\gamma)) \in \mathbb{Z}_p^{\times} \ltimes \mathbb{Z}_p^d$ for $\gamma \in \mathcal{G}$. For $1 \leq j \leq d$, let

$$\mathcal{G}_0 := \{\gamma \in \mathcal{G};\ \eta_j(\gamma) = 0 \text{ for all } j > 0\},$$
$$\mathcal{G}_j := \{\gamma \in \mathcal{G};\ \eta_0(\gamma) = 1, \eta_i(\gamma) = 0 \text{ for all positive } i \neq j\}.$$

Notation 4.3.1. Let (R, v) be a $\mathbb{Q}_p$-Banach algebra and M a finite free R-module endowed with an R-valuation v. Assume that $\mathcal{G}$ acts on R and M such that:

(i) The $\mathcal{G}$-action on R is $\mathbb{Q}_p$-linear and the action of $\mathcal{G}$ on M is R-semilinear.

(ii) We have $v \circ \gamma(x) = v(x)$ for all $x \in R$ and $\gamma \in \mathcal{G}$.

(iii) There exists an open subgroup $\mathcal{G}_o \leq_o \mathcal{G}$ such that

$$v((\gamma - 1)x) \geq v(x) + v(p)$$

for all $\gamma \in \mathcal{G}_o$ and $x \in R$.

(iv) For any $x \in M$, there exists an open subgroup $\mathcal{G}_x \leq_o \mathcal{G}_o$ such that

$$v((\gamma - 1)x) \geq v(x) + v(p)$$

for all $\gamma \in \mathcal{G}_x$.

Construction 4.3.2. Let the notation be as in Notation 4.3.1. We extend the construction of the differential operator ∇_V in [Berger 2002, §5.1] to this setting. By assumption, there exists an open subgroup $\mathcal{G}_M \leq_o \mathcal{G}_o$ such that

$$v((\gamma - 1)x) \geq v(x) + v(p)$$

for all $x \in M$ and $\gamma \in \mathcal{G}_M$. Hence, we can apply Berger's argument to the 1-parameter subgroup $\gamma^{\mathbb{Z}_p}$ for $\gamma \in \mathcal{G}_M$. Thus, we can define a continuous $\mathbb{Q}_p$-linear map

$$\begin{aligned} \log(\gamma) : M &\to M \\ x &\mapsto \log(\gamma)(x) := \sum_{n \geq 1} (-1)^{n-1} \frac{(\gamma - 1)^n}{n} x \end{aligned}$$

for $\gamma \in \mathcal{G}_M$. Moreover, the operators

$$\nabla_0(x) := \frac{\log(\gamma)(x)}{\log(\eta_0(\gamma))} \quad \text{for } \gamma \in \mathcal{G}_M \cap \mathcal{G}_0,$$
$$\nabla_j(x) := \frac{\log(\gamma)(x)}{\eta_j(\gamma)} \quad \text{for } \gamma \in \mathcal{G}_M \cap \mathcal{G}_j$$

for $1 \leq j \leq d$ are independent of the choice of γ.

Assume that N satisfies the conditions of Notation 4.3.1. Then, $M \otimes_R N$ satisfies the conditions of Notation 4.3.1, and we have

$$\log(\gamma \otimes \gamma) = \log(\gamma) \otimes \mathrm{id}_N + \mathrm{id}_M \otimes \log(\gamma) \quad \text{for } \gamma \in \mathcal{G}_M \cap \mathcal{G}_N$$

in $\mathrm{End}_{\mathbb{Q}_p}(M \otimes_R N)$. With $(M, N) = (R, R)$ or (M, R), $\nabla_j : R \to R$ is a continuous derivation and $\nabla_j : M \to M$ is a continuous derivation, compatible with $\nabla_j : R \to R$, that is, $\nabla_j(\lambda x) = \nabla_j(\lambda)x + \lambda\nabla_j(x)$ for $\lambda \in R$ and $x \in M$.

Lemma 4.3.3. *Let the notation be as in Construction 4.3.2. In* $\mathrm{End}_{\mathbb{Q}_p}(M)$, *we have*

$$[\nabla_i, \nabla_j] = -[\nabla_j, \nabla_i] = \begin{cases} \nabla_j & \text{if } i = 0,\ 1 \leq j \leq d, \\ 0 & \text{if } 1 \leq i, j \leq d. \end{cases}$$

Proof. Since $\mathcal{G}_i$ and $\mathcal{G}_j$ are commutative for $1 \leq i, j \leq d$, the assertion in the second case is trivial. We prove the other case. Fix $x \in M$. We regard $\mathcal{G}$ as a subgroup of $\mathrm{GL}_{d+1}(\mathbb{Z}_p)$ as in Section 1.3. For sufficiently small $u_0, u_j \in \mathbb{Z}_p$, put $\gamma_0 := 1 + u_0 E_{1,0} \in \mathcal{G}_0 \cap \mathcal{G}_M$, $\gamma_j := 1 + u_j E_{1,j} \in \mathcal{G}_j \cap \mathcal{G}_M$, where $E_{1,j}$ is the

$(1, j+1)$-th elementary matrix in $M_{d+1}(\mathbb{Z}_p)$. Then, the assertion is equivalent to the equality

$$\log(\gamma_0)\circ\log(\gamma_j)(x)-\log(\gamma_j)\circ\log(\gamma_0)(x)=\log(1+u_0)\log(\gamma_j)x.$$

In the group ring $\mathbb{Q}_p[\mathcal{G}]$, we have

$$\sum_{1\le i\le n}\frac{(-1)^{n-1}}{n}u_0^n u_j E_{1,j}=\sum_{1\le i\le n}\frac{(-1)^{n-1}}{n}(u_0E_{1,1})^n\sum_{1\le i\le n}\frac{(-1)^{n-1}}{n}(u_jE_{1,j})^n \\ -\sum_{1\le i\le n}\frac{(-1)^{n-1}}{n}(u_jE_{1,j})^n\sum_{1\le i\le n}\frac{(-1)^{n-1}}{n}(u_0E_{1,1})^n.$$

After applying both sides to x, the LHS converges to $\log(1+u_0)\log(\gamma_j)(x)$ and the RHS converges to $\log(\gamma_0)\circ\log(\gamma_j)(x)-\log(\gamma_j)\circ\log(\gamma_0)(x)$, which implies the assertion. □

In the following, we will use the Fréchet version of Construction 4.3.2.

Construction 4.3.4. Let $(R, \{w_r\})$ be a Fréchet algebra and M a finite free R-module endowed with R-valuations $\{w_r\}$. Assume that $\mathcal{G}$ acts on R and M and assume that the $\mathcal{G}$-actions on $(\widehat{R}_r, w_r)$ and $(\widehat{M}_r, w_r)$ satisfy the conditions of Notation 4.3.1 for all r, where $\widehat{R}_r$ and $\widehat{M}_r$ are the completions of R and M with respect to w_r. By applying Construction 4.3.2 to each $\widehat{R}_r$ and $\widehat{M}_r$ and passing to the limits, we obtain continuous derivations $\nabla_j : R\to R$ and $\nabla_j : M\to M$ for $0\le j\le d$, which are compatible with $\nabla_j : R\to R$, that satisfy

$$[\nabla_0,\nabla_j]=\nabla_j\quad\text{for } 1\le j\le d,\qquad[\nabla_i,\nabla_j]=0\quad\text{for } 1\le i,j\le d.$$

Thus, the actions of $\nabla_0,\dots\nabla_d$ give rise to a differential action of the Lie algebra $\mathrm{Lie}(\mathcal{G})\cong\mathbb{Q}_p\ltimes\mathbb{Q}_p^d$.

4.4. ***Differential action and differential conductor of*** $\mathbb{N}_{\mathbf{dR}}$**.** In Section 4.2, we constructed $\mathbb{N}_{\mathrm{dR}}(V)$ for de Rham representations V as a (φ,Γ_K)-module. The aim of this subsection is to endow $\mathbb{N}_{\mathrm{dR}}(V)$ with the structure of (φ,∇)-module in the sense of Definition 1.7.5 by using the results in Section 4.3. As a consequence, we can define the differential Swan conductor of $\mathbb{N}_{\mathrm{dR}}(V)$ (Definition 4.4.9). Throughout this subsection, let V denote a p-adic representation of G_K.

Lemma 4.4.1. *There exists an open normal subgroup $\Gamma_K^o\le_o\Gamma_K$ and $r_K>0$ such that for all $0<r\le r_K$, there exists $c_r>0$ such that*

$$w_r((1-\gamma)x)\ge w_r(x)+c_r,\qquad\forall x\in\mathbb{B}_K^{\dagger,r},\forall\gamma\in\Gamma_K^o.$$

Proof. We may assume $x\in\mathbb{A}_K^{\dagger,r}$. Recall that the ring $\Lambda_{m,\mathcal{O}_K}^{(i)}$ is a subring of $\widetilde{\mathbb{A}}_K^{\dagger,r}$ containing $\mathbb{A}_K^{\dagger,r}$ for $m\in\mathbb{N}$ by [Andreatta and Brinon 2008, page 82]. Hence, we only have to prove a similar assertion for $\Lambda^{(i)}_{m,\mathcal{O}_K}$. Then, the assertion follows from [Andreatta and Brinon 2008, Proposition 4.22] if we define Γ^o_K as the closed subgroup of Γ_K topologically generated by $\{\gamma_j^{p^m};\ 0\le j\le d\}$ for sufficiently large m. □

By shrinking Γ^o_K, if necessary, we may assume that Γ^o_K is an open subgroup of $(1+2p\mathbb{Z}_p)\ltimes\mathbb{Z}_p^d$ as in Section 1.3. In the rest of this paper, we assume that r_0 in Notation 4.2.1 is sufficiently small such that $r_0\le r_K$.

Lemma 4.4.2. *For $x\in\widetilde{\mathbb{B}}^{\dagger,r}$ and $c>0$, there exists an open subgroup $U_{x,c}\le_o G_K$ such that*

$$w_r((g-1)x)\ge c\quad \textit{for all } g\in U_{x,c}.$$

Proof. We may assume that x is of the form $[\bar{x}]$ with $\bar{x}\in\widetilde{\mathbb{E}}$. Indeed, if we write $x=\sum_{k\gg-\infty}p^k[x_k]$ with $x_k\in\widetilde{\mathbb{E}}$, then, by definition, there exists N such that $w_r(p^k[x_k])\ge c$ for all $k\ge N$. We choose $U_{x,c}$ such that $w_r((g-1)(p^k[x_k]))\ge c$ for all $k\le N$ and all $g\in U_{x,c}$. Then, $U_{x,c}$ satisfies the condition.

Let $x=[\bar{x}]$ with $\bar{x}\in\widetilde{\mathbb{E}}^{\times}$. Since the action of G_K on $\widetilde{\mathbb{E}}$ is continuous, there exists $U_{x,c}\le_o G_K$ such that $v_{\widetilde{\mathbb{E}}}((g-1)\bar{x})\ge p^{\lfloor c\rfloor}c/r$ (>0) for all $g\in U_{x,c}$. We prove that $U_{x,c}$ satisfies the desired condition. We can write

$$(g-1)[\bar{x}]=[(g-1)\bar{x}]+\sum_{k\ge1}p^k[x_k]$$

for some $x_k\in\widetilde{\mathbb{E}}$. Since

$$[\bar{x}]\Big(\Big[\frac{(g-1)\bar{x}}{\bar{x}}\Big]+1\Big)=(g(\bar{x}),-x_1^p,-x_2^{p^2},\dots),$$

$x_k^{p^k}/\bar{x}$ can be written as the value of a polynomial, with coefficients in $\mathbb{Z}$ with zero constant term, at $(g-1)\bar{x}/\bar{x}$. Indeed, let $S_m\in\mathbb{Z}[X_0,\dots,X_m,Y_0,\dots,Y_m]$ for $m\in\mathbb{N}$ be a family of polynomials defining the addition on the ring of Witt vectors, see [Bourbaki 2006, n°3, §1, IX]. Then, S_m is homogeneous of degree p^m, where $\deg(X_i)=\deg(Y_i)=p^i$. Since $S_0=X_0+Y_0$ and $\sum_{0\le i\le m}p^iS_i^{p^{m-i}}=\sum_{0\le i\le m}p^iX_i^{p^{m-i}}+\sum_{0\le i\le m}p^iY_i^{p^{m-i}}$ for $m\ge1$, the coefficients of both $X_0^{p^m},Y_0^{p^m}\in S_m$ are equal to zero, which implies the assertion. Hence, for $n\in\mathbb{N}$, we have

$$\begin{aligned}v_{\widetilde{\mathbb{E}}}^{\le n}((g-1)[\bar{x}])&=\inf_{1\le k\le n}\big\{v_{\widetilde{\mathbb{E}}}((g-1)\bar{x}),\,v_{\widetilde{\mathbb{E}}}(x_k)\big\}\\&\ge\inf_{1\le k\le n}\Big\{v_{\widetilde{\mathbb{E}}}((g-1)\bar{x}),\,\frac{1}{p^k}v_{\widetilde{\mathbb{E}}}((g-1)\bar{x})\Big\}=\frac{1}{p^n}v_{\widetilde{\mathbb{E}}}((g-1)\bar{x}).\end{aligned}$$

Note that $v_{\widetilde{\mathbb{E}}}^{\le n}\big((g-1)[\bar{x}]\big)=\infty$ for $n\in\mathbb{Z}_{<0}$. Hence, we have $w_r((g-1)[\bar{x}])=\inf_{n\in\mathbb{N}}(rv_{\widetilde{\mathbb{E}}}^{\le n}((g-1)[\bar{x}])+n)\ge\inf\big(r\cdot\frac{1}{p^{\lfloor c\rfloor}}v_{\widetilde{\mathbb{E}}}((g-1)\bar{x}),\lfloor c\rfloor\big)\ge c$, which implies the assertion. □

Lemma 4.4.3. *Let $\{e_i\}$ be a $\mathbb{B}^{\dagger,r}_K$-basis of $\mathbb{D}^{\dagger,r}(V)$. We endow $\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)$ with valuations $\{w_s\}_{0<s\le r}$ that are compatible with the $\{w_s\}_{0<s\le r}$ associated to $\{e_i\}$. Then, the actions of Γ^o_K on $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ and $\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)$ satisfy the conditions of Notation 4.3.1.*

Proof. Conditions (i) and (ii) follow from the definition. Condition (iii) follows from the formula $\gamma^p-1=\sum_{1\le i\le p}\binom{p}{i}(\gamma-1)^i$ and Lemma 4.4.1. To prove condition (iv), we may assume $x\in\mathbb{D}^{\dagger,r}(V)$. We choose a lattice T of V stable under the G_K-action. Let $\{f_i\}$ be a basis of T and endow $\widetilde{\mathbb{B}}^{\dagger,r}\otimes_{\mathbb{Q}_p}V$ with the valuations $\{w'_s\}_{0<s\le r}$, compatible with the $\{w_s\}_{0<s\le r}$, associated to the $\widetilde{\mathbb{B}}^{\dagger,r}$-basis $\{1\otimes f_i\}$. By the canonical isomorphism $\mathbb{B}^{\dagger,r}\otimes_{\mathbb{B}^{\dagger,r}_K}\mathbb{D}^{\dagger,r}(V)\cong\mathbb{B}^{\dagger,r}\otimes_{\mathbb{Q}_p}V$ following from Theorem 1.10.5, we regard $\{1\otimes e_i\}$ as a $\mathbb{B}^{\dagger,r}$-basis of $\mathbb{B}^{\dagger,r}\otimes_{\mathbb{Q}_p}V$. Then, w_s is equivalent to w'_s; therefore, we only have to prove that for any $x\in\widetilde{\mathbb{B}}^{\dagger,r}\otimes_{\mathbb{Q}_p}V$ and $0<s\le r$, there exists an open subgroup $G^o_{K,s,x}\le_o G_K$ such that $w'_s((g-1)x)\ge w'_s(x)+w'_s(p)$ for all $g\in G^o_{K,s,x}$. We may assume that x is of the form $\lambda\otimes v$ for $\lambda\in\widetilde{\mathbb{B}}^{\dagger,r}$ and $v\in T$. Since the action of G_K on T is continuous, there exists an open subgroup $U\le_o G_K$ such that U acts trivially on T/pT. We apply Lemma 4.4.2 after regarding $\lambda\in\widetilde{\mathbb{B}}^{\dagger,s}$, and get that there exists an open subgroup $U'\le_o G_K$ such that $w_s((g-1)\lambda)\ge w_s(\lambda)+w_s(p)$ for all $g\in U'$. If we put $G^o_{K,s,x}:=U\cap U'$, then the assertion follows from

$$(g-1)(\lambda\otimes v)=(g-1)(\lambda)\otimes g(v)+\lambda\otimes(g-1)v. \qquad\square$$

Definition 4.4.4. By Lemma 4.4.3, we can apply Construction 4.3.4 to $\mathcal{G}=\Gamma_K$, $R=\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ and $M=\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)$. Thus, we obtain continuous differentials operators ∇_j on $\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)$ for $0\le j\le d$. The operator ∇_j induces a continuous differential operator on $\mathbb{D}^{\dagger}_{\mathrm{rig}}(V)$, which is denoted by ∇_j again. Since the actions of Γ_K and φ commute, ∇_j commutes with φ by definition.

Until otherwise stated, let $V=\mathbb{Q}_p$ and regard $\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(\mathbb{Q}_p)$ as $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. Then, ∇_j can be regarded as a continuous derivation on $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$. In the following, we will describe this derivation explicitly.

Construction 4.4.5. As in [Andreatta and Brinon 2010, Propostion 4.3], the action of Γ_K on $K_n[\![t,u_1,\dots,u_d]\!]$ induces K_n-linear differentials

$$\begin{aligned}\widetilde{\nabla}_0&:=\frac{\log(\gamma_0)}{\log(\eta_0(\gamma_0))}=t(1+\pi)\frac{\partial}{\partial\pi},\\ \widetilde{\nabla}_j&:=\frac{\log(\gamma_j)}{\eta_j(\gamma_j)}=-t[\tilde{t}_j]\frac{\partial}{\partial u_j}\quad\text{for } 1\le j\le d\end{aligned}$$

for all sufficiently small $\gamma_0\in\Gamma_{K,0}$ and $\gamma_j\in\Gamma_{K,j}$. Note that these are continuous with respect to the canonical topology. Since the action of Γ_K commutes with ∇^{geom} by definition, $\widetilde{\nabla}_j$ acts on $K_n[\![t,u_1,\dots,u_d]\!]^{\nabla}$.

We assume $K = \widetilde{K}$ until otherwise stated. By the isomorphism $\mathbb{A}_K^{\dagger,r} \cong \mathcal{O}((\pi))^{\dagger,r}$, we have derivations

$$\partial_0 := \frac{\partial}{\partial \pi}, \ \partial_1 := \frac{\partial}{\partial[\tilde{t}_1]}, \ldots, \ \partial_d := \frac{\partial}{\partial[\tilde{t}_d]},$$

on $\mathbb{A}_K^{\dagger,r}$ (see Section 1.7), which are continuous with respect to the Fréchet topology defined by $\{w_s\}_{0<s\leq r}$. By passing to the completion, we obtain continuous derivations $\partial_j : \mathbb{B}^{\dagger,r}_{\mathrm{rig},K} \to \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ for $0 \leq j \leq d$. The derivation ∂_j also extends to a derivation $\partial_j : \mathbb{B}^{\dagger}_{\mathrm{rig},K} \to \mathbb{B}^{\dagger}_{\mathrm{rig},K}$. By Lemma 4.2.7, we may regard $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ as a dense subring of $K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla}$ via ι_n. Hence, we can extend any continuous derivation ∂ on $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ to a continuous derivation on $K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla}$, which is denoted by $\iota_n(\partial)$. Note that we have a formula

$$\iota_n(\partial)(\iota_n(x)) = \iota_n(\partial(x)) \text{ for } x \in \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}. \tag{7}$$

Lemma 4.4.6. *For $n \geq n(r)$, we have*

$$\iota_n(t(1+\pi)\partial_0) = \widetilde{\nabla}_0, \quad \iota_n(t[\tilde{t}_j]\partial_j) = \widetilde{\nabla}_j \ \textit{for } 1 \leq j \leq d.$$

Proof. Let $1 \leq j \leq d$ and put $\delta_0 := \iota_n(t(1+\pi)\partial_0) - \widetilde{\nabla}_0$ and $\delta_j := \iota_n(t[\tilde{t}_j]\partial_j) - \widetilde{\nabla}_j$. Let $f : K_n[\![t, u_1, \ldots, u_d]\!] \to K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla}$ be the map defined in the proof of Lemma 4.1.4, which is continuous by Lemma 4.1.6. Since f induces a surjection on the residue fields by definition, $f(K_n[t])$ is a dense subring of $K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla}$ by Lemmas 4.1.4 and 4.1.5. Hence, we only have to prove that $\delta_0 \circ f(K_n[t]) = \delta_j \circ f(K_n[t]) = 0$. We view $\delta_0 \circ f|_{K_n}, \ \delta_j \circ f|_{K_n} \in \mathrm{Der}_{\mathrm{cont}}(K_n, K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla})$, which is isomorphic to $\mathrm{Hom}_{K_n}(\widehat{\Omega}^1_{K_n}, K_n[\![t, u_1, \ldots, u_d]\!]^{\nabla})$ by Lemma 1.2.3. Since $\widehat{\Omega}^1_{K_n} \cong K_n \otimes_K \widehat{\Omega}^1_K$ has a K_n-basis $\{dt_i;\ 1 \leq i \leq d\}$ and since we have $f(t) = t$ and $f(t_i) = [\tilde{t}_i]$ by definition, we only have to prove $\delta_0(t) = \delta_j(t) = 0$ and $\delta_0([\tilde{t}_i]) = \delta_j([\tilde{t}_i]) = 0$ for all $1 \leq i \leq d$. By using formula (7), we get

$$\iota_n(t(1+\pi)\partial_0)(t) = t = \widetilde{\nabla}_0(t), \quad \iota_n(t(1+\pi)\partial_0)[\tilde{t}_i] = 0,$$
$$\iota_n(t[\tilde{t}_j]\partial_j)(t) = 0 = \widetilde{\nabla}_j(t), \quad \iota_n(t[\tilde{t}_j]\partial_j)[\tilde{t}_i] = \delta_{ij} t[\tilde{t}_j]$$

for all $1 \leq i \leq d$. Since $(\partial/\partial u_j)[\tilde{t}_i] = -(\partial/\partial u_j)u_i = -\delta_{ij}$ for all $1 \leq i \leq d$, we obtain the assertion. □

For the rest of this section, we drop the assumptions $K = \widetilde{K}$ and $V = \mathbb{Q}_p$.

Corollary 4.4.7. *The derivation*

$$d' : \mathbb{B}^{\dagger}_{\mathrm{rig},K} \to \Omega^1_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}$$
$$x \mapsto \nabla_0(x)\frac{1}{t(1+\pi)}d\pi + \sum_{1\leq j\leq d} \nabla_j(x)\frac{1}{t}d[\tilde{t}_j]$$

coincides with the canonical derivation $d : \mathbb{B}^{\dagger}_{\mathrm{rig},K} \to \Omega^1_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}$.

Proof. Since the canonical map $\mathbb{B}^{\dagger}_{\mathrm{rig},\widetilde{K}} \to \mathbb{B}^{\dagger}_{\mathrm{rig},K}$ is finite étale by [Kedlaya 2005, Proposition 2.4.10], we can reduce to the case $K = \widetilde{K}$. Let the notation be as in Lemma 4.4.6. Obviously, ∇_j extends to $\widetilde{\nabla}_j$ by passing to the completion. Since ι_n is injective, we have

$$\nabla_0 = t(1+\pi)\partial_0, \quad \nabla_j = t[\tilde{t}_j]\partial_j \ \text{ for } 1 \leq j \leq d.$$

as derivations of $\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$ by Lemma 4.4.6, which implies the assertion. □

Lemma 4.4.8. *Let $V \in \mathrm{Rep}_{\mathrm{dR}}(G_K)$.*

(i) *We have $\nabla_j(\mathbb{N}_{\mathrm{dR}}(V)) \subset t\mathbb{N}_{\mathrm{dR}}(V)$ for all $0 \leq j \leq d$. We put $\nabla'_j := 1/t\nabla_j$, which is a continuous differential operator on $\mathbb{N}_{\mathrm{dR}}(V)$.*

(ii) *For all $0 \leq i, j \leq d$, we have*

$$[\nabla'_i, \nabla'_j] = 0$$

(iii) *For all $0 \leq i, j \leq d$, we have*

$$\nabla'_j \circ \varphi = p\varphi \circ \nabla'_j$$

Proof.

(i) By Tate twist, we may assume that the Hodge–Tate weights of V are sufficiently small. Let the notation be as in Construction 4.4.5 and Proposition 4.2.12 (with $B = \mathbb{B}^{\dagger,r}_{\mathrm{rig},K}$). By viewing $t\mathbb{N}_{\mathrm{dR},r}(V)$ and $t\mathbb{D}_{\mathrm{dR}}(V)$ as $\mathbb{N}_{\mathrm{dR},r}(V(1))$ and $\mathbb{D}_{\mathrm{dR}}(V(1))$, respectively, we only have to prove that $\iota_n(\nabla_j(x)) \in tD_n$ for all $n \geq n(r)$ and $x \in \mathbb{N}_{\mathrm{dR},r}(V)$. For sufficiently small $\gamma_j \in \Gamma_{K,j}$, we have $\iota_n \circ \log(\gamma_j)(x) = \log(\gamma_j)(\iota_n(x))$ and $\iota_n(x) \in D_n \subset B_n \otimes_K \mathbb{D}_{\mathrm{dR}}(V)$. Since Γ_K acts trivially on $\mathbb{D}_{\mathrm{dR}}(V)$, $\log(\gamma_j)$ acts on $B_n \otimes_K \mathbb{D}_{\mathrm{dR}}(V)$ as $\log(\gamma_j) \otimes 1$. Since $\log(\gamma_j)(B_n) \subset tB_n$ (see Construction 4.4.5), we have $\iota_n \circ \log(\gamma_j)(x) \in (B_n \otimes_K \mathbb{D}_{\mathrm{dR}}(V(1)))^{\nabla^{\mathrm{geom}}=0} = tD_n$, which implies the assertion.

(ii) This follows from a straightforward calculation using Lemma 4.3.3, $\nabla_0(t) = t$, and $\nabla_i(t) = \nabla_j(t) = 0$.

(iii) Since ∇_j commutes with φ, we have $t\nabla'_j \circ \varphi = \nabla_j \circ \varphi = \varphi \circ \nabla_j = \varphi(t)\varphi \circ \nabla'_j = pt\varphi \circ \nabla'_j$. By dividing by t, we obtain the assertion since $\mathbb{N}_{\mathrm{dR}}(V)$ is torsion free. □

Definition 4.4.9. Let the notation be as in Lemma 4.4.8. For $V \in \mathrm{Rep}_{\mathrm{dR}}(G_K)$, put

$$\begin{aligned}\nabla : \mathbb{N}_{\mathrm{dR}}(V) &\to \mathbb{N}_{\mathrm{dR}}(V) \otimes_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} \Omega^1_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} \\ x &\mapsto \nabla'_0(x) \otimes \frac{1}{1+\pi} d\pi + \sum_{1 \leq j \leq d} \nabla'_j(x) \otimes d[\tilde{t}_j],\end{aligned}$$

which defines a ∇-structure on $\mathbb{N}_{\mathrm{dR}}(V)$ by Corollary 4.4.7. Furthermore, this ∇-structure is compatible with the φ-structure on $\mathbb{N}_{\mathrm{dR}}(V)$ by Lemma 4.4.8(iii) and $\varphi((1+\pi)^{-1}d\pi) = p(1+\pi)^{-1}d\pi$ and $\varphi(d[\tilde{t}_j]) = pd[\tilde{t}_j]$. Thus, $\mathbb{N}_{\mathrm{dR}}(V)$ is endowed with a (φ, ∇)-module structure and we obtain the differential Swan conductor $\mathrm{Swan}^{\nabla}(\mathbb{N}_{\mathrm{dR}}(V))$ of $\mathbb{N}_{\mathrm{dR}}(V)$. The slope filtration of $\mathbb{N}_{\mathrm{dR}}(V)$ as a (φ, ∇)-module (Theorem 1.7.6) is Γ_K-stable by the commutativity of the Γ_K- and φ-actions, and the uniqueness of the slope filtration ([Kedlaya 2007, Theorem 6.4.1]).

4.5. ***Comparison of pure objects.*** In this subsection, we will study "pure" objects in various categories.

Notation 4.5.1. Let G be a topological group and R a topological ring on which G acts. Let $\phi : R \to R$ be a continuous ring homomorphism that commutes with the action of G. A (ϕ, G)-module over R is a finite free R-module with continuous and semilinear action of G and a semilinear endomorphism ϕ, both of which are commutative. We denote the category of (ϕ, G)-modules over R by $\mathrm{Mod}_R(\phi, G)$. The morphisms in $\mathrm{Mod}_R(\phi, G)$ consist of R-linear maps commuting with ϕ and G.

Definition 4.5.2 [Berger 2008a, Definition 3.2.1]. Let $h \geq 1$ and $a \in \mathbb{Z}$ be relatively prime. Let $\mathrm{Rep}_{a,h}(G_K)$ be the category with objects $V_{a,h} \in \mathrm{Rep}_{\mathbb{Q}_{p^h}}(G_K)$, endowed with a semilinear Frobenius action $\varphi : V_{a,h} \to V_{a,h}$ that commutes with the G_K-action such that $\varphi^h = p^a$. The morphisms of this category are $\mathbb{Q}_{p^h}$-linear maps that commute with (φ, G_K)-actions. When $h = 1$ and $a = 0$, $\mathrm{Rep}_{a,h}(G_K) = \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Let $s := a/h \in \mathbb{Q}$. We denote by $D_{[s]}$ the $\mathbb{Q}_p$-vector space $\bigoplus_{1\leq i\leq h} \mathbb{Q}_p e_i$ endowed with a trivial G_K-action and with φ-actions via $\varphi(e_i) := e_{i+1}$ if $i \neq h$ and $\varphi(e_h) := p^a e_1$. Then, $\mathbb{Q}_{p^h} \otimes_{\mathbb{Q}_p} D_{[s]}$ belongs to $\mathrm{Rep}_{a,h}(G_K)$.

Definition 4.5.3. For $s \in \mathbb{Q}$, we define

$$\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}}(\varphi, G_K), \quad \mathrm{Mod}^s_{\mathbb{B}^\dagger_{\mathrm{rig},K}}(\varphi, \Gamma_K), \quad \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^\dagger}(\varphi, G_K), \quad \mathrm{Mod}^s_{\mathbb{B}^\dagger_K}(\varphi, \Gamma_K)$$

to be the full subcategories of $\mathrm{Mod}_{\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}}(\varphi, G_K)$, $\mathrm{Mod}_{\mathbb{B}^\dagger_{\mathrm{rig},K}}(\varphi, \Gamma_K)$, $\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^\dagger}(\varphi, G_K)$ and $\mathrm{Mod}^s_{\mathbb{B}^\dagger_K}(\varphi, \Gamma_K)$, whose objects are pure of slope s as φ-modules.

Lemma 4.5.4. (i) *For any $r > 0$, there exists a canonical injection*

$$\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \to \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}},$$

which is (φ, G_K)-equivariant. In the following, we regard $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$ as a subring of $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ and we endow $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$ with a Fréchet topology induced by the family of valuations $\{w_r\}_{r>0}$.

(ii) *For $h \in \mathbb{N}_{>0}$,*

$$(\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}})^{\varphi^h=1} = (\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}})^{\varphi^h=1} = \mathbb{Q}_{p^h}.$$

Proof. By definition, $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$ and $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ depend only on $\mathbb{C}_p$ and not on K. By regarding $\mathbb{C}_p$ as the p-adic completion of the algebraic closure of K^{pf}, we can reduce to the perfect residue field case. Assertion (i) follows from [Berger 2002, Exemple 2.8(2), Definition 2.16]. Assertion (ii) for $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$ is due to Colmez, see [Ohkubo 2013, Lemma 6.2], and (ii) for $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ is a consequence of [Berger 2002, Proposition 3.2]. □

Definition 4.5.5. For $s \in \mathbb{Q}$, an object $M \in \mathrm{Mod}_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi, G_K)$ is said to be pure of slope s if M is isomorphic to $(\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} D_{[s]})^m$ as a φ-module for some $m \in \mathbb{N}$. Denote by $\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi, G_K)$ the category of (φ, G_K)-modules over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$, which are pure of slope s.

Lemma 4.5.6. *Let the notation be as in Notation 1.6.2 and Definition 1.7.5. For $s \in \mathbb{Q}$, the forgetful functor*

$$\mathrm{Mod}^s_{\mathcal{R}}(\varphi, \nabla) \to \mathrm{Mod}^s_{\mathcal{R}}(\varphi).$$

is fully faithful.

Proof. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}^s_{\Gamma[1/p]}(\varphi, \nabla) \arrow[r, "\alpha_1"] & \mathrm{Mod}^s_{\Gamma[1/p]}(\varphi) \\
\mathrm{Mod}^s_{\Gamma^\dagger[1/p]}(\varphi, \nabla) \arrow[r, "\alpha_2"] \arrow[u, "\beta_1"] \arrow[d, "\beta_2"'] & \mathrm{Mod}^s_{\Gamma^\dagger[1/p]}(\varphi) \arrow[u, "\gamma_1"'] \arrow[d, "\gamma_2"] \\
\mathrm{Mod}^s_{\mathcal{R}}(\varphi, \nabla) \arrow[r, "\alpha_3"] & \mathrm{Mod}^s_{\mathcal{R}}(\varphi)
\end{tikzcd}$$

where $\alpha_\bullet$ is a forgetful functor, and $\beta_\bullet$ and $\gamma_\bullet$ are base change functors. We first note that γ_1 (resp. γ_2) is fully faithful (resp. an equivalence) by [Kedlaya 2005, Theorem 6.3.3(a)] (resp. [Kedlaya 2005, Theorem 6.3.3(b)]). Let $M, N \in \mathrm{Mod}^s_{\Gamma^\dagger[1/p]}(\varphi, \nabla)$ and let $\widehat{M}, \widehat{N}$ be the base changes of M, N via the canonical map $\Gamma^\dagger[1/p] \to \Gamma[1/p]$. Then, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}^s_{\Gamma^\dagger[1/p]}(\varphi,\nabla)}(M, N) &= \mathrm{Hom}_{\Gamma^\dagger[1/p]}(M, N)^{\varphi=1,\nabla=0} \\
&= \mathrm{Hom}_{\Gamma[1/p]}(\widehat{M}, \widehat{N})^{\varphi=1,\nabla=0},
\end{aligned}$$

where the first equality follows by definition and the second equality follows because γ_1 is fully faithful. Therefore, β_1 is fully faithful. For the same reason, since γ_2 is fully faithful, so is β_2. Note that α_1 is an equivalence in the étale case, i.e., $s = 0$ ([Kedlaya 2007, Proposition 3.2.8]). Let $M, N \in \mathrm{Mod}^s_{\Gamma[1/p]}(\varphi, \nabla)$. Since $\mathrm{Hom}_{\Gamma[1/p]}(M, N) \cong M^\vee \otimes_{\Gamma[1/p]} N$ can be regarded as an étale (φ, ∇)-module over

$\Gamma[1/p]$, where $M^\vee$ denotes the dual of M, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}^s_{\Gamma[1/p]}(\varphi,\nabla)}(M,N) &= \mathrm{Hom}_{\Gamma[1/p]}(M,N)^{\varphi=1,\nabla=0} = \mathrm{Hom}_{\Gamma[1/p]}(M,N)^{\varphi=1}\\ &= \mathrm{Hom}_{\mathrm{Mod}^s_{\Gamma[1/p]}(\varphi)}(M,N),\end{aligned}$$

where the first and third equalities follow from the definition and the second equality follows since α_1 is fully faithful in the étale case. Therefore, α_1 is an equivalence. Since α_1, β_1 and γ_1 are fully faithful, so is α_2. Since α_2, β_2 and γ_2 are fully faithful, so is α_3. $\square$

Lemma 4.5.7. *Let $s\in\mathbb{Q}$ and let $h\in\mathbb{N}_{\geq 1}$, $a\in\mathbb{Z}$ be relatively prime with $s=a/h$.*

(i) *There exist equivalences of categories*

$$\begin{aligned}
\widetilde{\mathbb{D}}^{\nabla+}_{\mathrm{rig}}&:\mathrm{Rep}_{a,h}(G_K)\to \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi,G_K); && V_{a,h}\mapsto \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}\otimes_{\mathbb{Q}_{p^h}} V_{a,h},\\
\widetilde{\mathbb{D}}^{\dagger}_{\mathrm{rig}}&:\mathrm{Rep}_{a,h}(G_K)\to \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}}(\varphi,G_K); && V_{a,h}\mapsto \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}\otimes_{\mathbb{Q}_{p^h}} V_{a,h},\\
\mathbb{D}^{\dagger}_{\mathrm{rig}}&:\mathrm{Rep}_{a,h}(G_K)\to \mathrm{Mod}^s_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}(\varphi,\Gamma_K); && V_{a,h}\mapsto \mathbb{B}^{\dagger}_{\mathrm{rig},K}\otimes_{\mathbb{B}^{\dagger}_K}(\mathbb{B}^{\dagger}\otimes_{\mathbb{Q}_{p^h}} V_{a,h})^{H_K},\\
\widetilde{\mathbb{D}}^{\dagger}&:\mathrm{Rep}_{a,h}(G_K)\to \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\dagger}}(\varphi,G_K); && V_{a,h}\mapsto \widetilde{\mathbb{B}}^{\dagger}\otimes_{\mathbb{Q}_{p^h}} V_{a,h},\\
\mathbb{D}^{\dagger}&:\mathrm{Rep}_{a,h}(G_K)\to \mathrm{Mod}^s_{\mathbb{B}^{\dagger}_K}(\varphi,\Gamma_K); && V_{a,h}\mapsto (\mathbb{B}^{\dagger}\otimes_{\mathbb{Q}_{p^h}} V_{a,h})^{H_K}.
\end{aligned}$$

More precisely, quasi-inverses of $\widetilde{\mathbb{D}}^{\nabla+}_{\mathrm{rig}}$, $\widetilde{\mathbb{D}}^{\dagger}_{\mathrm{rig}}$ and $\widetilde{\mathbb{D}}^{\dagger}$ are given by $M\mapsto M^{\varphi^h=p^a}$.

(ii) *We denote by α_i for $1\leq i\leq 5$ the following canonical morphisms of rings:*

$$\begin{tikzcd}
\mathbb{B}^{\dagger}_K \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"] & \mathbb{B}^{\dagger}_{\mathrm{rig},K} \arrow[d, "\alpha_4"] & \\
\mathbb{B}^{\dagger}_{\mathrm{rig}} \arrow[r, "\alpha_3"] & \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} & \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}, \arrow[l, "\alpha_5"']
\end{tikzcd}$$

where the left square is commutative. Then, the α_i's induce the following base change functors $\alpha^*_\bullet$:

$$\begin{tikzcd}
\mathrm{Mod}^s_{\mathbb{B}^{\dagger}_K}(\varphi,\Gamma_K) \arrow[r, "\alpha_1^*"] \arrow[d, "\alpha_2^*"] & \mathrm{Mod}^s_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}(\varphi,\Gamma_K) \arrow[d, "\alpha_4^*"] & \\
\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\dagger}}(\varphi,G_K) \arrow[r, "\alpha_3^*"] & \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}}(\varphi,G_K) & \mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi,G_K), \arrow[l, "\alpha_5^*"']
\end{tikzcd}$$

where the left square is commutative. Moreover, the functors $\alpha^*_\bullet$'s are compatible with the functor defined in (i), i.e., $\alpha_1^*\circ\mathbb{D}^{\dagger}=\mathbb{D}^{\dagger}_{\mathrm{rig}}$, etc. In particular, the $\alpha^*_\bullet$'s are equivalences.

Proof.

(i) We prove the assertion for $\widetilde{\mathbb{D}}^{\nabla+}_{\mathrm{rig}}$. Let $\mathcal{D} := \widetilde{\mathbb{D}}^{\nabla+}_{\mathrm{rig}}$ and let $\mathcal{V}$ be, as before, the functor in the other direction. Let $V \in \mathrm{Rep}_{a,h}(G_K)$. Then, there exists a functorial morphism $V \to \mathcal{V} \circ \mathcal{D}(V)$, which is bijective by Lemma 4.5.4(ii). Hence, we have a natural equivalence $\mathcal{V} \circ \mathcal{D} \simeq \mathrm{id}$. For $M \in \mathrm{Mod}^{s}_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi, G_K)$, we get a functorial morphism $\mathcal{D} \circ \mathcal{V}(M) \to M$ that is bijective by the isomorphism $M \cong (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} D_{[s]})^m$ of φ-modules and Lemma 4.5.4(ii). Hence, we have a natural equivalence $\mathcal{D} \circ \mathcal{V} \simeq \mathrm{id}$.

The assertions for $\widetilde{\mathbb{D}}^{\dagger}_{\mathrm{rig}}$ and $\widetilde{\mathbb{D}}^{\dagger}$ follow similarly: instead of using the isomorphism $M \cong (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} D_{[s]})^m$, we use Kedlaya's Dieudonné–Manin decomposition theorems over $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ and $\widetilde{\mathbb{B}}^{\dagger}$, see Propositions 4.5.3 and 4.5.10 and Definition 4.6.1; Theorem 6.3.3(b) of [Kedlaya 2005], respectively. These assert that any object M in $\mathrm{Mod}^{s}_{\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}}(\varphi)$ or $\mathrm{Mod}^{s}_{\widetilde{\mathbb{B}}^{\dagger}}(\varphi)$ is isomorphic to a direct sum of $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} D_{[s]}$ or of $\widetilde{\mathbb{B}}^{\dagger} \otimes_{\mathbb{Q}_p} D_{[s]}$, respectively.

We next prove the assertion for $\mathbb{D}^{\dagger}$. For $M \in \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{K}}(\varphi, \Gamma_K)$, let $\mathcal{V}(M) := (\mathbb{B}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_{K}} M)^{\varphi^h = p^a}$. We will check that $\mathcal{V}$ gives a quasi-inverse of $\mathbb{D}^{\dagger}$. Let $V_{a,h} \in \mathrm{Rep}_{a,h}(G_K)$. By forgetting the action of φ on $V_{a,h}$ and applying Theorem 1.10.5 to $V = V_{a,h}$, we get a canonical bijection $\mathbb{B}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_{K}} \mathbb{D}^{\dagger}(V_{a,h}) \to \mathbb{B}^{\dagger} \otimes_{\mathbb{Q}_{p^h}} V_{a,h}$. Since this map is φ-equivariant, we have canonical isomorphisms $\mathcal{V} \circ \mathbb{D}^{\dagger}(V_{a,h}) \cong (\mathbb{B}^{\dagger})^{\varphi^h=1} \otimes_{\mathbb{Q}_{p^h}} V_{a,h} \cong V_{a,h}$ by Lemma 4.5.4(ii). Thus, we obtain a natural equivalence $\mathcal{V} \circ \mathbb{D}^{\dagger} \simeq \mathrm{id}$. We prove $\mathbb{D}^{\dagger} \circ \mathcal{V} \simeq \mathrm{id}$. Let $M \in \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{K}}(\varphi, \Gamma_K)$. From [Kedlaya 2005, Proposition 6.3.5], we obtain the existence of an $\mathbb{A}^{\dagger}_{K}$-lattice N of M such that $p^{-a}\varphi^h$ maps some basis of N to another basis of N. Let M' denote M with the φ^h-action given by $x \mapsto p^{-a}\varphi^h(x)$ and with the same Γ_K-action as M. By the existence of the above lattice N, we have $M' \in \mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}^{\dagger}_{K}}(\varphi^h, \Gamma_K)$. Since we have G_K-equivariant isomorphisms $\mathcal{V}(M) = (\mathbb{B}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_{K}} M)^{\varphi^h=p^a} \cong (\mathbb{B}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_{K}} M')^{\varphi^h=1} = \mathbb{V}(M')$, the assertion follows from the étale case (Theorem 1.10.5).

Finally, we prove the assertion for $\mathbb{D}^{\dagger}_{\mathrm{rig}}$. By the base change equivalence

$$\alpha_1^* : \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{K}}(\varphi) \to \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}(\varphi),$$

see [Kedlaya 2005, Theorem 6.3.3(b)], we also have the base change equivalence $\alpha_1^* : \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{K}}(\varphi, \Gamma_K) \to \mathrm{Mod}^{s}_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}(\varphi, \Gamma_K)$. Hence, the assertion follows from the $\mathbb{D}^{\dagger}$-case.

(ii) To check that the $\alpha_{\bullet}^*$'s are well-defined, we have only to prove that pure objects are preserved by base change. For α_1 and α_3, this follows from [Kedlaya 2005, Theorem 6.3.3(b)]. For α_2, α_4, this follows from the definitions: $M \in \mathrm{Mod}_{\mathbb{B}^{\dagger}_{K}}(\varphi)$ and $\mathrm{Mod}_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}}(\varphi)$ are pure if $\widetilde{\mathbb{B}}^{\dagger} \otimes_{\mathbb{B}^{\dagger}_{K}} M$ and $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} M$, respectively, are pure by [Kedlaya 2005, Definitions 4.6.1 and 6.3.1] . For α_5, it follows from [Kedlaya 2005, Proposition 4.5.10 and Definition 4.6.1].

The commutativity of the diagram is trivial. The compatibility follows from the definition. □

4.6. ***Swan conductor for de Rham representations.*** In this subsection, we define Swan conductors of de Rham representations. In this subsection, Assumption 1.9.1 is not necessary since we do not use the results of [Andreatta and Brinon 2008].

We first recall the canonical slope filtration associated to a Dieudonné–Manin decomposition.

Definition 4.6.1 [Colmez 2008b, Remarque 3.3]. A φ-module M over $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}$ is a finite free $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}$-module together with a semilinear φ-action. A φ-module M over $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}$ admits a Dieudonné–Manin decomposition if there exists an isomorphism $f : M \cong \bigoplus_{1\le i\le m} \widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+} \otimes_{\mathbb{Q}_p} D_{[s_i]}$ of φ-modules over $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}$ with $s_1 \le \cdots \le s_m \in \mathbb{Q}$. We define the slope multiset of M as the multiset of cardinality $\operatorname{rank}(M)$, consisting of the s_i, together with its multiplicity $\dim_{\mathbb{Q}_p} D_{[s_i]}$. Let $s'_1 < \cdots < s'_{r'}$ be the distinct elements in the slope multiset of M. Then, we define $\mathrm{Fil}^0_f(M) := 0$ and $\mathrm{Fil}^i_f(M) := f^{-1}(\bigoplus_{j;s_j\le s'_i} \widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+} \otimes_{\mathbb{Q}_p} D_{[s_j]})$ for $1 \le i \le r'$. Note that the filtration and the slope multiset are independent of the choice of f above.

Definition 4.6.2. Let $V \in \mathrm{Rep}_{\mathrm{dR}}(G_K)$. First, we assume that the Hodge–Tate weights of V are negative. By assumption, we have $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. As in [Ohkubo 2013, Proposition 5.3], we define

$$\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V) := \bigl\{x \in \widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+} \otimes_{\mathbb{Q}_p} V;\ \iota_n(x) \in (\mathbb{B}^+_{\mathrm{dR}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V))^{\nabla^{\mathrm{geom}}=0} \text{ for all } n \in \mathbb{Z}\bigr\},$$

where $\iota_n : \widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+} \otimes_{\mathbb{Q}_p} V \to \mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is defined by $x \otimes v \mapsto \varphi^{-n}(x) \otimes v$. Since $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$ admits a Dieudonné–Manin decomposition due to Colmez ([Ohkubo 2013, Proposition 6.2]), $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$ is endowed with a canonical slope filtration $\mathrm{Fil}^\bullet(\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V))$ of φ-modules by Definition 4.6.1. Let $s_1 < \cdots < s_r$ be the distinct elements in the slope multiset of $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$. Write $s_i = a_i/h_i$ with $a_i \in \mathbb{Z}$, $h_i \in \mathbb{N}_{>0}$ relatively prime. By the uniqueness of slope filtrations, Fil^i is G_K-stable and the graded piece $\mathrm{gr}^i(\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V))$ lies in $\mathrm{Mod}^{s_i}_{\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}}(\varphi, G_K)$. Hence, by Lemma 4.5.7, there exists a unique $\mathcal{V}_i \in \mathrm{Rep}_{a_i,h_i}(G_K)$, up to isomorphism, such that $\mathrm{gr}^i(\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)) \cong \widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+} \otimes_{\mathbb{Q}_{p^{h_i}}} \mathcal{V}_i$. It is proved in Step 1 of the proof of the main theorem of [Ohkubo 2013] that the inertia I_K acts on $\mathcal{V}_i$ via a finite quotient, i.e., $\mathcal{V}_i \in \mathrm{Rep}^{f.g.}_{\mathbb{Q}_{p^{h_i}}}(G_K)$ (in the reference, Fil^i and $\mathcal{V}_i$ are denoted by $\mathcal{M}_i$ and W_i). Hence, we can define

$$\mathrm{Swan}(V) := \sum_i \mathrm{Swan}^{\mathrm{AS}}(\mathcal{V}_i).$$

In the general Hodge–Tate weights case, we define $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) := \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V(-n))(n)$ and $\mathrm{Swan}(V) := \mathrm{Swan}(V(-n))$ for sufficiently large n. The definition is independent of the choice of n since the above construction is compatible with Tate twist.

Remark 4.6.3. As in [Colmez 2008a], we should consider an appropriate contribution of "monodromy action" to define the Artin conductor. To avoid complication, we do not define Artin conductors for de Rham representations in this paper.

The lemma below easily follows from Hilbert 90.

Lemma 4.6.4. *Let $V \in \mathrm{Rep}_{\mathrm{dR}}(G_K)$.*

(i) *If L is the p-adic completion of an unramified extension of K, then we have* $\mathrm{Swan}(V|_L) = \mathrm{Swan}(V)$.

(ii) *Assume $V \in \mathrm{Rep}^f_{\mathbb{Q}_p}(G_K)$. Then, we have* $\mathrm{Swan}(V) = \mathrm{Swan}^{\mathrm{AS}}(V)$.

Though the following result will not be used in the proof of the main theorem, we remark that when k_K is perfect, our definition is compatible with the classical definition.

Lemma 4.6.5 (Compatibility of usual Swan conductor in the perfect residue field case). *Assume that k_K is perfect. Then, we have* $\mathrm{Swan}(V) = \mathrm{Swan}(\mathbb{D}_{\mathrm{pst}}(V))$ *(see* [*Colmez 2008a*, §0.4] *for the definition of* $\mathbb{D}_{\mathrm{pst}}$*).*

Proof. Let the notation be as in Definition 4.6.2. By Tate twist, we may assume that all Hodge–Tate weights of V are negative. By $\mathrm{Swan}(\mathbb{D}_{\mathrm{pst}}(V)) = \mathrm{Swan}(\mathbb{D}_{\mathrm{pst}}(V|_{K^{\mathrm{ur}}}))$ and Lemma 4.6.4(i), we may assume that k_K is algebraically closed by replacing K by K^{ur}. Since $\mathbb{B}^+_{\mathrm{dR}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V)$ is a lattice of $\mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$, we may identify $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)[1/t]$ with $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V[1/t]$. By the p-adic monodromy theorem, there exists a finite Galois extension L/K such that $\mathbb{D}_{\mathrm{st},L}(V) := (\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_L}$ has dimension $\dim_{\mathbb{Q}_p} V$. Moreover, we may assume that G_L acts trivially on each $\mathcal{V}_i$. Put $D_i := (\mathbb{B}_{\mathrm{st}} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \mathrm{Fil}^i(\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)))^{G_L}$. This forms an increasing filtration of $\mathbb{D}_{\mathrm{st},L}(V)$. Then, we have canonical morphisms

$$D_i/D_{i+1} \hookrightarrow (\mathbb{B}_{\mathrm{st}} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \mathrm{gr}^i(\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)))^{G_L} \cong (\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_{p^{h_i}}} \mathcal{V}_i)^{G_L} \cong W(k_L)[1/p] \otimes_{\mathbb{Q}_{p^{h_i}}} \mathcal{V}_i,$$

where the first injection is an isomorphism by counting dimensions. By the additivity of Swan conductors, we have $\mathrm{Swan}(\mathbb{D}_{\mathrm{pst}}(V)) = \mathrm{Swan}(\mathbb{D}_{\mathrm{st},L}(V)) = \sum_i \mathrm{Swan}(D_i/D_{i+1}) = \sum_i \mathrm{Swan}(\mathcal{V}_i) = \mathrm{Swan}(V)$. □

4.7. ***Main theorem.*** The aim of this subsection is to prove the following theorem, which generalizes Marmora's formula in Remark 4.7.2:

Main Theorem 4.7.1. *Let V be a de Rham representation of G_K. Then, the sequence* $\{\mathrm{Swan}(V|_{K_n})\}_{n>0}$ *is eventually stationary and we have*

$$\mathrm{Swan}^{\nabla}(\mathbb{N}_{\mathrm{dR}}(V)) = \lim_n \mathrm{Swan}(V|_{K_n}).$$

Remark 4.7.2. When k_K is perfect, we explain that our formula coincides with the following formula from [Marmora 2004, Théorème 1.1]:

$$\mathrm{Irr}(\mathbb{N}_{\mathrm{dR}}(V)) = \lim_{n\to\infty} \mathrm{Swan}(\mathbb{D}_{\mathrm{pst}}(V|_{K_n})).$$

Here, the LHS means the irregularity of $\mathbb{N}_{\mathrm{dR}}(V)$ regarded as a p-adic differential equation. By Lemma 4.6.5, the RHS is equal to the RHS in Main Theorem 4.7.1. Therefore, we only have to prove $\mathrm{Irr}(D) = \mathrm{Swan}^{\nabla}(D)$ for a (φ, ∇)-module D over the Robba ring. Since D is endowed with a slope filtration and since both irregularity and the differential Swan conductor are additive, we may assume that D is étale by dévissage. Let V be the corresponding p-adic representation of finite local monodromy. Then, the differential Swan conductor $\mathrm{Swan}^{\nabla}(D)$ coincides with the usual Swan conductor of V ([Kedlaya 2007, Proposition 3.5.5]). On the other hand, $\mathrm{Irr}(D)$ coincides with the usual Swan conductor of V ([Tsuzuki 1998, Theorem 7.2.2]), which implies the assertion.

We will deduce Theorem 4.7.1 from Lemma 3.5.4(ii) by dévissage. In the following, we use the notation as in Definition 4.6.2.

Lemma 4.7.3. *Let V be a de Rham representation of G_K with nonpositive Hodge–Tate weights.*

(i) *The (φ, G_K)-modules*

$$\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger}_{\mathrm{rig},K}} \mathbb{N}_{\mathrm{dR}}(V), \quad \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$$

coincide with each other in $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$. Moreover, the two filtrations induced by the slope filtrations of $\mathbb{N}_{\mathrm{dR}}(V)$ and $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ also coincide with each other.

(ii) *Let the notation be as in Construction 1.7.7. Then, there exists a canonical isomorphism*

$$\mathrm{gr}^i(\mathbb{N}_{\mathrm{dR}}(V)) \cong D^{\dagger}_{\mathrm{rig}}(\mathcal{V}_i|_{\mathbb{E}_K})$$

as (φ, ∇)-modules over $\mathbb{B}^{\dagger}_{\mathrm{rig},K}$.

Proof. (i) We prove the first assertion. By Lemma 4.2.11 (with $B = \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$), we only have to prove that $D^{(1)} := \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbb{N}_{\mathrm{dR},r}(V)$, $D^{(2)} := \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ and $D := \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$ satisfy the conditions in the lemma. We have $\mathbb{N}_{\mathrm{dR},r}(V)[1/t] = \mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V)[1/t]$ by definition and

$$\begin{aligned}\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbb{D}^{\dagger,r}_{\mathrm{rig}}(V) &\cong \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,r}} \mathbb{B}^{\dagger,r} \otimes_{\mathbb{B}^{\dagger,r}} \mathbb{D}^{\dagger,r}(V)\\ &\cong \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{B}^{\dagger,r}} \mathbb{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V \cong \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V.\end{aligned}$$

As we have $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)[1/t] = \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$ by definition, we obtain a canonical isomorphism $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)[1/t] \cong \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$, which implies condition (i).

By Proposition 4.2.12(ii), we have a canonical isomorphism $\mathbb{B}^+_{\mathrm{dR}}\otimes_{\iota_n,\mathbb{B}^{\dagger,r}_{\mathrm{rig},K}}\mathbb{N}_{\mathrm{dR},r}(V)\cong\mathbb{B}^+_{\mathrm{dR}}\otimes_K\mathbb{D}_{\mathrm{dR}}(V)$. On the other hand, we have canonical isomorphisms

$$\mathbb{B}^+_{\mathrm{dR}}\otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)\cong\mathbb{B}^+_{\mathrm{dR}}\otimes_{\mathbb{B}^{\nabla+}_{\mathrm{dR}}}(\mathbb{B}^+_{\mathrm{dR}}\otimes_K\mathbb{D}_{\mathrm{dR}}(V))^{\nabla^{\mathrm{geom}}=0}\cong\mathbb{B}^+_{\mathrm{dR}}\otimes_K\mathbb{D}_{\mathrm{dR}}(V),$$

where the first isomorphism follows from [Ohkubo 2013, Proposition 5.3(ii)] and the second isomorphism follows from [Ohkubo 2013, Proposition 5.4]. Since the canonical map $\mathbb{B}^{\nabla+}_{\mathrm{dR}}\to\mathbb{B}^+_{\mathrm{dR}}$ is faithfully flat, condition (ii) is verified. The second assertion follows from the uniqueness of the slope filtration [Kedlaya 2005, Theorem 6.4.1].

(ii) By (i), there exists canonical isomorphisms

$$\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{B}^\dagger_{\mathrm{rig},K}}\mathrm{gr}^i(\mathbb{N}_{\mathrm{dR}}(V))\cong\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}\otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}\mathrm{gr}^i(\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V))\cong\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{Q}_{p^{h_i}}}\mathcal{V}_i$$

as (φ, G_K)-modules. By Lemma 4.5.7, we obtain a canonical isomorphism between $\mathrm{gr}^i(\mathbb{N}_{\mathrm{dR}}(V))$ and $\mathbb{D}^\dagger_{\mathrm{rig}}(\mathcal{V}_i)$ as (φ,Γ_K)-modules. Since $\mathcal{V}_i$ is of finite local monodromy, so is $\mathcal{V}_i|_{\mathbb{E}_K}$. So, $\dim_{\mathbb{B}^\dagger_K}D^\dagger(\mathcal{V}_i|_{\mathbb{E}_K})=\dim_{\mathbb{Q}_{p^h}}\mathcal{V}_i$; in particular, the canonical injection $D^\dagger(\mathcal{V}_i|_{\mathbb{E}_K})\hookrightarrow(\mathbb{B}^\dagger\otimes_{\mathbb{Q}_{p^{h_i}}}\mathcal{V}_i)^{H_K}$ is an isomorphism. Therefore, we have canonical isomorphisms $D^\dagger_{\mathrm{rig}}(\mathcal{V}_i|_{\mathbb{E}_K})\cong\mathbb{D}^\dagger_{\mathrm{rig}}(\mathcal{V}_i)\cong\mathrm{gr}^i(\mathbb{N}_{\mathrm{dR}}(V))$ as (pure) φ-modules over $\mathbb{B}^\dagger_{\mathrm{rig},K}$; hence, the assertion follows from Lemma 4.5.6. □

Remark 4.7.4. One can prove that there exist canonical isomorphisms

$$\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{B}^\dagger_{\mathrm{rig},K}}\mathbb{N}_{\mathrm{dR}}(V)\cong\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}\otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)\cong\mathbb{N}^\dagger_{\mathrm{rig}}(V).$$

Lemma 4.7.5. *We have*

$$\mathrm{Swan}^\nabla(\mathbb{N}_{\mathrm{dR}}(V))=\sum_{1\le i\le r}\mathrm{Swan}^{\mathrm{AS}}(\mathcal{V}_i|_{\mathbb{E}_K}).$$

Proof. We have

$$\begin{aligned}\mathrm{Swan}^\nabla(\mathbb{N}_{\mathrm{dR}}(V))&=\sum_{1\le i\le r}\mathrm{Swan}^\nabla\big(\mathrm{gr}^i(\mathbb{N}_{\mathrm{dR}}(V))\big)\\&=\sum_{1\le i\le r}\mathrm{Swan}^\nabla\big(D^\dagger_{\mathrm{rig}}(\mathcal{V}_i|_{\mathbb{E}_K})\big)=\sum_{1\le i\le r}\mathrm{Swan}^{\mathrm{AS}}(\mathcal{V}_i|_{\mathbb{E}_K}),\end{aligned}$$

where the first equality follows from the additivity of the differential Swan conductor (Lemma 1.7.9), the second one follows from Lemma 4.7.3(ii), and the third one follows from Xiao's comparison theorem (Theorem 1.7.10). □

Proof of Main Theorem 4.7.1. By Lemma 4.7.5 and the definition of the Swan conductor (Definition 4.6.2), we only have to prove $\mathrm{Swan}^{\mathrm{AS}}(\mathcal{V}_i|_{\mathbb{E}_K})=\mathrm{Swan}^{\mathrm{AS}}(\mathcal{V}_i|_{K_n})$ for all sufficiently large n. This follows from Lemma 3.5.4(ii). □

Appendix: list of notation

The following is a list of notation in order defined.

1.2: $\widehat{\Omega}^1_K$, ∂_j, $\partial/\partial t_j$.

1.3: $\widetilde{K}_n$, $\widetilde{K}_\infty$, $\Gamma_{\widetilde{K}}$, $H_{\widetilde{K}}$, γ_a, γ_b, $\eta=(\eta_0,\dots,\eta_d)$, $\mathfrak{g}$, L_n, L_∞, Γ_L, H_L, $\Gamma_{L,j}$.

1.4: $\widetilde{\mathbb{E}}^{(+)}$, $v_{\widetilde{\mathbb{E}}}$, $\widetilde{\mathbb{A}}^{(+)}$, $\widetilde{\mathbb{B}}^{(+)}$, ε, $\tilde{t}_j$, π, q, $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{dR}}^{(+)}$, u_j, t, $\mathbb{D}_{\mathrm{dR}}(\,\cdot\,)$, ∇^{geom}, $\mathbb{B}_{\mathrm{dR}}^{\nabla(+)}$, $\mathbb{A}_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{cris}}$, $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}$.

1.5: $as^a_{L/K,Z}$, $\mathcal{F}^a(L)$, $b(L/K)$, $as^a_{L/K,Z,P}$, $\mathcal{F}^a_{\log}(L)$, $b_{\log}(L/K)$, $\mathrm{Art}^{\mathrm{AS}}(\,\cdot\,)$, $\mathrm{Swan}^{\mathrm{AS}}(\,\cdot\,)$.

1.6: $v^{\le n}$, w_r, $W(E)_r$, $W_{\mathrm{con}}(E)$, Γ_r, Γ_{con}, $\Gamma_{\mathrm{an},r}$, $\Gamma_{\mathrm{an,con}}$, $\mathcal{O}\{\{S\}\}$, $\mathcal{O}((S))^{\dagger,r}$, $\mathcal{O}((S))^{\dagger}$, $\mathcal{R}$, $\mathrm{Mod}_\bullet(\sigma)$, $\mathrm{Mod}^{\mathrm{et}}_\bullet(\sigma)$, $\mathrm{Mod}^{s}_\bullet(\sigma)$.

1.7: Ω^1_R, $\Omega^1_{\mathcal{R}}$, $d:\mathcal{R}\to\Omega^1_{\mathcal{R}}$, $\mathrm{Mod}^s_\bullet(\varphi^h,\nabla)$ D, $D^\dagger$, $\mathrm{Swan}^\nabla(\,\cdot\,)$.

1.8: $X^{(+)}_{\mathfrak{K}}=X^{(+)}(\mathfrak{K},\xi,n_0)$.

1.9: $\mathbb{E}^{(+)}_L$, $\widetilde{\mathbb{E}}^{(+)}_K$, $\widetilde{\mathbb{A}}^{(+)}_L$, $\widetilde{\mathbb{B}}_L$, $\mathbb{A}$, $\mathbb{B}_L$, $\mathbb{B}$, $\mathrm{Mod}^{\mathrm{et}}_{\mathbb{B}_L}(\varphi^h,\Gamma_L)$, $\mathbb{D}(\,\cdot\,)$, $\mathbb{V}(\,\cdot\,)$.

1.10: $\widetilde{\mathbb{A}}^{\dagger,r}$, $\widetilde{\mathbb{A}}^{\dagger}$, $\widetilde{\mathbb{B}}^{\dagger,r}$, $\widetilde{\mathbb{B}}^{\dagger}$, $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$, $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$, $\mathbb{A}^{\dagger,r}$, $\mathbb{A}^{\dagger}$, $\mathbb{B}^{\dagger,r}$, $\mathbb{B}^{\dagger}$, $\mathbb{B}^{\dagger,r}_{\mathrm{rig}}$, $\mathbb{B}^{\dagger}_{\mathrm{rig}}$, $\widetilde{\mathbb{A}}^{\dagger,r}_L$, $\widetilde{\mathbb{A}}^{\dagger}_L$, $\widetilde{\mathbb{B}}^{\dagger,r}_L$, $\widetilde{\mathbb{B}}^{\dagger}_L$, $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},L}$, $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},L}$, $\mathbb{A}^{\dagger,r}_L$, $\mathbb{A}^{\dagger}_L$, $\mathbb{B}^{\dagger,r}_L$, $\mathbb{B}^{\dagger}_L$, $\mathbb{B}^{\dagger,r}_{\mathrm{rig},L}$, $\mathbb{B}^{\dagger}_{\mathrm{rig},L}$, $\mathbb{D}^{\dagger,r}(\,\cdot\,)$, $\mathbb{D}^{\dagger}(\,\cdot\,)$, $\mathbb{D}^{\dagger,r}_{\mathrm{rig}}(\,\cdot\,)$, $\mathbb{D}^{\dagger}_{\mathrm{rig}}(\,\cdot\,)$.

3.1: $R\langle\underline{X}\rangle$, $\mathcal{O}((S))^{\dagger,r}_0$, $|\cdot|_r$, $\mathcal{O}[\![S]\!]\langle\underline{X}\rangle$, $\mathcal{O}((S))^{\dagger,r}_0\langle\underline{X}\rangle$, $\mathcal{O}((S))^{\dagger,r}\langle\underline{X}\rangle$, $\deg(\mathfrak{p})$, $\kappa(\mathfrak{p})$, $\kappa(p)$, $\pi_{\mathfrak{p}}$.

3.2: $\succeq$, $\succ$, $\succeq_{\mathrm{lex}}$, $\underline{v}_R$, $\underline{\deg}_R$, $\mathrm{LT}_R(\,\cdot\,)$, $|\cdot|_{\mathrm{qt}}$.

3.3: A, $I^{\dagger,r}$, $A^{\dagger,r}$, $|\cdot|_{r,\mathrm{qt}}$.

3.4: $\mathrm{Idem}(\,\cdot\,)$, $as\ \cdot_{\mathfrak{p},\mathrm{qt}}$, $|\cdot|_{\mathfrak{p},\mathrm{sp}}$ $A_{\kappa(\mathfrak{p})}$.

3.5: AS^r, $AS^r_{\log}$.

4.1: $K_n[\![u_1,\dots,u_d]\!]^\nabla$.

4.2: ι_n, $t_{n,w}$, $\mathbb{N}_{\mathrm{dR},r}(\,\cdot\,)$, $\mathbb{N}_{\mathrm{dR}}(\,\cdot\,)$, $\widetilde{\mathbb{N}}^{\dagger,r}_{\mathrm{rig}}(\,\cdot\,)$, $\widetilde{\mathbb{N}}^{\dagger}_{\mathrm{rig}}(\,\cdot\,)$.

4.3: ∇_j.

4.4: $\widetilde{\nabla}_j$, ∇'_j.

4.5: $\mathrm{Rep}_{a,h}(G_K)$, $D_{[s]}$, $\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig}}}(\varphi,G_K)$, $\mathrm{Mod}^s_{\mathbb{B}^\dagger_{\mathrm{rig},K}}(\varphi,\Gamma_K)$, $\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^\dagger}(\varphi,G_K)$, $\mathrm{Mod}^s_{\mathbb{B}^\dagger_K}(\varphi,\Gamma_K)$, $\mathrm{Mod}^s_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}}(\varphi,G_K)$, $\widetilde{\mathbb{D}}^{\nabla+}_{\mathrm{rig}}(\,\cdot\,)$, $\widetilde{\mathbb{D}}^{+}_{\mathrm{rig}}(\,\cdot\,)$, $\mathbb{D}^{\dagger}_{\mathrm{rig}}(\,\cdot\,)$, $\widetilde{\mathbb{D}}^{\dagger}(\,\cdot\,)$, $\mathbb{D}^{\dagger}(\,\cdot\,)$.

4.6: $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(\,\cdot\,)$, $\mathcal{V}_i$, $\mathrm{Swan}(\,\cdot\,)$.

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