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We study the condition on a field that any triple of (bilinear) Pfister forms of a given dimension are linked. This is a strengthening of the condition of linkage investigated by Elman and Lam, which asks the same for pairs of Pfister forms. In characteristic different from two this condition for triples of 2-fold Pfister forms is related to the Hasse number.

1. Introduction

Milnor's seminal article [1969/1970] on K -theory of fields had an enormous impact on quadratic form theory. In a series of articles Elman and Lam explored the correspondence between Pfister forms and symbols (canonical generators) in the K -theory modulo 2 of a field. The notion of *linkage* for Pfister forms was introduced in [Elman and Lam 1972a]. With the definition from [Elman and Lam 1972a, Section 4] one can consider linkage of a finite number of Pfister forms. However, the study of linkage has mostly been limited to pairs of Pfister forms. Initially, this study was restricted to fields of characteristic different from 2, where quadratic forms are characterised by their associated (symmetric bilinear) polar forms.

When trying to extend notions and statements to cover the case of characteristic 2, one has to choose between quadratic forms or symmetric bilinear forms. In this article we work mainly in the setup of Milnor K -theory over a field of arbitrary characteristic. We study linkage of symbols in the Milnor K -groups modulo 2, or equivalently, of symmetric bilinear Pfister forms. In particular, we study the condition that a certain Milnor K -group modulo 2 has *triple linkage*, i.e., that any three symbols have a common linkage. This condition turns out to have stronger consequences than usual linkage, in particular on the vanishing of higher K -groups. In the last section we focus on fields of characteristic different from 2 and relate the condition of triple linkage to quadratic forms and the Hasse number \tilde{u} (the u -invariant if the field is nonreal).

For a recent study of triple linkage of quadratic Pfister forms covering fields of characteristic 2, we refer the reader to [Chapman et al. 2018].

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2. Symbols and Pfister forms

We refer to [Elman et al. 2008] for standard results used from quadratic form theory. Let E be a field. By a *form over E* we mean a pair (V, b) of a finite-dimensional E -vector space V and a nondegenerate symmetric bilinear form b on V . We use an equality sign to indicate that two forms are isometric.

Let n always denote a nonnegative integer. Given $a_1, \dots, a_n \in E^\times$ we denote the bilinear n -fold Pfister form $\langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$ over E by $\langle\langle a_1, \dots, a_n \rangle\rangle$. In the sequel we refer to bilinear Pfister forms simply as *Pfister forms*. Given a Pfister form π , the orthogonal complement of the subform $\langle 1 \rangle$ in π is called the *pure part of π* .

Theorem 2.1 (Elman–Lam). *Let $r \in \mathbb{N}$. Let ρ be an anisotropic r -fold Pfister form over E and let ρ' denote its pure part. Let π be a Pfister form over E such that $\pi \otimes \rho$ is anisotropic and let $c_1 \in E^\times$ be such that $-c_1$ is represented by $\pi \otimes \rho'$. Then there exist $c_2, \dots, c_r \in E^\times$ such that $\pi \otimes \rho = \pi \otimes \langle\langle c_1, \dots, c_r \rangle\rangle$.*

Proof. See [Elman et al. 2008, Proposition 6.15] or [Elman and Lam 1972a, Theorem 2.6]. □

We denote by $k_n E$ the n -th Milnor K -group of E modulo 2; this is the abelian group generated by symbols $\{a_1, \dots, a_n\}$, with $a_1, \dots, a_n \in E^\times$, subject to the defining relations that the map $(E^\times)^n \rightarrow k_n E$ given by $(a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$ is multilinear and further that $\{a_1, \dots, a_n\} = 0$ whenever $a_i \in E^{\times 2}$ for some $i \leq n$ or $a_i + a_{i+1} = 1$ for some $i < n$. The direct sum $\bigoplus_{n \in \mathbb{N}} k_n E$ is a graded ring with the multiplication induced by concatenation of symbols.

We recall some results from [Elman et al. 2008] on the relation of symbols and Pfister forms, which for fields of characteristic different from 2 go back to [Elman and Lam 1972a]. We begin with the one-to-one correspondence between symbols and Pfister forms.

Theorem 2.2 (Elman–Lam). *For $a_1, \dots, a_n, b_1, \dots, b_n \in E^\times$, we have*

$$\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \quad \text{if and only if} \quad \langle\langle a_1, \dots, a_n \rangle\rangle = \langle\langle b_1, \dots, b_n \rangle\rangle.$$

Proof. See [Elman et al. 2008, Theorem 6.20]. □

We denote by ΣE^2 the subgroup of E^\times consisting of the nonzero sums of squares in E . Recall that the field E is *real* if $-1 \notin \Sigma E^2$, *nonreal* otherwise. For $m \in \mathbb{N}$ we denote by $D_E(m)$ the subset of ΣE^2 consisting of the elements that are sums of m squares in E .

Corollary 2.3. *For $a \in E^\times$ and the symbol $\tau = \{-1, \dots, -1\}$ in $k_n E$, we have:*

- (a) $a \in D_E(2^n)$ if and only if $\tau \cdot \{a\} = 0$ in $k_{n+1} E$.
- (b) $a \in D_E(2^n - 1)$ if and only if $\tau = \{-a, a_2, \dots, a_n\}$ for certain $a_2, \dots, a_n \in E^\times$.

Proof. This follows from [Theorem 2.1](#) and [Theorem 2.2](#). \square

Lemma 2.4. *Let τ and τ' be symbols in $k_n E$ and let $a, a' \in E^\times$ be such that $\tau \cdot \{a\} = \tau' \cdot \{a'\}$. Then $\tau \cdot \{a\} = \tau \cdot \{c\} = \tau' \cdot \{c\} = \tau' \cdot \{a'\}$ for some $c \in E^\times$.*

Proof. This follows from [\[Elman et al. 2008, Corollary 6.16 and Theorem 6.20\]](#). \square

3. Linkage

Assume from now on that $n \geq 2$. Given two symbols $\sigma_1, \sigma_2 \in k_n E$, the sum $\sigma_1 + \sigma_2 \in k_n E$ is equal to a symbol if and only if there exists a symbol $\sigma' \in k_{n-1} E$ and $b_1, b_2 \in E^\times$ such that $\sigma_i = \sigma' \cdot \{b_i\}$ for $i = 1, 2$ (see [\[Elman and Lam 1972a, Lemma 5.4\]](#)); in this case, we say that σ_1 and σ_2 are *linked*.

We say that $k_n E$ is *linked* if any two symbols in $k_n E$ are linked (which in the terminology of [\[Elman and Lam 1973\]](#) corresponds to saying that $l^n E$ is linked.) Obviously, if $k_n E$ is linked, then so is $k_m E$ for any integer $m \geq n$.

The following statement was obtained in [\[Elman and Lam 1973, Corollary 2.8 and Corollary 2.9\]](#). For convenience of the reader we include a compact proof, whose first lines follow [\[Elman and Lam 1972b, Section 3, Example 3\]](#). The statement should be compared with [Theorem 5.1](#).

Theorem 3.1 (Elman–Lam). *Assume that $k_n E$ is linked. Then $\Sigma E^2 = D_E(2^{n+1})$ and $k_{n+2} E = \{-1, \dots, -1\} \cdot k_1 E$. In particular, if E is nonreal then $k_{n+2} E = 0$.*

Proof. Consider an arbitrary symbol $\tau \in k_{n-2} E$ and $a_1, a_2, b_1, b_2 \in E^\times$. Since $k_n E$ is linked and by [Lemma 2.4](#), there exist $c_1, c_2 \in E^\times$ and a symbol $\sigma \in k_{n-1} E$ such that $\tau \cdot \{a_i, b_i\} = \tau \cdot \{a_i, c_i\} = \sigma \cdot \{c_i\}$ for $i = 1, 2$. It follows that

$$\tau \cdot \{a_1, c_1, c_2\} = \sigma \cdot \{c_1, c_2\} = \tau \cdot \{a_2, c_1, c_2\},$$

whereby

$$\tau \cdot \{a_1, b_1, a_2, b_2\} = \tau \cdot \{a_1, c_1, a_2, c_2\} = \tau \cdot \{a_2, c_1, a_2, c_2\} = \{-1, c_1\} \cdot \tau \cdot \{a_2, b_2\}.$$

This argument shows that for any $a_1, \dots, a_{n+2} \in E^\times$, there exists $c \in E^\times$ such that

$$\{a_1, \dots, a_{n+2}\} = \{a_2, \dots, a_{n+1}, -1, c\}.$$

Applying this rule $n + 1$ times, we conclude that every symbol in $k_{n+2} E$ is of the form $\{-1, \dots, -1, c\}$ with $c \in E^\times$. Hence $k_{n+2} E = \{-1, \dots, -1\} \cdot k_1 E$.

Moreover, if $a_1, \dots, a_{n+2} \in E^\times$ are such that $a_2 \in D_E(2)$, then $\{-1, a_2\} = 0$ and we obtain from the above rule that $\{a_1, \dots, a_{n+2}\} = 0$. Thus we have $\{a\} \cdot k_{n+1} E = 0$ for any $a \in D_E(2)$.

Consider an element $c \in D_E(2^{n+1} + 1)$. We write $c = a + b$ with $a \in D_E(2)$ and $b \in D_E(2^{n+1} - 1)$. In $k_{n+1} E$ we obtain that $\{-1, \dots, -1\} = \{-b\} \cdot \tau$ for a

symbol τ in $k_n E$, by [Corollary 2.3](#). Since $c - b = a$ we have $\{-b, c\} = \{a, bc\}$. As $\{a\} \cdot k_{n+1} E = 0$, we obtain in $k_{n+2} E$ that

$$\{-1, \dots, -1, c\} = \{-b, c\} \cdot \tau = \{a, bc\} \cdot \tau = 0,$$

which shows that $c \in D_E(2^{n+1})$. This argument shows that $\Sigma E^2 = D_E(2^{n+1})$.

Assume finally that E is nonreal. If $-1 \in E^{\times 2}$ then $\{-1\} = \{1\} = 0$ in $k_1 E$. If $\text{char}(E) \neq 2$ then $E^\times = \Sigma E^2 = D_E(2^{n+1})$. Hence, in any case we obtain that $k_{n+2} E = \{-1, \dots, -1\} \cdot k_1 E = 0$. \square

If E is nonreal and $k_n E$ is linked with $n \geq 2$, then $k_{n+2} E$ vanishes by [Theorem 3.1](#), but we may have that $k_{n+1} E \neq 0$, as the following well-known example shows.

Example 3.2. For $E = \mathbb{C}((t_1)) \cdots ((t_{n+1}))$, $k_n E$ is linked and $k_{n+1} E \simeq \mathbb{Z}/2\mathbb{Z}$.

4. The linkage pairing

We are going to investigate an operation on linked symbols. Let $n \geq 2$. To any pair of linked symbols in $k_n E$ we associate a symbol in $k_{n+1} E$.

Proposition 4.1. *Let $\sigma_1, \sigma_2 \in k_n E$ be two linked symbols. There is a unique symbol $\rho \in k_{n+1} E$ such that, for any symbol $\tau \in k_{n-1} E$ and any $a_1, a_2 \in E^\times$ with $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$, we have $\rho = \tau \cdot \{a_1, a_2\}$.*

Proof. By the hypothesis there exist a symbol $\tau \in k_{n-1} E$ and elements $a_1, a_2 \in E^\times$ with $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$. Suppose we have another symbol $\tau' \in k_{n-1} E$ and $a'_1, a'_2 \in E^\times$ with $\sigma_i = \tau' \cdot \{a'_i\}$ for $i = 1, 2$. By [Lemma 2.4](#) there exist $c_1, c_2 \in E^\times$ such that $\tau \cdot \{c_i\} = \sigma_i = \tau' \cdot \{c_i\}$ holds for $i = 1, 2$. We obtain that

$$\tau \cdot \{a_1, a_2\} = \tau \cdot \{c_1, c_2\} = \tau' \cdot \{c_1, c_2\} = \tau' \cdot \{a'_1, a'_2\}. \quad \square$$

Corollary 4.2. *Suppose $a, b, c \in E^\times$ and let τ be a symbol in $k_{n-1} E$ such that $\tau \cdot \{-1, a\} = 0$. Assume there exist a symbol ρ in $k_n E$ and $x, y, z \in E^\times$ such that $\tau \cdot \{a, b\} = \rho \cdot \{x\}$, $\tau \cdot \{a, c\} = \rho \cdot \{y\}$ and $\tau \cdot \{b, c\} = \rho \cdot \{z\}$. Then $\tau \cdot \{a, b, c\} = 0$.*

Proof. By [Proposition 4.1](#) we obtain that $\rho \cdot \{x, z\} = \tau \cdot \{a, b, c\} = \rho \cdot \{y, z\}$, whereby $\rho \cdot \{xy, z\} = 0$. We have $\tau \cdot \{b, -bc\} = \tau \cdot \{b, c\} = \rho \cdot \{z\}$. Since $\tau \cdot \{-1, a\} = 0$ we further have $\tau \cdot \{a, -bc\} = \tau \cdot \{a, bc\} = \rho \cdot \{xy\}$. We conclude with [Proposition 4.1](#) that $\tau \cdot \{a, b, c\} = \tau \cdot \{a, b, -bc\} = \rho \cdot \{xy, z\} = 0$. \square

Corollary 4.3. *Assume that $k_n E$ is linked. We obtain a surjective pairing*

$$\langle \cdot, \cdot \rangle : k_n E \times k_n E \rightarrow k_{n+1} E$$

by letting $\langle \tau \cdot \{a_1\}, \tau \cdot \{a_2\} \rangle = \tau \cdot \{a_1, a_2\}$ for any symbol $\tau \in k_{n-1} E$ and any $a_1, a_2 \in E^\times$.

Proof. Let $\sigma_1, \sigma_2 \in k_n E$ be given. As $k_n E$ is linked, there exist $a_1, a_2 \in E^\times$ and a symbol $\tau \in k_{n-1} E$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$. By [Proposition 4.1](#) the symbol $\rho = \tau \cdot \{a_1, a_2\} \in k_{n+1} E$ only depends on σ_1 and σ_2 but not on the choice of τ and $a_1, a_2 \in E^\times$. Hence, the pairing is well-defined. As $k_n E$ is linked, so is $k_{n+1} E$, and it follows that the pairing is surjective. \square

If $k_n E$ is linked then we call the pairing in [Corollary 4.3](#) the *linkage pairing on $k_n E$* .

Theorem 4.4. *Assume that $k_n E$ is linked. Then the following are equivalent:*

- (i) *The linkage pairing on $k_n E$ is bilinear.*
- (ii) $\Sigma E^2 = D_E(2^n)$ and $k_{n+1} E = \{-1, \dots, -1\} \cdot k_1 E$.
- (iii) *Either $k_{n+1} E = 0$, or E is real and the rule $c \mapsto \{-1, \dots, -1, c\}$ determines an isomorphism $E^\times / \Sigma E^2 \rightarrow k_{n+1} E$.*

Proof. As a consequence of the definition of the linkage pairing

$$\langle \cdot, \cdot \rangle : k_n E \times k_n E \rightarrow k_{n+1} E,$$

we have for any $\rho, \rho' \in k_n E$ that $\langle \rho, \rho' \rangle = \rho \cdot \{d\}$ for some $d \in E^\times$.

(i) \Rightarrow (ii): Consider an arbitrary symbol $\tau \in k_{n-2} E$ and $a, b, c \in E^\times$. Set $\sigma_1 = \{a, b\}$, $\sigma_2 = \{a, c\}$, $\sigma_3 = \{b, c\}$ and $\sigma_4 = \{-ab, c\}$. We obtain that $\langle \tau \cdot \sigma_i, \tau \cdot \sigma_1 \rangle = \tau \cdot \{a, b, c\}$ for $i = 2, 3, 4$. Assuming that the pairing is bilinear, we get that

$$\langle \tau \cdot (\sigma_2 + \sigma_3 + \sigma_4), \tau \cdot \sigma_1 \rangle = \tau \cdot \{a, b, c\}.$$

Since $\sigma_2 + \sigma_3 + \sigma_4 = \{-1, c\}$, we conclude that

$$\tau \cdot \{a, b, c\} = \langle \tau \cdot \{-1, c\}, \tau \cdot \{a, b\} \rangle = \tau \cdot \{-1, c, d\}$$

for some $d \in E^\times$. This argument shows that, for any $c_1, \dots, c_n, c_{n+1} \in E^\times$, there exists $d \in E^\times$ such that $\{c_1, \dots, c_n, c_{n+1}\} = \{-1, c_1, \dots, c_{n-1}, d\}$ in $k_{n+1} E$. Using this rule n times we obtain for any $c_1, \dots, c_n, c_{n+1} \in E^\times$ that

$$\{c_1, \dots, c_{n+1}\} = \{-1, \dots, -1, c_1, d\} = \{-1, \dots, -1, d'\}$$

for some $d, d' \in E^\times$. This shows that $k_{n+1} E = \{-1, \dots, -1\} \cdot k_1 E$ and that $\{a\} \cdot k_n E = 0$ for any $a \in D_E(2^{n-1})$.

Consider now an element $c \in D_E(2^n + 1)$. We choose $a, b \in D_E(2^{n-1})$ such that $c - a - b$ is a square in E . Then we have $\{-a, -b, c\} = 0$ in $k_3 E$, whereby $\{a, -b, c\} = \{-1, -b, c\}$. For the symbol $\epsilon = \{-1, \dots, -1\}$ in $k_{n-2} E$ we obtain that $\epsilon \cdot \{-1, b\} = 0$ and conclude that

$$\epsilon \cdot \{-1, -1, c\} = \epsilon \cdot \{-1, -b, c\} = \epsilon \cdot \{a, -b, c\} = 0$$

for $a \in D_E(2^{n-1})$. Hence $\{-1, \dots, -1, c\} = 0$ in $k_{n+1}E$, whereby $c \in D_E(2^n)$. This shows that $\Sigma E^2 = D_E(2^n)$.

(ii) \Rightarrow (iii): This implication is obvious.

(iii) \Rightarrow (i): Let $\varepsilon = \{-1, \dots, -1\}$ in $k_{n-1}E$. For any symbol $\tau \in k_{n-1}E$ and any $a, b \in E^\times$ we have $\tau \cdot \{a\} \cdot \tau \cdot \{b\} = \varepsilon \cdot \tau \cdot \{a, b\} = \varepsilon \cdot \langle \tau \cdot \{a\}, \tau \cdot \{b\} \rangle$. Since k_nE is linked, this means that

$$\rho \cdot \rho' = \varepsilon \cdot \langle \rho, \rho' \rangle \text{ in } k_{2n}E \text{ for any } \rho, \rho' \in k_nE.$$

Hence, the pairing $k_nE \times k_nE \rightarrow k_{2n}E$, $(\rho, \rho') \mapsto \varepsilon \cdot \langle \rho, \rho' \rangle$ is bilinear. On the other hand, (iii) implies that $k_{n+1}E \rightarrow k_{2n}E$, $\xi \mapsto \varepsilon \cdot \xi$ is an isomorphism. Therefore the pairing $\langle \cdot, \cdot \rangle$ is bilinear. \square

5. Triple linkage

Let $n \geq 2$. We say k_nE has *triple linkage* if for any three symbols $\sigma_1, \sigma_2, \sigma_3 \in k_nE$ there exist a symbol $\tau \in k_{n-1}E$ and $a_1, a_2, a_3 \in E^\times$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2, 3$. Note that this implies that k_nE is linked.

Theorem 5.1. *Assume that k_nE has triple linkage. Then $\Sigma E^2 = D_E(2^n)$ and $k_{n+1}E = \{-1, \dots, -1\} \cdot k_1E$. In particular, if E is nonreal then $k_{n+1}E = 0$.*

Proof. Let $\langle \cdot, \cdot \rangle : k_nE \times k_nE \rightarrow k_{n+1}E$ be the linkage pairing. Consider three symbols $\sigma_1, \sigma_2, \sigma_3 \in k_nE$. By the hypothesis there exist a symbol $\tau \in k_{n-1}E$ and $a_1, a_2, a_3 \in E^\times$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2, 3$. As $\sigma_1 + \sigma_2 = \tau \cdot \{a_1 a_2\}$ we obtain that

$$\langle \sigma_1 + \sigma_2, \sigma_3 \rangle = \tau \cdot \{a_1 a_2, a_3\} = \tau \cdot \{a_1, a_3\} + \tau \cdot \{a_2, a_3\} = \langle \sigma_1, \sigma_3 \rangle + \langle \sigma_2, \sigma_3 \rangle.$$

Hence the linkage pairing is bilinear and [Theorem 4.4](#) yields the statement. \square

Question 5.2. If k_nE has triple linkage, do then any finite number of symbols in k_nE have a common linkage (by a symbol in $k_{n-1}E$)?

Triple linkage holds for k_nE if E is a \mathcal{C}_n -field, in the terms of Tsen–Lang theory (see [\[Pfister 1995, Chapter 5\]](#)). This is a direct consequence of the next statement. For $n = 1$ and $|\mathcal{S}| = 3$ the statement corresponds to [\[Sivatski 2014, Proposition 9\]](#).

Proposition 5.3. *Assume that there exists a finite system \mathcal{S} of nonzero symbols in k_nE that do not have a common linkage. Then there exists an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms in $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables over E .*

Proof. Let $m \in \mathbb{N}$ be as large as possible such that there exist $a_1, \dots, a_m \in E^\times$ for which the symbol $\{a_1, \dots, a_m\}$ factors every element of \mathcal{S} . By the hypothesis, $m \leq n - 2$. We set $\pi = \langle \langle a_1, \dots, a_m \rangle \rangle$. Using [Theorem 2.1](#) and the one-to-one

correspondence between Pfister forms and symbols, we choose for $\sigma \in \mathcal{S}$ an $(n-m)$ -fold Pfister form ρ_σ over E such that σ corresponds to the n -fold Pfister form $\pi \otimes \rho_\sigma$ over E , and we denote by ρ'_σ the pure part of ρ_σ . Note that $\dim(\pi \otimes \rho'_\sigma) = 2^n - 2^m$ for any $\sigma \in \mathcal{S}$. By [Theorem 2.1](#) and by the maximality of m , there exists no element $c \in E^\times$ such that $-c$ is represented by all the forms $\pi \otimes \rho'_\sigma$ with $\sigma \in \mathcal{S}$. We fix $\sigma_0 \in \mathcal{S}$ and set $\mathcal{S}' = \mathcal{S} \setminus \{\sigma_0\}$. Considering each of the forms $\pi \otimes \rho'_\sigma$ for $\sigma \in \mathcal{S}$ with its own variables, we obtain an anisotropic system of quadratic forms $(\pi \otimes \rho'_{\sigma_0} - \pi \otimes \rho'_\sigma)_{\sigma \in \mathcal{S}'}$ in $|\mathcal{S}| \cdot (2^n - 2^m)$ variables over E . If $m < n - 2$ we may substitute zero for some of these variables. So in any case we obtain an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms over E in exactly $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables. \square

Let $\tilde{u}(E)$ denote the *Hasse number of E* , which is defined as the supremum in $\mathbb{N} \cup \{\infty\}$ on the dimension of anisotropic totally indefinite quadratic forms over E ; see [[Pfister 1995](#), Chapter 8, Section 3]. The study of this invariant was initiated in [[Elman et al. 1973](#)], and the notation was introduced in [[Elman 1977](#)]. The definition of the Hasse number captures one of several possibilities to study bounds on the dimension of anisotropic quadratic forms in a meaningful way without restriction to nonreal fields. The results below have their main interest in the case where E is nonreal, and in this case $\tilde{u}(E)$ is the usual u -invariant; see [[Pfister 1995](#), Chapter 8; [Elman et al. 2008](#), Chapter VI].

Corollary 5.4. *If $\tilde{u}(E(t)) \leq 2^{n+1}$ then $k_n E$ has triple linkage.*

Proof. Suppose first that E is real and $\tilde{u}(E(t)) < \infty$. It follows by [[Elman et al. 1973](#), Theorem I] that E is hereditarily euclidean. Hence $k_n E \simeq \mathbb{Z}/2\mathbb{Z}$, whereby $k_n E$ has triple linkage.

Assume now that E is nonreal and $\tilde{u}(E(t)) \leq 2^{n+1} < 9 \cdot 2^{n-2}$. By the Amer-Brumer theorem [[Pfister 1995](#), Chapter 9, Proposition 1.10], it follows that every pair of quadratic forms in $9 \cdot 2^{n-2}$ variables over E is isotropic. Hence $k_n E$ has triple linkage, by [Proposition 5.3](#). \square

The next example shows that the converse to the statement in [Corollary 5.4](#) does not hold.

Example 5.5. Let E_0 be a quadratically closed field of characteristic not 2 having a finite field extension of even degree. (One can for example take E_0 as the quadratic closure of \mathbb{Q} : any polynomial over \mathbb{Q} having as Galois group a dihedral group of order $2m$ for an odd positive integer m will have as splitting field over E_0 an extension of order $2m$. See also the discussion of finite extensions of quadratically closed fields in [[Lam 2005](#), Chapter VII, §7].) It follows from this choice of E_0 that $\tilde{u}(E_0) = 1$ while $\tilde{u}(F_0) \geq 2$ for some finite separable extension F_0/E_0 . Thus $k_1 F_0 \neq 0$. Consider the fields of iterated power series in n variables

$$E = E_0((u_1)) \cdots ((u_n)) \quad \text{and} \quad F = F_0((u_1)) \cdots ((u_n)).$$

We obtain $\tilde{u}(E) = 2^n$ and $k_{n+1}F \neq 0$. Since F/E is a finite separable extension, F is the residue field of a discrete valuation on $E(t)$. We conclude that $k_{n+2}E(t) \neq 0$. After translation to Pfister forms via [Elman et al. 2008, Theorem 6.20], the Arason–Pfister Hauptsatz [Elman et al. 2008, Theorem 6.18] yields $\tilde{u}(E(t)) \geq 2^{n+2}$.

Note that $\{u_1, \dots, u_n\}$ is the only nonzero symbol in $k_n E$. In particular $k_n E$ has triple linkage. Finally, $k_n F$ does not have triple linkage, for $k_{n+1}F \neq 0$.

The following reformulates and enhances [Elman et al. 1973, Theorem G]. The notation $l^n E$ refers to the n -th power of the fundamental ideal lE in the Witt ring (of symmetric bilinear forms) of E ; recall that $l^n E$ is additively generated by the classes of the n -fold Pfister forms over E .

Theorem 5.6 (Elman–Lam–Prestel). *Assume that $\text{char}(E) \neq 2$. The following are equivalent:*

- (i) $\tilde{u}(E) \leq 4$.
- (ii) E is linked and $l^3 E$ is torsion-free.
- (iii) E is linked and the linkage pairing $k_2 E \times k_2 E \rightarrow k_3 E$ is bilinear.
- (iv) $\Sigma E^2 = D_E(4)$ and $k_3 E = \{-1, -1\} \cdot k_1 E$.
- (v) $k_2 E$ is linked and either $k_3 E = 0$, or E is real and $c \mapsto \{-1, -1, c\}$ defines an isomorphism $E^\times / \Sigma E^2 \rightarrow k_3 E$.

Proof. Conditions (iii)–(v) are equivalent by Theorem 4.4. The equivalence of (i) and (ii) is shown in [Elman et al. 1973, Theorem G] and in [Elman 1977, Theorem 4.7]. By [Elman and Lam 1973, Corollary 2.9], (ii) implies that $l^3 E = 4 \cdot lE$. Therefore, the equivalence of (ii) and (iv) follows using Theorem 2.2. \square

Corollary 5.7. *Assume that $\text{char}(E) \neq 2$ and that $k_2 E$ has triple linkage. Then $\tilde{u}(E) \leq 4$.*

Proof. This follows from Theorem 5.1 together with Theorem 5.6. \square

In the case where E is nonreal, one can show the converse of Corollary 5.7 by using the following statement, which is a direct consequence of a deep result of Peyre [1995, Proposition 6.1] combined with an observation by Sivatski [2014, Corollary 11]. This was pointed out to the author of the present article by Adam Chapman and David Leep.

Proposition 5.8 (Peyre–Sivatski). *Assume that $\text{char}(E) \neq 2$. Let \mathcal{H} be a subgroup of $k_2 E$ with $|\mathcal{H}| \leq 8$. Assume that every element of \mathcal{H} is a symbol and that $l^3 E = 0$. Then there exists $a \in E^\times$ such that for every $\sigma \in \mathcal{H}$ one has $\sigma = \{a, b_\sigma\}$ for some $b_\sigma \in E^\times$.*

Proof. By the hypothesis, every $\sigma \in \mathcal{H}$ corresponds to an E -quaternion algebra. Let $\sigma_1, \sigma_2, \sigma_3 \in k_2 E$ be three symbols that generate \mathcal{H} and let Q_1, Q_2, Q_3 denote the

corresponding quaternion algebras. Since $l^3 E = 0$ we have $H^3(E, \mathbb{Z}/\mathbb{Q}(2)) = 0$. It follows by [Peyre 1995, Proposition 6.1] that there exists a field extension F/E with $[F : E] = 2m$ for an odd integer m and such that $(Q_i)_F$ is split for $i = 1, 2, 3$. Then, by [Sivatski 2014, Corollary 11], Q_1, Q_2, Q_3 have a common slot $a \in E^\times$. It follows that any $\sigma \in \mathcal{H}$ is of the form $\sigma = \{a, b_\sigma\}$ with $b_\sigma \in E^\times$. \square

In [Sivatski 2014], Sivatski seems to be unaware that he provides a proof of Proposition 5.8. In [Sivatski 2014, Corollary 12] he comes to a closely related conclusion, but at the end of his article he asks whether fields of cohomological 2-dimension 2 satisfy the conclusion stated in Proposition 5.8.

Corollary 5.9. *Assume that E is nonreal with $\text{char}(E) \neq 2$. Then $\tilde{u}(E) \leq 4$ if and only if $k_2 E$ has triple linkage.*

Proof. One of the implications is Corollary 5.7, while the converse follows from Proposition 5.8. \square

The hypothesis in Proposition 5.8 on E can be weakened. Instead of assuming $l^3 E = 0$, which requires E to be nonreal, it is sufficient to assume that $l^3 E$ is torsion-free and that E has the so-called *ED-property* about field orderings introduced in [Prestel and Ware 1979]. This can be proven by using algebras with involution and skew-hermitian forms over quaternion algebras. In this way Proposition 5.8 is recovered with a different proof, which in particular is independent of the algebraic geometry behind Peyre's result [1995, Proposition 6.1].

Since $\tilde{u}(E) < \infty$ implies that E satisfies the *ED-property*, the generalisation carries over to Corollary 5.9 and makes the condition that E is nonreal superfluous: if $\text{char}(E) \neq 2$, then $\tilde{u}(E) \leq 4$ if and only if $k_2 E$ has triple linkage.

The author is planning to give details on this argument in a future article.

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